meson resonances from lattice QCD

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Scattering processes and resonances from lattice QCD

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The vast majority of hadrons observed in nature are not stable under the strong interaction; rather they are resonances whose existence is deduced from enhancements in the energy dependence of scattering amplitudes. The study of hadron resonances offers a window into the workings of quantum chromodynamics (QCD) in the low-energy nonperturbative region, and in addition many probes of the limits of the electroweak sector of the standard model consider processes which feature hadron resonances. From a theoretical standpoint, this is a challenging field: the same dynamics that binds quarks and gluons into hadron resonances also controls their decay into lighter hadrons, so a complete approach to QCD is required. Presently, lattice QCD is the only available tool that provides the required nonperturbative evaluation of hadron observables. This article reviews progress in the study of few-hadron reactions in which resonances and bound states appear using lattice QCD techniques. The leading approach is described that takes advantage of the periodic finite spatial volume used in lattice QCD calculations to extract scattering amplitudes from the discrete spectrum of QCD eigenstates in a box. An explanation is given of how from explicit lattice QCD calculations one can rigorously garner information about a variety of resonance properties, including their masses, widths, decay couplings, and form factors. The challenges which currently limit the field are discussed along with the steps being taken to resolve them.

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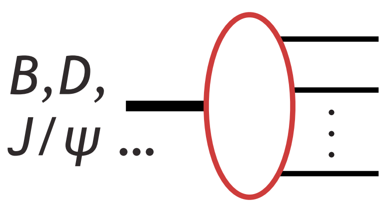
producing meson resonances

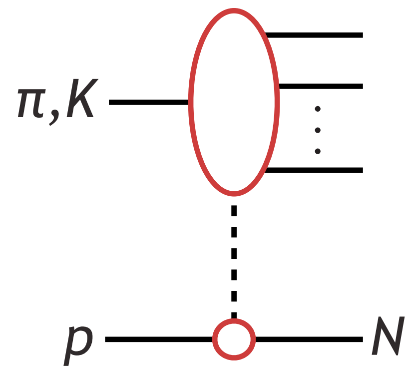
some example processes:

heavy flavour decays



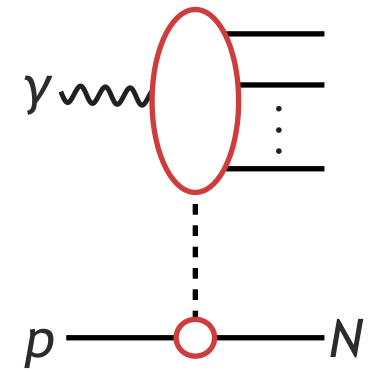
e.g. LHCb

peripheral meson hadroproduction



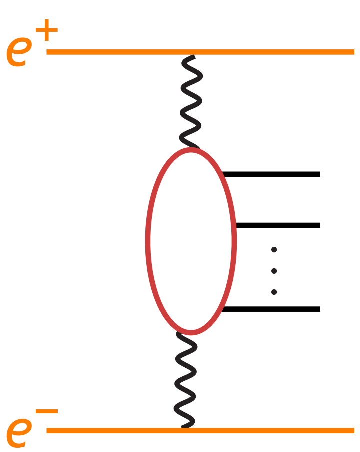
e.g. COMPASS

peripheral meson photoproduction



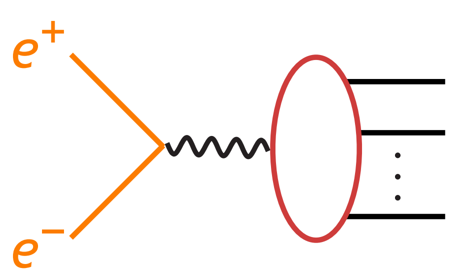
e.g. GlueX

two photon fusion



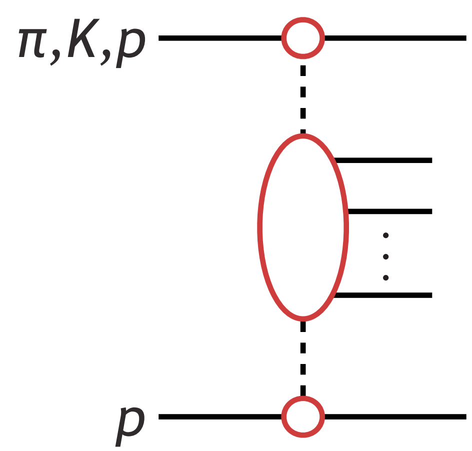
e.g. Belle

e^+e^- annihilation



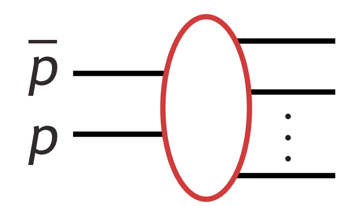
e.g. BES III

central production



e.g. WA102

$p\bar{p}$ annihilation

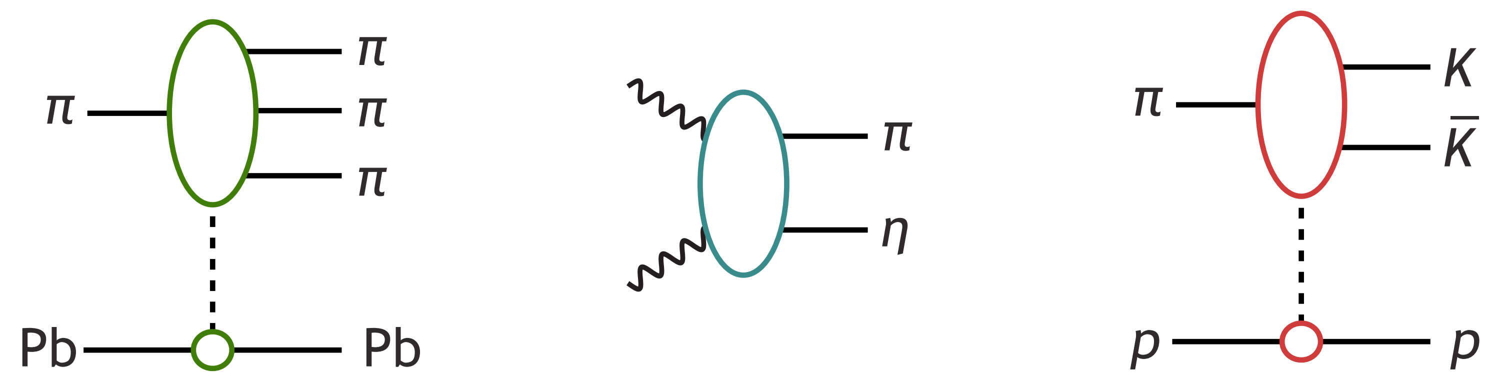


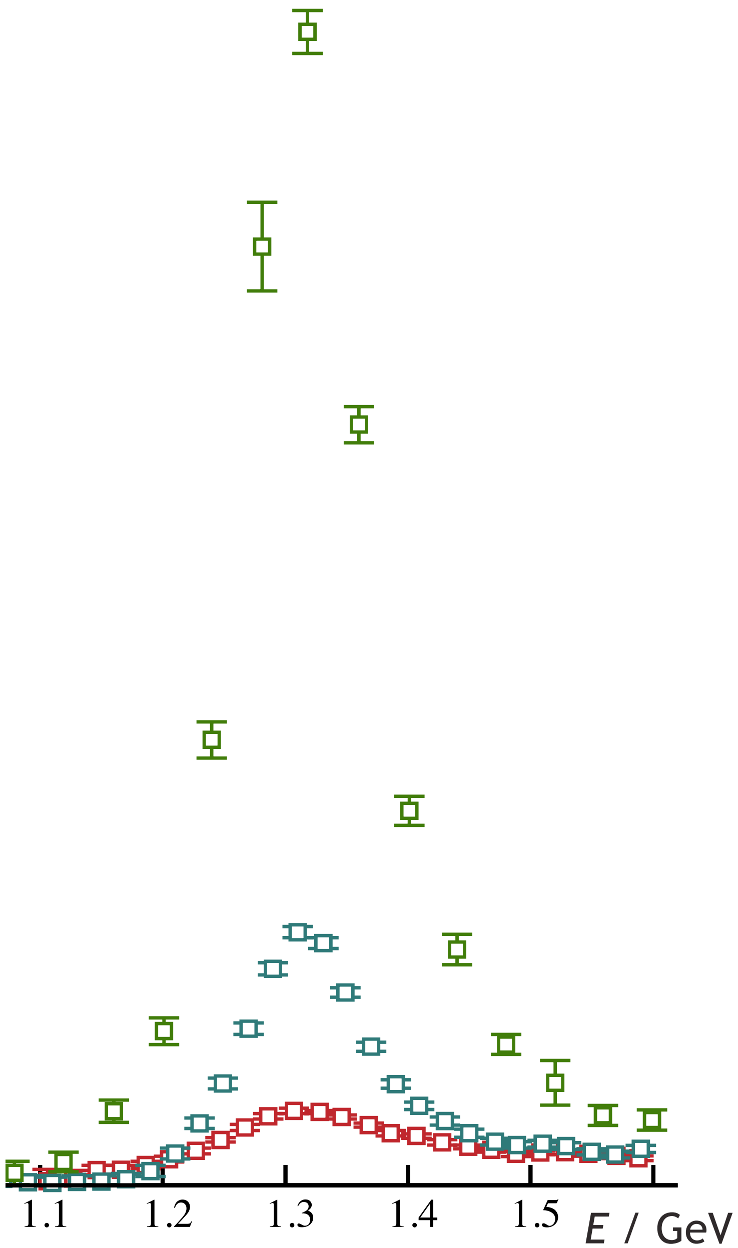
e.g. Crystal Barrel

many decades of accumulated data ...

‘straightforward’ coupled-channel resonances

same ‘bump’ appears in multiple different processes





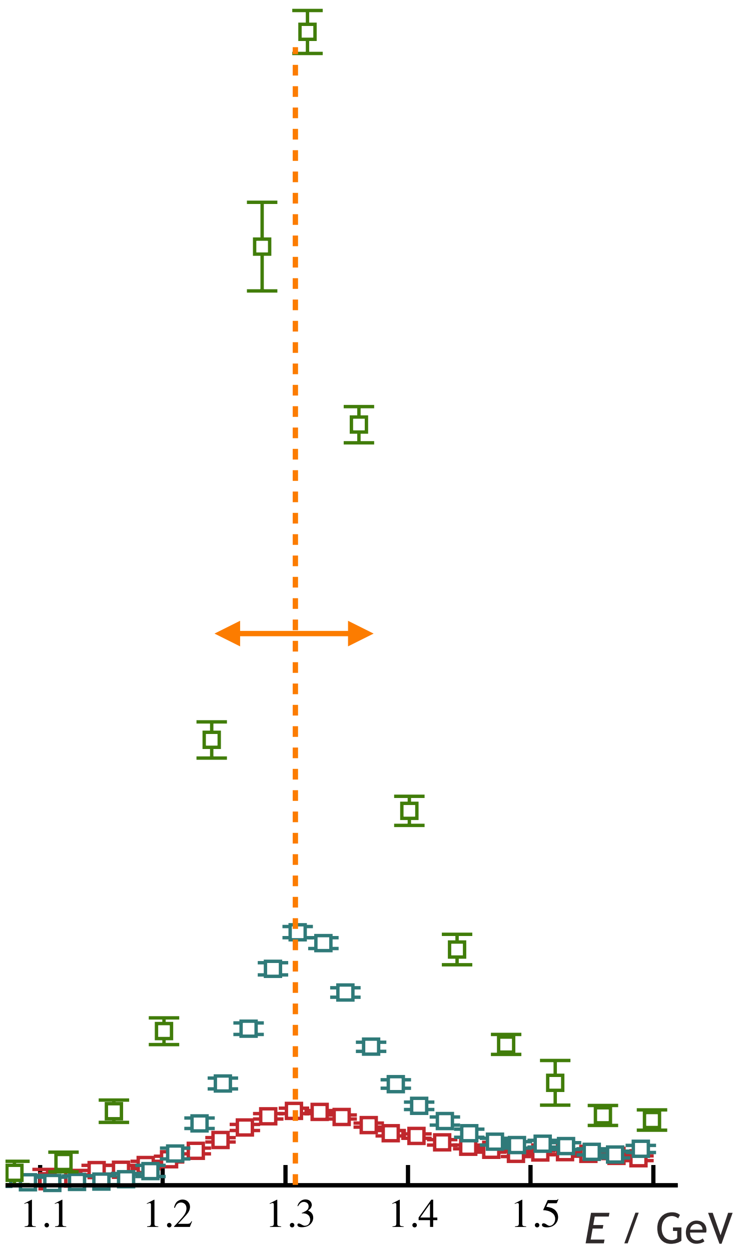
$\pi\,\mathrm{Pb} \to \pi\rho\,\mathrm{Pb}$ COMPASS

$\gamma\gamma \to \pi\eta$ Belle

$\pi p \to K\overline{K}\,p$ CERN SPS

‘straightforward’ coupled-channel resonances

$a_2(1320)$



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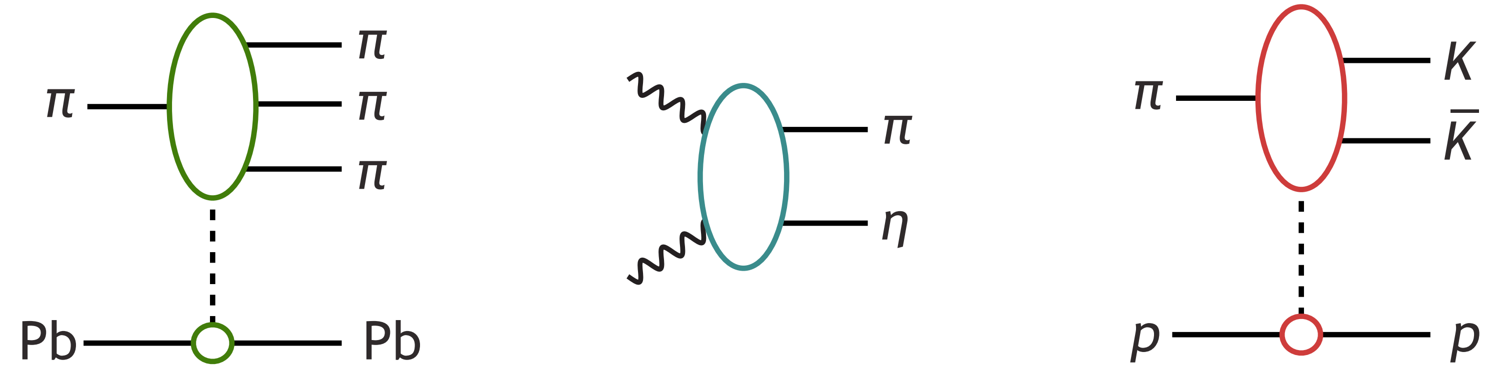
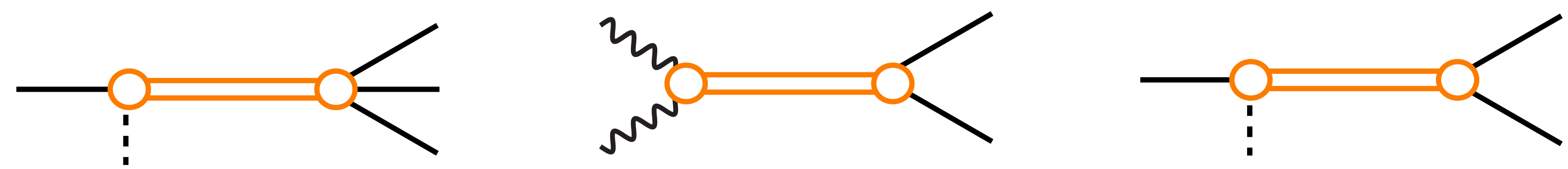
same ‘bump’ appears in multiple different processes …



… due to same **a_2 resonance**



pdg summary entry

$a_2(1320)$ $I^G(J^{PC}) = 1^-(2^{++})$

Mass $m = 1318.3^{+0.5}_{-0.6}$ MeV

Full width $\Gamma = 107 \pm 5$ MeV

$a_2(1320)$ DECAY MODES	Fraction (Γ_i/Γ)
3π	(70.1 ± 2.7) %
$\eta\pi$	(14.5 ± 1.2) %
$\omega\pi\pi$	(10.6 ± 3.2) %
$K\overline{K}$	(4.9 ± 0.8) %
$\eta'(958)\pi$	$(5.5 \pm 0.9) \times 10^{-3}$
$\pi^\pm\gamma$	$(2.91 \pm 0.27) \times 10^{-3}$
$\gamma\gamma$	$(9.4 \pm 0.7) \times 10^{-6}$

the experimental excited meson spectrum

pdg meson listings

LIGHT UNFLAVORED ($S = C = B = 0$)	$I^G(J^{PC})$		$I^G(J^{PC})$
• $\pi^\pm$	$1^-(0^-)$	• $\rho_3(1690)$	$1^+(3^{--})$
• π^0	$1^-(0^{-+})$	• $\rho(1700)$	$1^+(1^{--})$
• η	$0^+(0^{-+})$	$a_2(1700)$	$1^-(2^{++})$
• $f_0(500)$	$0^+(0^{++})$	• $f_0(1710)$	$0^+(0^{++})$
• $\rho(770)$	$1^+(1^{--})$	$\eta(1760)$	$0^+(0^{-+})$
• $\omega(782)$	$0^-(1^{--})$	• $\pi(1800)$	$1^-(0^{-+})$
• $\eta'(958)$	$0^+(0^{-+})$	$f_2(1810)$	$0^+(2^{++})$
• $f_0(980)$	$0^+(0^{++})$	$X(1835)$	$?^?(0^{-+})$
• $a_0(980)$	$1^-(0^{++})$	$X(1840)$	$?^?(?^{??})$
• $\phi(1020)$	$0^-(1^{--})$	$a_1(1420)$	$1^-(1^{++})$
• $h_1(1170)$	$0^-(1^{+-})$	• $\phi_3(1850)$	$0^-(3^{--})$
• $b_1(1235)$	$1^+(1^{+-})$	$\eta_2(1870)$	$0^+(2^{-+})$
• $a_1(1260)$	$1^-(1^{++})$	• $\pi_2(1880)$	$1^-(2^{-+})$
• $f_2(1270)$	$0^+(2^{++})$	$\rho(1900)$	$1^+(1^{--})$
• $f_1(1285)$	$0^+(1^{++})$	$f_2(1910)$	$0^+(2^{++})$
• $\eta(1295)$	$0^+(0^{-+})$	$a_0(1950)$	$1^-(0^{++})$
• $\pi(1300)$	$1^-(0^{-+})$	• $f_2(1950)$	$0^+(2^{++})$
• $a_2(1320)$	$1^-(2^{++})$	$\rho_3(1990)$	$1^+(3^{--})$
• $f_0(1370)$	$0^+(0^{++})$	• $f_2(2010)$	$0^+(2^{++})$
$h_1(1380)$	$?^-(1^{+-})$	$f_0(2020)$	$0^+(0^{++})$
• $\pi_1(1400)$	$1^-(1^{-+})$	• $a_4(2040)$	$1^-(4^{++})$
• $\eta(1405)$	$0^+(0^{-+})$	• $f_4(2050)$	$0^+(4^{++})$
• $f_1(1420)$	$0^+(1^{++})$	$\pi_2(2100)$	$1^-(2^{-+})$
• $\omega(1420)$	$0^-(1^{--})$	$f_0(2100)$	$0^+(0^{++})$
$f_2(1430)$	$0^+(2^{++})$	$f_2(2150)$	$0^+(2^{++})$
• $a_0(1450)$	$1^-(0^{++})$	$\rho(2150)$	$1^+(1^{--})$

STRANGE ($S = \pm1$, $C = B = 0$)	$I(J^P)$
• $K^\pm$	$1/2(0^-)$
• K^0	$1/2(0^-)$
• K^0_S	$1/2(0^-)$
• K^0_L	$1/2(0^-)$
$K^*_0(800)$	$1/2(0^+)$
• $K^*(892)$	$1/2(1^-)$
• $K_1(1270)$	$1/2(1^+)$
• $K_1(1400)$	$1/2(1^+)$
• $K^*(1410)$	$1/2(1^-)$
• $K^*_0(1430)$	$1/2(0^+)$
• $K^*_2(1430)$	$1/2(2^+)$
$K(1460)$	$1/2(0^-)$
$K_2(1580)$	$1/2(2^-)$
$K(1630)$	$1/2(?^?)$
$K_1(1650)$	$1/2(1^+)$
• $K^*(1680)$	$1/2(1^-)$
• $K_2(1770)$	$1/2(2^-)$
• $K^*_3(1780)$	$1/2(3^-)$
• $K_2(1820)$	$1/2(2^-)$
$K(1830)$	$1/2(0^-)$
$K^*_0(1950)$	$1/2(0^+)$
$K^*_2(1980)$	$1/2(2^+)$
• $K^*_4(2045)$	$1/2(4^+)$
$K_2(2250)$	$1/2(2^-)$
$K_3(2320)$	$1/2(3^+)$

CHARMED, STRANGE ($C = S = \pm1$)	$I(J^P)$
• $D^\pm_s$	$0(0^-)$
• $D^{*\pm}_s$	$0(?^?)$
• $D^*_{s0}(2317)^\pm$	$0(0^+)$
• $D_{s1}(2460)^\pm$	$0(1^+)$
• $D_{s1}(2536)^\pm$	$0(1^+)$
• $D_{s2}(2573)$	$0(2^+)$
• $D^*_{s1}(2700)^\pm$	$0(1^-)$
$D^*_{s1}(2860)^\pm$	$0(1^-)$
$D^*_{s3}(2860)^\pm$	$0(3^-)$
$D_{sJ}(3040)^\pm$	$0(?^?)$

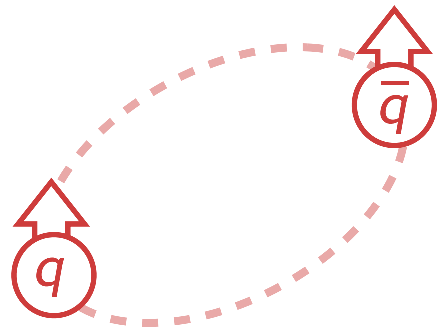
BOTTOM ($B = \pm1$)	
• $B^\pm$	$1/2(0^-)$
• B^0	$1/2(0^-)$
• $B^\pm/B^0$ ADMIXTURE	
• $B^\pm/B^0/B^0_s/b$-baryon ADMIXTURE	
V_{cb} and V_{ub} CKM Matrix Elements	
• B^*	$1/2(1^-)$
• $B_1(5721)^+$	$1/2(1^+)$
• $B_1(5721)^0$	$1/2(1^+)$
$B^*_J(5732)$	$?(?^?)$
• $B^*_2(5747)^+$	$1/2(2^+)$
• $B^*_2(5747)^0$	$1/2(2^+)$

$c\bar{c}$	$I^G(J^{PC})$
• $\eta_c(1S)$	$0^+(0^{-+})$
• $J/\psi(1S)$	$0^-(1^{--})$
• $\chi_{c0}(1P)$	$0^+(0^{++})$
• $\chi_{c1}(1P)$	$0^+(1^{++})$
• $h_c(1P)$	$?^?(1^{+-})$
• $\chi_{c2}(1P)$	$0^+(2^{++})$
• $\eta_c(2S)$	$0^+(0^{-+})$
• $\psi(2S)$	$0^-(1^{--})$
• $\psi(3770)$	$0^-(1^{--})$
• $\psi(3823)$	$?^?(2^{--})$
• $X(3872)$	$0^+(1^{++})$
• $X(3900)$	$1^+(1^{+-})$
• $X(3915)$	$0^+(0/2^{++})$
• $\chi_{c2}(2P)$	$0^+(2^{++})$
$X(3940)$	$?^?(?^{??})$
• $X(4020)$	$1(?^?)$
• $\psi(4040)$	$0^-(1^{--})$
$X(4050)^\pm$	$?(?^?)$
$X(4055)^\pm$	$?(?^?)$
• $X(4140)$	$0^+(1^{++})$
• $\psi(4160)$	$0^-(1^{--})$
$X(4160)$	$?^?(?^{??})$
$X(4200)^\pm$	$?(1^+)$
$X(4230)$	$?^?(1^{--})$
$X(4240)^\pm$	$?^?(0^-)$
$X(4250)^\pm$	$?(?^?)$
• $X(4260)$	$?^?(1^{--})$

pdg.lbl.gov

hadron spectroscopy

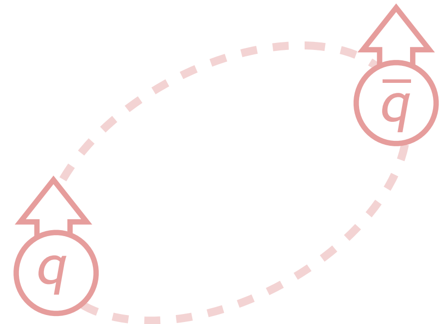
the canonical view of the meson spectrum provided by the $q\bar{q}$ constituent quark model



$^{2S+1}\ell_J$	J^{PC}	light mesons	charmonium
$^1S_0$	0^{-+}	π, η, η'	η_c
$^3S_1$	1^{--}	ρ, ω, ϕ	J/ψ
$^1P_1$	1^{+-}	b_1, h_1	h_c
$^3P_{0,1,2}$	$(0,1,2)^{++}$	a_J, f_J	χ_{cJ}
$^1D_2$	2^{-+}	π_2, η_2	?
$^3D_{1,2,3}$	$(1,2,3)^{--}$	$\rho, \rho_3, \omega, \omega_3 \ldots$	ψ'

gets the gross features of the spectrum right …

but treats excited hadrons as essentially stable the canonical view of the meson spectrum provided by the $q\bar{q}$ constituent quark model



${}^{2S+1}\ell_J$	J^{PC}	light mesons	charmonium
${}^1S_0$	0^{-+}	π, η, η'	η_c
${}^3S_1$	1^{--}	ρ, ω, ϕ	J/ψ
${}^1P_1$	1^{+-}	b_1, h_1	h_c
${}^3P_{0,1,2}$	$(0,1,2)^{++}$	a_J, f_J	χ_{cJ}
${}^1D_2$	2^{-+}	π_2, η_2	?
${}^3D_{1,2,3}$	$(1,2,3)^{--}$	$\rho, \rho_3, \omega, \omega_3 \ldots$	ψ'

gets the gross features of the spectrum right …

but treats excited hadrons as essentially stable

is there more than this ?

why doesn't QCD have meson states where the gluonic field is 'active' ?

glueballs = states where quarks are not required

hybrids = states where quark colour neutralized by gluonic field

lattice QCD can be used to study such speculations rigorously …

hadron spectroscopy — ‘rigorously’

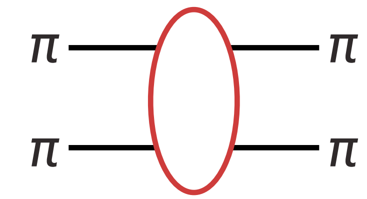
want to study excited hadrons as they really are — rapidly decaying resonances

same dynamics that binds them also causes their decay

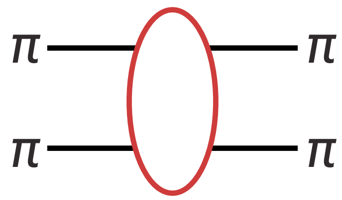
we need to compute **scattering amplitudes** and see if they resonate

start with the simplest case: **elastic scattering** …

e.g.



elastic partial-waves & unitarity



$$\frac{d\sigma}{d\Omega} \propto \left|t(E,\cos\theta)\right|^2$$

elastic scattering amplitude
can be expanded in partial-waves

$$t(E,\cos\theta) = \sum_\ell (2\ell+1)\, t_\ell(E)\, P_\ell(\cos\theta)$$

partial-wave amplitude

resonances appear in
a single partial-wave

ℓ	0	1	2	...
J^P	0^+	1^-	2^+	

conservation of probability
a.k.a **elastic unitarity**

$$\mathrm{Im}\, t_\ell(E) = \rho(E)\left|t_\ell(E)\right|^2 \quad \text{or} \quad \mathrm{Im}\,\frac{1}{t_\ell(E)} = -\rho(E)$$

‘phase-space’ $\rho(E) = \dfrac{2k(E)}{E}$

c.m. momentum $k(E) = \frac{1}{2}\sqrt{E^2 - 4m^2}$

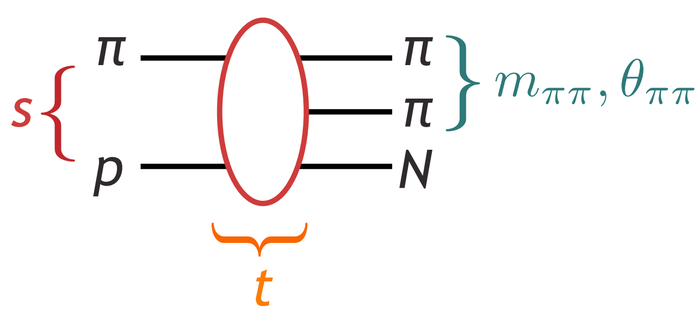
can parameterise elastic scattering
in terms of a single real parameter

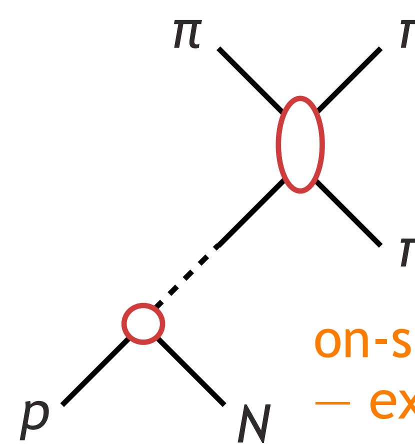
$$t_\ell(E) = \frac{1}{\rho(E)}\, e^{i\delta_\ell(E)} \sin\delta_\ell(E)$$

‘phase-shift’

the simplest case: $\pi\pi$ elastic scattering

extract from charged pion beams on nucleon targets

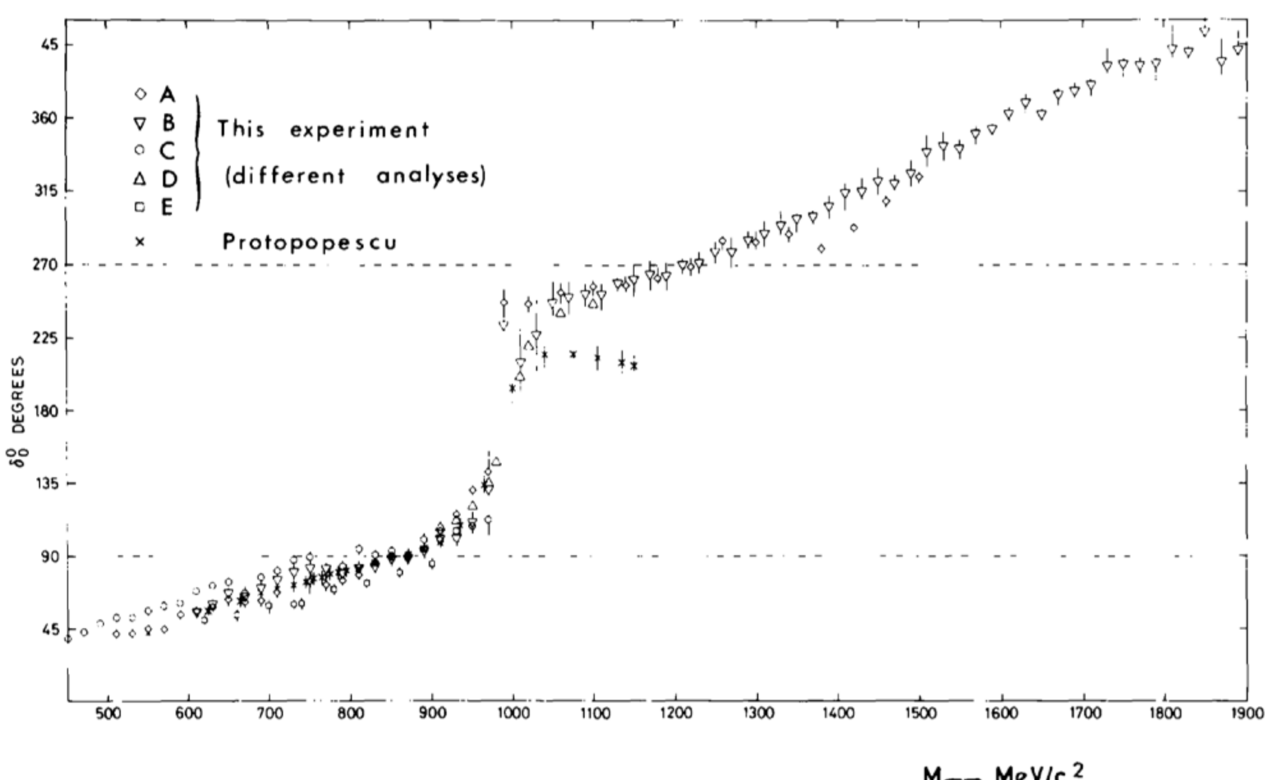




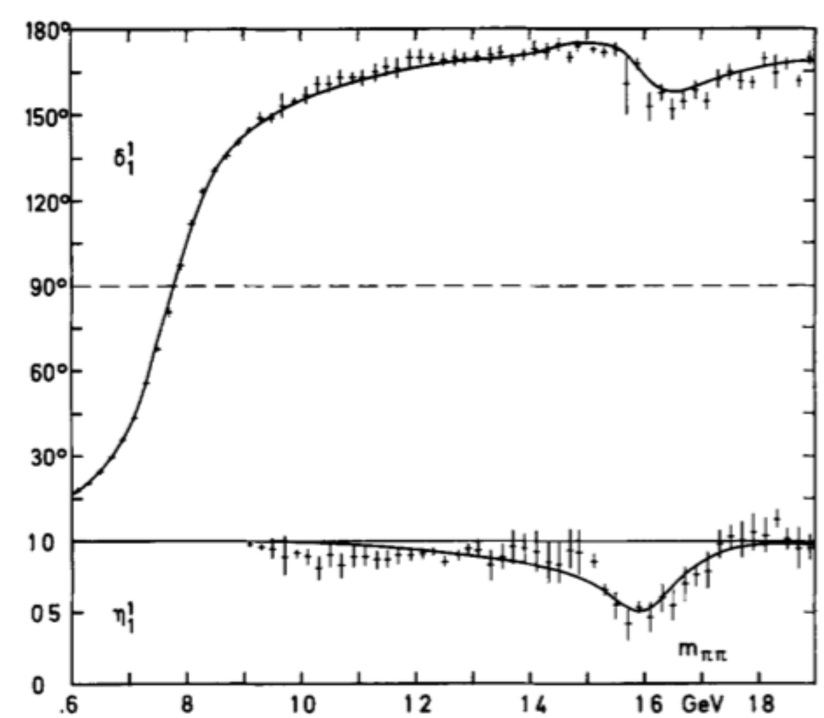
$\pi\pi$ partial-waves
— project $P_\ell(\cos\theta)$

on-shell π exchange
— extrapolate to $t = -m_\pi^2$

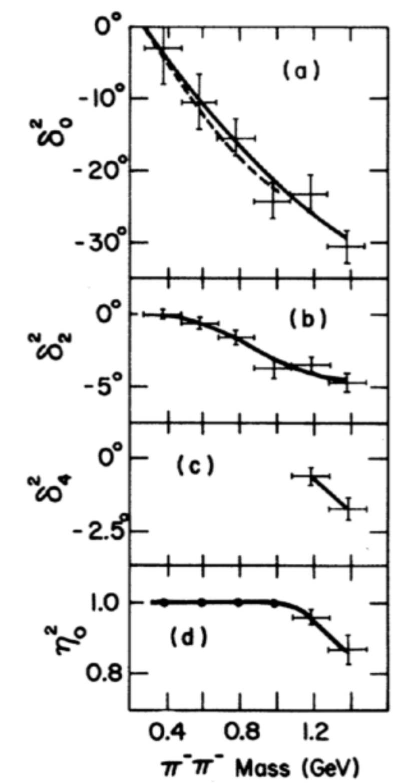
isospin=0	isospin=1	isospin=2



Grayer 1974



Hyams 1973



Cohen 1972

the simplest case: $\pi\pi$ elastic scattering

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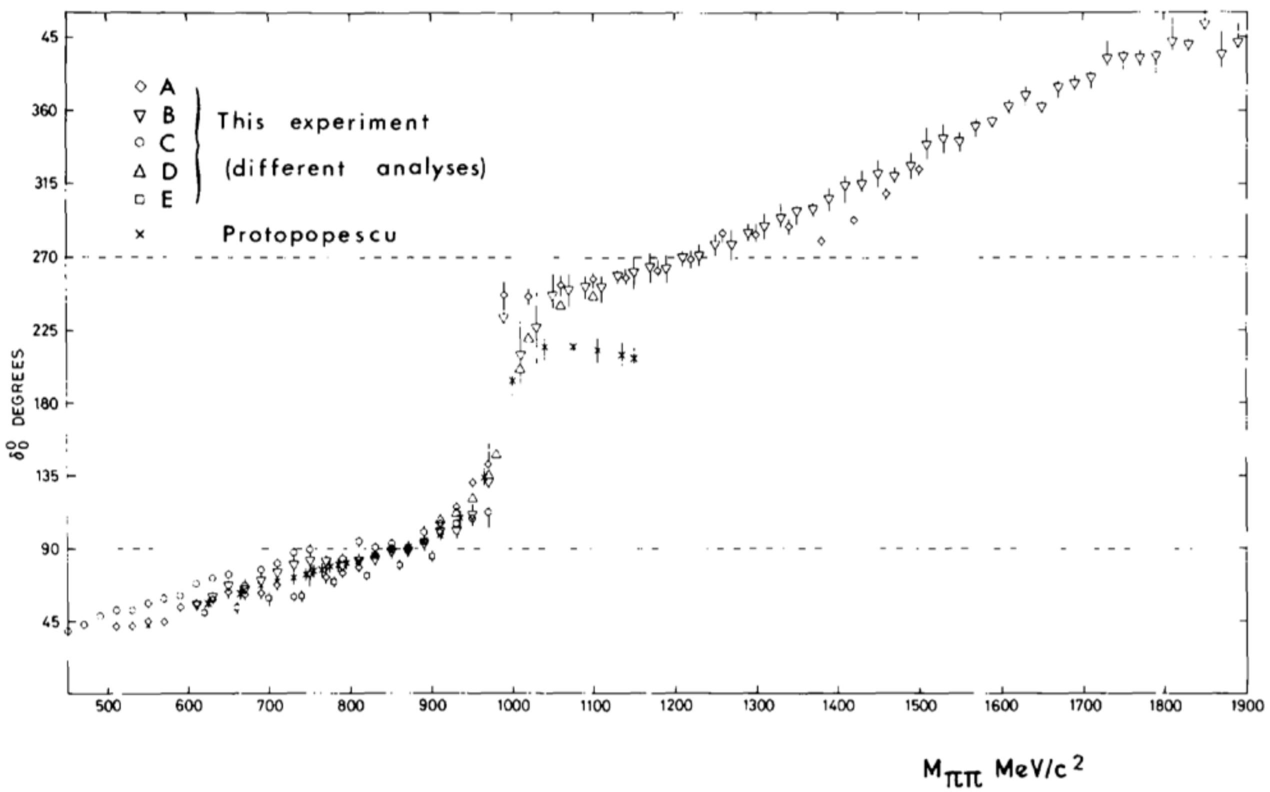

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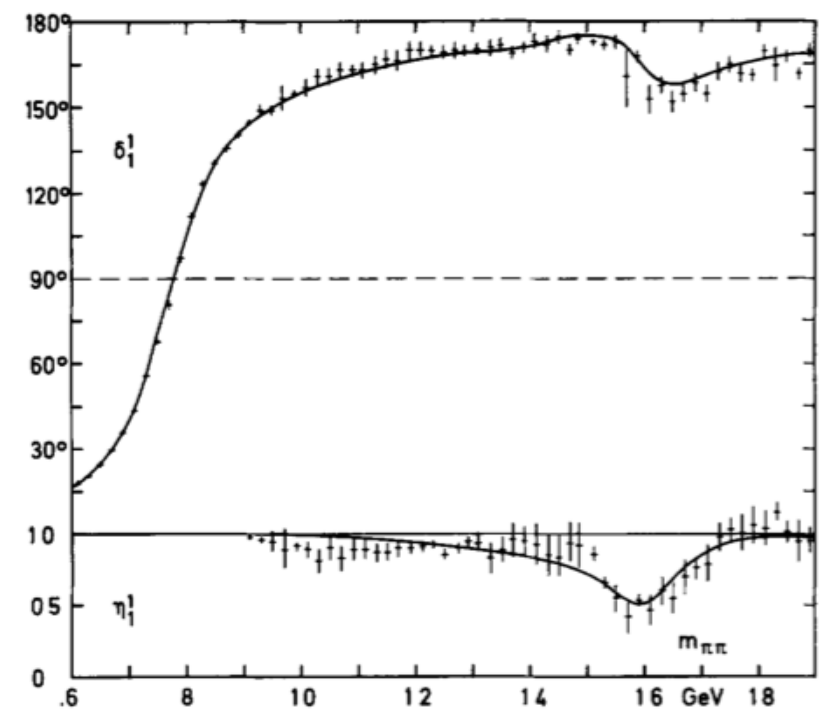

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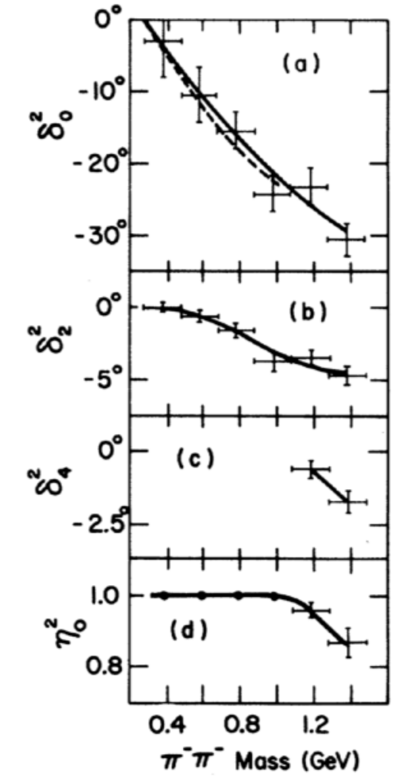


a first target: can a first-principles QCD calculation lead to this kind of behaviour ?

a next target: can we understand these behaviours in terms of resonances ?

an ultimate target: can we understand the quark-gluon make-up of these resonances ?

I'm going to assume you're familiar with the basic idea:

discretize the QCD action in Euclidean space-time i.e. $\bar{\psi}_{\mathbf{x}'t'} \, M_{\mathbf{x}'t',\,\mathbf{x}t}[U] \, \psi_{\mathbf{x}t}$
integrate out the quark fields
sample gauge field configurations according to a probability $\det M[U] \, e^{-S[U]}$

parameters:
– lattice spacing (just assume fine here)
– lattice volume (very important here!)
– quark masses (might not take physical values)

evaluate **correlation functions** on each configuration in the ensemble

energy eigenstates of the QCD Hamiltonian ?

embedded within two-point correlation functions $C_{ij}(t) = \langle 0 | \mathcal{O}_i(t)\, \mathcal{O}_j^\dagger(0) | 0 \rangle$

e.g. we expect the pion to be a QCD eigenstate with $E = m_\pi$ (in the rest frame)

compute $C_{ij}(t) = \langle 0 | \mathcal{O}_i(t)\, \mathcal{O}_j^\dagger(0) | 0 \rangle$

operator with pion quantum numbers
(color singlet, isospin=1, $J^P = 0^-$)
constructed from quark, gluon fields

$$\mathcal{O}_i(t) = e^{Ht}\, \mathcal{O}_i(0)\, e^{-Ht}$$
$$1 = \sum_{\mathfrak{n}} |\mathfrak{n}\rangle\langle\mathfrak{n}|$$

$$C_{ij}(t) = \sum_{\mathfrak{n}} e^{-E_{\mathfrak{n}} t} \langle 0 | \mathcal{O}_i(0) | \mathfrak{n} \rangle \langle \mathfrak{n} | \mathcal{O}_j^\dagger(0) | 0 \rangle$$

lowest energy eigenstate will be the pion

examine the time-dependence of the correlation function …

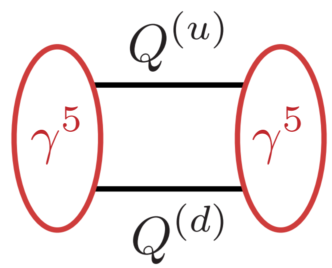
computing the pion mass

e.g. compute $C_{ij}(t) = \langle 0 | \mathcal{O}_i(t)\, \mathcal{O}_j^\dagger(0) | 0 \rangle$ with $\mathcal{O}(t) = \sum_{\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{x}}\, \bar{u}_{\mathbf{x},t}\, \gamma^5\, d_{\mathbf{x},t}$

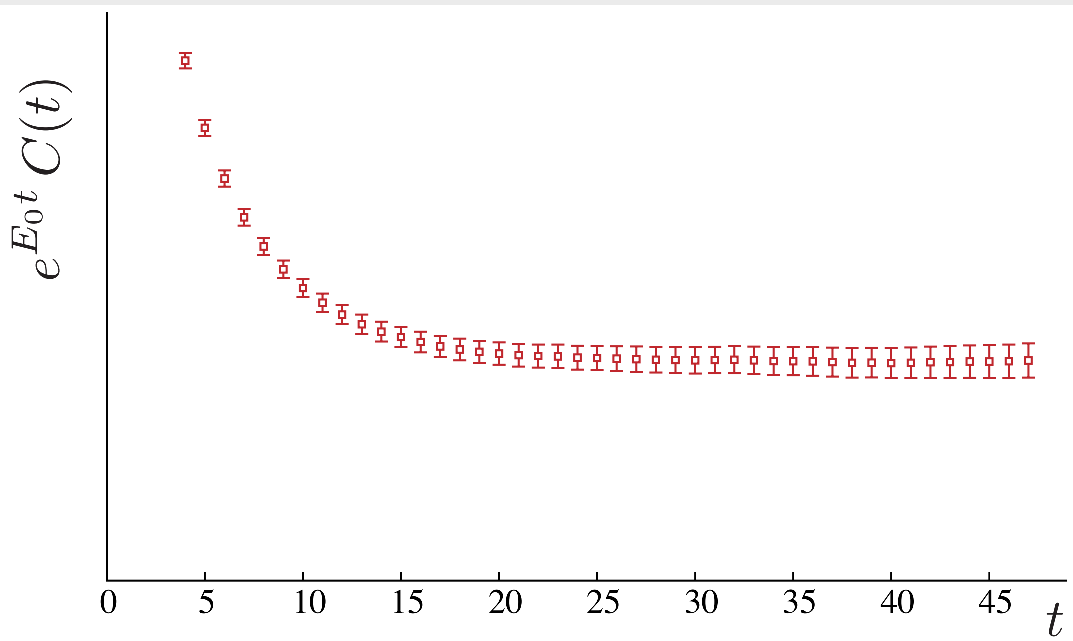
requires evaluation of $\mathrm{tr}\left[Q^{(u)}_{\mathbf{y}0;\mathbf{x}t}\, \gamma^5\, Q^{(d)}_{\mathbf{x}t;\mathbf{y}0}\, \gamma^5 \right]$ averaged over gauge-field configurations

propagator $Q = M^{-1}$

$\bar{\psi}_{\mathbf{x}'t'}\, M_{\mathbf{x}'t',\,\mathbf{x}t}[U]\, \psi_{\mathbf{x}t}$

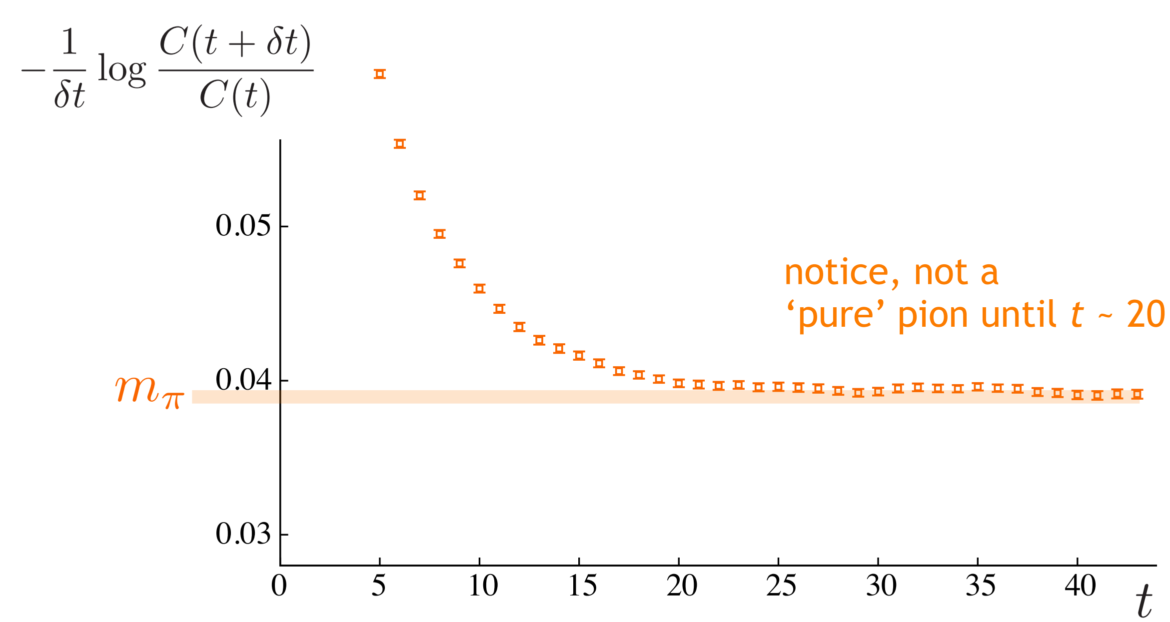


correlation function



$$C(t) = W_0\, e^{-E_0 t} + \ldots$$

effective mass plot



relatively straightforward to determine the 'ground-state' mass ...

energy eigenstates of the QCD Hamiltonian ?

what about 'excited' eigenstates ?

they're present in the sum: $C_{ij}(t) = \sum_{\mathfrak{n}} e^{-E_{\mathfrak{n}} t} \langle 0 | \mathcal{O}_i(0) | \mathfrak{n} \rangle \langle \mathfrak{n} | \mathcal{O}_j^\dagger(0) | 0 \rangle$

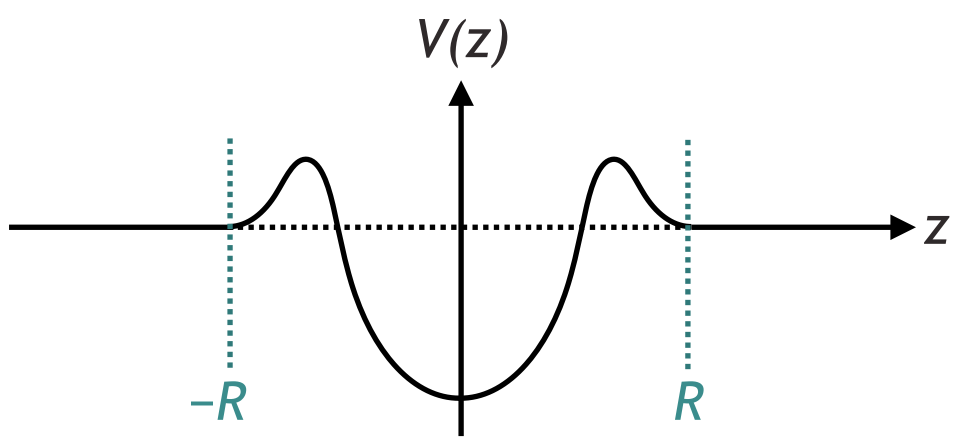
but why did we assume a **discrete** spectrum of states ? $1 = \sum_{\mathfrak{n}} |\mathfrak{n}\rangle\langle\mathfrak{n}|$

for **scattering**, the spectrum should be **continuous** !

in fact the assumption of a discrete spectrum is correct …

most easily illustrated considering **one-dimensional quantum mechanics**

imagine two identical bosons separated by a distance *z* interacting through a finite-range potential *V(z)*



solve the Schrödinger equation

$$-\frac{1}{m}\frac{d^2\psi}{dz^2} + V(z)\psi(z) = E\psi(z)$$

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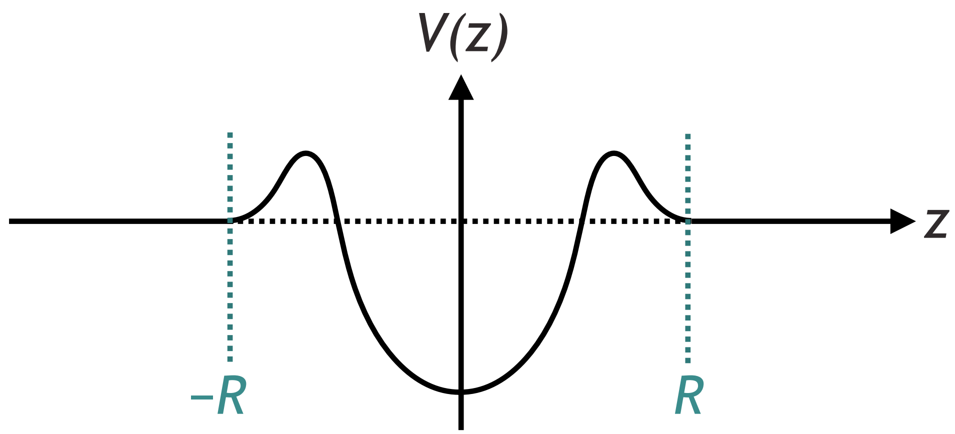

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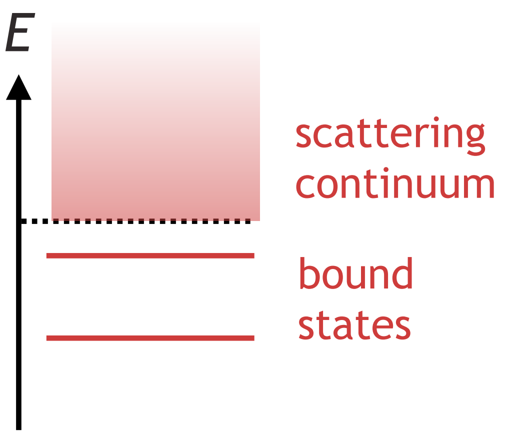
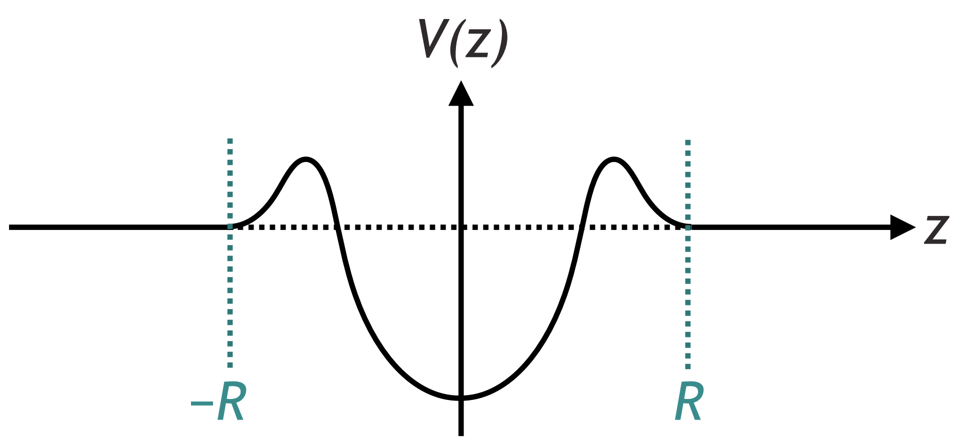


$$\psi(|z| > R) \sim \cos\big(\, p\,|z| + \delta(p)\,\big)$$

scattering

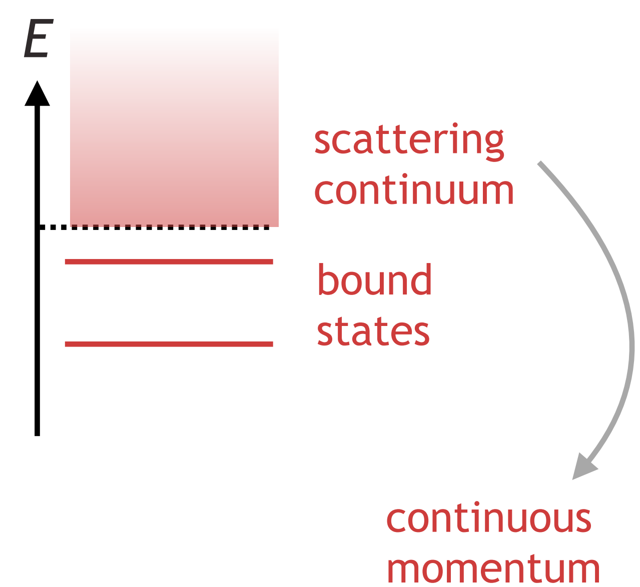
most easily illustrated considering **one-dimensional quantum mechanics**

imagine two identical bosons separated by a distance *z* interacting through a finite-range potential *V(z)*



solve the Schrödinger equation

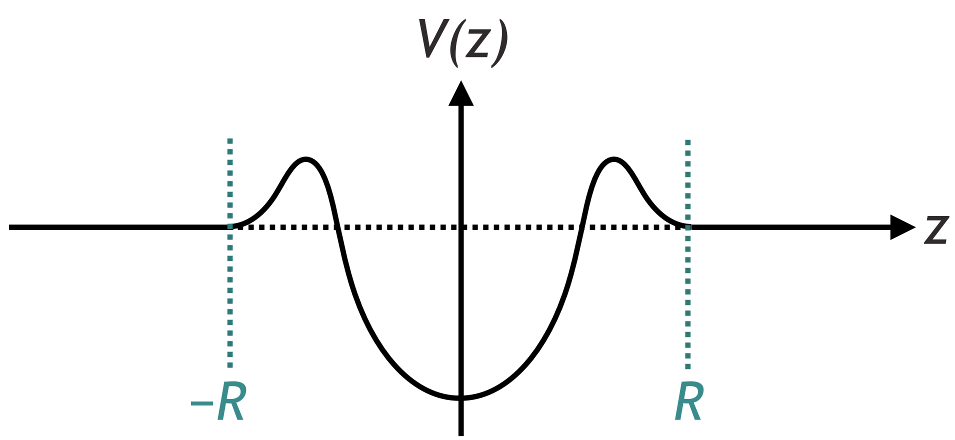
$$-\frac{1}{m}\frac{d^2\psi}{dz^2} + V(z)\psi(z) = E\psi(z)$$



$$\psi(|z|>R) \sim \cos\big(p|z| + \delta(p)\big)$$

most easily illustrated considering **one-dimensional quantum mechanics**

imagine two identical bosons separated by a distance z interacting through a finite-range potential *V(z)*



solve the Schrödinger equation

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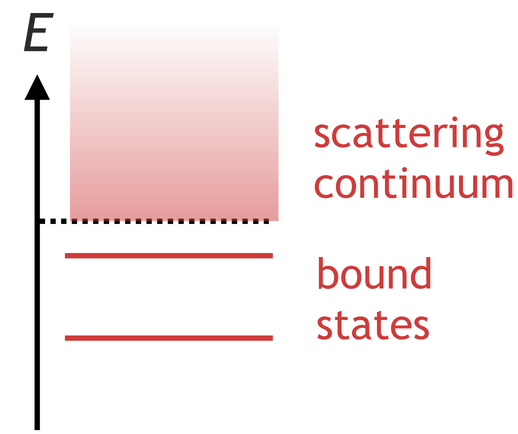
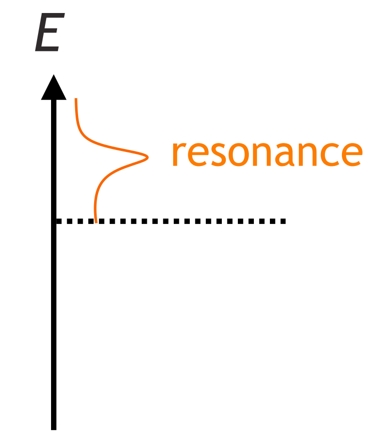
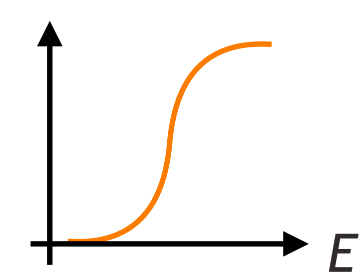




$$\psi(|z| > R) \sim \cos\left(p\,|z| + \boxed{\delta(p)}\right)$$

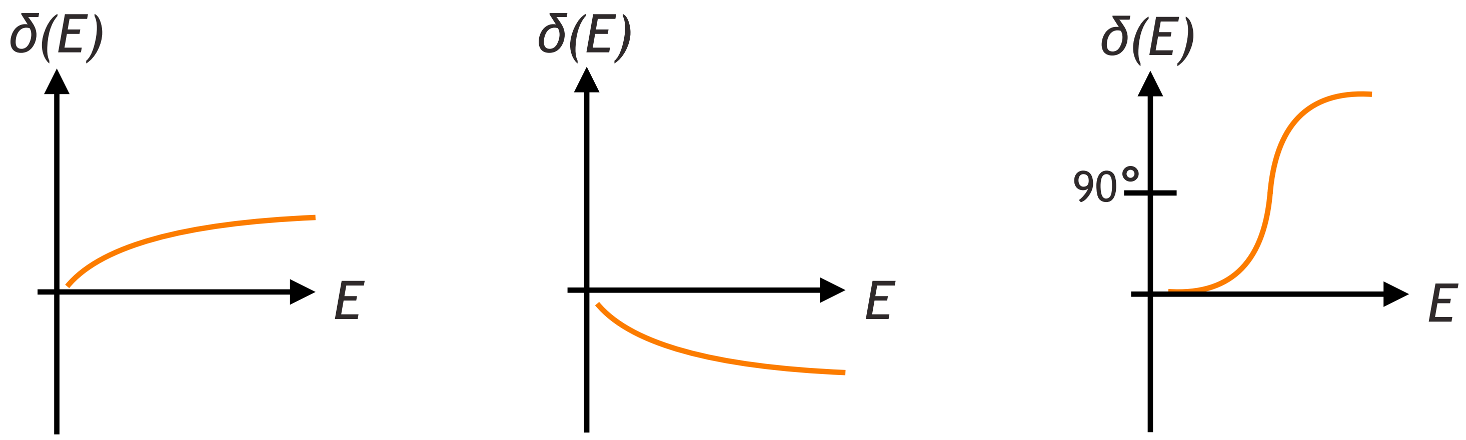
phase-shift



$$\psi(|z| > R) \sim \cos\left(p\,|z| + \delta(p)\right)$$

phase-shift

e.g.



'weak' attraction | 'weak' repulsion | resonance

generally, consider S-matrix

$$|\text{in}\rangle = S\,|\text{out}\rangle$$

elastic scattering

$$S(E) = e^{2i\delta(E)}$$

‘scattering’ in a finite-volume

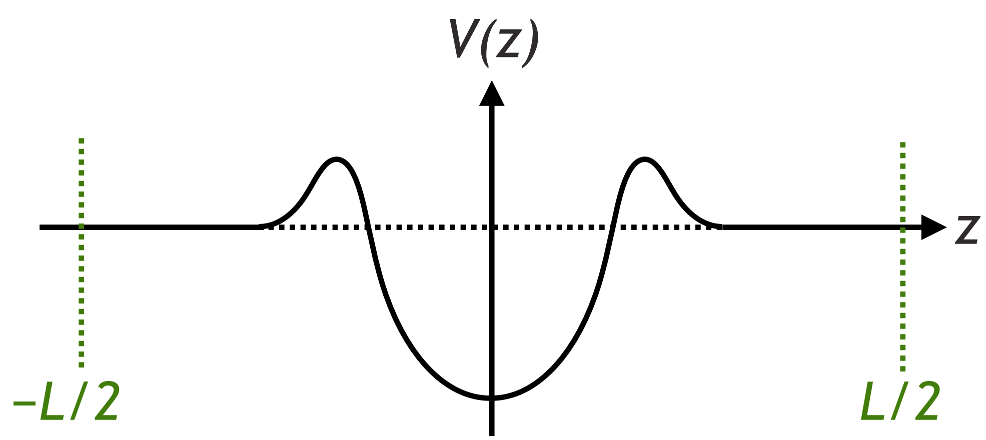
now put the system in a ‘box’ — periodic boundary condition at $z = \pm L/2$

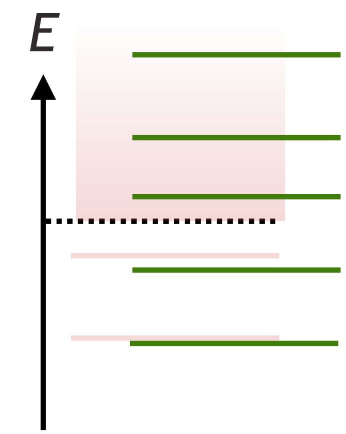
$$\psi(|z| > R) \sim \cos\big(\, p\,|z| + \delta(p)\,\big)$$

$$\psi(L/2) = \psi(-L/2)$$
$$\frac{d\psi}{dz}(L/2) = \frac{d\psi}{dz}(-L/2)$$

momentum quantization condition

$$p = \frac{2\pi}{L} n - \frac{2}{L}\delta(p)$$





3+1 dim quantum field theory result

for elastic scattering in a cube the corresponding relationship is $\cot \delta_\ell(E) = \mathcal{M}_\ell\big(E(L), L\big)$

in the simplest case of a single partial wave being non-zero

Lüscher 1986

⋮

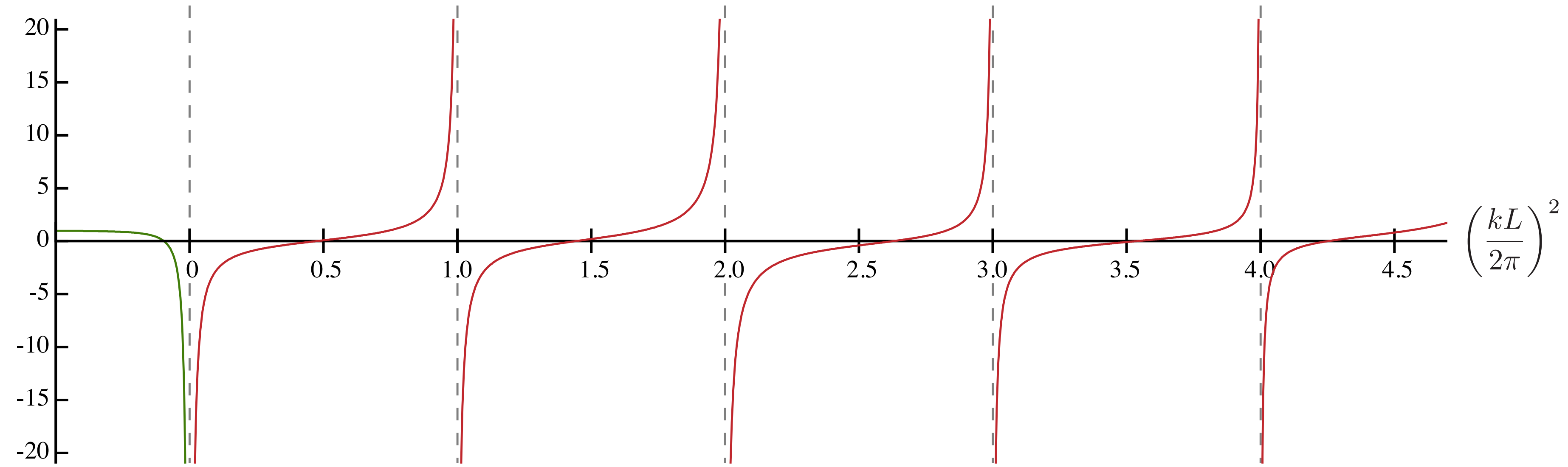
many subsequent works
see the RMP for a complete list

will present some complications later ...

$$\cot \delta_\ell(E) = \boxed{\mathcal{M}_\ell\big(E(L), L\big)}$$

known function expressing the 'kinematics' of the finite-volume

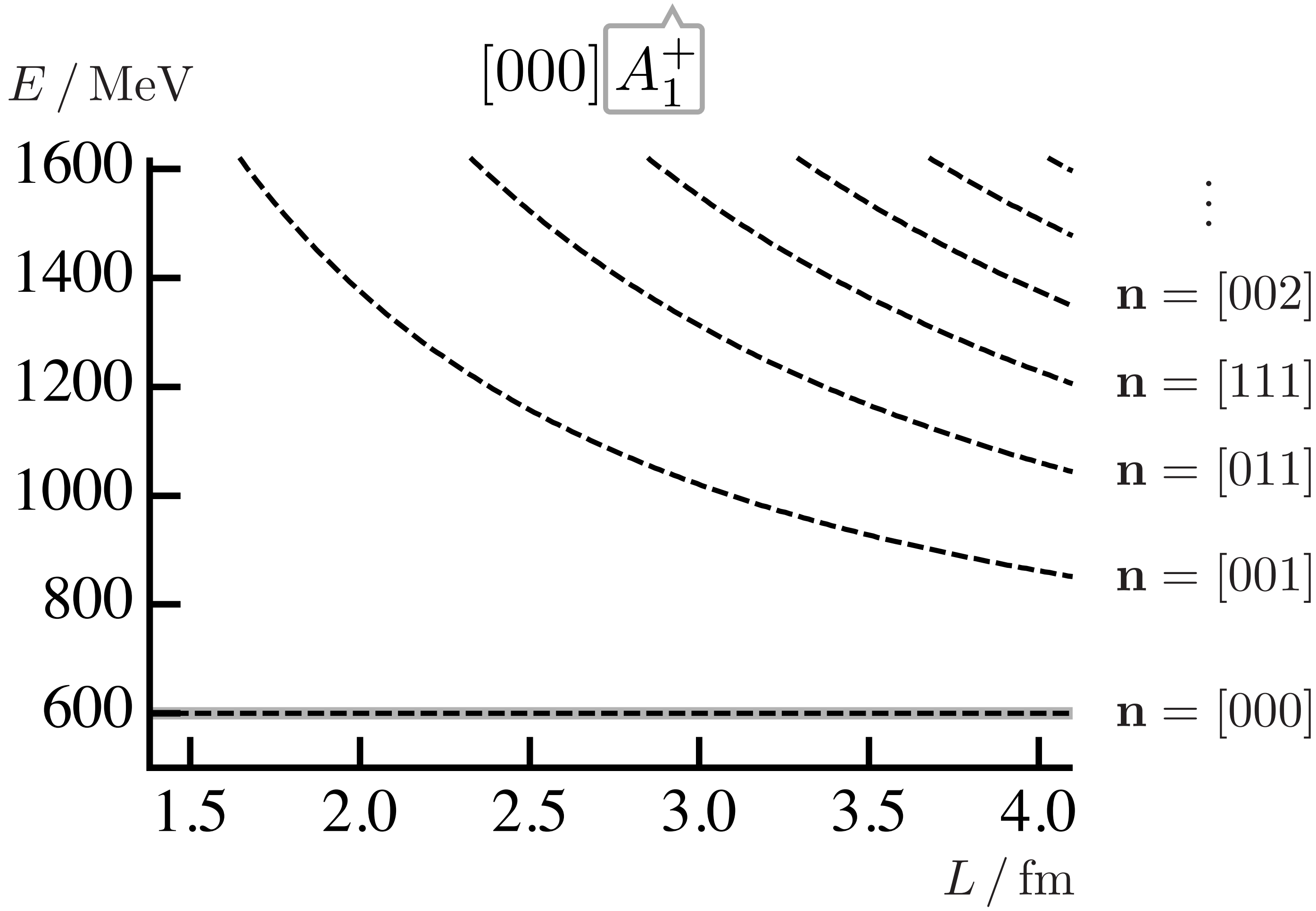
e.g. $\mathcal{M}_0$

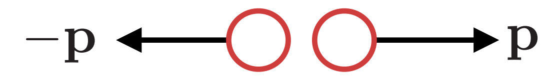


no interaction

scattering particles m = 300 MeV

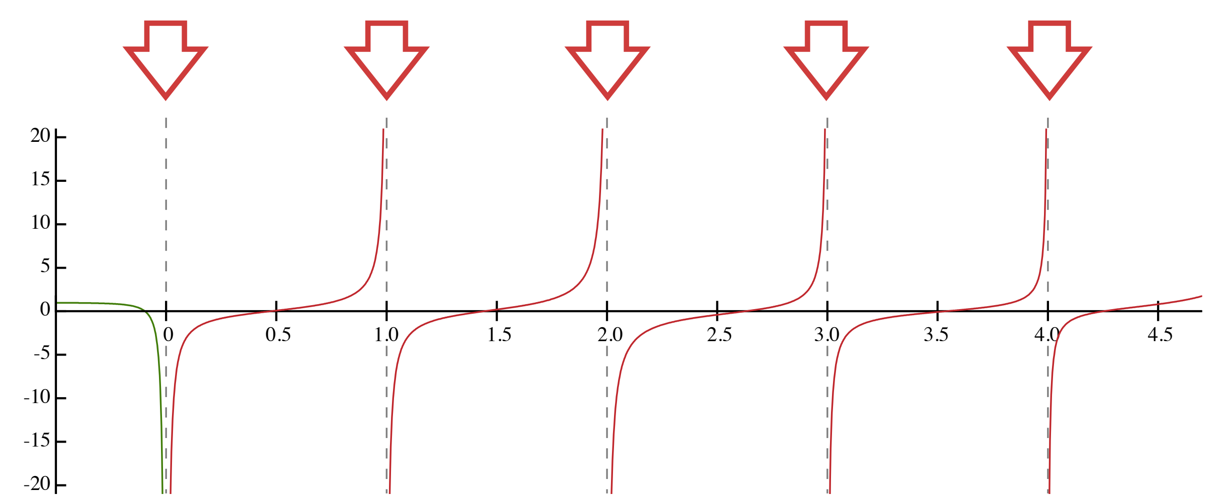
for now, think of this as *S*-wave





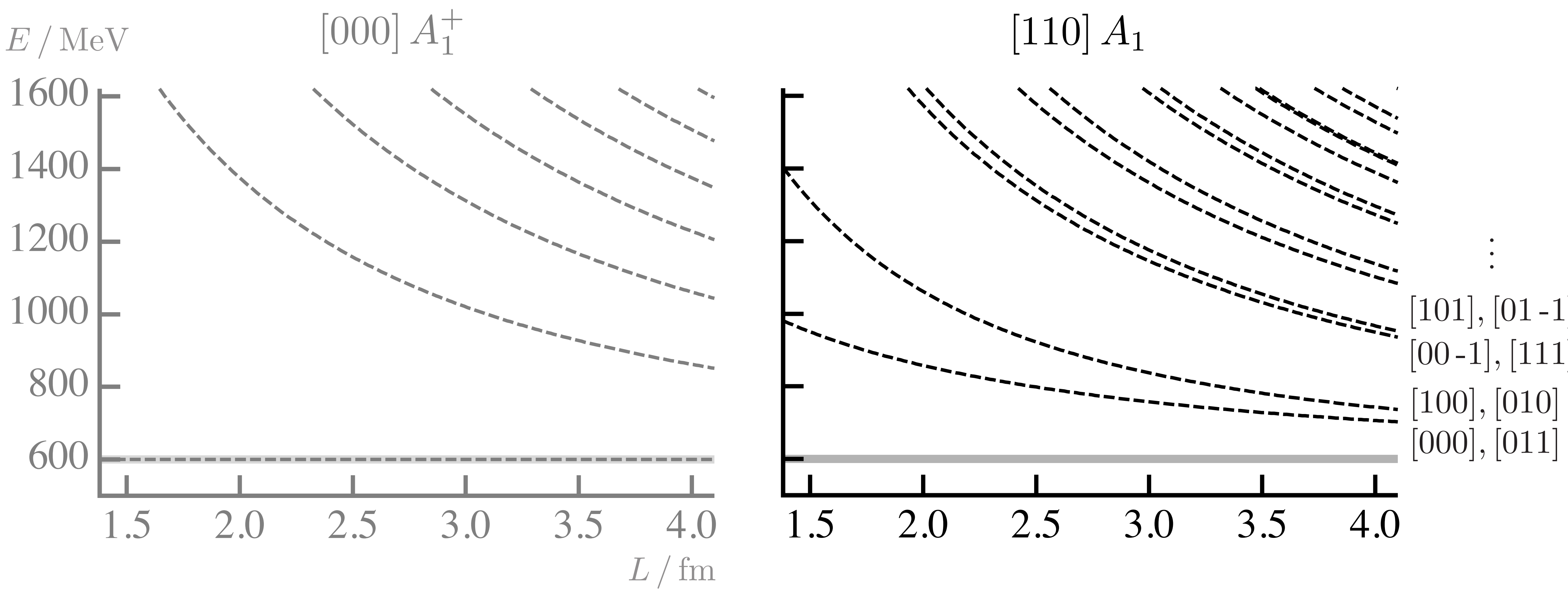
$$E_{\mathrm{ni}} = 2\sqrt{m^2 + \mathbf{p}^2} \qquad \mathbf{p} = \frac{2\pi}{L}\mathbf{n}$$

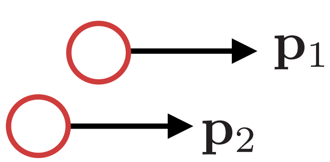
$$\cot\delta(E) = \mathcal{M}\big(E(L), L\big) \qquad \delta \to 0,\ \cot\delta \to \infty$$



no interaction — in a moving frame

$\cot \delta_\ell(E) = \mathcal{M}_\ell^{\mathbf{P}}\left(E(L), L\right)$





$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$$

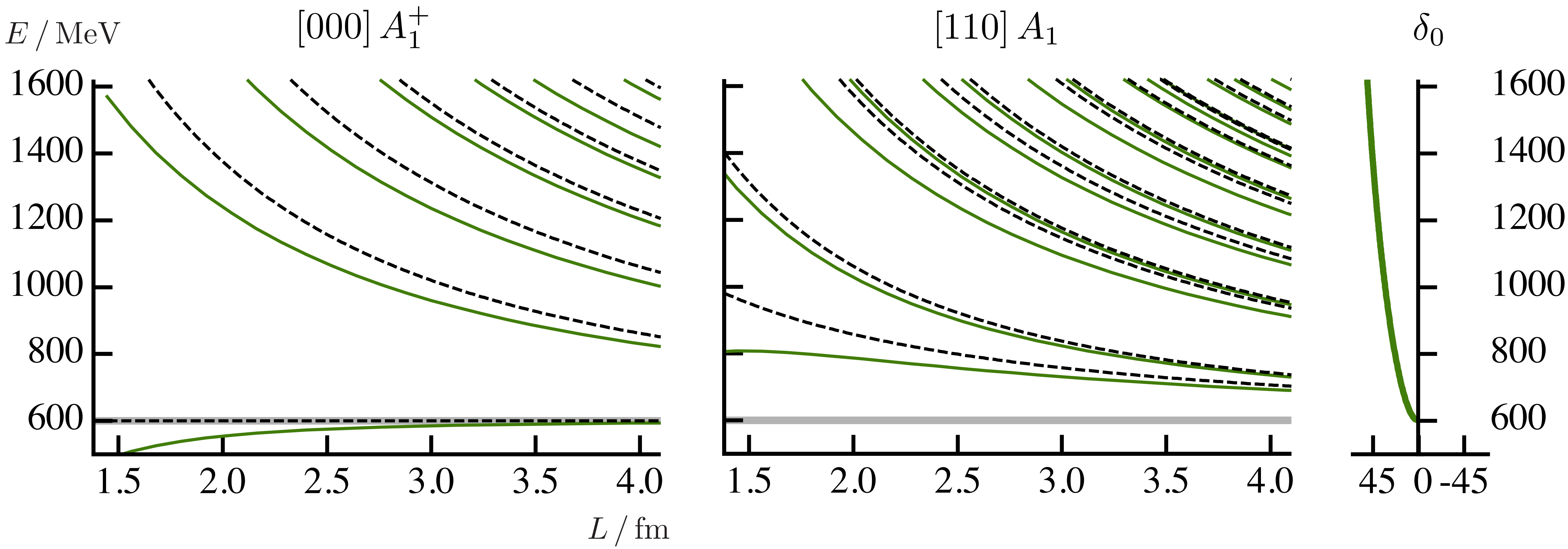
$$\mathbf{P} = \frac{2\pi}{L}\mathbf{n}$$

$$E_{\mathrm{ni}} = \sqrt{m^2 + \mathbf{p}_1^2} + \sqrt{m^2 + \mathbf{p}_2^2}$$

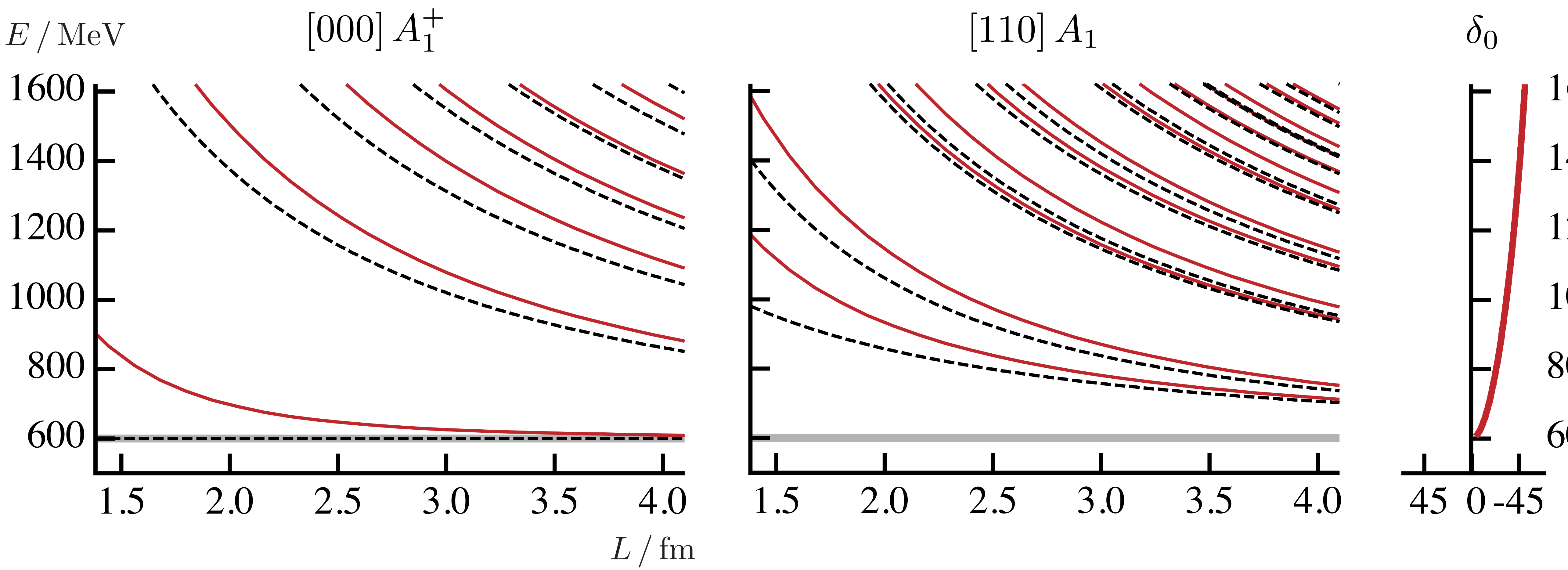
$$\mathbf{p}_{1,2} = \frac{2\pi}{L}\mathbf{n}_{1,2}$$

in the cm frame $E_{\mathrm{cm}} = \sqrt{E^2 - \mathbf{P}^2}$

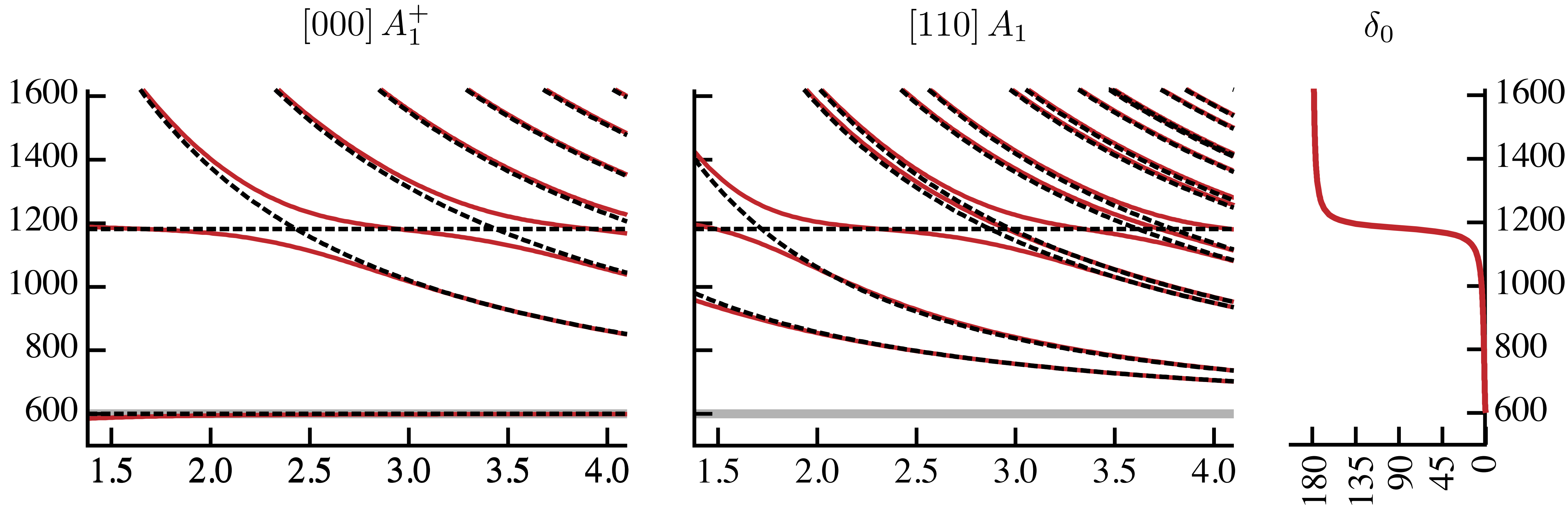
weak attraction



weak repulsion



an elastic resonance

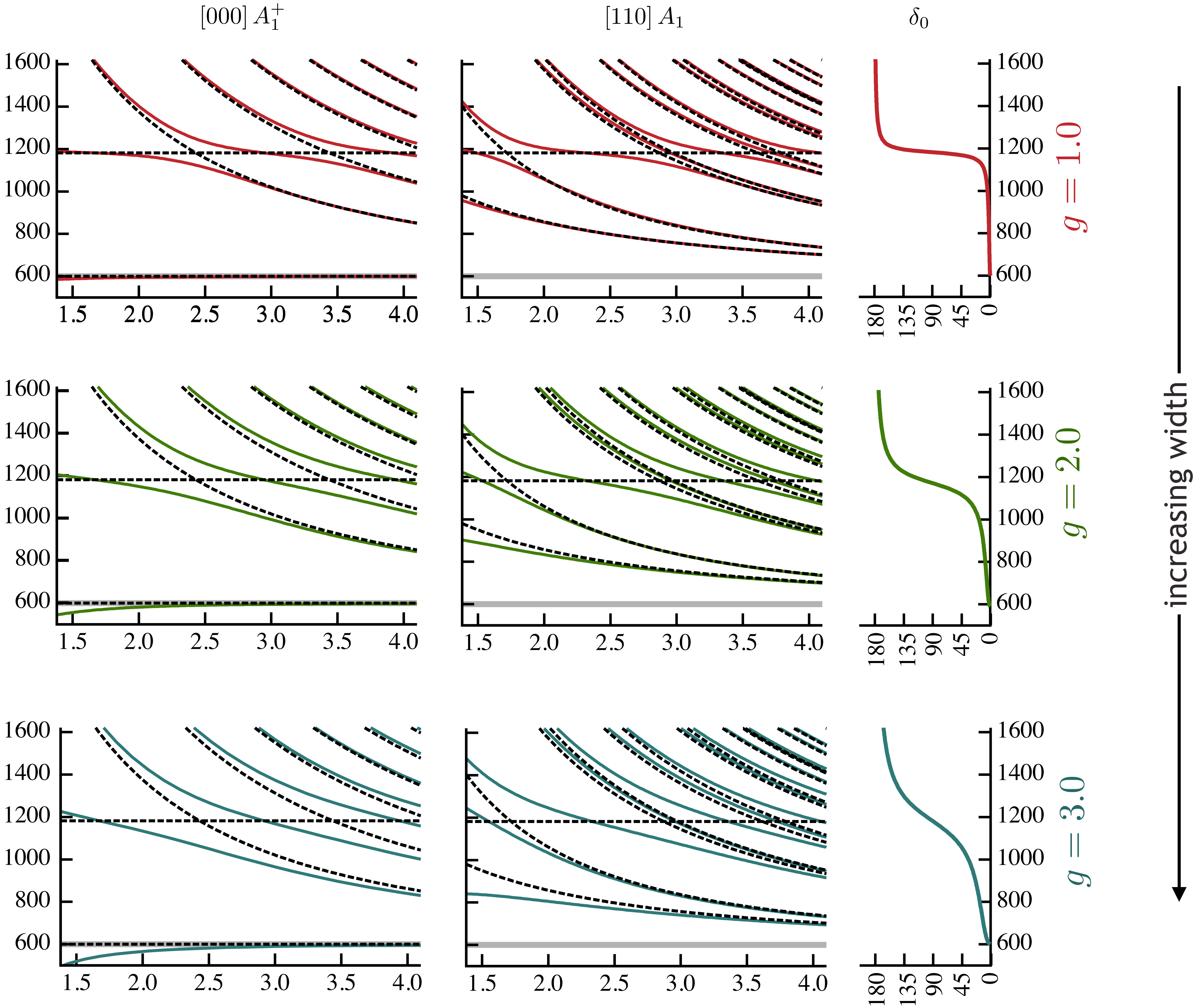


note the avoided level crossings

$$\tan\delta = \frac{E\,\Gamma(E)}{m_R^2 - E^2}$$

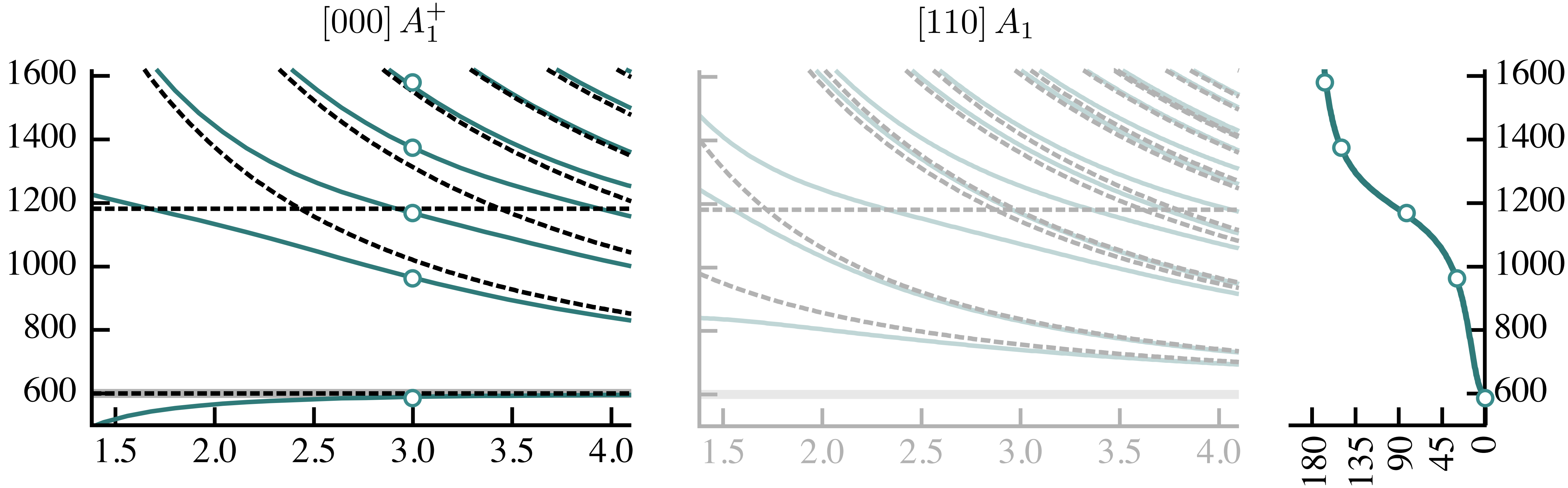
$$\Gamma(E) = \frac{g^2}{6\pi}\frac{m_R^2}{E^2}k(E)$$

elastic resonance

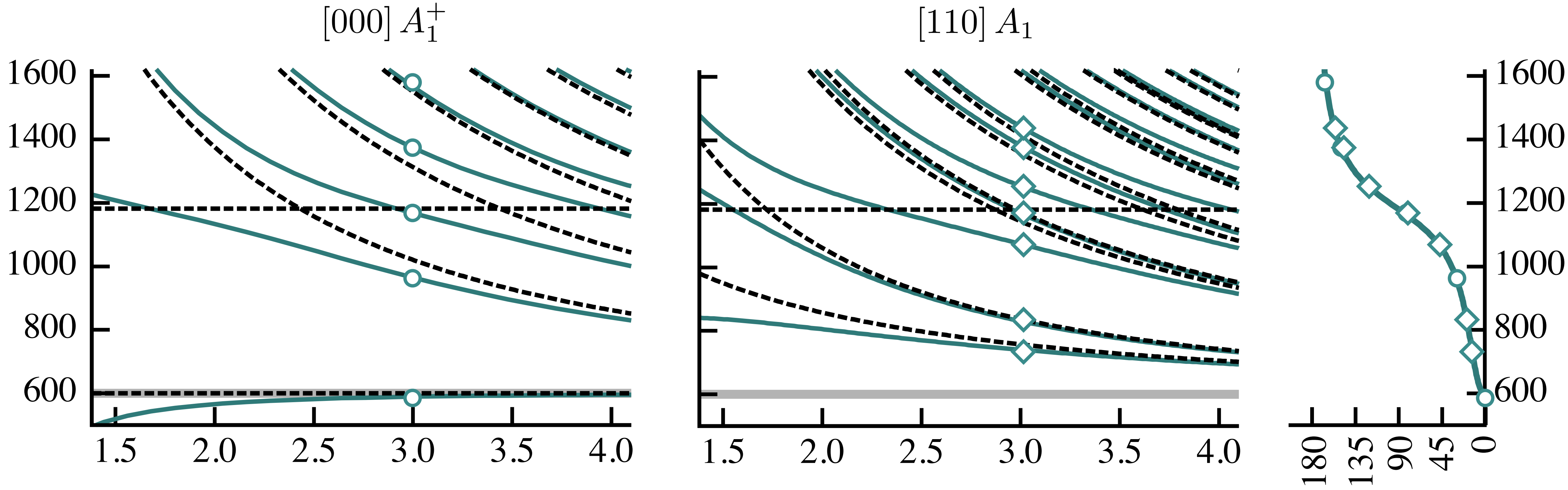


$$\Gamma(E) = \frac{g^2}{6\pi}\frac{m_R^2}{E^2}k(E)$$

an elastic resonance — finite-volume mapping



an elastic resonance — finite-volume mapping



determining the moving-frame spectrum
provides much more information

job of lattice QCD: determine the discrete spectrum

we need to reliably determine excited state spectra in multiple volumes / in moving frames

spectrum information is in two-point correlation functions $\langle 0 | \mathcal{O}_i(t)\, \mathcal{O}_j(0) | 0 \rangle = \sum_{\mathfrak{n}} Z_i^{\mathfrak{n}}\, Z_j^{\mathfrak{n}*}\, e^{-E_{\mathfrak{n}} t}$

two-point correlation functions & the excited spectrum

conceptually straightforward, consider a single correlation function

$$\langle 0 | \mathcal{O}_i(t)\, \mathcal{O}_j(0) | 0 \rangle = \sum_{\mathfrak{n}} Z_i^{\mathfrak{n}}\, Z_j^{\mathfrak{n}*}\, e^{-E_{\mathfrak{n}} t}$$

provided the 'overlaps' $Z_i^{\mathfrak{n}} = \langle 0 | \mathcal{O}_i(0) | \mathfrak{n} \rangle$ are non-zero, every state in the spectrum contributes

fit to a sum of exponentials ?

in practice fitting to a sum of exponentials is unreliable:

- overlaps for some states might be very small (operator dependent)
- don't know how many states required in the sum
- (nearly) degenerate states can't be distinguished by t-dependence alone
- limited number of timeslices & statistical noise

a more powerful approach makes use of a basis of operators & linear algebra …

try to build an ‘optimum’ operator for state *n* as a linear superposition in some basis of operators

$$\Omega^\dagger = \sum_i v_i \, \mathcal{O}_i \qquad \Omega^\dagger |0\rangle = |\mathfrak{n}\rangle + \sum_{\mathfrak{m}\neq\mathfrak{n}} \epsilon_\mathfrak{m} |\mathfrak{m}\rangle$$

with the ε_m as small as possible

corresponding correlation function would be $\langle 0 | \Omega(t)\, \Omega^\dagger(0) | 0 \rangle = e^{-E_\mathfrak{n} t} + \sum_{\mathfrak{m}\neq\mathfrak{n}} |\epsilon_\mathfrak{m}|^2 \, e^{-E_\mathfrak{m} t}$

and we want to minimize this, by varying the v_i

$$\langle 0 | \Omega(t)\, \Omega^\dagger(0) | 0 \rangle = \sum_{i,j} v_i^* \, \langle 0 | \mathcal{O}_i(t)\, \mathcal{O}_j^\dagger(0) | 0 \rangle \, v_j = \sum_{i,j} v_i^* \, C_{ij}(t)\, v_j$$

need to avoid the trivial minimum $v_i = 0$ ⇒ constrain normalization $\sum_{i,j} v_i^* \, C_{ij}(t_0)\, v_j = N$

e.g. $N = 1$

implement constraint via a Lagrange multiplier

⇒ minimize $\Lambda = \sum_{i,j} v_i^* \, C_{ij}(t)\, v_j - \lambda \Big[\sum_{i,j} v_i^* \, C_{ij}(t_0)\, v_j - 1 \Big]$

which leads to a generalized eigenvalue problem for $\mathbf{v}$

$$\mathbf{C}(t)\, \mathbf{v} = \lambda \, \mathbf{C}(t_0)\, \mathbf{v}$$

n^{th} eigenvalue

$\lambda_\mathfrak{n}(t) \sim e^{-E_\mathfrak{n}(t-t_0)}$

job of lattice QCD: determine the discrete spectrum

we need to reliably determine excited state spectra — in multiple volumes / in moving frames

spectrum information is in two-point correlation functions $\langle 0 | \mathcal{O}_i(t)\, \mathcal{O}_j(0) | 0 \rangle = \sum_{\mathfrak{n}} Z_i^{\mathfrak{n}}\, Z_j^{\mathfrak{n}*}\, e^{-E_{\mathfrak{n}} t}$

but what operators $\mathcal{O}$ should we consider ?

must be constructed
out of quark/gluon fields

easiest constructions with meson quantum numbers — fermion bilinears $\bar{\psi}\mathbf{\Gamma}\psi$

well motivated by
success of quark model

'looks' like a $q\bar{q}$ system

Wick
contractions

'annihilation'
required for isospin=0

quark propagation from t to t
⇒ matrix inversions on many t

job of lattice QCD: determine the discrete spectrum

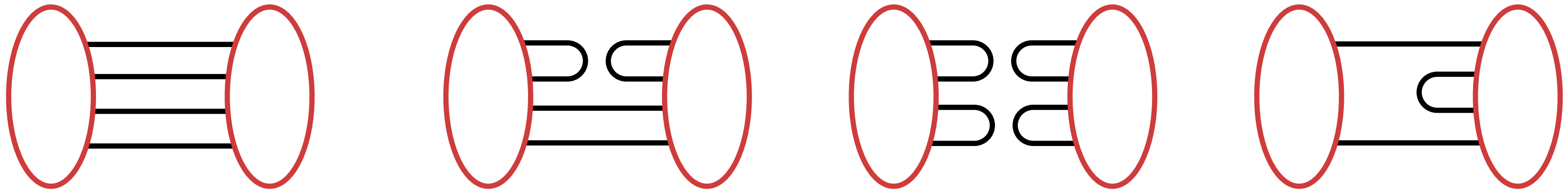
easiest constructions with meson quantum numbers – fermion bilinears $\bar{\psi}\mathbf{\Gamma}\psi$

but can also construct operators with more quark fields

e.g. 'local' tetraquark operators $\bar{\psi}_{\mathbf{x}}\bar{\psi}_{\mathbf{x}}\psi_{\mathbf{x}}\psi_{\mathbf{x}}$

e.g. 'meson-meson'-like operators $\sum_{\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{x}}\,\bar{\psi}_{\mathbf{x}}\mathbf{\Gamma}\psi_{\mathbf{x}}\,\sum_{\mathbf{y}} e^{i\mathbf{q}\cdot\mathbf{y}}\,\bar{\psi}_{\mathbf{y}}\mathbf{\Gamma}'\psi_{\mathbf{y}}$

schematic
Wick
contractions



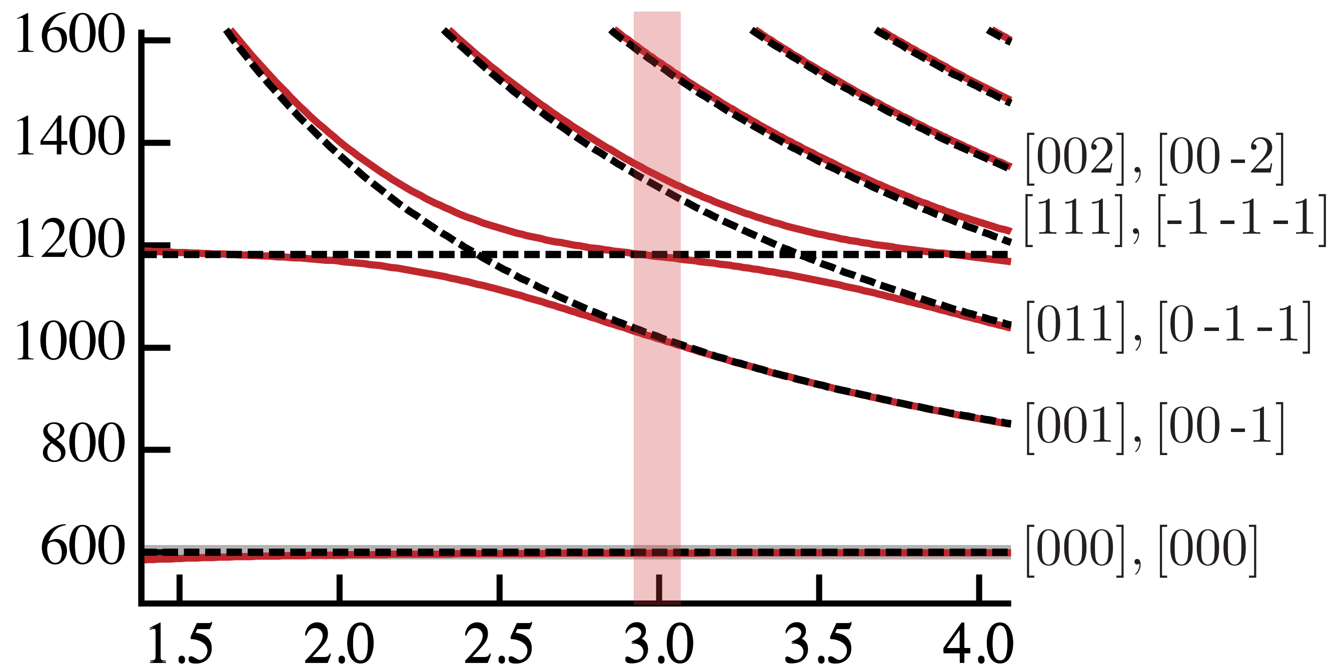
'annihilation'
generally
required

and can clearly include still more quark fields ad infinitum ...

... is there some organizing principle
which suggests what operator basis we should use ?

the non-interacting spectrum as an operator basis guide

e.g. narrow resonance (in rest frame)



suppose we want to determine all states up to 1500 MeV on a 3 fm lattice

we might try an operator basis featuring 'meson-meson'-like operators with back-to-back momentum up to [111]

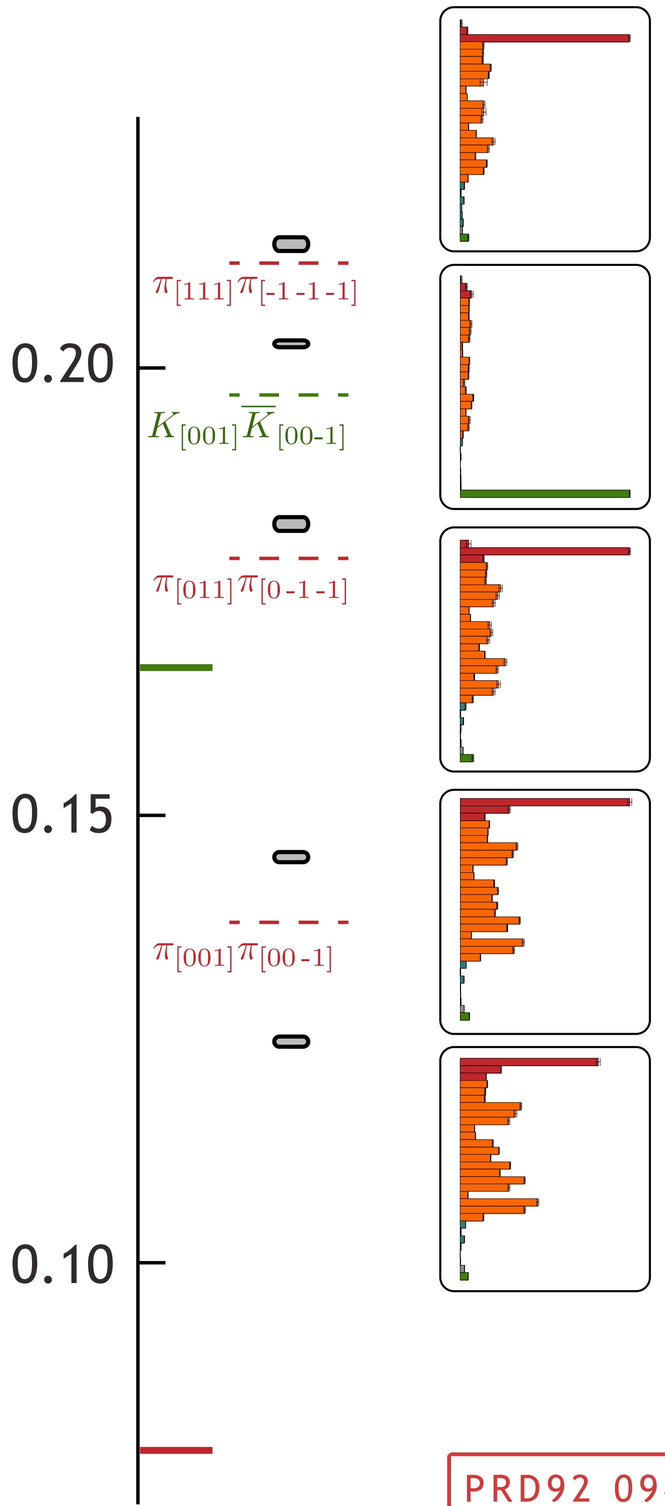
'look like' the expected meson-meson basis states

plus a set of $\bar{\psi}\Gamma\psi$ operators

'look like' a bound $q\bar{q}$-like basis state

isospin=1 T_1^- irrep spectrum

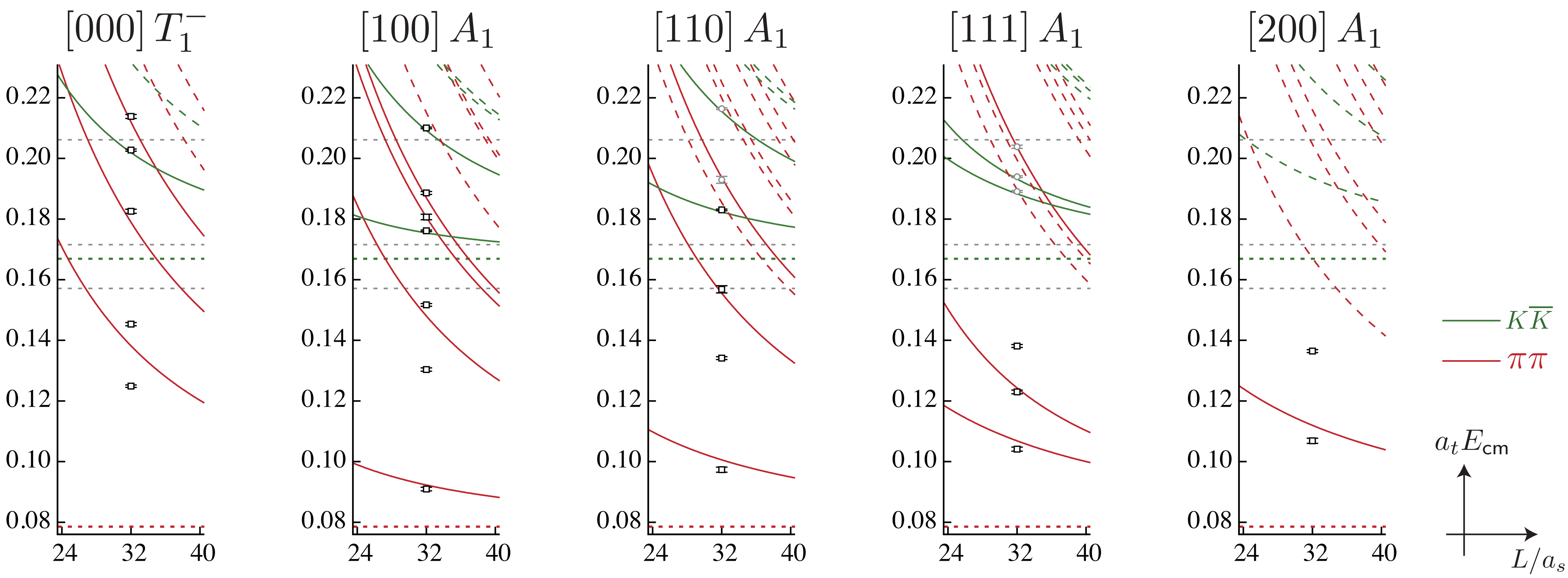
variational analysis of 30×30 correlation matrix: $3\times\pi\pi$, $26\times\bar{\psi}\Gamma\psi$, $1\times K\overline{K}$



PRD92 094502 (2015)

$m_\pi = 0.039$
$m_K = 0.083$
$L \sim 3.8$ fm

… and in moving frames …



… giving a phase-shift