Cut-off effects in gradient flow observables

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HMI meeting

Scattering from the Lattice

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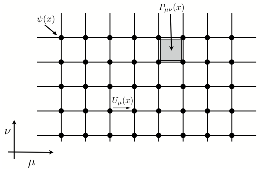
OUTLINE

INTRO: LATTICE QCD AND CUTOFF EFFECTS



LQCD

- is a regularised version of a QFT

  fundamental variables $A_\mu(x) \to U_\mu(x)$

- It allows one to do non-perturbative computations from first principles using stochastic methods, i.e. Monte Carlo simulations

$$\langle O \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}U \, O(U) e^{-S_G[U]} \xrightarrow{\text{numerical computation}} \langle O \rangle \approx \frac{1}{N} \sum_{i=1}^{N} O(\{U\}_i)$$

- Minimising cut-off effects in order to make reliable extrapolations of numerical data to the continuum limit

GRADIENT FLOW IN THE CONTINUUM

- Gradient flow observables have many interesting applications because they are easy to measure on the lattice with high statistical precision. We
  1. define $B_\mu(x,t)$
  2. built observables with the field $B_\mu(x,t)$
  3. consider expectation value of these observables.

- The gradient flow is defined by a mapping $A_\mu(x) \to B_\mu(x,t)$

$$\frac{dB_\mu(x,t)}{dt} = D_\nu G_{\nu\mu}(x,t) \quad \sim \left(-\frac{\delta S_{YM}[B]}{\delta B_\mu}\right)$$

$$B_\mu(x,t=0) = A_\mu(x)$$

defined for $t \geqslant 0$, where

- $B_\mu(x,t)$ is a new gauge field depending on the flow time
- differentiation with respect to flow time t
- $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]$
- $D_\mu = \partial_\mu + [B_\mu, \cdot]$
- A_μ the fundamental gauge field in QCD

Gradient flow equation: first order solution

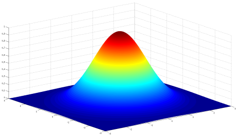
- Interpretation from perturbation theory
  - $$B_\mu(x,t) = \sum_n B_{\mu,n}(x,t) g_0^n$$

    At leading order in g_0 , after gauge fixing, we get:

    $$\frac{\partial B_\mu}{\partial t} = \partial^2 B_\mu + \text{ non linear terms}$$
  - the flow equation is the heat equation with solution

    $$B_\mu(x,t) = \int d^D y\, A_\mu(y)\, K_t(x-y) + \text{ non linear terms}$$
  - where

    $$K_t(x) = \frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{\frac{D}{2}}}.$$



  - smearing radius $\sqrt{2Dt}$

[M. Lüscher 2010]

- It is a smoothing process (smearing of gauge links known in LQCD)

[C. Morningstar M. Peardon 2003]

DEFINITION OF A GRADIENT FLOW OBSERVABLE

- The easiest gauge invariant object we can define is the action density

$$\boldsymbol{E(x,t) = -\frac{1}{2}\,\mathrm{tr}\,\{G_{\mu\nu}(x,t)G_{\mu\nu}(x,t)\}}$$

- significant advantage: at $t > 0$ this observable is renormalised ! (unlike $E(x,0)$ which has divergences)
- In perturbation theory

$$\langle E(x,t)\rangle = \frac{3}{16\pi^2 t^2}(g^2_{\overline{MS}} + O(g^4_{\overline{MS}}))$$

[M. Lüscher 2010]

- Non-perturbatively

$$\langle E(x,t)\rangle = \frac{1}{4\mathcal{Z}}\int \mathcal{D}A_\mu\, G^a_{\mu\nu}(x,t)G^a_{\mu\nu}(x,t)\, e^{-S[A]}$$

- Non-perturbative definition of the coupling

$$\bar{g}^2_{GF} = \mathcal{N}^{-1}t^2\langle E(x,t)\rangle$$

$\mathcal{N}$ normalisation of the coupling $\bar{g}^2_{GF} = g_0^2 + O(g_0^4)$

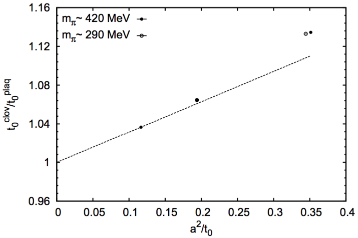
USES OF THE GRADIENT FLOW OBSERVABLES

- computation of the **coupling** and quark masses
  [M. Dalla Brida, P. Fritzsch, T. Korzec, A. Ramos, S. Sint, R. Sommer 2018]
  [A. Hasenfratz 2014]
- definition of the energy momentum tensor
  [F. Capponi, L. Del Debbio, A. Patella, A. Rago 2016]
- small flow time expansion
  [N. Husung, M. Koren, P. Krah, and R. Sommer 2017] **[H. Suzuki 2015]**
- topological susceptibility
  [M. Cè, M. García Vera, L. Giusti, S. Schaefer 2016]
- scale setting $t^2\langle E(t)\rangle|_{t=t_0} = 0.3$
  [M. Lüscher 2010]

  $\langle E^{clov}(t)\rangle$ or $\langle E^{pl}(t)\rangle$

Advantage: avoiding many renormalisation problems and having higher statistical precision

Problem to solve: large cutoff effects



[M. Bruno et al. 2016]

LATTICE DISCRETISATIONS AND SYMANZIK IMPROVEMENT

- Different discretisations on the lattice correspond to the same quantity in the continuum (i.e. taking the limit $a \to 0$)

- $\Rightarrow$ use the universality of continuum limit to minimise the lattice artifact (for ex. the action on the lattice is not unique)

- A systematic way to build improved quantities is the so-called Symanzik improvement program: adding local counterterms to the action S

$$S_{eff} = S_0 + a^2 S_2 + \ldots \qquad \text{(pure gauge)}$$

  and the same procedure applies to local composite fields ϕ

$$O_{eff} = O_0 + a^2 O_2 + \ldots$$

  $\Rightarrow \langle O \rangle^{lat}$ in such a way that the leading cutoff effects are eliminated in all observables **[K. Symanzik 1983]**

- Balance between the complexity of the expression and the behaviour in the limit $a \to 0$

SOURCES OF CUTOFF EFFECTS AND IMPROVEMENT

- The theory with the flowed field $B_\mu(x,t)$ is not local and Symanzik improvement for a LOCAL theory $\Rightarrow$ reformulate the theory in $4+1$ dimensions to apply standard machinery for renormalisation and power counting.
- Removing cutoff effects coming from the sources:
  1. action
  2. gradient flow equation
  3. observable
  4. **single additional counterterm** compared to the pure gauge theory in 4 dim

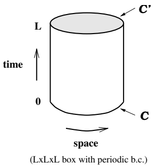
     It corresponds to a **modified initial condition** for the flow equation

     $$V_\mu(t,x)|_{t=0} = e^{c_b g_0^2 \partial_{x,\mu} S_g[U]} U_\mu(x)$$

     this introduces the c_b dependence we want to study numerically

[A. Ramos and S. Sint 2015]

- On the lattice $V = L^3 T$ with Schrödinger Functional bc's



[M. Luscher, R. Narayanan, P. Weisz, and U. Wolff 1992]

Action and gradient flow equation on the lattice

- 1. discretisation of the action
  → Wilson action → improved LW action

$$S_W[U] = \frac{1}{g_0^2}\sum_p \{1 - U(p)\} \qquad S[U] = \frac{1}{g_0^2}\sum_{k=0}^{1} c_k \sum_{C\in S_k} w(C)\, tr\{1 - U(C)\}$$

- 2. discretisation of the flow equation
  → Wilson flow

$$\partial_t V_\mu(t,x) = -\partial_{x,\mu}(g_0^2 S_W[V])V_\mu(t,x), \qquad V_\mu(0,x) = U_\mu(x)$$

→ improved Zeuthen flow

$$a^2\partial_t V_\mu(t,x) = -g_0^2(1+\frac{a^2}{12}D_\mu D_\mu^*)\partial_{x,\mu}(S_{LW}[V])V_\mu(t,x), \qquad V_\mu(0,x) = U_\mu(x)$$

$\partial_{x,\mu}$ differential operator with respect to the link variable $V_\mu(t,x) = exp\{aB_\mu\}$

COMPONENTS OF THE OBSERVABLE

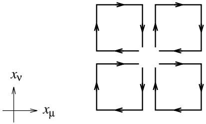
- 3. discretisation of the observable

$$\text{continuum} \qquad E(x,t) = -\frac{1}{2}\,\mathrm{tr}\,\{G_{\mu\nu}(x,t)G_{\mu\nu}(x,t)\}$$

→clover definition of the field strength tensor

$$E^{cl}(t,x) = -\frac{1}{2}\sum_{\mu\nu}\mathrm{tr}\,\{G^{cl}_{\mu\nu}G^{cl}_{\mu\nu}\}$$

$$\text{at L.O.} \quad G^{cl}_{\mu\nu} = \tilde{\partial}_\mu(1-\frac{1}{2}a\,\partial^*_\nu)B_\nu - \tilde{\partial}_\nu(1-\frac{1}{2}a\,\partial^*_\mu)B_\mu$$



→plaquette definition of the field strength tensor

$$E^{pl}(t,x) = -\frac{a^{-4}}{2}\sum_{\mu\nu}[\mathrm{tr}(P_{\mu\nu}(t,x)+P_{\mu\nu}(t,x)^\dagger) - 2N]$$

$$\text{at L.O.} \quad G^{pl}_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

lattice derivatives: $\tilde{\partial}$ symm combination, ∂^* backward, ∂ forward

OBSERVABLE ON THE LATTICE

Splitting colour magnetic (spatial) and colour electric (mixed) components

$$E(t,x) = E_{MAG}(t,x) + E_{EL}(t,x)$$

$$E_{MAG}(t,x) = -\frac{1}{2}\sum_{k,l}\mathrm{tr}(G_{kl}G_{kl})$$

$$E_{EL}(t,x) = -\frac{1}{2}\sum_{k}\left(\mathrm{tr}(G_{0k}G_{0k}) + \mathrm{tr}(G_{0k}G_{0k})\right)$$

→ improved observables

- magnetic $E^{imp}_{MAG} = \frac{4}{3}E^{pl}_{MAG} + \frac{1}{3}E^{cl}_{MAG}$ **[A. Ramos and S. Sint 2015]**
- electric $E^{imp}_{EL} = \tilde{E}_{EL} - \frac{1}{6}a^2\partial_0^2\tilde{E}_{EL}$ where $\tilde{E}_{EL} = \frac{4}{3}E^{pl-sym}_{EL} + \frac{1}{3}E^{cl}_{EL}$

4. What about c_b ?

PERTURBATIVE MODEL: COUPLING AT ORDER g_0^2

- Non-perturbative definition of the coupling $\bar{g}^2_{GF} = \mathcal{N}^{-1} t^2 \langle E(x_0, t)\rangle$
- Perturbative expansion of $\langle E(x_0, t)\rangle$

$$\langle E(x_0, t)\rangle = E_0 g_0^2 + O(g_0^4)$$

$$E_0 = \frac{g_0^2}{2} \langle \partial_\mu B^a_{\nu,1} \partial_\mu B^a_{\nu,1} - \partial_\mu B^a_{\nu,1} \partial_\nu B^a_{\mu,1}\rangle$$

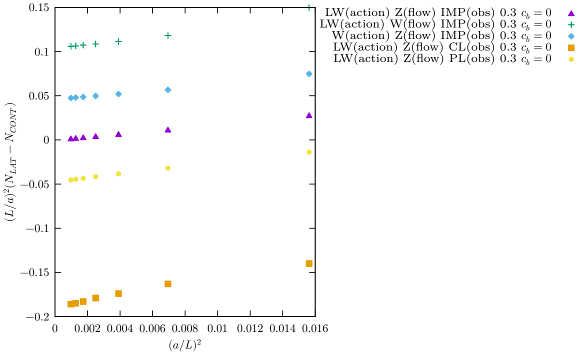
- **numerical computation** of the normalisation factor $\mathcal{N} = t^2 E_0$ at L.O., which means computing the coupling at L.O.

$$t^2 \langle E_{mag}(t, x)\rangle|_{\sqrt{8t}=cL, x_0=\frac{T}{2}} = \mathcal{N}_{LAT}(c, \frac{a}{L}) \bar{g}^2_{GF}(L)$$

[P. Fritzsch A. Ramos 2013]

- study of not only Wilson flow but also improved Zeuthen flow, use of advantageous setup -expectation:
  - $c_b^* = 0$ improved (LW) action - improved (Z) flow - improved (OBS) mag,el
  - behavoiur within the **improvement**: $\mathcal{N}_{LAT} = \mathcal{N}_{CONT} + O(a^4)$

PERTURBATIVE MODEL: COLOUR MAGNETIC COMPONENT



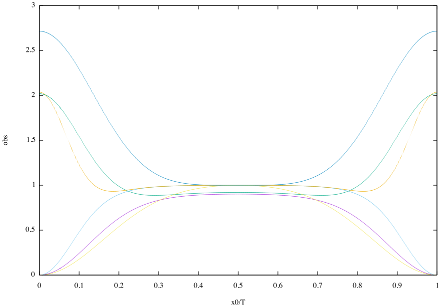
$\mathcal{N}_{LAT}(c, \frac{a}{L}) = t^2 \langle E_{mag}(t,x)\rangle|_{c=0.3, x_0=\frac{T}{2}}$ **LW (action) Z(flow) IMP(obs)** $O(a^2)$ improved

improved (action)- improved (flow)- improved (observable) $\Rightarrow$

$\Rightarrow c_b^* = 0$ realises full $O(a^2)$ improvement

PERTURBATIVE MODEL: WILSON ACTION NOT IMPROVED

- Wilson action, NOT improved ⇒ counterterm basis incomplete!
- Consider the Zeuthen improved flow and a set of 12 improved observables:



$\langle E_{mag}(t, x_0)\rangle$ and $\langle E_{el}(t, x_0)\rangle$

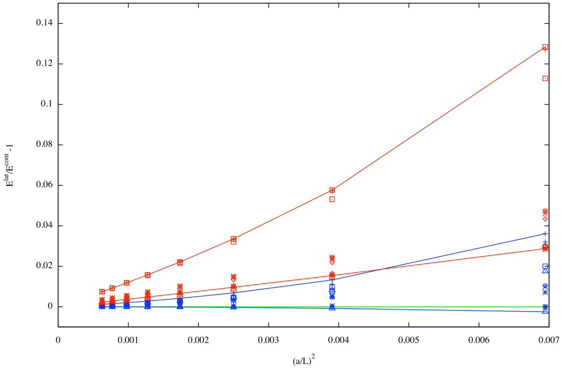
both at $c = 0.2, 0.3, 0.4$

where $c = \frac{\sqrt{8t}}{L} = \frac{\text{spreading radius}}{\text{lattice size}}$

and at $x_0 = \frac{T}{2}, \frac{T}{4}$

- Let's fix $c^*_{b(mag@0.3)}$ that minimise cutoff effects in one observable E_{mag} at $c = 0.3$.
  Does this $c^*_{b(mag@0.3)}$ **reduce the $O(a^2)$ effects in the other flow observables?**

PERTURBATIVE MODEL: $c^*_{b(mag@0.3)}$ EFFECT ON ALL OTHER OBSERVABLES



W(act)-Z(flow)-IMP(obs) $c^*_{b(mag@0.3)}$ by definition cancel cutoff effects in $E_{mag@0.3}$, green line. It reduces the $O(a^2)$ effetcs in all other observables: comparing $c_b = 0$ (red) with $c^*_{b(mag@0.3)}$ (blue) we see both the spread and the slope are reduced.

Does this work beyond perturbation theory?

NON-PERTURBATIVE STUDY

- Analysis set of data **[data set obtained from A. Ramos and M. Dalla Brida]**
  The GF coupling is sensitive to different topological charge sectors
  $Q(t) = \frac{1}{16\pi^2} \sum_x G_{\mu\nu}(x,t)\tilde{G}_{\mu\nu}(x,t)$

  $\Rightarrow$ Modified definition of the GF coupling

$$\bar{g}^2_{GF,0} = \mathcal{N}^{-1} t^2 \frac{\langle E(t)\delta_{Q,0}\rangle}{\langle \delta_{Q,0}\rangle}\bigg|_{t=\frac{c^2L^2}{8}}$$

[P. Fritzsch A. Ramos and F. Stollenwerk 2013]

- Extrapolations to the continuum of the observables at different values of $c = \frac{8t}{L} = 0.2, 0.3, 0.4$
  - $E_{MAG}(t,x)$, $E_{EL}(t,x)$, adding more observables $\partial_0^2 T^2 E_{MAG}(t,x)$, $\partial_0^2 T^2 E_{EL}(t,x)$

- Tuning c_b coefficient in the simulations...

Summary and ongoing studies

- Gradient flow observables have many useful applications because they are finite at $t > 0$, after the usual renormalisation of bare parameters. (One of the application is the study of the strong coupling)
- The drawback is that they have large cutoff effects. We study how to remove/minimise using Symanzik improvement program. In particular we test numerically the 4th source of them tuning the c_b **parameter**
- Perturbative study: we confirm numerically
  - the expected behaviour within the **improvement** for both magnetic $E_{MAG}(t,x)$ and electric $E_{EL}(t,x)$ components: $\mathcal{N}_{LAT} = \mathcal{N}_{CONT} + O(a^4)$
  - the theoretical expectation for the c_b **value** at L.O. with improved quantities
  - we study what the effect of $c^*_{b(mag@0.3)}$ on other observables when using Wilson action and we see reduced cutoff effects!
- Non-perturbative study [currently running...]
  - Analysis of both magnetic and electric components for $c_b = 0$ and extrapolation to the **continuum limit**
  - We are simulating other values c_b to see if the hypothesis of **using** $c^*_{b(fixed\ obs)}$ **in other observable to realise the improvement** works **beyond perturbation theory**.

Thank You!