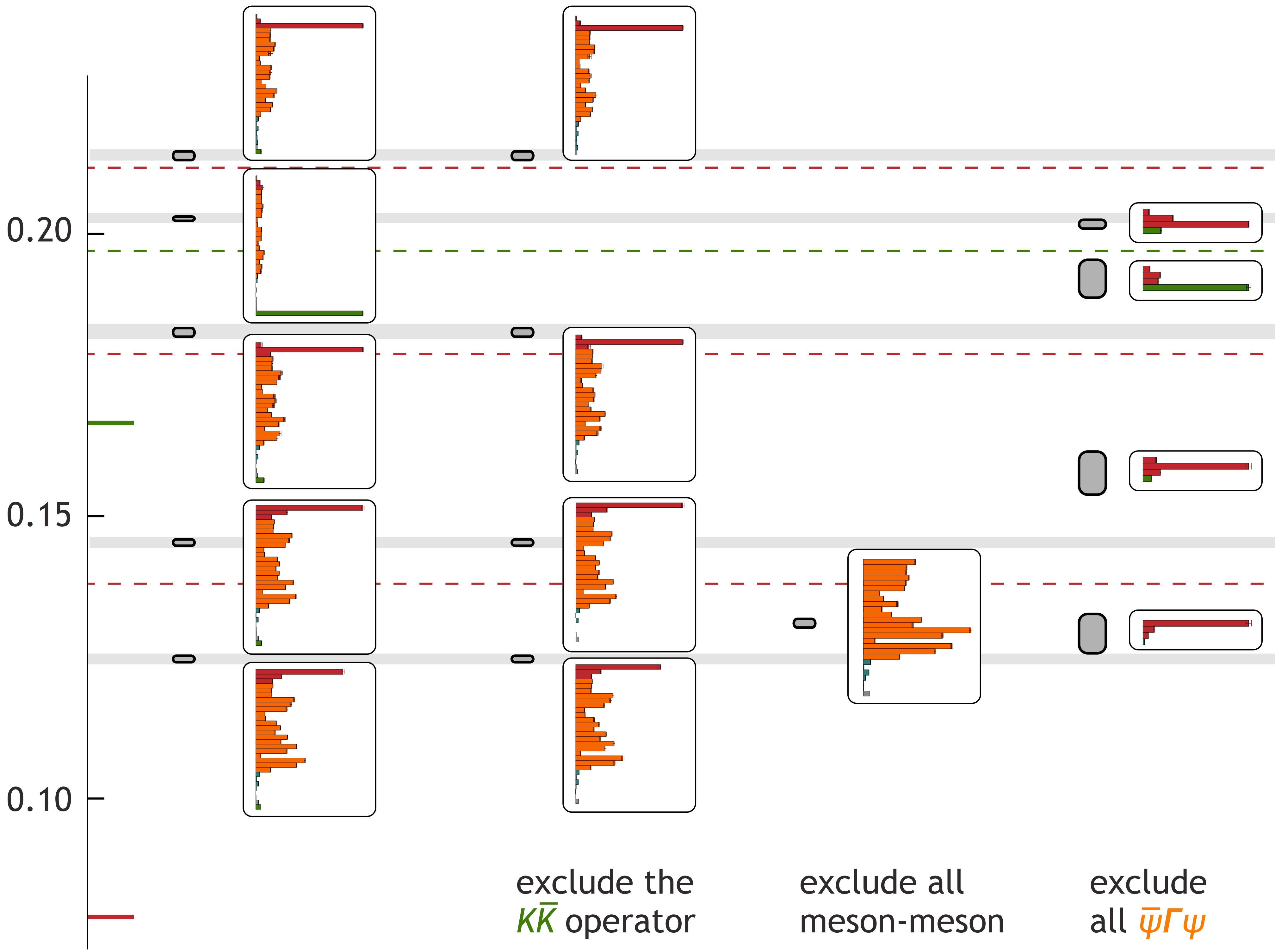
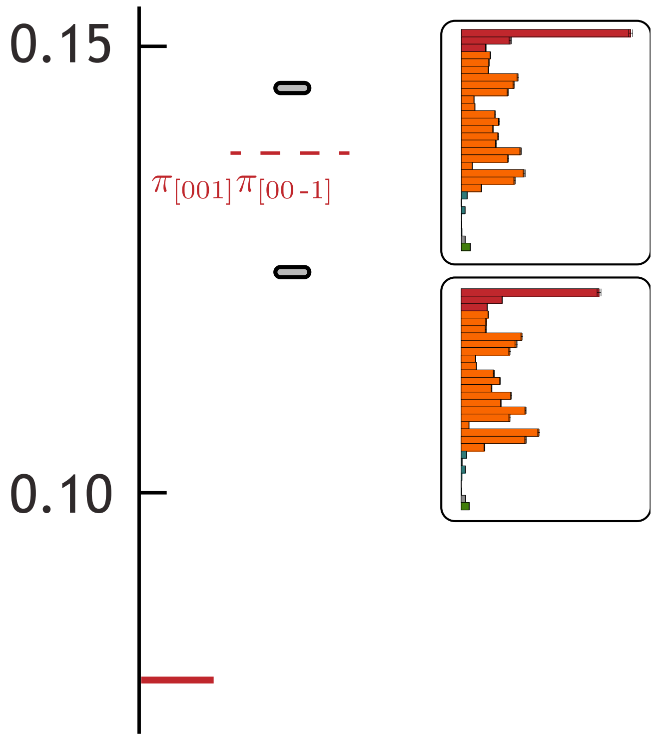
what happens if we vary the operator basis ?

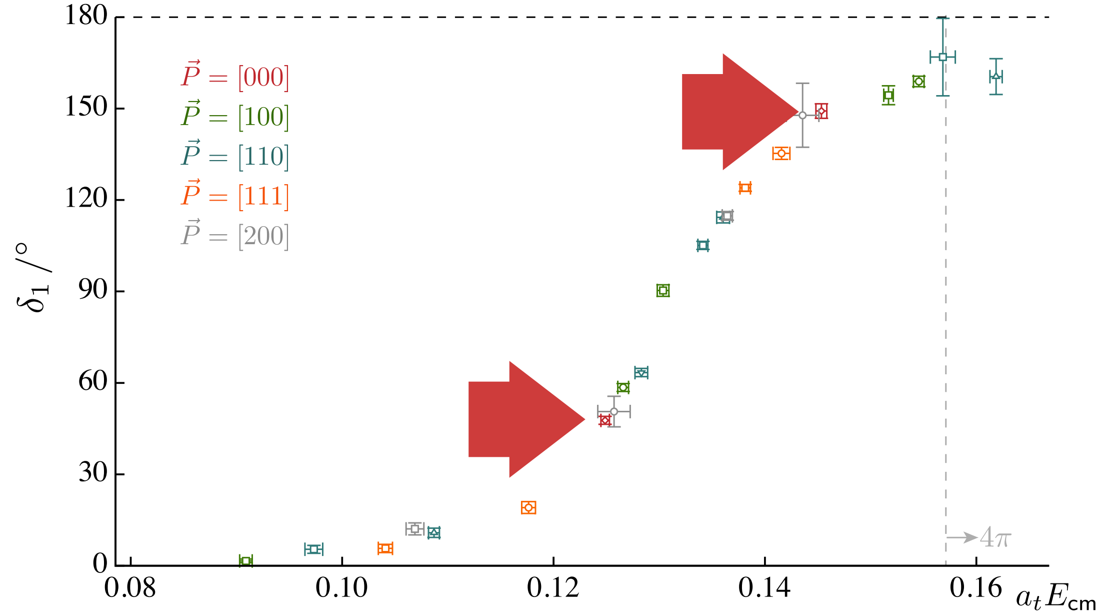


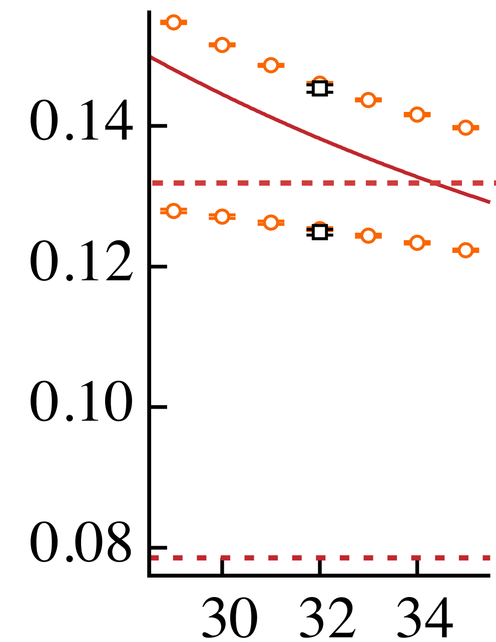
$m_\pi = 0.039$
$m_K = 0.083$
$L \sim 3.8$ fm

what’s happening here ?

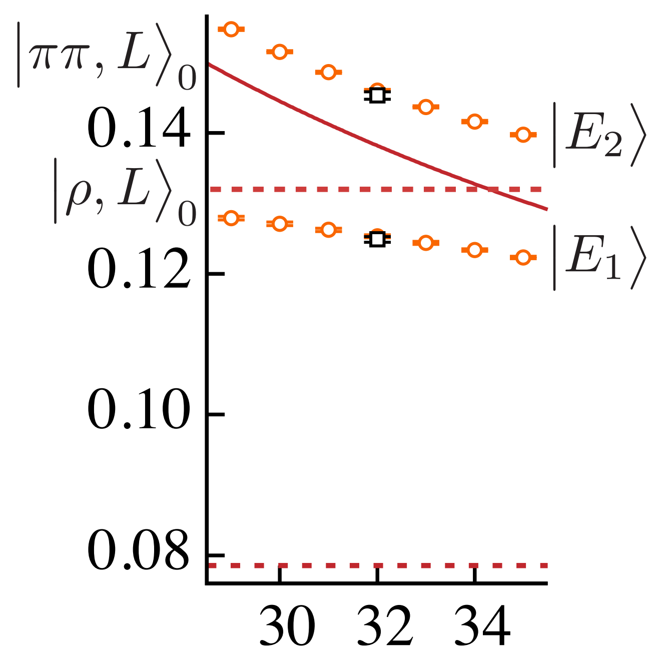
focus on the lowest two states







an avoided level crossing



think about this as a **two-state problem**

imagine we could turn off the coupling so
a 'bound-state' and a 'meson-meson' state were eigenstates

$$|\rho, L\rangle_0 \qquad\qquad |\pi\pi, L\rangle_0$$

with the coupling turned on, the eigenstates are admixtures

$$|E_1\rangle = \cos\theta\,|\rho, L\rangle_0 + \sin\theta\,|\pi\pi, L\rangle_0$$
$$|E_2\rangle = -\sin\theta\,|\rho, L\rangle_0 + \cos\theta\,|\pi\pi, L\rangle_0$$

with operators that 'look-like' $|\rho, L\rangle_0$ and $|\pi\pi, L\rangle_0$ in the basis, the variational method separates $|E_1\rangle$,$|E_2\rangle$

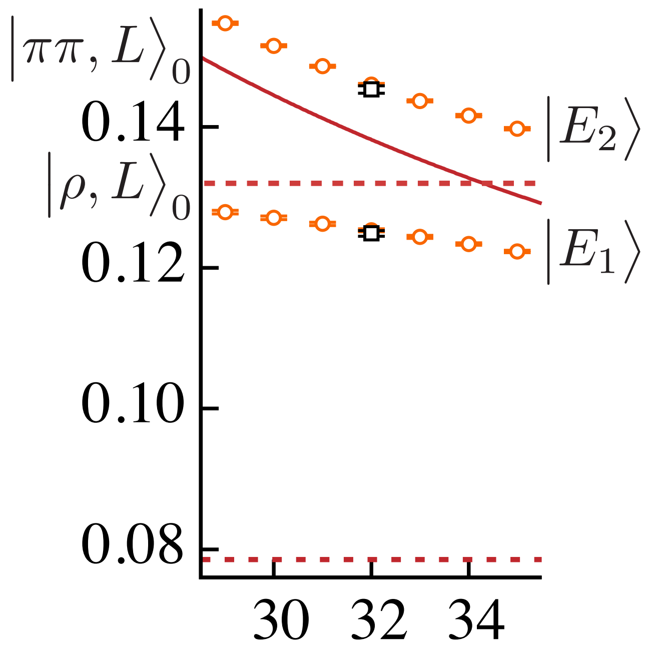
$$\begin{pmatrix} C_{\rho,\rho}(t) & C_{\rho,\pi\pi}(t) \\ C_{\pi\pi,\rho}(t) & C_{\pi\pi,\pi\pi}(t) \end{pmatrix} = \begin{pmatrix} Z_\rho & 0 \\ 0 & Z_{\pi\pi} \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} e^{-E_1 t} & 0 \\ 0 & e^{-E_2 t} \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} Z_\rho & 0 \\ 0 & Z_{\pi\pi} \end{pmatrix}$$

$$\mathcal{O}_\rho|0\rangle = Z_\rho|\rho, L\rangle_0 + \epsilon|\pi\pi, L\rangle_0$$
$$\mathcal{O}_{\pi\pi}|0\rangle = Z_{\pi\pi}|\pi\pi, L\rangle_0 + \epsilon|\rho, L\rangle_0$$

GEVP eigenvectors will find the rotation

and the principal correlators
$$\lambda_1(t) \sim e^{-E_1 t}$$
$$\lambda_2(t) \sim e^{-E_2 t}$$

what's happening here ?



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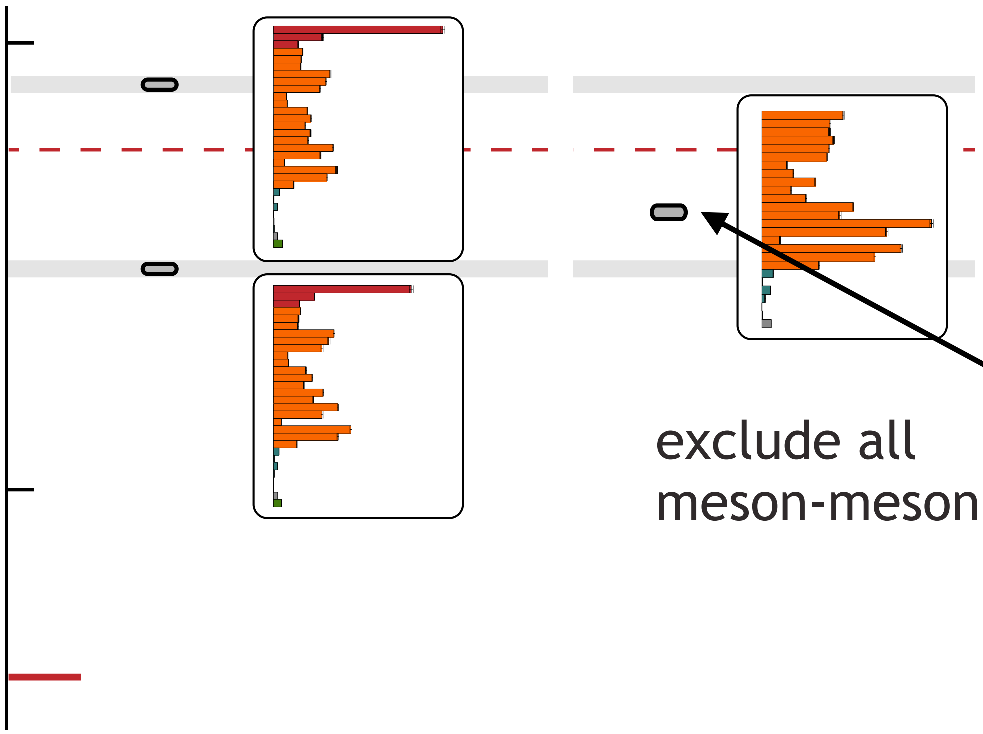
now suppose we used only the $\mathcal{O}_\rho$ operators

then $C(t) \propto \cos^2\theta\, e^{-E_1 t} + \sin^2\theta\, e^{-E_2 t}$ and there'll be two energies present …

… and they're very hard to separate

what's happening here ?

it looks like this is what's happening

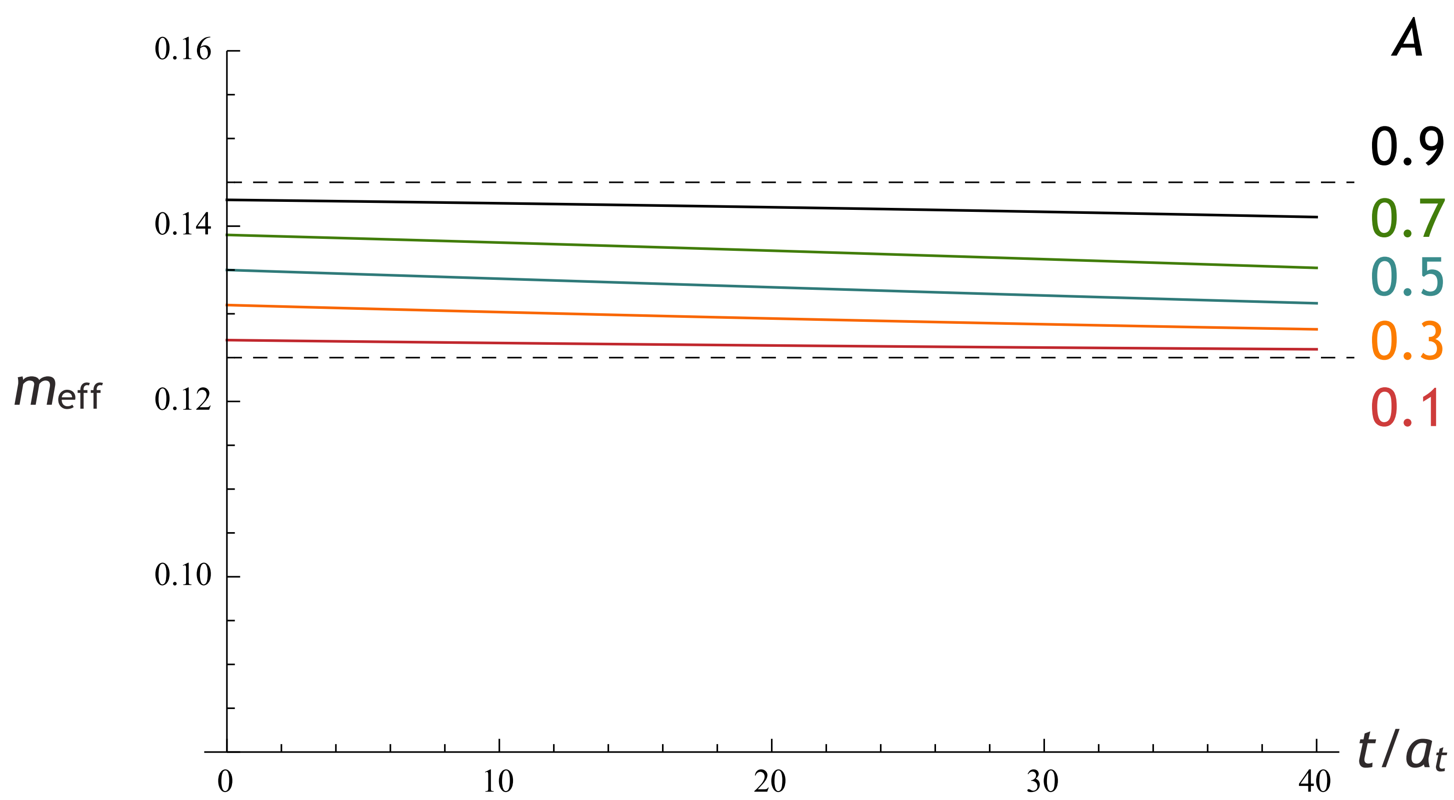


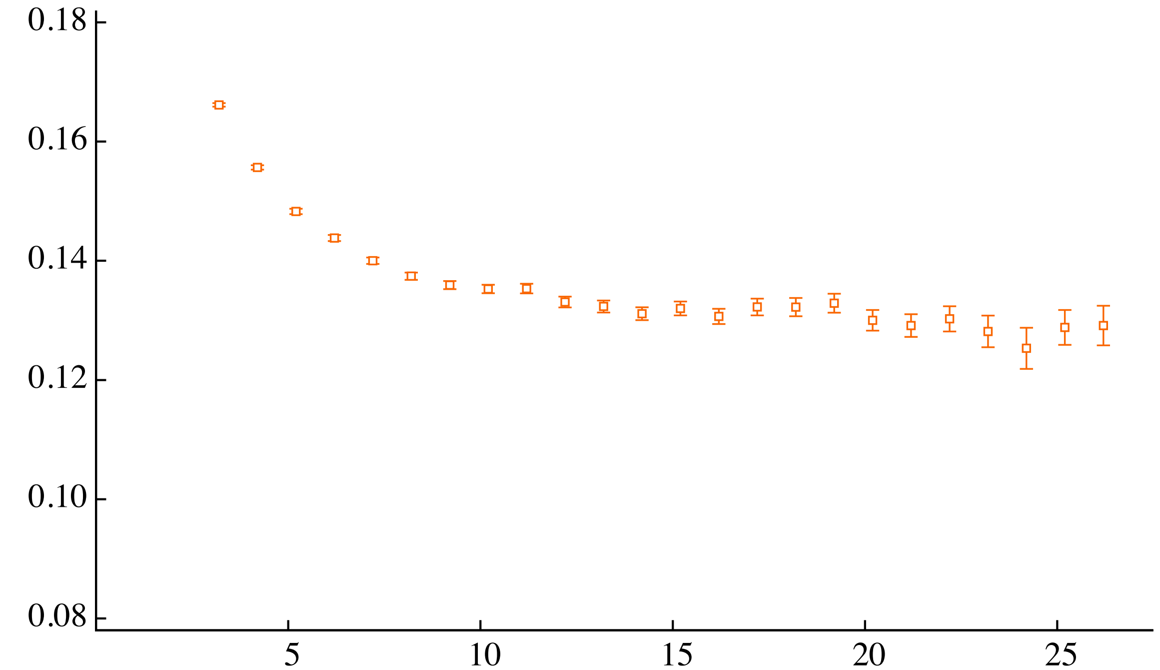
exclude all meson-meson

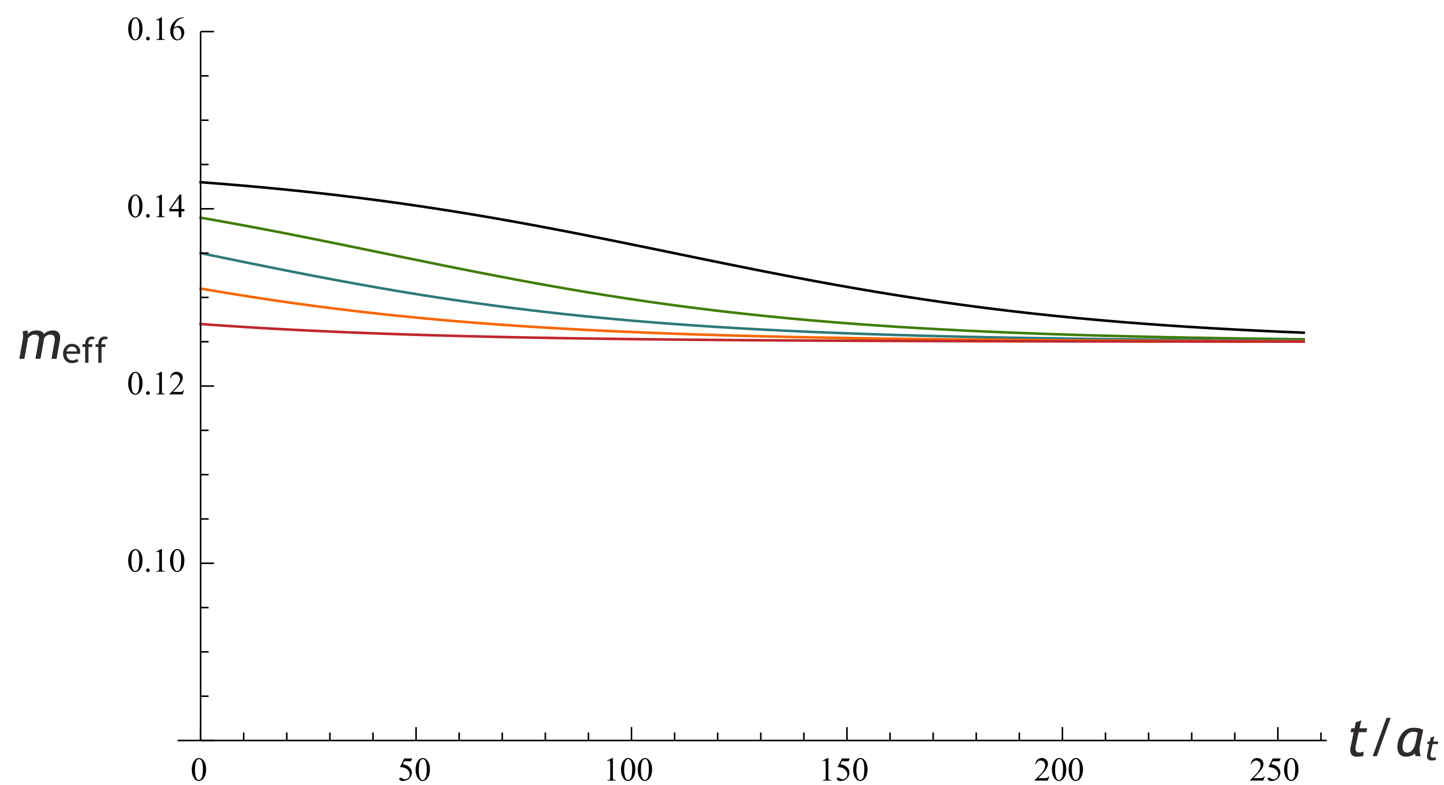
fitting gives a crude average of E_1, E_2

two-state admixture

$$C(t) = A\, e^{-0.125\, t/a_t} + (1-A)\, e^{-0.145\, t/a_t}$$







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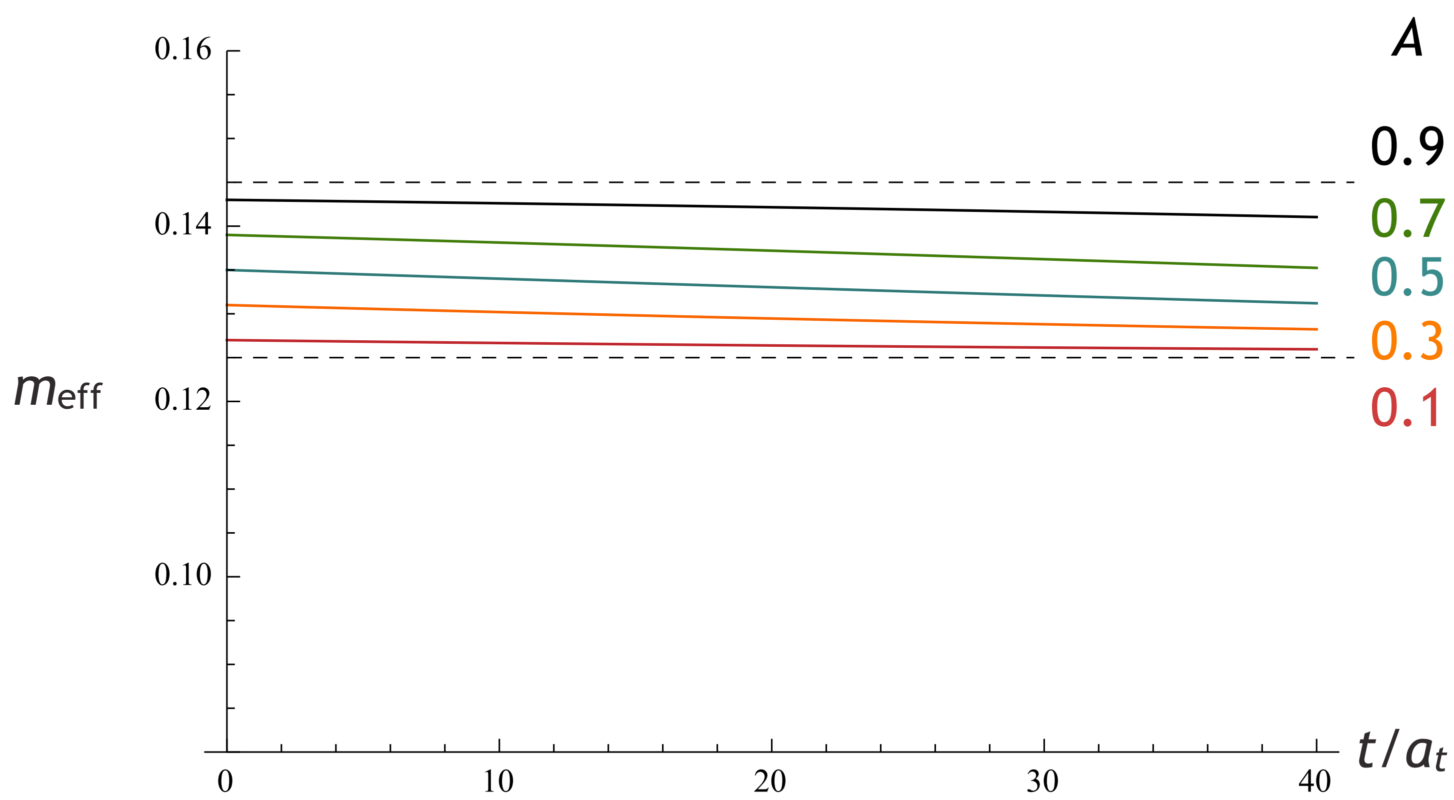
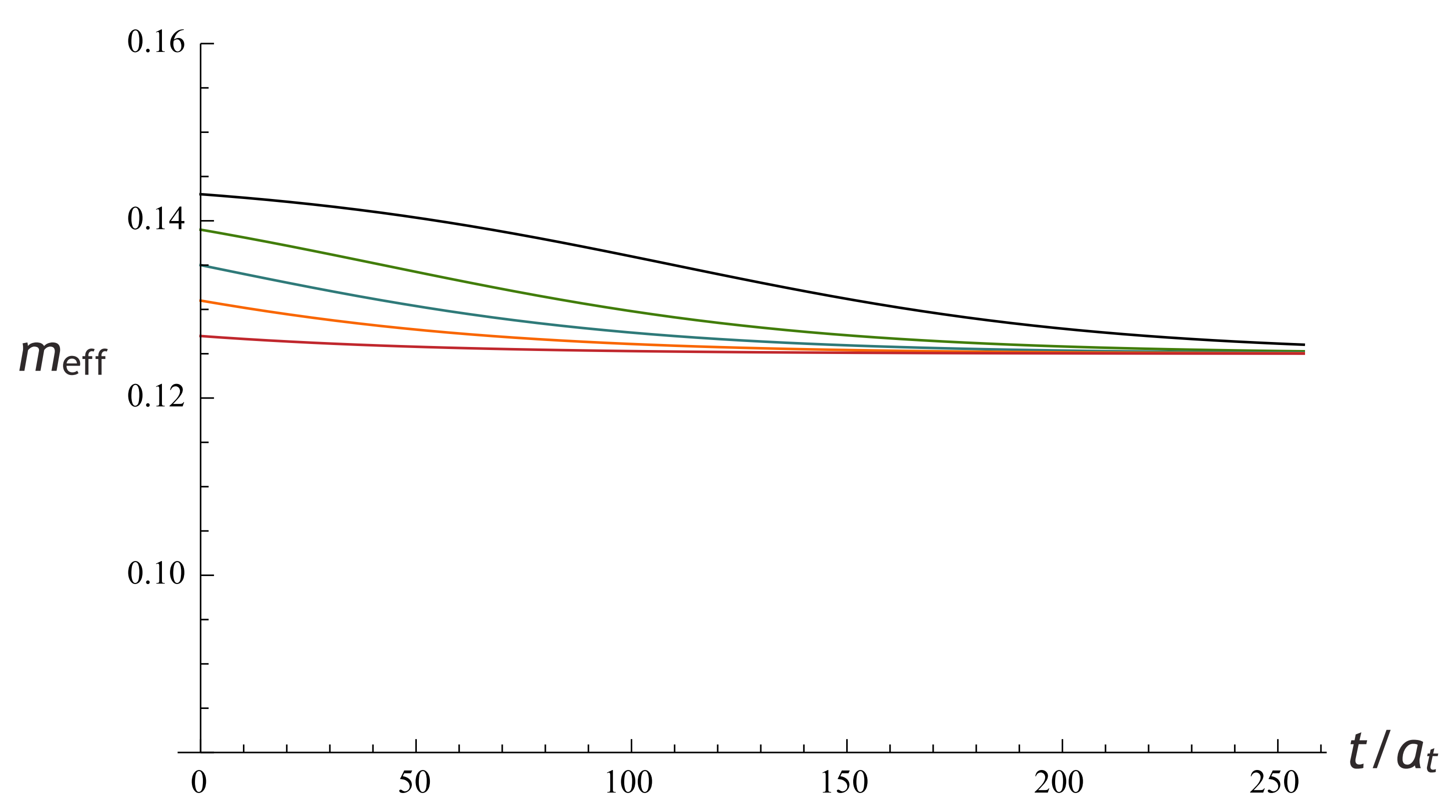
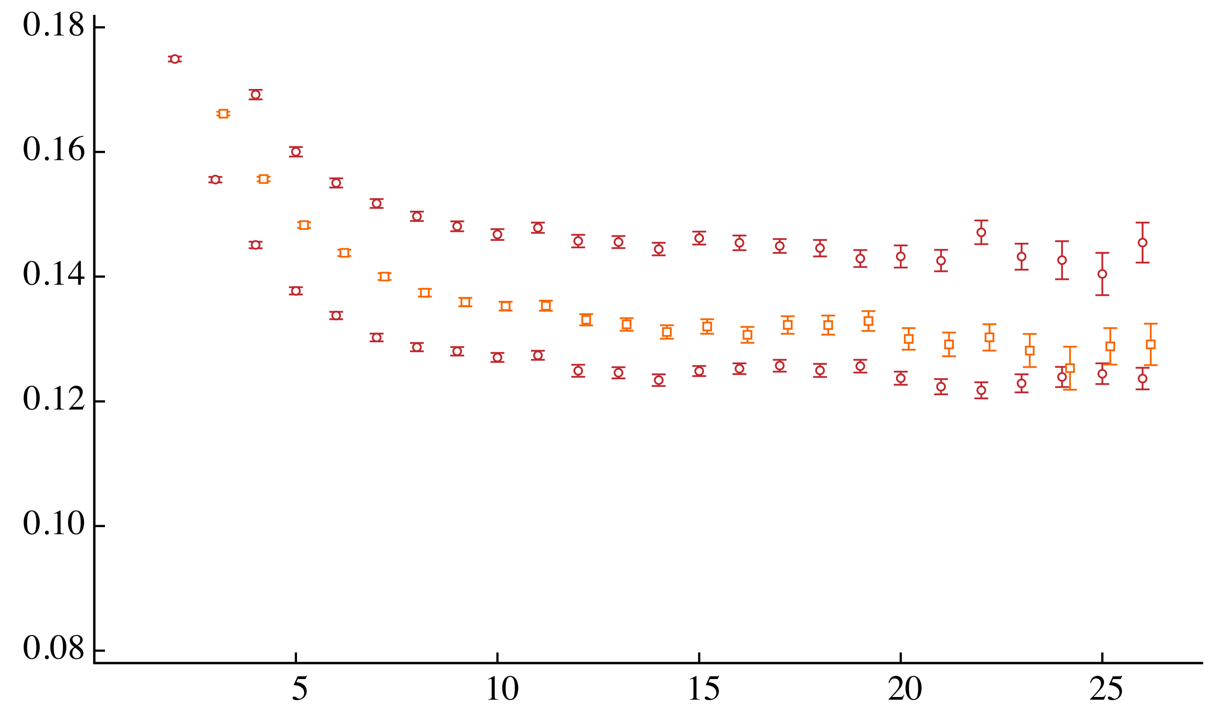






operator types ?

this explanation requires ‘single-meson’-like operators
to have negligible overlap onto ‘meson-meson’ basis states …

… why would that be ?

volume dependence !

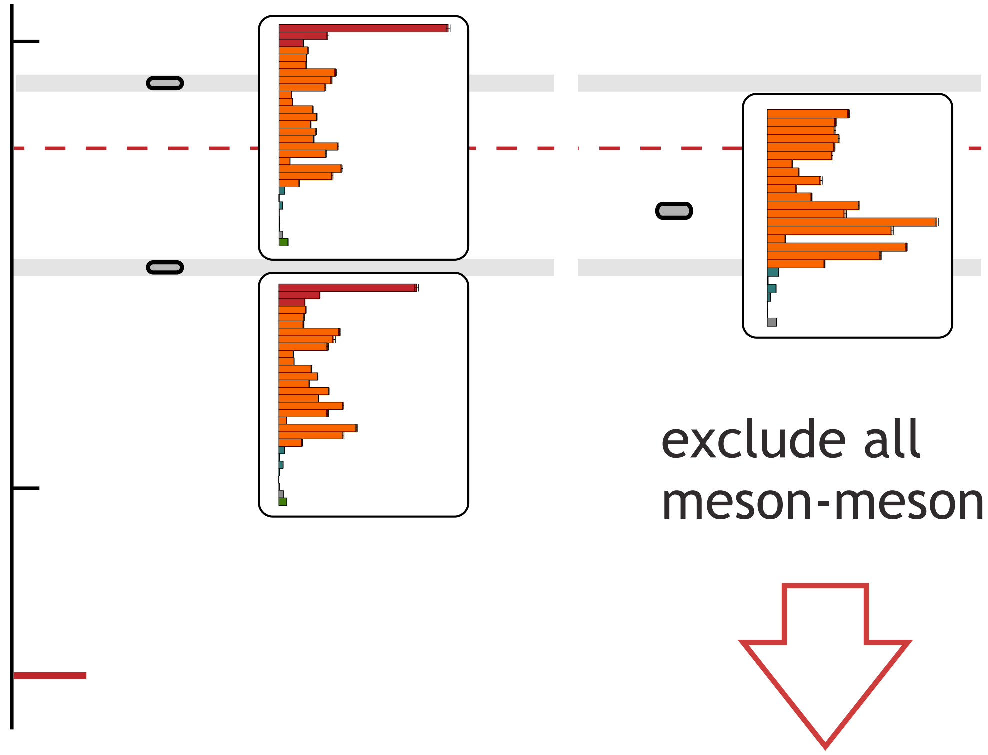
‘meson-meson’-like $\sum_{\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{x}}\,\bar{\psi}_{\mathbf{x}}\mathbf{\Gamma}\psi_{\mathbf{x}} \sum_{\mathbf{y}} e^{i\mathbf{q}\cdot\mathbf{y}}\,\bar{\psi}_{\mathbf{y}}\mathbf{\Gamma}'\psi_{\mathbf{y}}$ samples the whole volume of the lattice

‘single-meson’-like $\sum_{\mathbf{x}} e^{i\mathbf{P}\cdot\mathbf{x}}\,\bar{\psi}_{\mathbf{x}}\mathbf{\Gamma}\psi_{\mathbf{x}}$ samples a single point (translated)

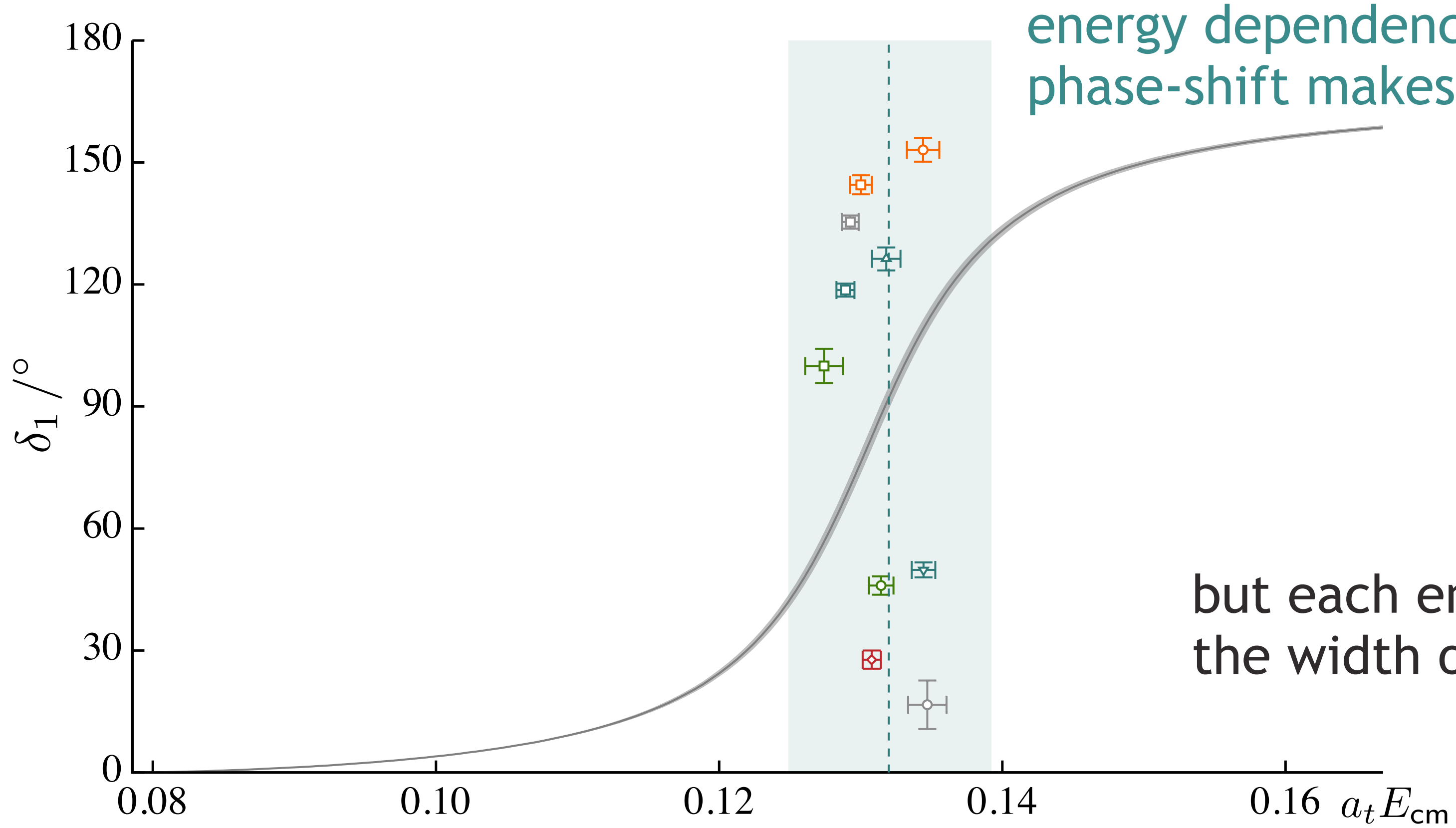
so: ‘looks-like’ = ‘has the same volume sampling as’

interesting side note:
tetraquark operators won’t work well for interpolating meson-meson components — wrong volume sampling

how bad is it really to get the wrong energies ?



exclude all meson-meson



energy dependence of the phase-shift makes no sense

but each energy within the width of the resonance ... ?

a finite cubic lattice has a smaller rotational symmetry group than an infinite continuum

simpler example of the problem: a rotationally symmetric two-dim system $\psi(r,\theta) = R_m(r)\, e^{im\theta}$

now considered on a square grid — minimum rotation is by $\pi/2$

m and *m*+4*n* transform the same !

back in 3D — **irreducible representations** of the reduced symmetry group contain multiple spins

cubic symmetry

$\Lambda(\text{dim})$	$A_1(1)$	$T_1(3)$	$T_2(3)$	$E(2)$	$A_2(1)$
J	$\mathbf{0}, 4\ldots$	$\mathbf{1}, 3, 4\ldots$	$\mathbf{2}, 3, 4\ldots$	$\mathbf{2}, 4\ldots$	$\mathbf{3}\ldots$

subduction $|\Lambda, \rho\rangle = \sum_m S^{\Lambda,\rho}_{J,m}\, |J, m\rangle$

for non-zero momentum it's even worse
— in continuum have **little group**, those rotations which don't change ***p***

⇒ label by **helicity**

can subduce helicity states into irreps of the reduced cubic symmetry

PRD85 014507 (2012)

reduction of rotational symmetry is an important feature of the quantization condition too

for elastic scattering, what we previously presented as $\cot\delta_\ell(E) = \mathcal{M}_\ell\big(E(L), L\big)$

should actually be $0 = \det\Big[\cot\delta_\ell\, \delta_{\ell,\ell'}\, \delta_{m,m'} - \mathcal{M}_{\ell m;\ell' m'}\Big]$

which when subduced becomes $0 = \det\Big[\cot\delta_\ell\, \delta_{\ell,\ell'}\, \delta_{n,n'} - \mathcal{M}^{\Lambda}_{\ell n;\ell' n}\Big]$

features **all** $\boldsymbol{\ell}$ subduced into irrep Λ

n = embedding of ℓ into Λ

what allows us to make progress is that $\delta_\ell(E) \sim k^{2\ell+1}$ at energies not too far from threshold

so higher angular momenta are naturally suppressed

in practice, truncate at some $\ell_{\max}$ ...

some technical stuff — ‘meson-meson’-like operators

what actually goes into a ‘$\pi\pi$’-like operator ?

one option for construction is to use products of single-meson operators in lattice irreps

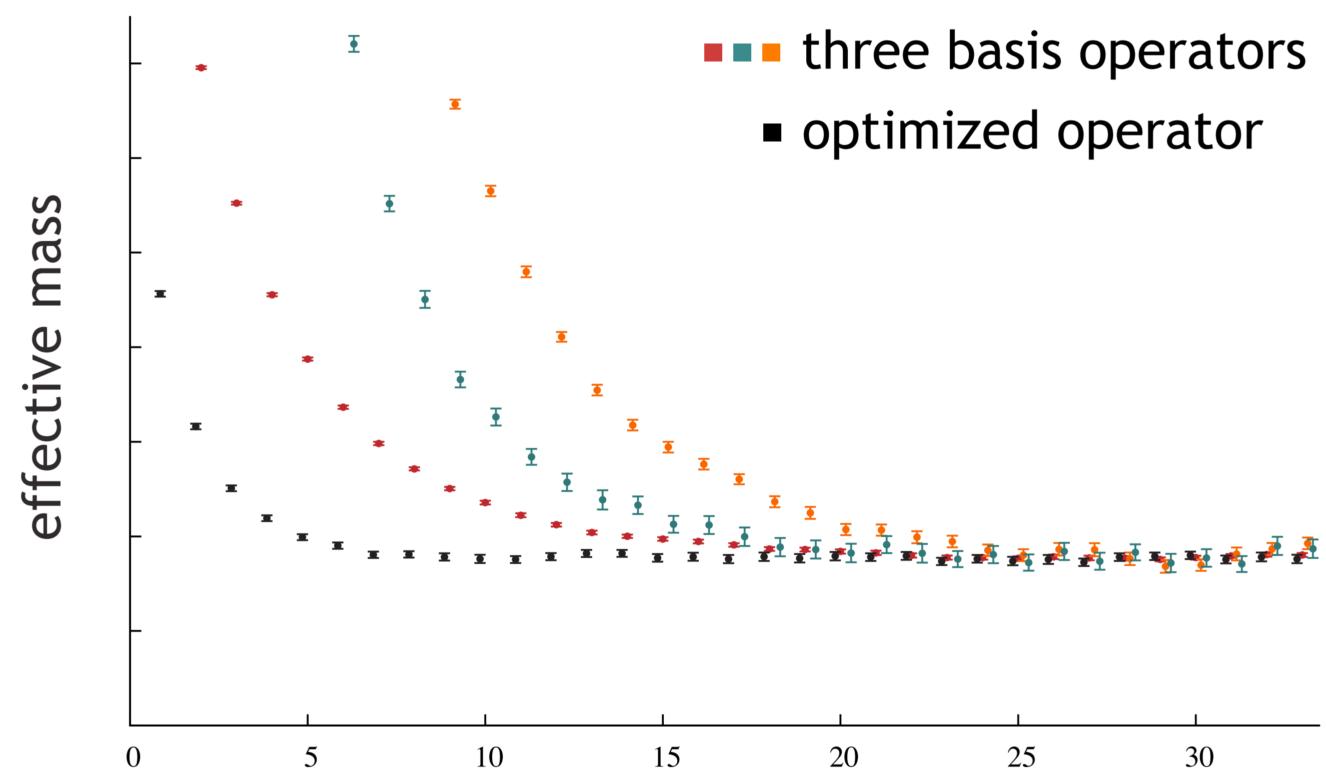
$$\sum_{\substack{\hat{\mathbf{p}}_1,\hat{\mathbf{p}}_2 \\ \mathbf{p}_1+\mathbf{p}_2=\mathbf{P}}} C_{\Lambda_1\otimes\Lambda_2\to\Lambda}(\hat{\mathbf{p}}_1,\hat{\mathbf{p}}_2)\,\pi(\mathbf{p}_1;\Lambda_1)\,\pi(\mathbf{p}_2;\Lambda_2)$$

‘lattice’ Clebsch-Gordan coefficients

some group theory to work them out — ask Christopher Thomas

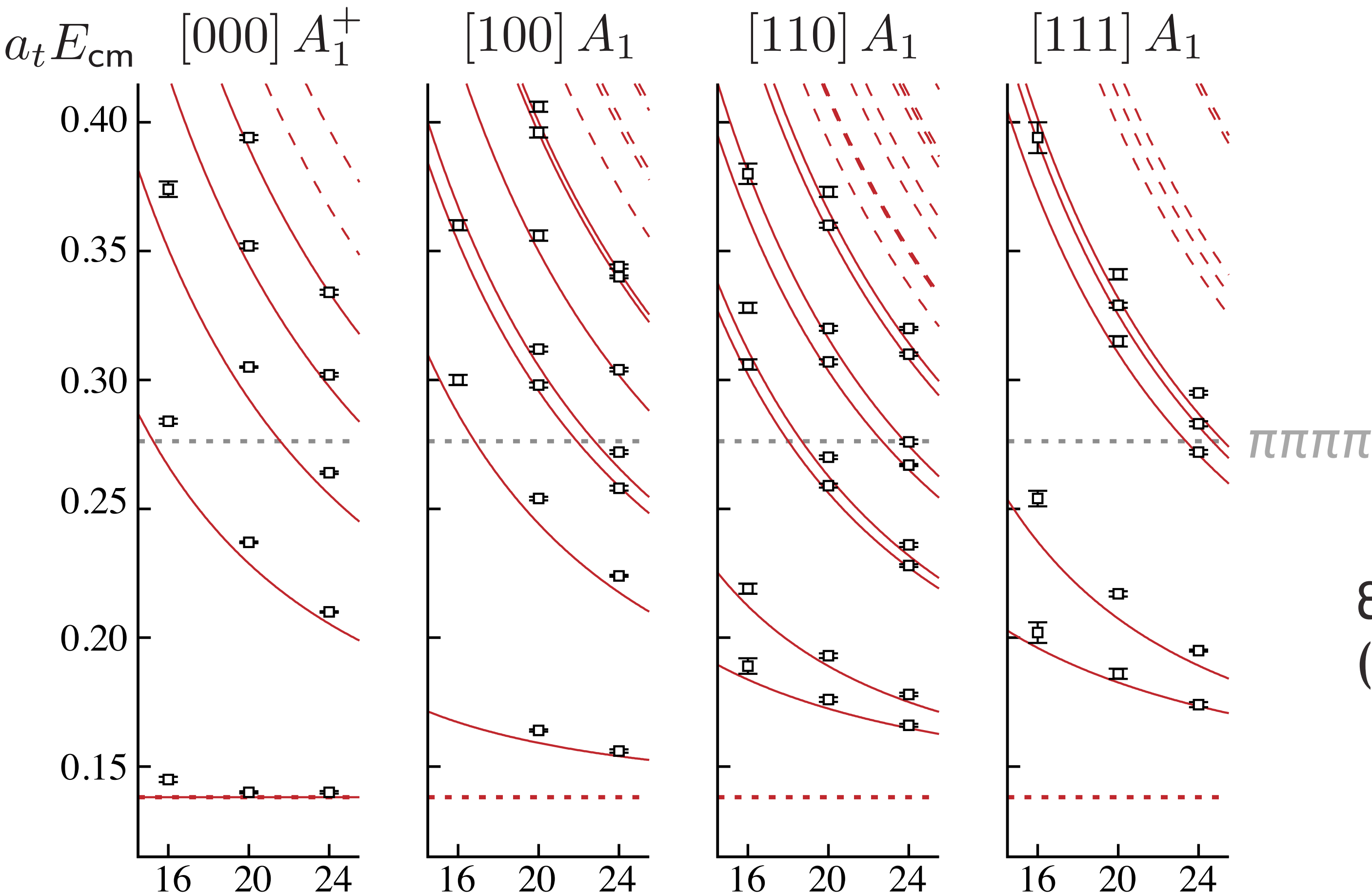
then each single-meson operator can be the **variationally optimized** one for that ***p***, *Λ*

[000] A_1^+ diagonal correlators



optimized operator saturated by the pion by timeslice 7

$\pi\pi$ isospin=2 — m_π~391 MeV



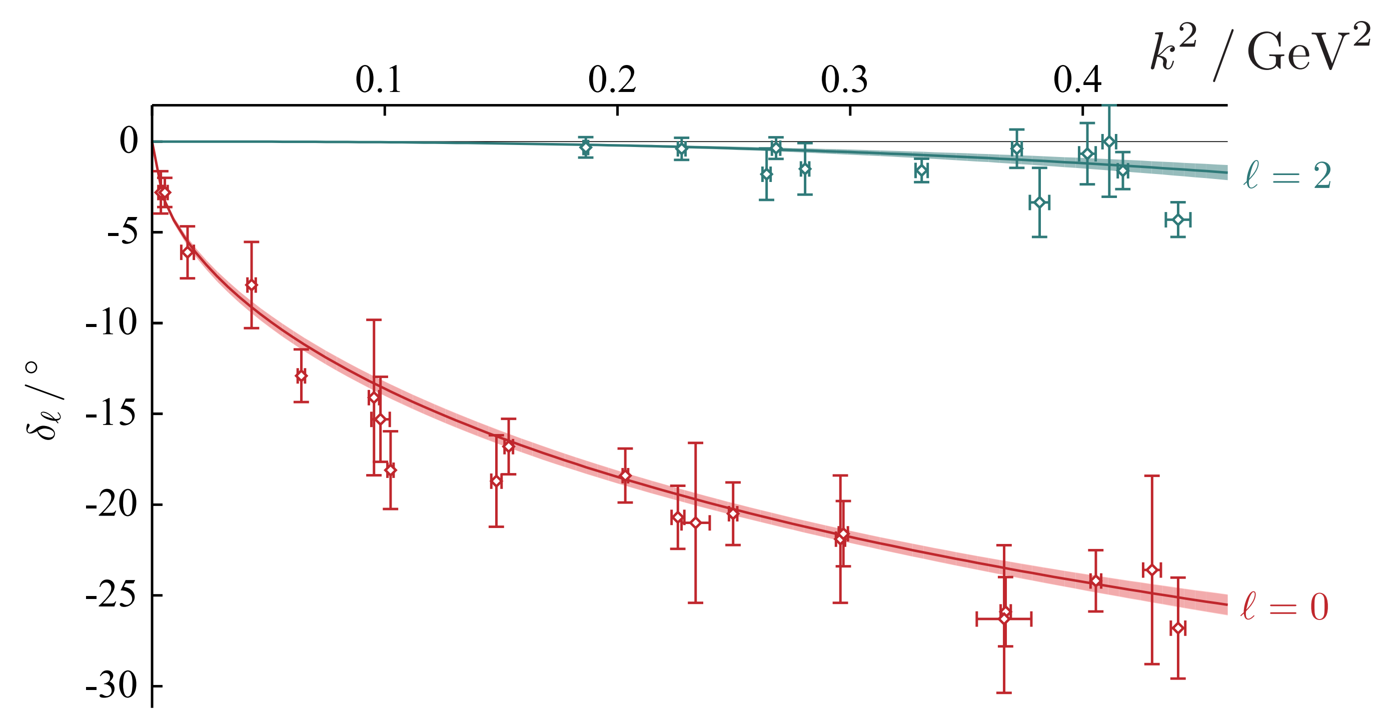
basis of $\pi\pi$ operators only

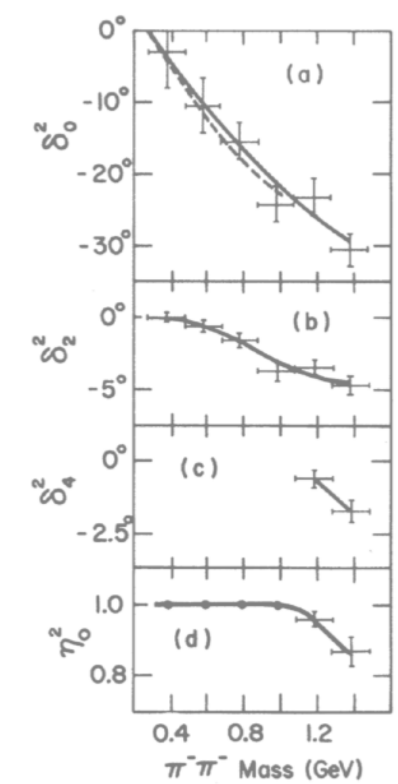
& spectra in irreps with lowest ℓ=2 (not shown here)

effective range expansion

$$k \cot\delta_0 = \frac{1}{a_0} + \frac{1}{2} r_0\, k^2 + \ldots$$

$$t \propto \frac{1}{k \cot\delta_0 - ik}$$

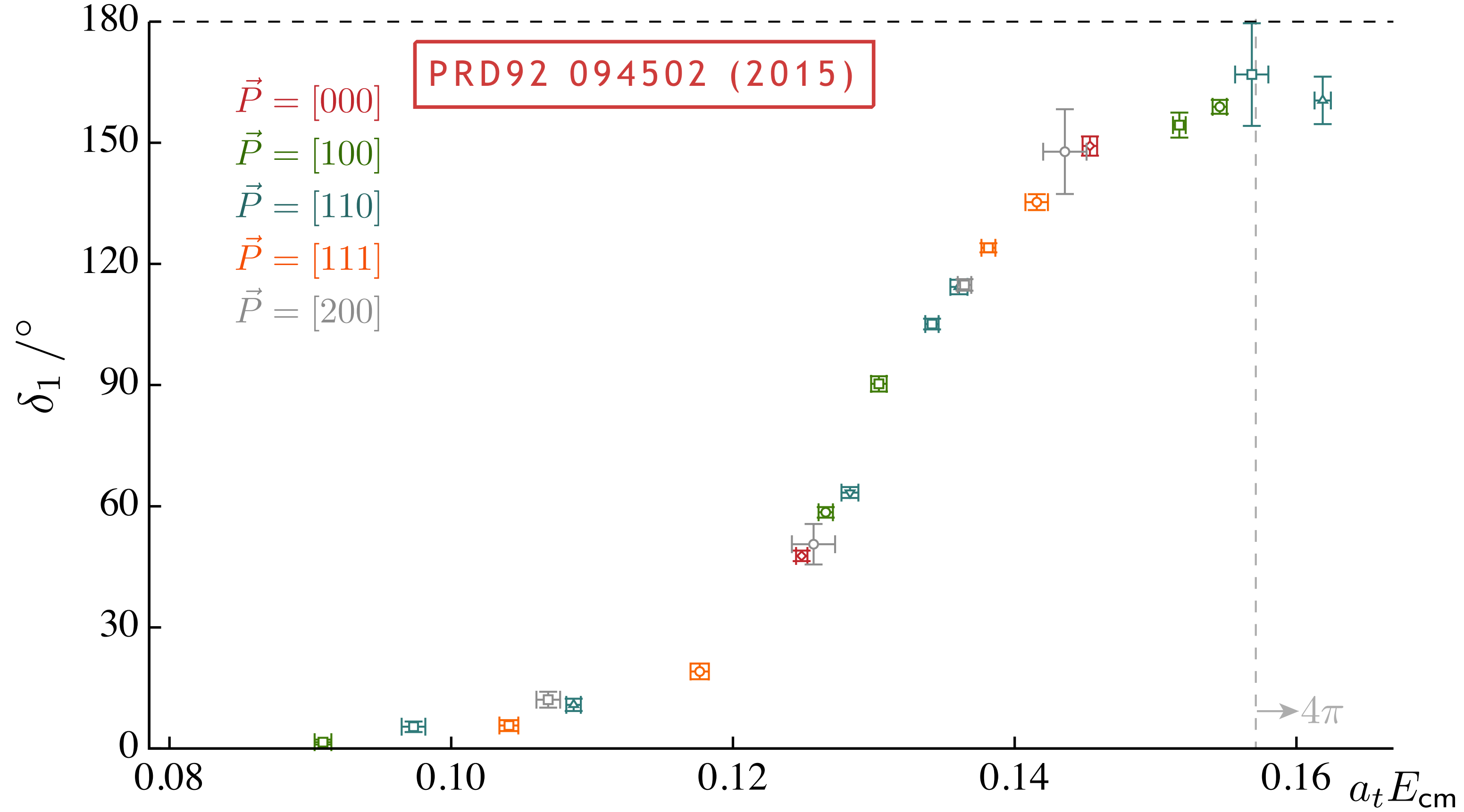




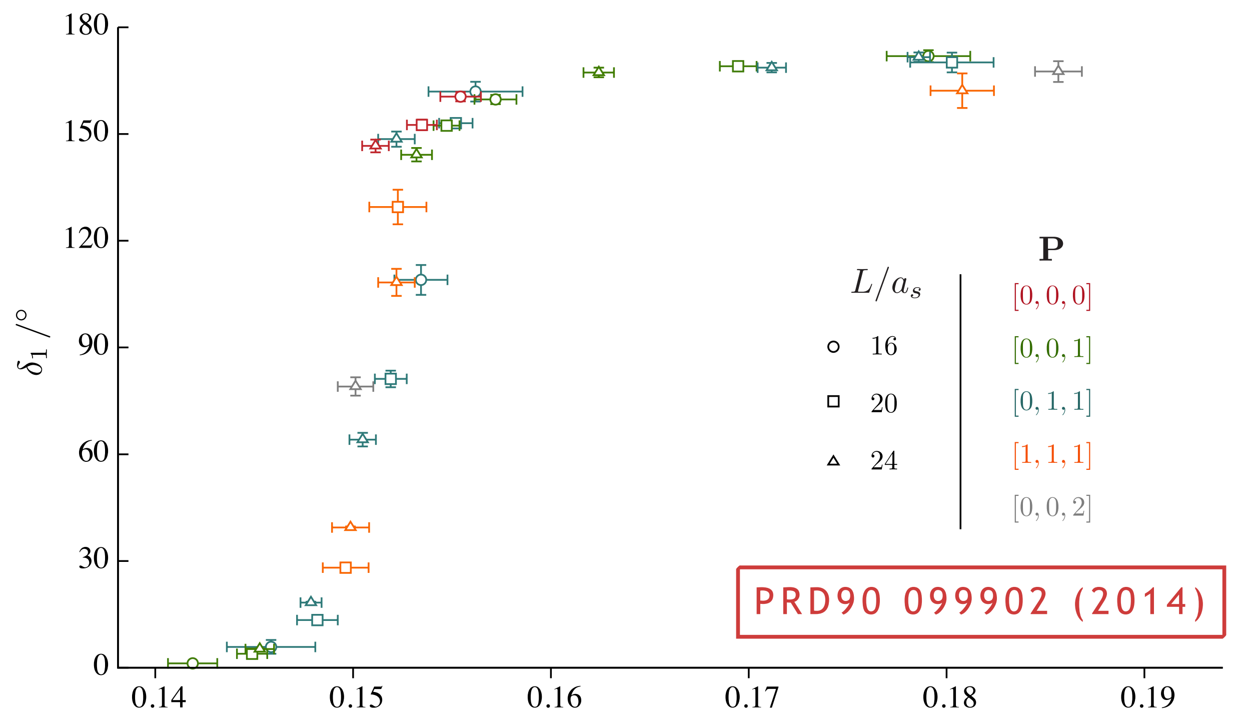
Cohen 1972

ππ isospin=1 — m_π~236 MeV, m_π~391 MeV

you saw this earlier ...



and a similar calculation at a heavier pion mass



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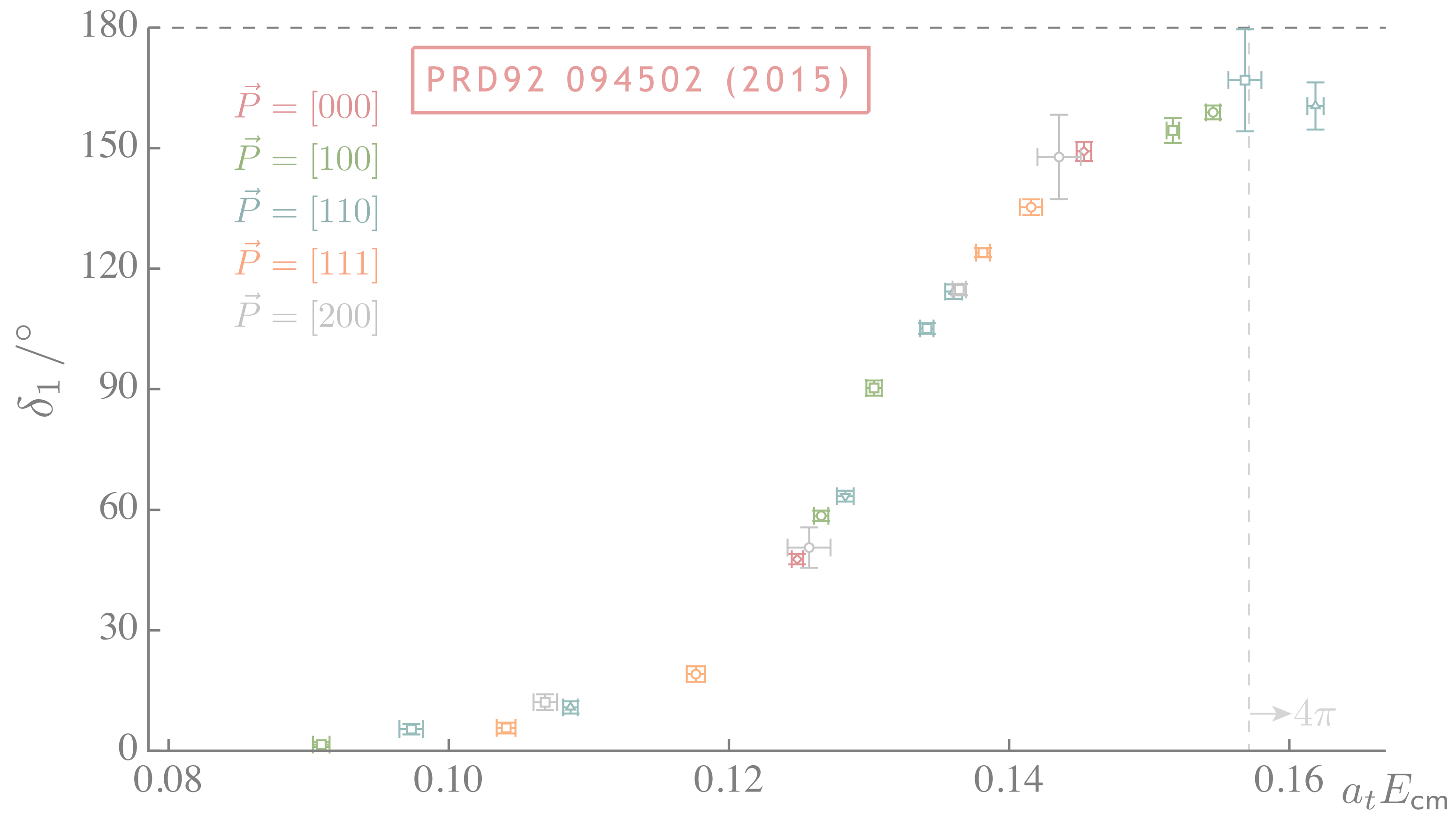
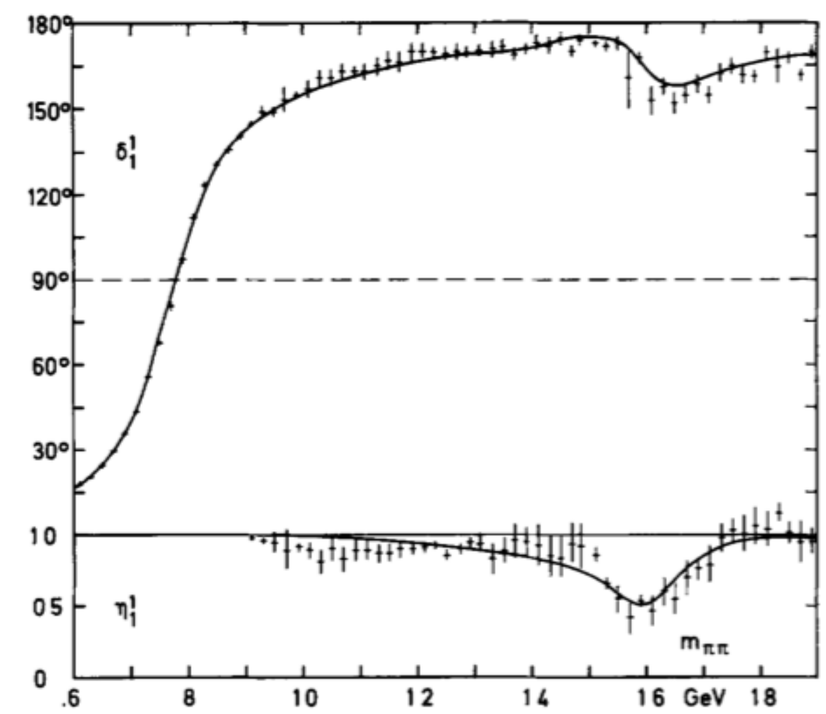
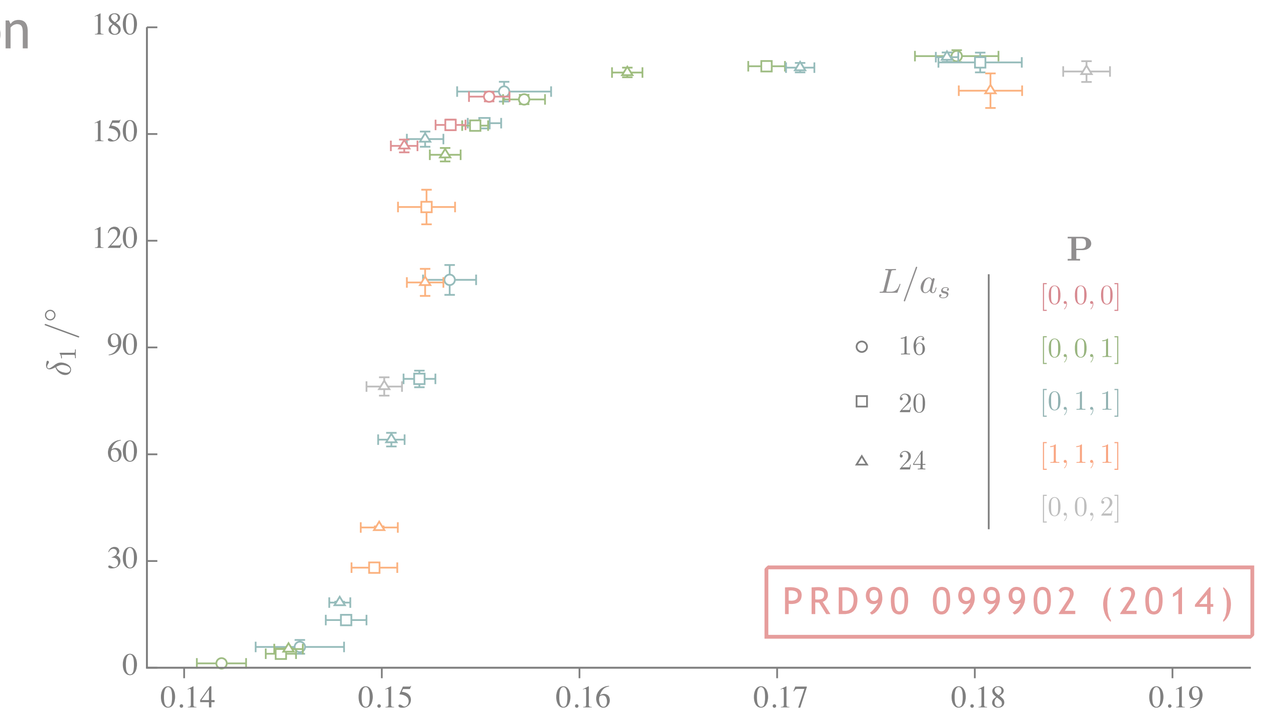




Hyams 1973

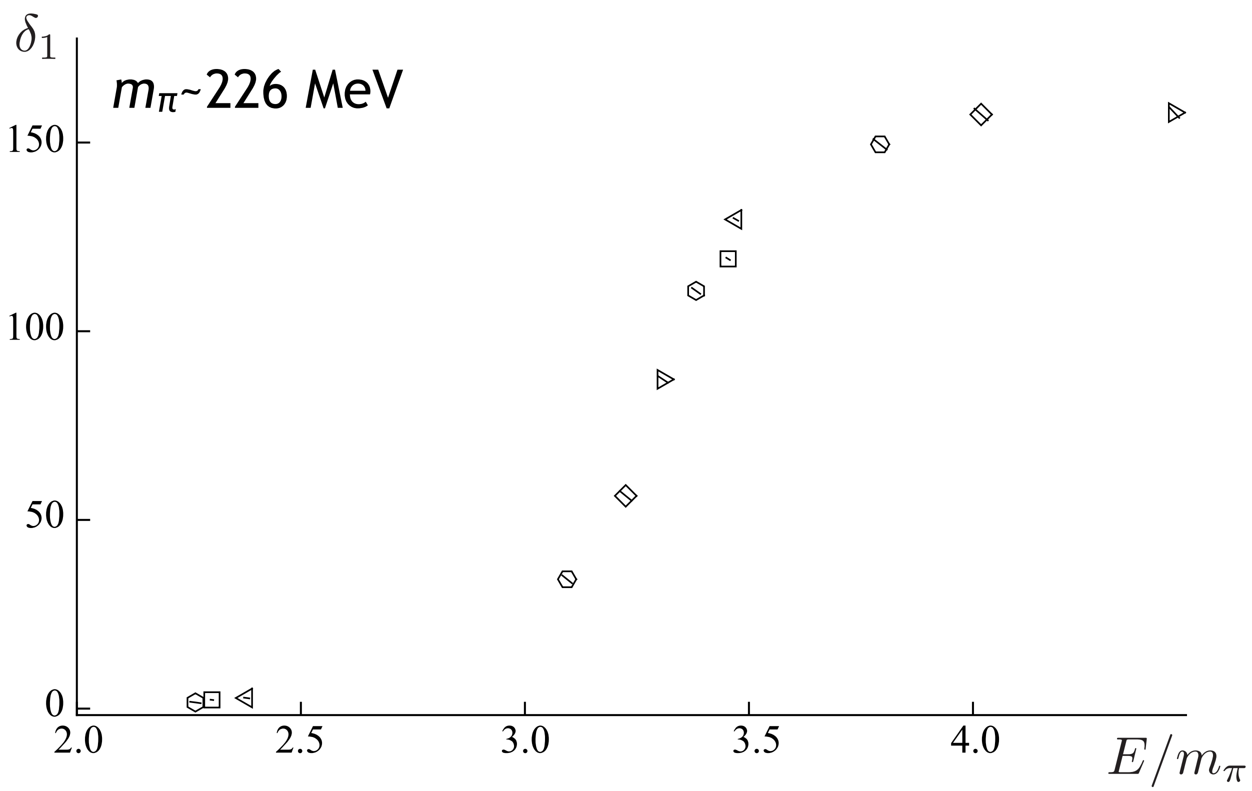
and a similar calculation at a heavier pion mass

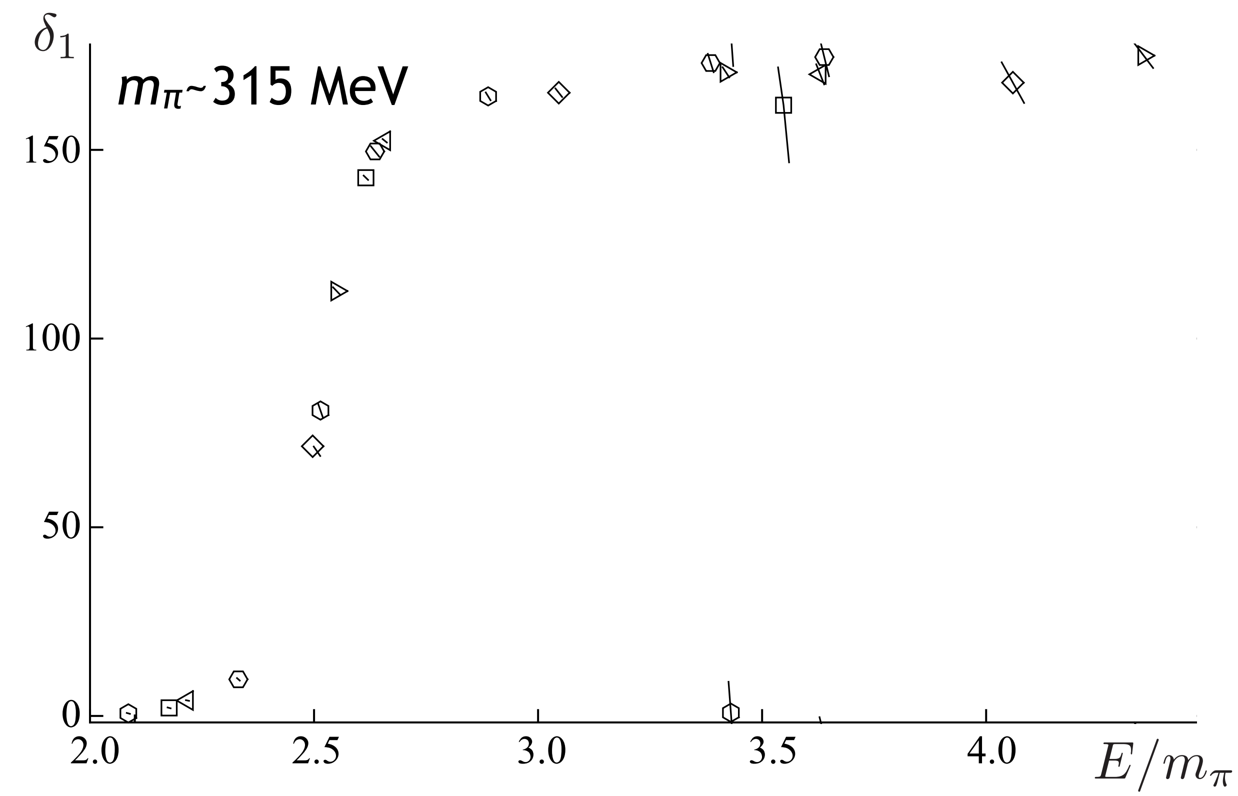


another approach uses lattices with one direction elongated

PRD94 034501 (2016)

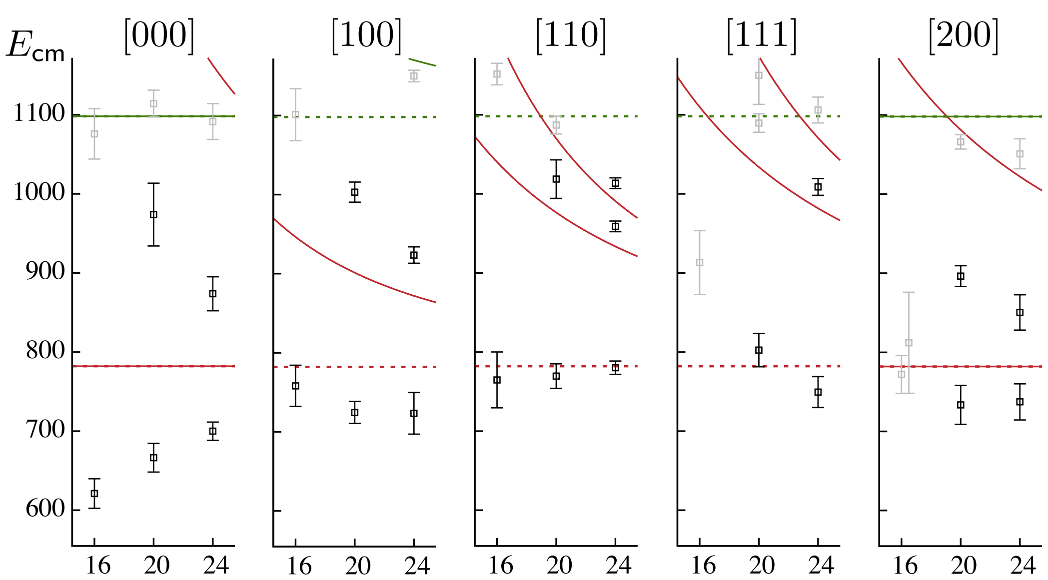
GWU group

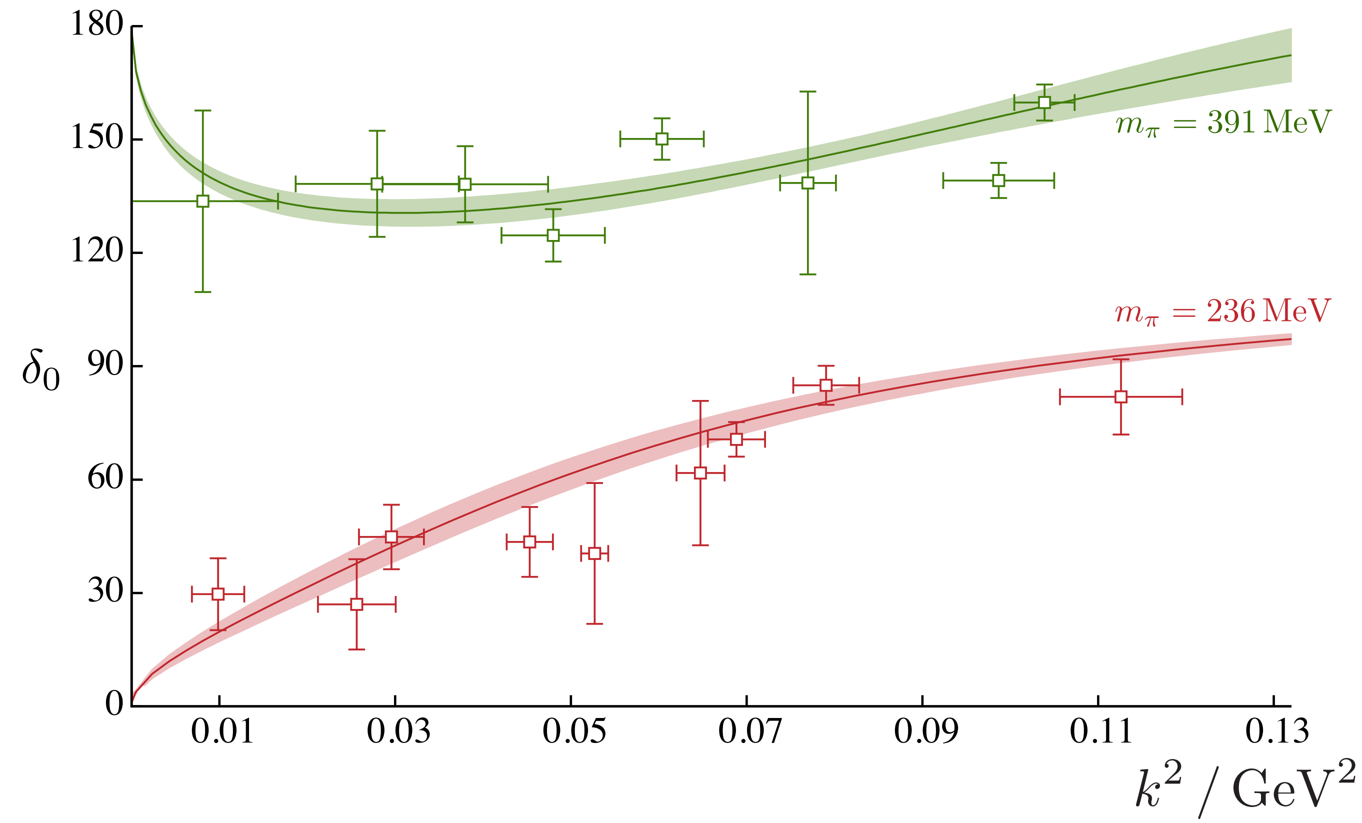




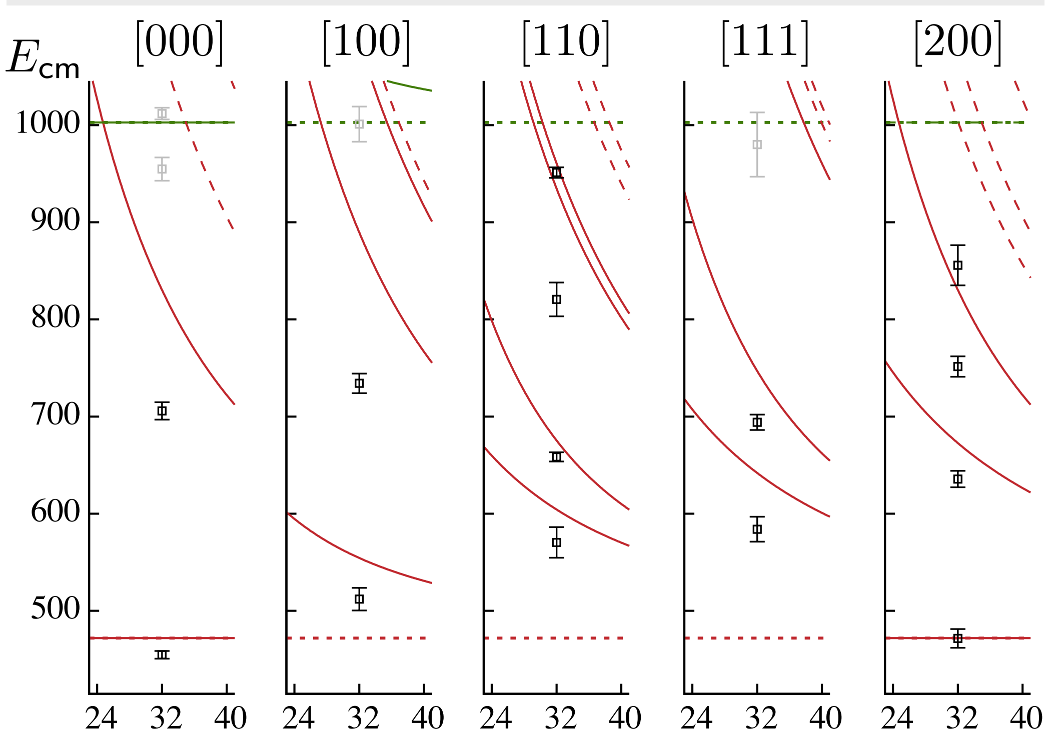
ππ isospin=0

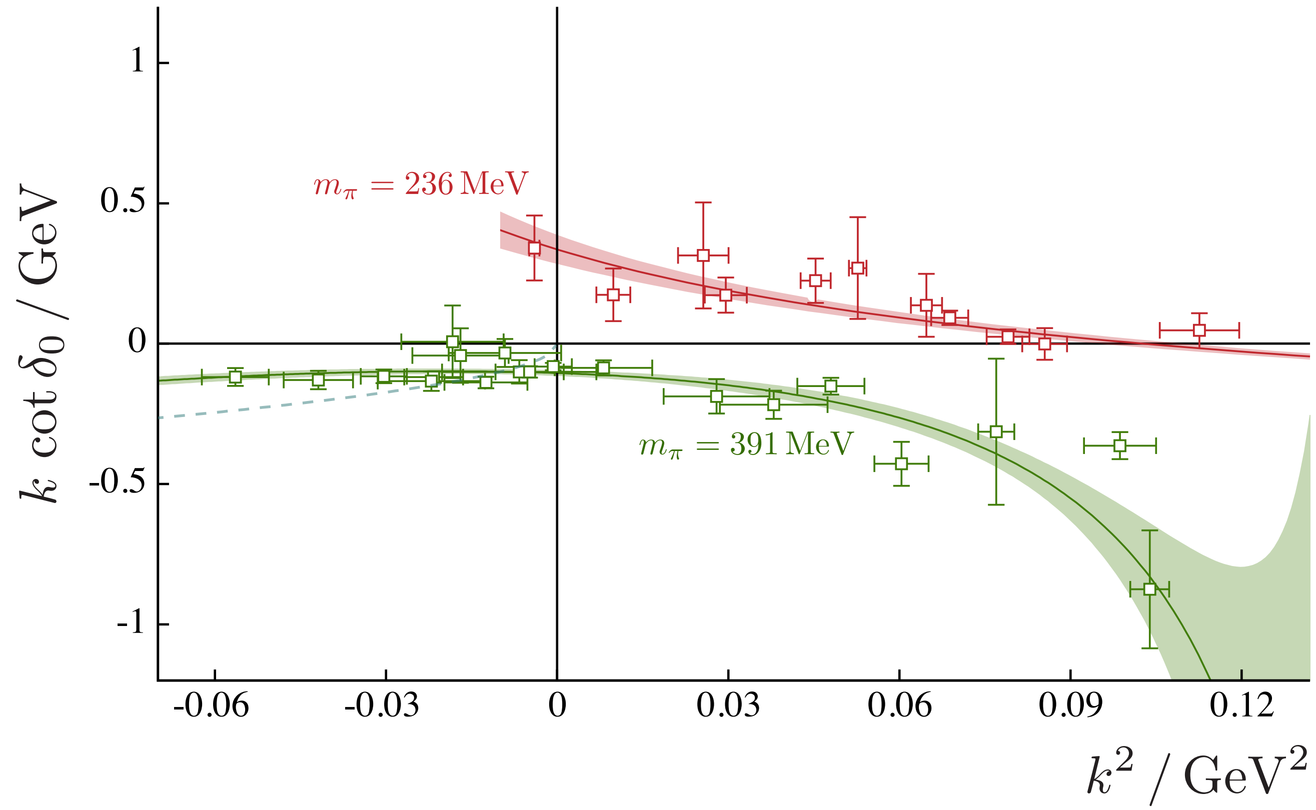
m π ~ 391 MeV



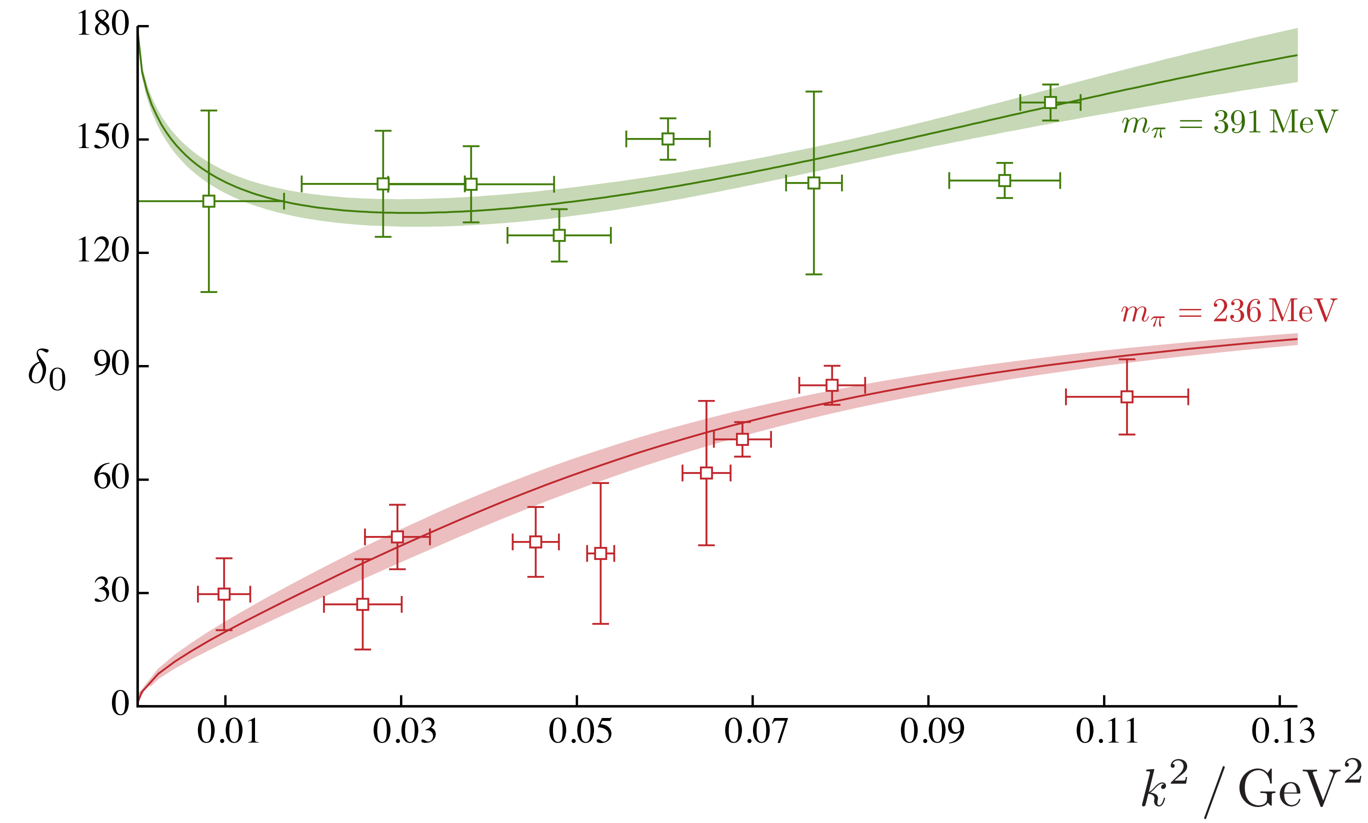


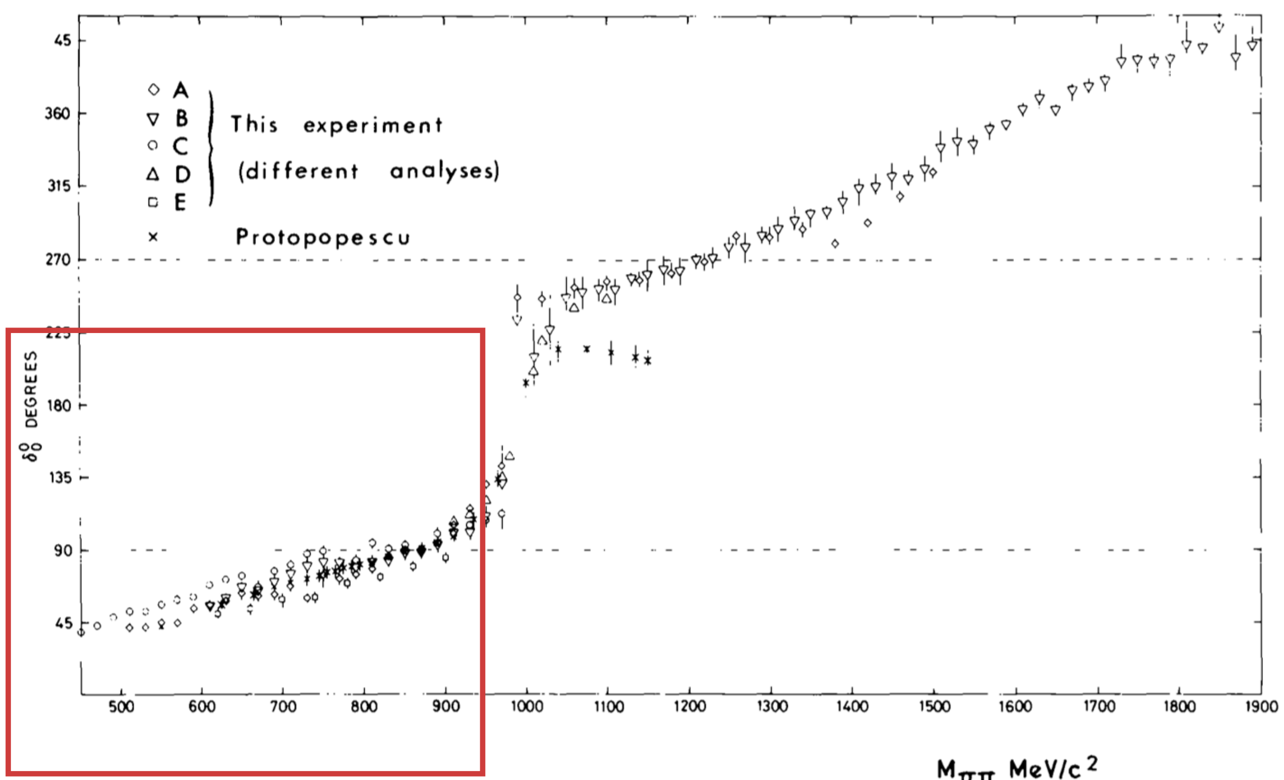
m π ~ 236 MeV



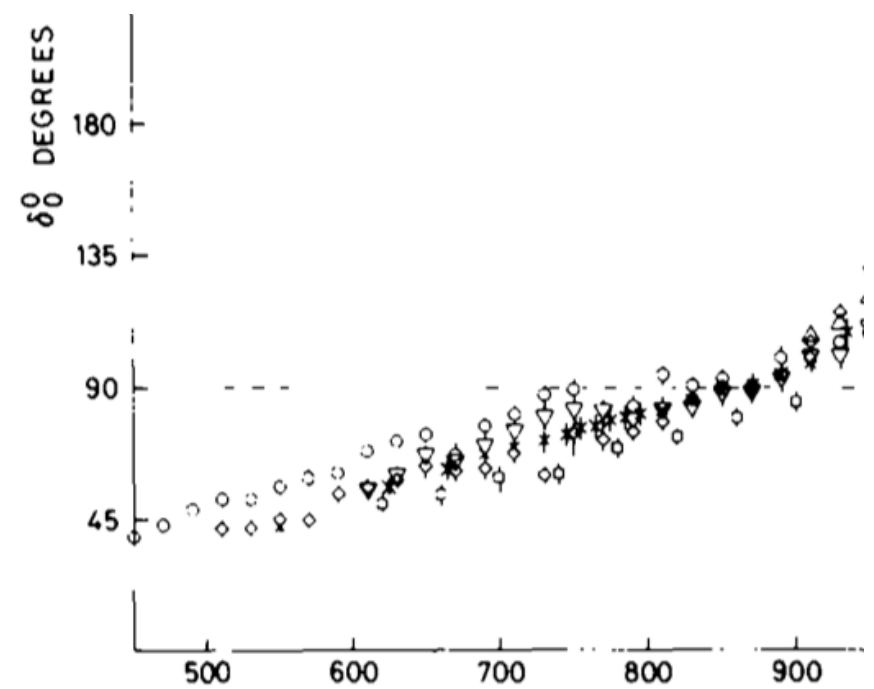


ππ isospin=0





Grayer 1974



coupled-channel scattering

Jozef Dudek

hadron spectrum collaboration
hadspec.org

William & Mary
CHARTERED 1693

Jefferson Lab
Thomas Jefferson National Accelerator Facility

evolution from scattering ‘in’ state to scattering ‘out’ state given by *S*-matrix elements $S_{ij} = \langle \mathrm{out}, i \,|\, \mathrm{in}, j \rangle$

e.g. in coupled $\pi\pi$, $K\bar{K}$ scattering $$\mathbf{S} = \begin{pmatrix} S_{\pi\pi,\pi\pi} & S_{\pi\pi,K\overline{K}} \\ S_{K\overline{K},\pi\pi} & S_{K\overline{K},K\overline{K}} \end{pmatrix}$$

more convenient to work with *t*-matrix $\mathbf{S} = \mathbf{1} + 2i\sqrt{\boldsymbol{\rho}}\cdot\mathbf{t}\cdot\sqrt{\boldsymbol{\rho}}$ typically in partial-waves $t^{(\ell)}_{ij}(E)$

in time-reversal invariant theories, ***t*** is symmetric $\Rightarrow \frac{1}{2}N(N+1)$ complex numbers at each energy?

conservation of probability, a.k.a. unitarity is an important constraint

$$\left(S^\dagger S\right)_{ij} = \sum_k \langle \mathrm{in}, i \,|\, \mathrm{out}, k\rangle\langle \mathrm{out}, k \,|\, \mathrm{in}, j\rangle = \delta_{ij}$$

completeness of outgoing states $1 = \sum_k \big|\, \mathrm{out}, k\rangle\langle \mathrm{out}, k \,\big|$

$$\mathrm{Im}\, t_{ij} = \sum_k t^*_{ik}\, \rho_k\, t_{kj}$$ sum over channels kinematically open

or $$\boxed{\mathrm{Im}\left(t^{-1}(E)\right)_{ij} = -\delta_{ij}\,\rho_i(E)\,\Theta(E - E_i^{\mathrm{thr.}})}$$

$\Rightarrow \frac{1}{2}N(N+1)$ real numbers at each energy a common parameterization uses two phase-shifts, δ_1, δ_2, and an inelastiticity, η

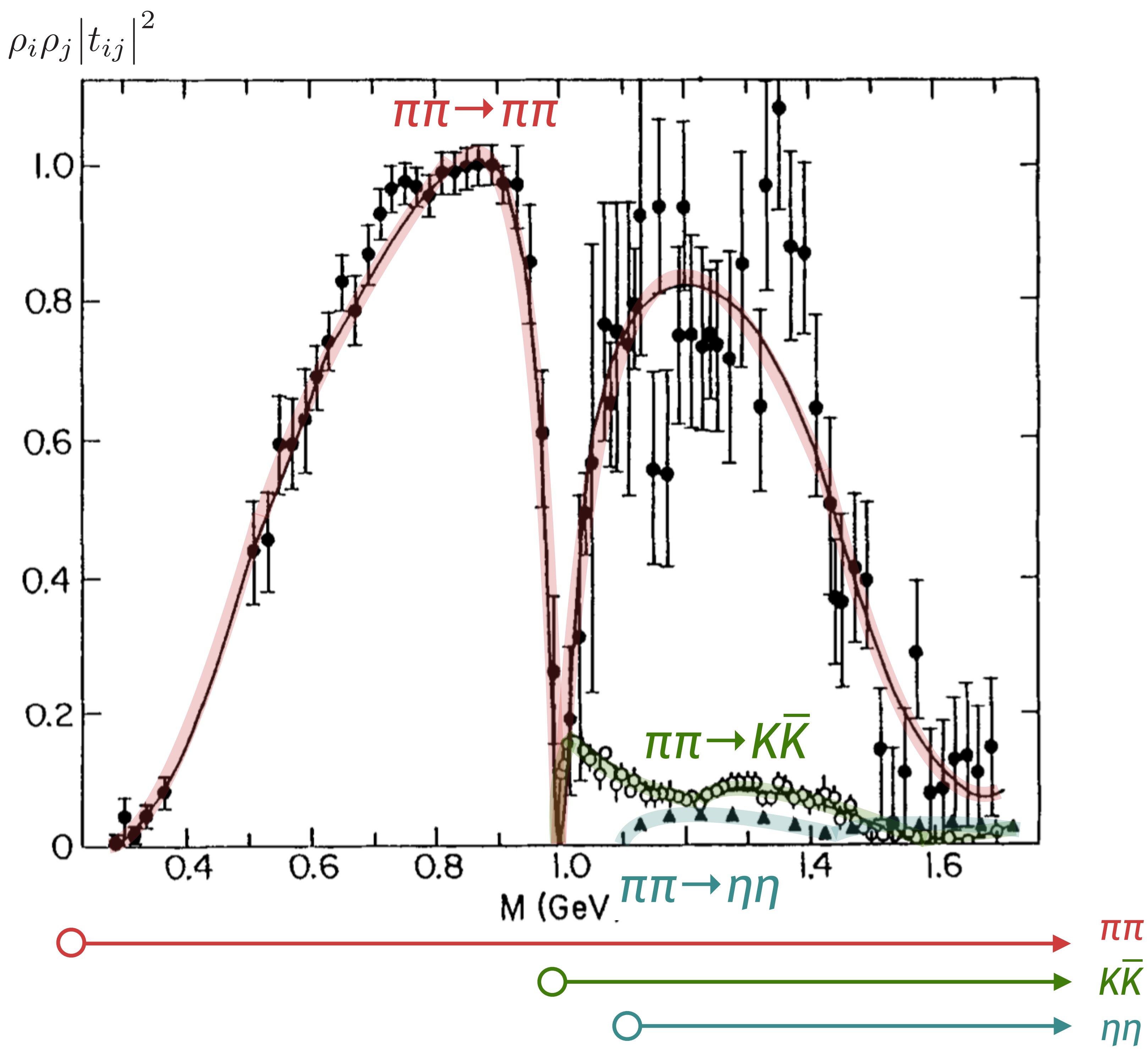
$$S = \begin{pmatrix} \eta\, e^{2i\delta_1} & i\sqrt{1-\eta^2}\, e^{i(\delta_1+\delta_2)} \\ i\sqrt{1-\eta^2}\, e^{i(\delta_1+\delta_2)} & \eta\, e^{2i\delta_2} \end{pmatrix}$$

$$t_{11} = \frac{1}{\rho_1} e^{i\delta_1} \left[\tfrac{1}{2}(\eta+1)\sin\delta_1 - \tfrac{i}{2}(\eta-1)\cos\delta_1 \right]$$

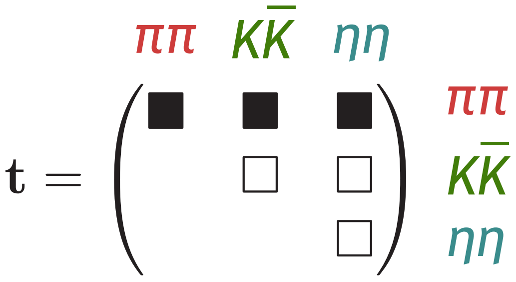
elastic form regained if $\eta \to 1$

$$\rho_1 \rho_2 \left| t_{12} \right|^2 = 1 - \eta^2$$

$\pi\pi$, $K\bar{K}$, $\eta\eta$ S-wave scattering



experimentally
quite difficult to fill out
the whole matrix

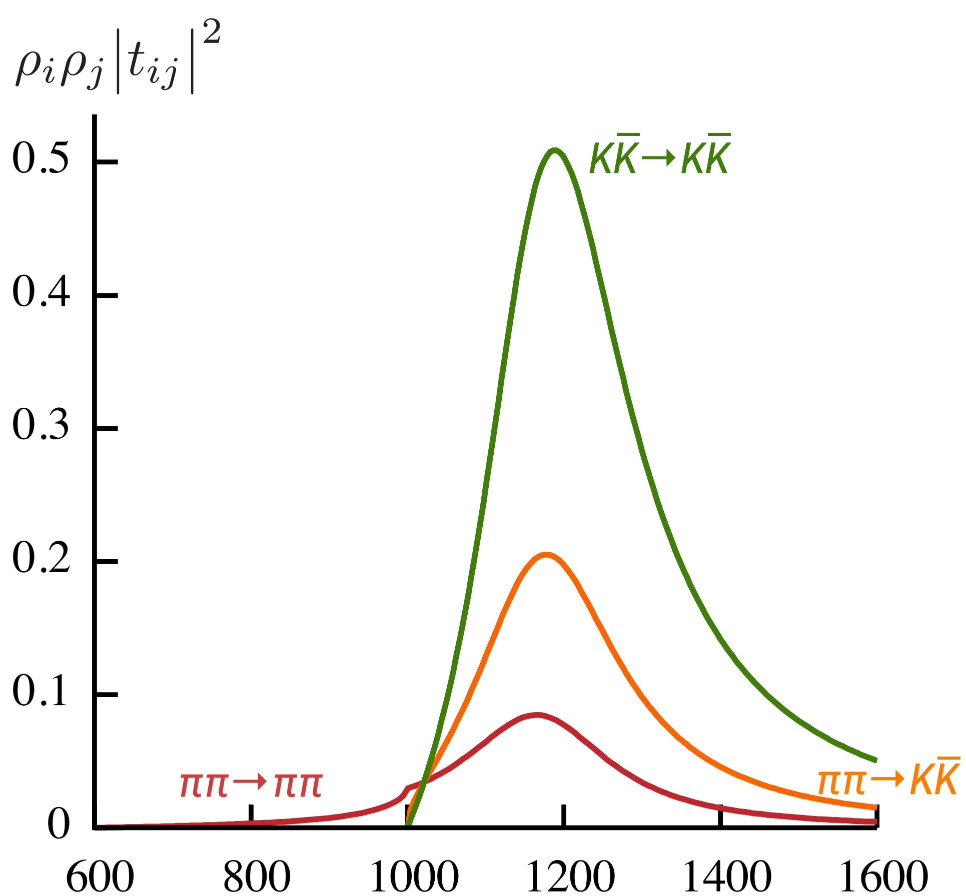


coupled-channel scattering — a simple resonance model

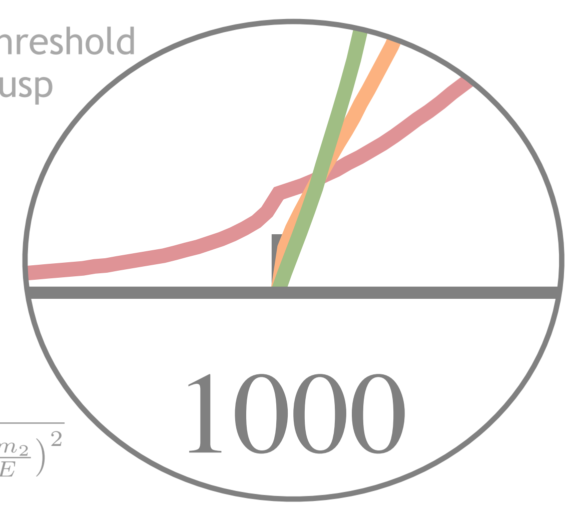
Flatté form — coupled-channel generalisation of Breit-Wigner

m_π = 300 MeV
m_K = 500 MeV

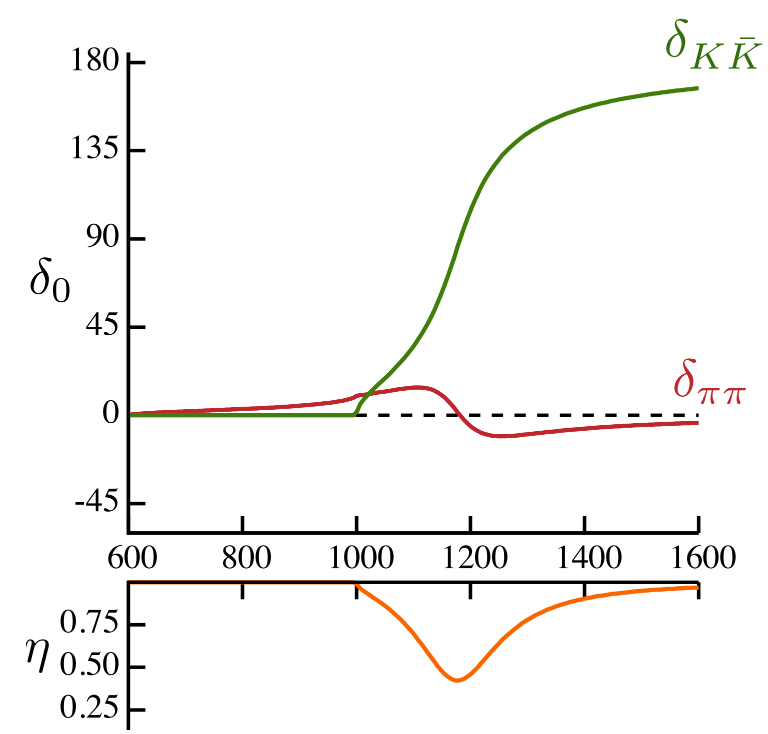
$$t_{ij}(E) = \frac{g_i\, g_j}{m^2 - E^2 - i g_1^2\, \rho_1 - i g_2^2\, \rho_2}$$



threshold cusp



$\rho_2(E) = \sqrt{1 - \left(\frac{2m_2}{E}\right)^2}$



m = 1182 MeV
$g_{\pi\pi}$ = 296 MeV
g_{KK} = 592 MeV

coupled-channel scattering in a finite-volume

the quantization condition generalizes to

$$0 = \det\left[\mathbf{1} + i\boldsymbol{\rho}\cdot\mathbf{t}\cdot\left(\mathbf{1} + i\boldsymbol{\mathcal{M}}\right)\right]$$

e.g. in A_1^+ irrep ($\ell = 0, 4 \ldots$)

$$\mathbf{t} = \begin{pmatrix} \begin{pmatrix} t^{(0)}_{11} & t^{(0)}_{12} \\ t^{(0)}_{12} & t^{(0)}_{22} \end{pmatrix} & \mathbf{0} & \cdots \\ \mathbf{0} & \begin{pmatrix} t^{(4)}_{11} & t^{(4)}_{12} \\ t^{(4)}_{12} & t^{(4)}_{22} \end{pmatrix} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

dense in channel space
— infinite-volume dynamics mixes channels

diagonal in angular momentum space
— ℓ good q.n. in infinite-volume

$$\boldsymbol{\mathcal{M}} = \begin{pmatrix} \begin{pmatrix} \mathcal{M}^{A_1^+}_{00}(k_1) & 0 \\ 0 & \mathcal{M}^{A_1^+}_{00}(k_2) \end{pmatrix} & \begin{pmatrix} \mathcal{M}^{A_1^+}_{04}(k_1) & 0 \\ 0 & \mathcal{M}^{A_1^+}_{04}(k_2) \end{pmatrix} & \cdots \\ \begin{pmatrix} \mathcal{M}^{A_1^+}_{40}(k_1) & 0 \\ 0 & \mathcal{M}^{A_1^+}_{40}(k_2) \end{pmatrix} & \begin{pmatrix} \mathcal{M}^{A_1^+}_{44}(k_1) & 0 \\ 0 & \mathcal{M}^{A_1^+}_{44}(k_2) \end{pmatrix} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

diagonal in channel space
— no dynamics

dense in angular momentum
— cubic symmetry lives here

$$k_1 = \tfrac{1}{2}\sqrt{E^2 - 4m_1^2}$$

$$k_2 = \tfrac{1}{2}\sqrt{E^2 - 4m_2^2}$$

the quantization condition generalizes to

$$0 = \det\left[\mathbf{1} + i\boldsymbol{\rho}\cdot\mathbf{t}\cdot\left(\mathbf{1} + i\boldsymbol{\mathcal{M}}\right)\right]$$

can also be expressed as $0 = \det\left[\mathbf{t}^{-1} + i\boldsymbol{\rho} - \boldsymbol{\mathcal{M}}\cdot\boldsymbol{\rho})\right]$

which exposes the role of unitarity $\mathrm{Im}\left(t^{-1}(E)\right)_{ij} = -\delta_{ij}\,\rho_i(E)\,\Theta(E - E_i^{\mathrm{thr.}})$

the quantization condition is a single real condition:

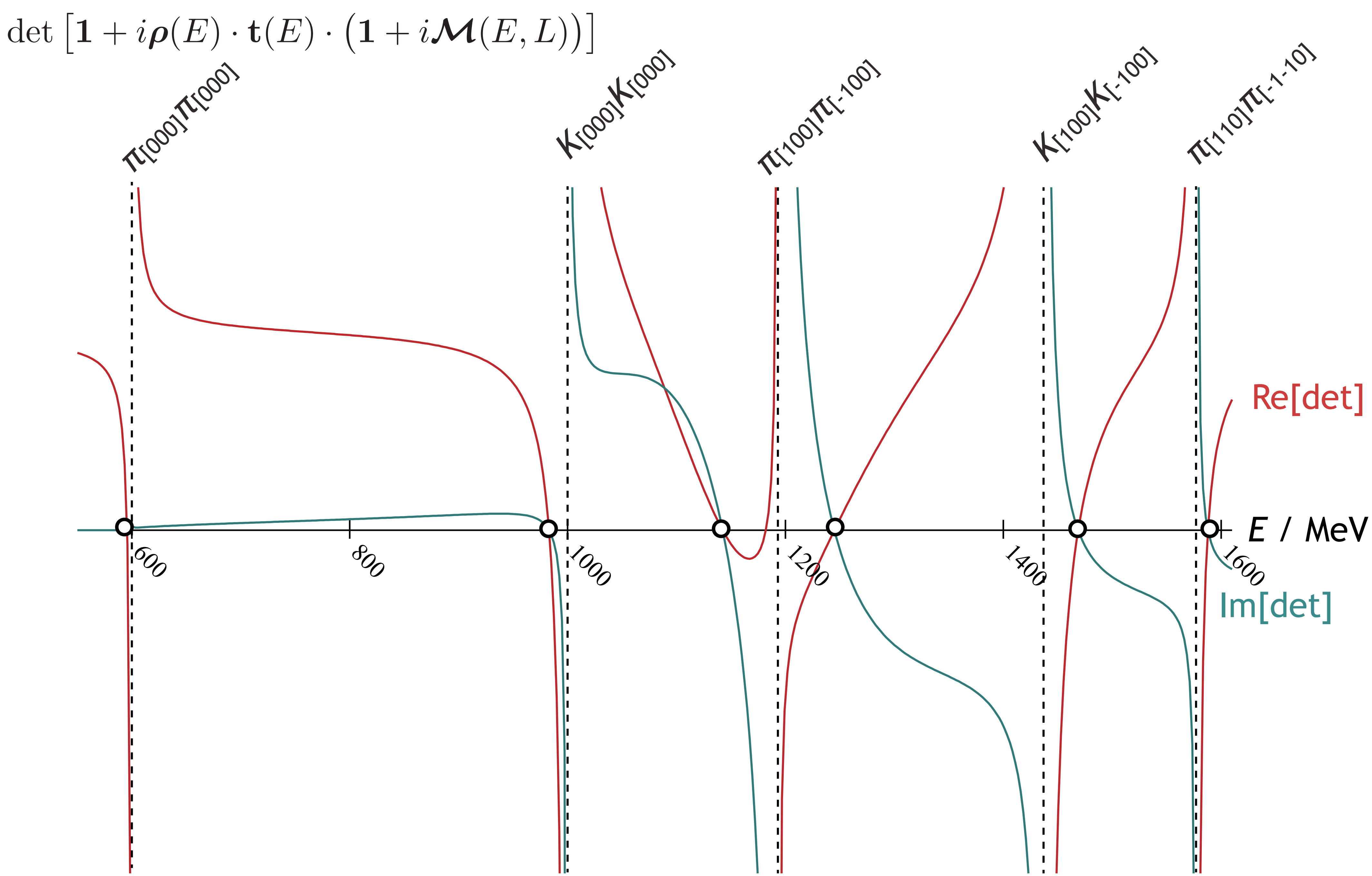
the zeroes $E=E_n(L)$ of the function $\det\left[\mathbf{1} + i\boldsymbol{\rho}(E)\cdot\mathbf{t}(E)\cdot\left(\mathbf{1} + i\boldsymbol{\mathcal{M}}(E,L)\right)\right]$

correspond to the spectrum in an $L\times L\times L$ volume

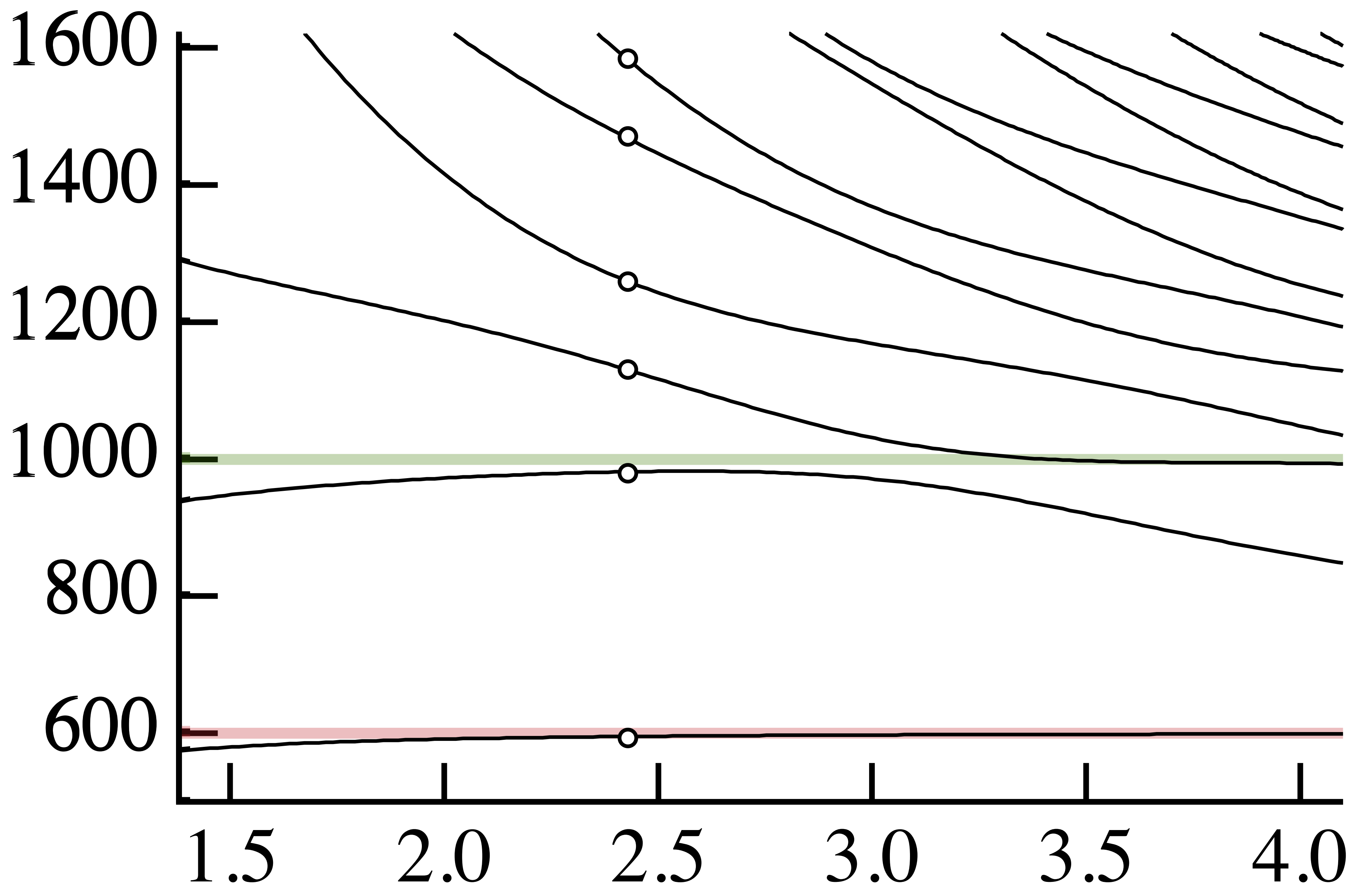
zeroes of the determinant

e.g. previously presented two-channel Flatté form — [000] A_1^+ irrep in L=2.4 fm box

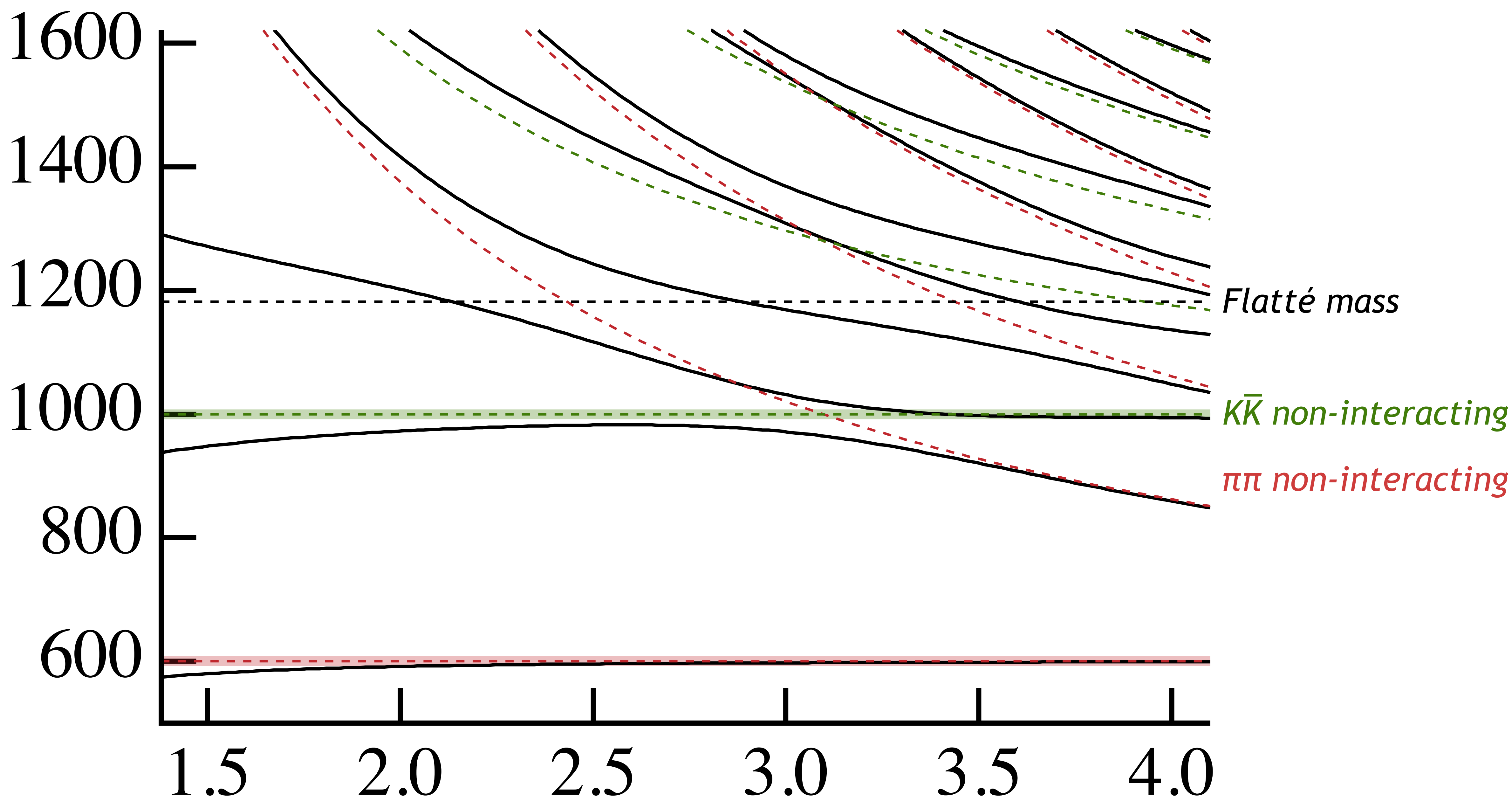
m_π = 300 MeV
m_K = 500 MeV



zeroes of the determinant

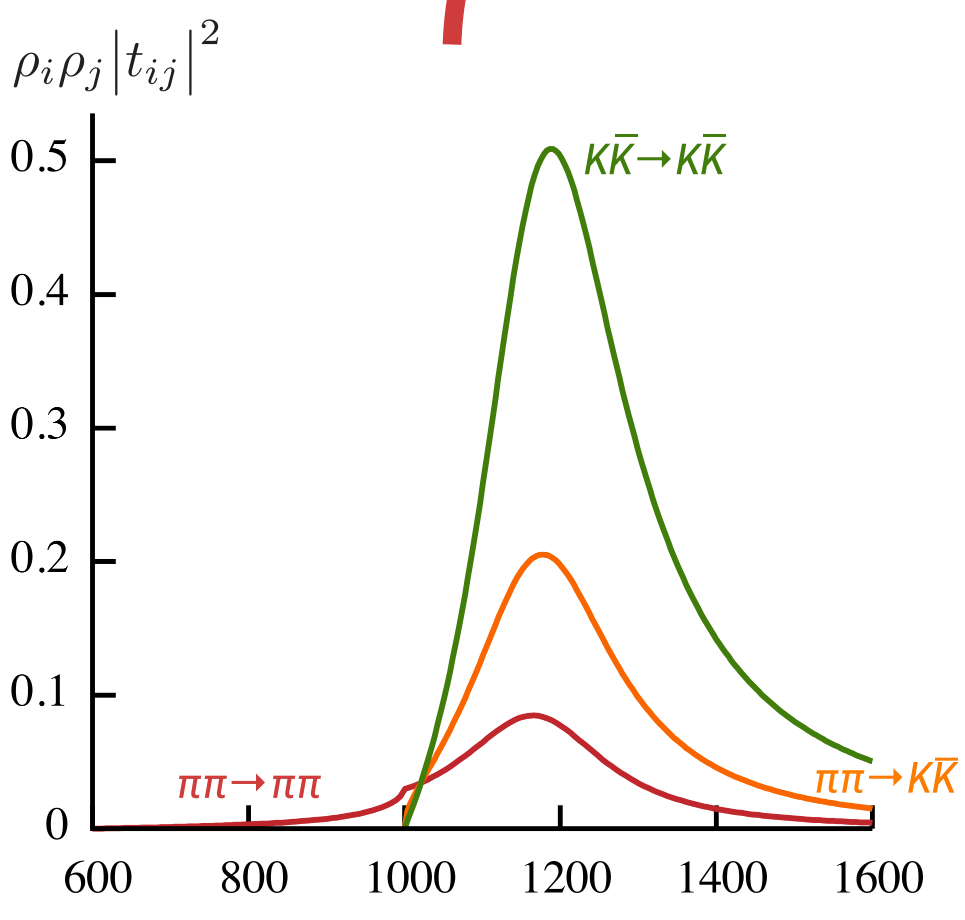


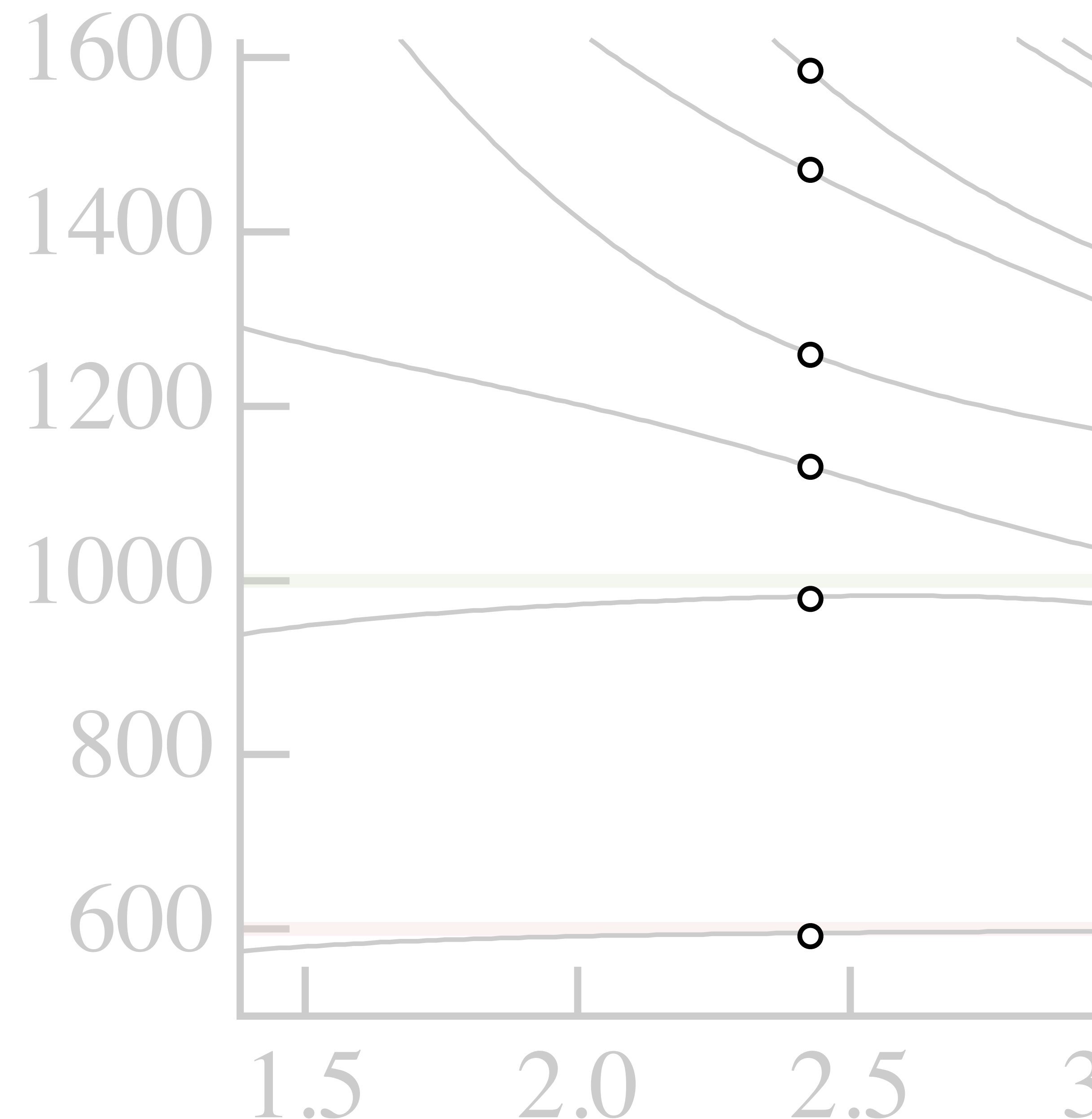
finite-volume spectrum



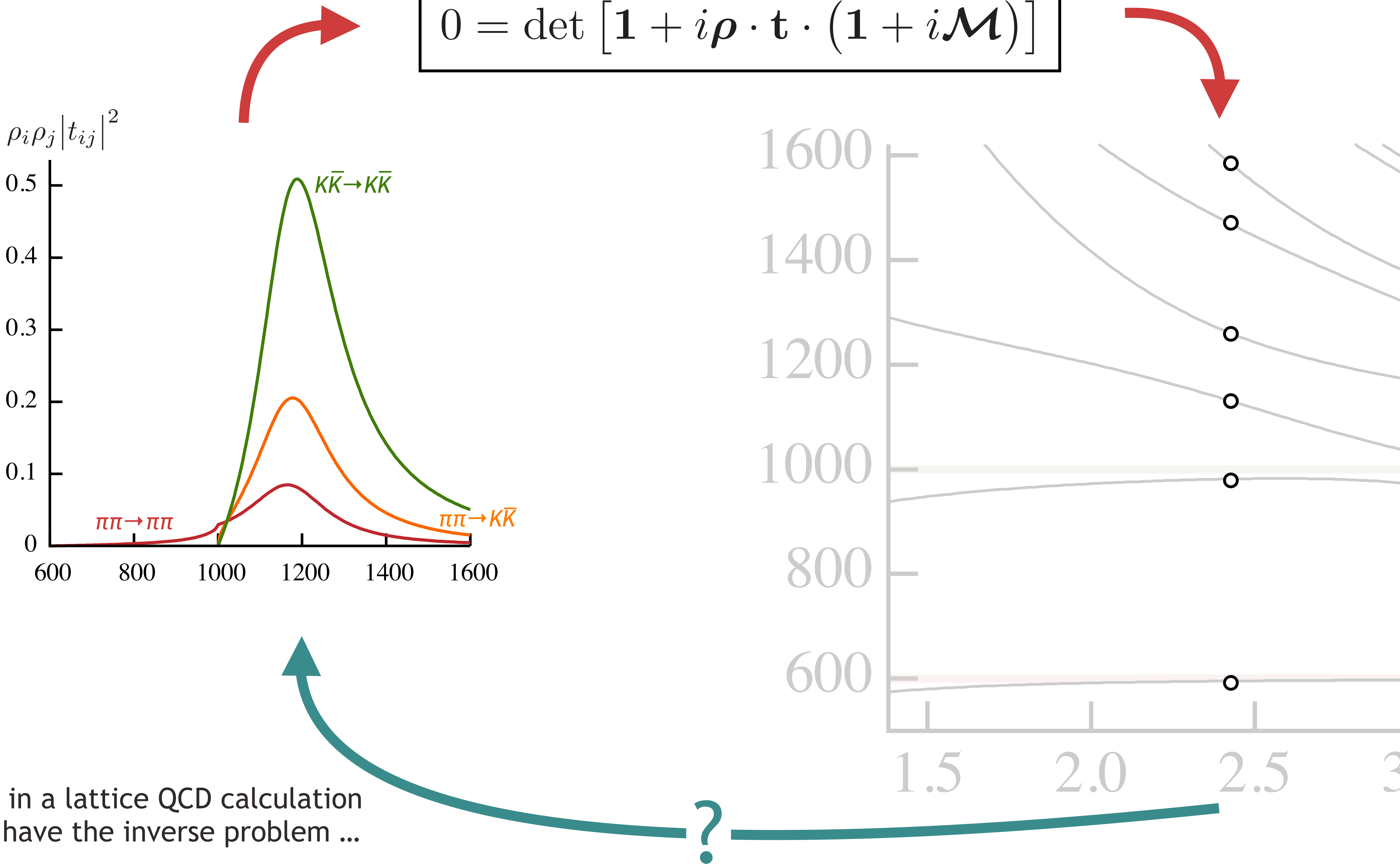
finite-volume approach

$$0 = \det\left[\mathbf{1} + i\boldsymbol{\rho}\cdot\mathbf{t}\cdot\left(\mathbf{1}+i\boldsymbol{\mathcal{M}}\right)\right]$$



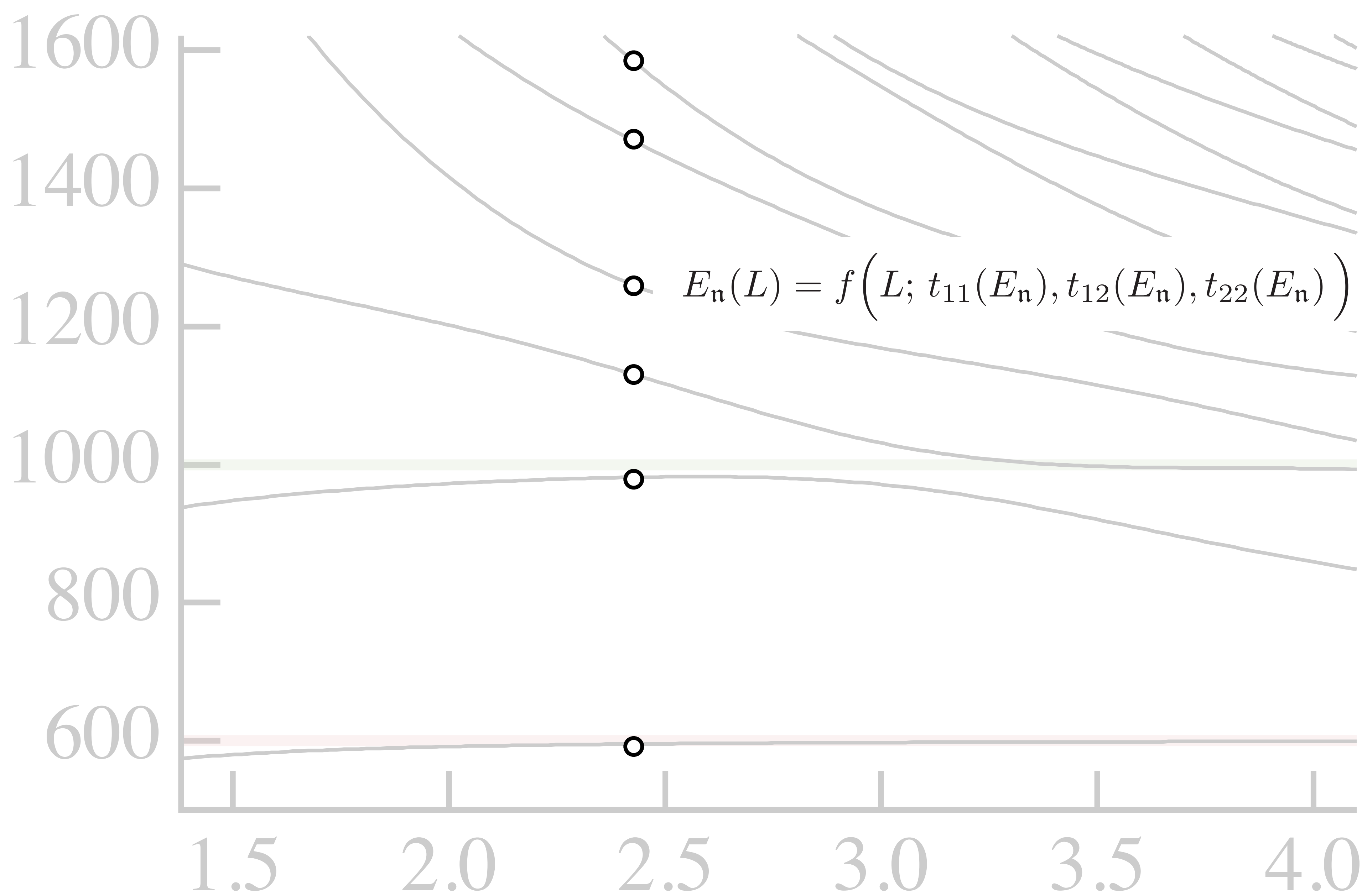


$$0 = \det\left[\mathbf{1} + i\boldsymbol{\rho}\cdot\mathbf{t}\cdot\left(\mathbf{1} + i\boldsymbol{\mathcal{M}}\right)\right]$$



but in a lattice QCD calculation
we have the inverse problem ...

position of each energy level depends upon all elements of the $\boldsymbol{t}$-matrix



$$0 = \det\big[\mathbf{1} + i\boldsymbol{\rho}\cdot\mathbf{t}\cdot\big(\mathbf{1} + i\boldsymbol{\mathcal{M}}\big)\big]$$

at $E = E_n(L)$
is one equation in three unknowns ...

parameterizing the *t*-matrix

a solution is to propose that different energies are not unrelated – parameterize $\mathbf{t}\big(E;\{a_i\}\big)$

then can use many energy levels to constrain the parameters by minimising a χ^2

$$\chi^2\big(\{a_i\}\big) = \sum_{\mathfrak{n},\mathfrak{n}'} \Big(E^{\text{lat.}}_{\mathfrak{n}} - E^{\text{par.}}_{\mathfrak{n}}(L;\{a_i\})\Big)\, \mathbb{C}^{-1}_{\mathfrak{n},\mathfrak{n}'} \Big(E^{\text{lat.}}_{\mathfrak{n}'} - E^{\text{par.}}_{\mathfrak{n}'}(L;\{a_i\})\Big)$$

inverse data covariance

energy levels solving
$0 = \det\big[\mathbf{1} + i\boldsymbol{\rho}\cdot\mathbf{t}\cdot\big(\mathbf{1} + i\boldsymbol{\mathcal{M}}\big)\big]$
for $\mathbf{t}\big(E;\{a_i\}\big)$

parameterizing the *t*-matrix

a solution is to propose that different energies are not unrelated – parameterize $\mathbf{t}\big(E;\{a_i\}\big)$

need to ensure multi-channel unitarity $\mathrm{Im}\,\big(t^{-1}(E)\big)_{ij} = -\delta_{ij}\,\rho_i(E)\,\Theta(E - E_i^{\mathrm{thr.}})$

– ***K***-matrix approach

$\mathbf{t}^{-1}(E) = \mathbf{K}^{-1}(E) + \mathbf{I}(E)$ with $\mathrm{Im}\,\big(I(E)\big)_{ij} = -\delta_{ij}\,\rho_i(E)$

simplest choice has $\mathrm{Re}\,\mathbf{I}(E) = 0$

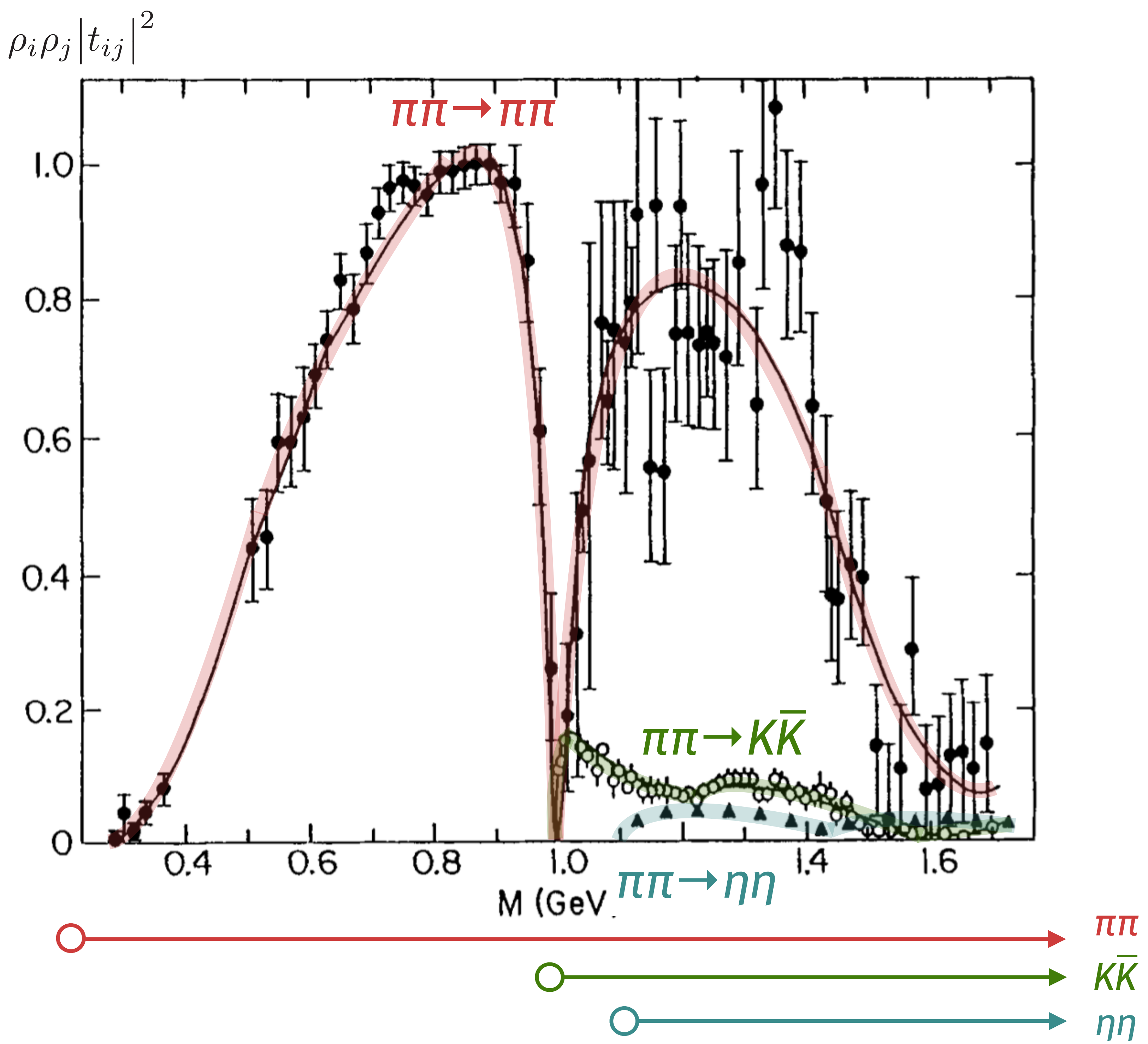
a more sophisticated approach =
"Chew-Mandelstam" phase-space

K(E) should be a real symmetric matrix

for reasons you'll see later,
better to parameterize in terms of $s = E^2$

e.g. $K_{ij} = \dfrac{g_i\,g_j}{m^2 - s}$ gives the Flatté form

$\pi\pi$, $K\bar{K}$, $\eta\eta$ S-wave scattering



explore this non-trivial system ...

... at a higher quark mass ...

$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering with m_π~391 MeV

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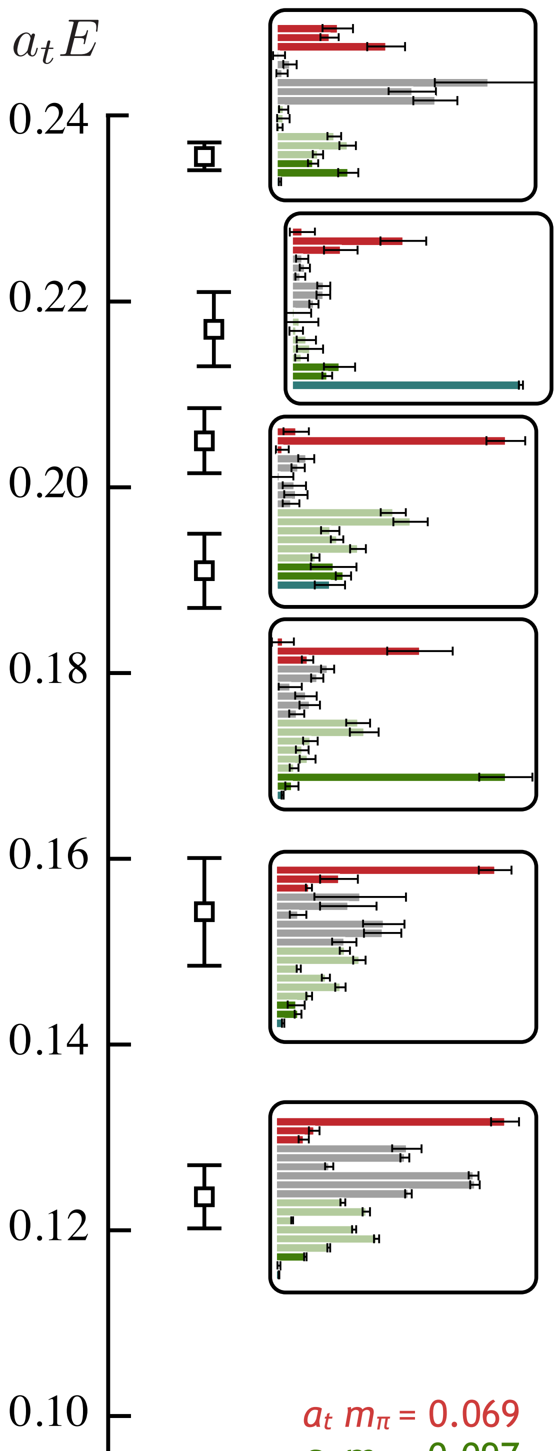
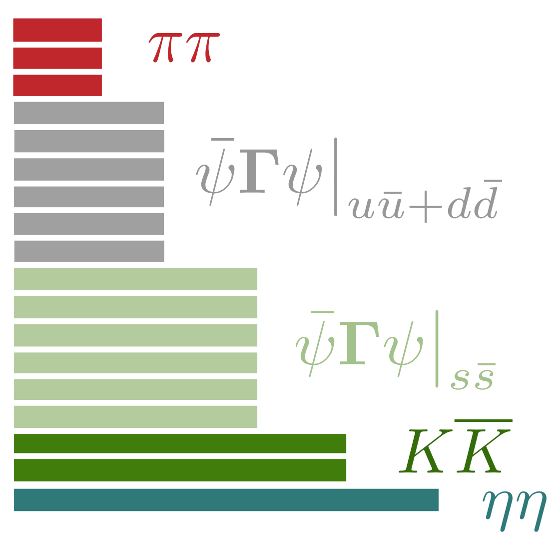
[000] A_1^+ 24^3

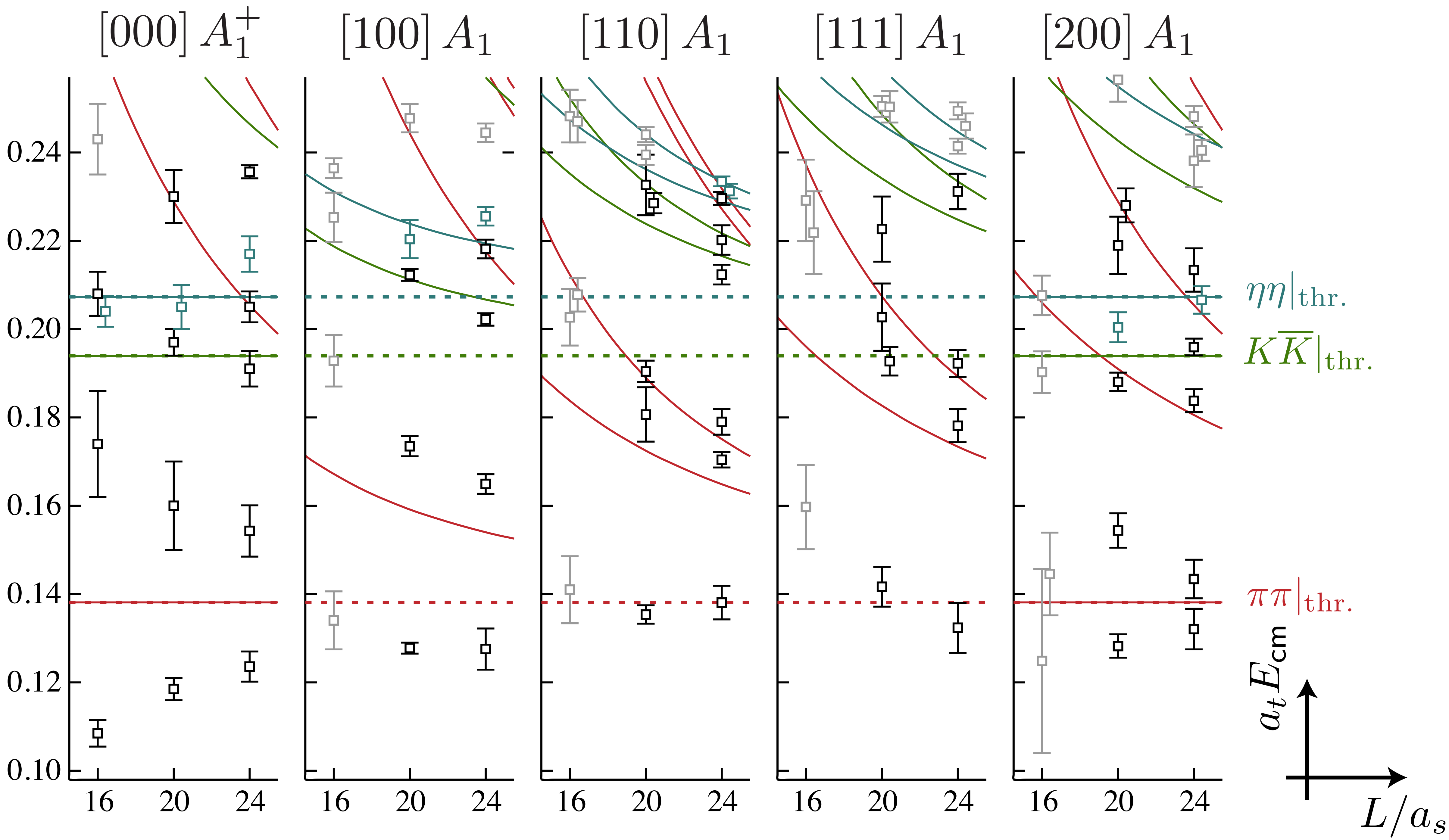


$a_t\, m_\pi = 0.069$
$a_t\, m_K = 0.097$
$a_t\, m_\eta = 0.104$

operator basis



$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering with m_π~391 MeV



what ***t***-matrix gives these spectra ?

$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering with m_π~391 MeV

not obvious what amplitude parameterization likely to describe the spectra well – try many ...

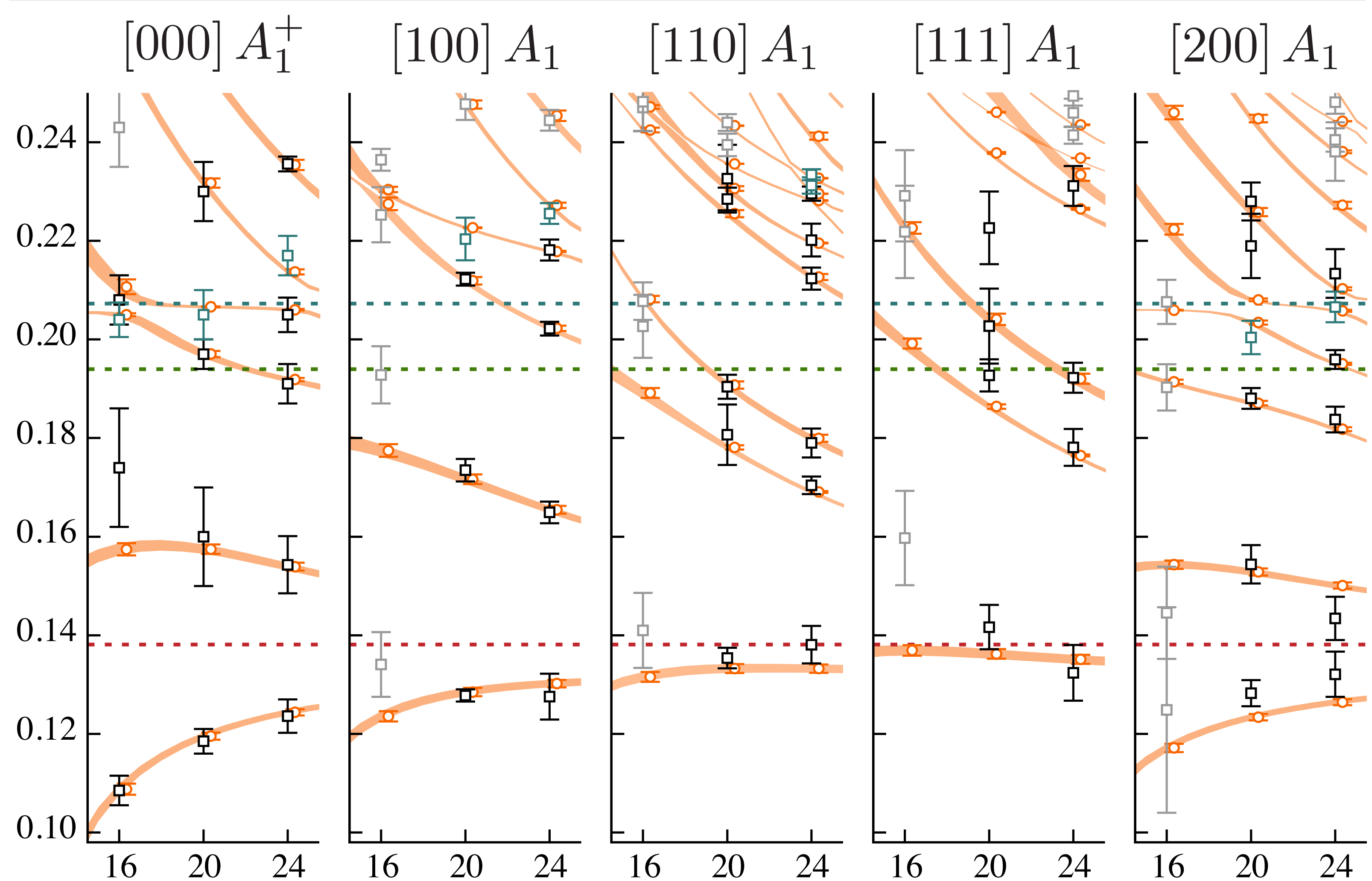
e.g. $$\mathbf{K}^{-1}(s) = \begin{pmatrix} a + b\,s & c + d\,s & e \\ c + d\,s & f & g \\ e & g & h \end{pmatrix}$$

$\{ a \ldots h \}$ are free parameters

with Chew-Mandelstam phase-space

$$I(s) = -\frac{\rho(s)}{\pi} \log\left[\frac{\rho(s) - 1}{\rho(s) + 1}\right]$$

best fit to lattice spectra



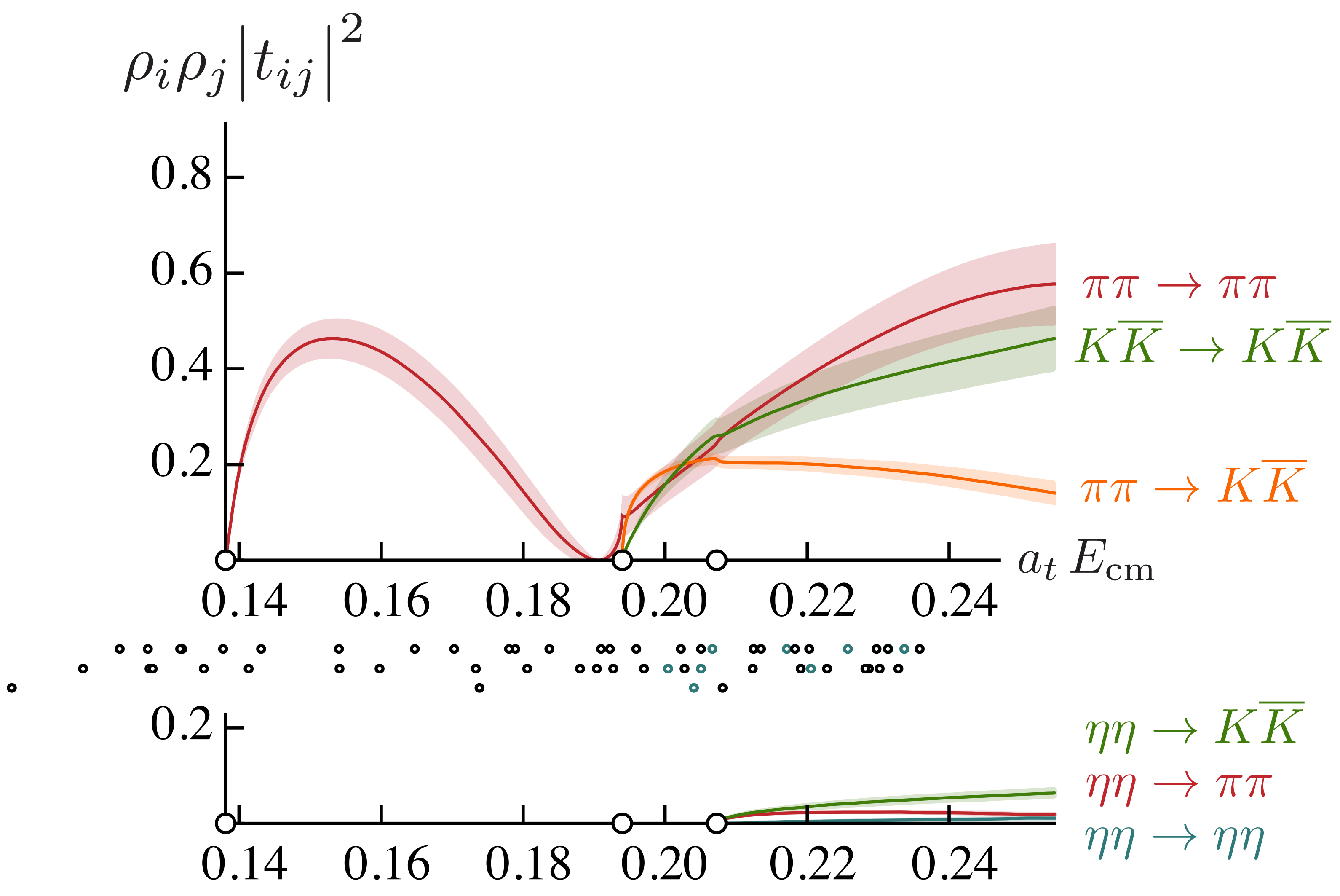
$$\frac{\chi^2}{N_{\rm dof}} = \frac{44.0}{57-8} = 0.90$$

$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering with m_π~391 MeV

e.g. $$\mathbf{K}^{-1}(s) = \begin{pmatrix} a + b\,s & c + d\,s & e \\ c + d\,s & f & g \\ e & g & h \end{pmatrix}$$

{ a ... h } are free parameters

S-wave amplitudes



$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering with m_π~391 MeV

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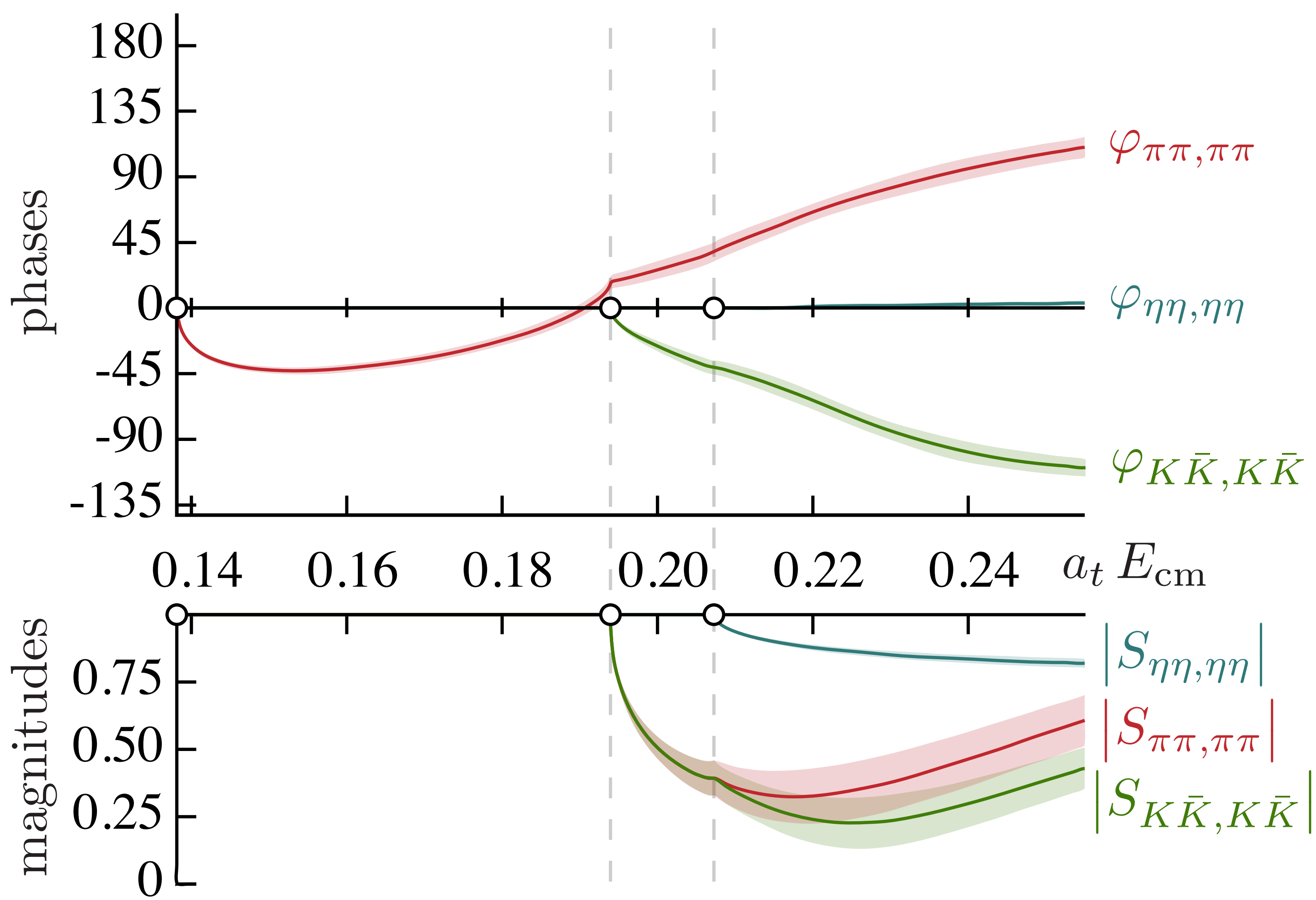
{ a … h } are free parameters

S-wave S-matrix diagonals

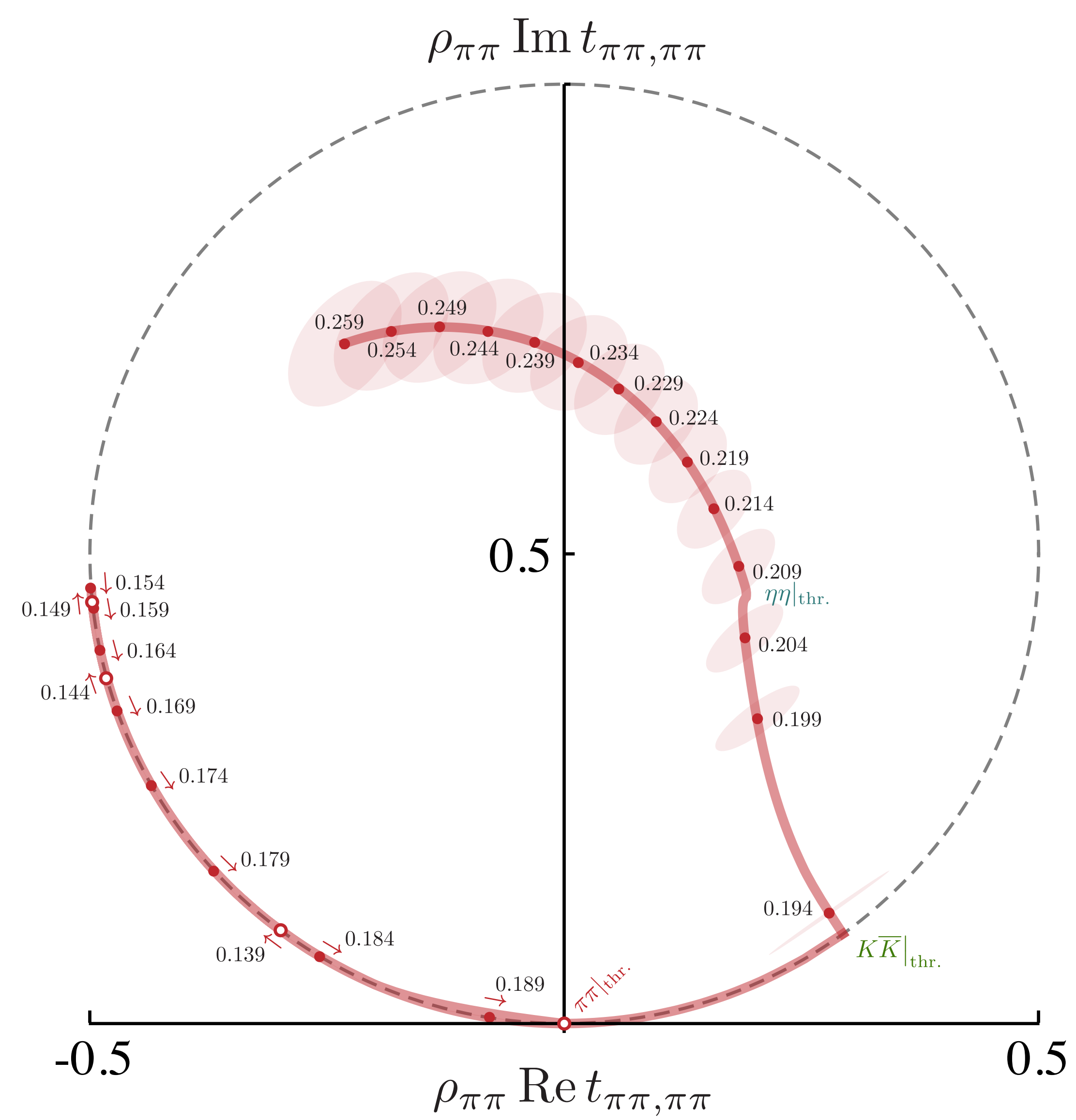


$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering with m_π~391 MeV

e.g. $$\mathbf{K}^{-1}(s) = \begin{pmatrix} a + b\,s & c + d\,s & e \\ c + d\,s & f & g \\ e & g & h \end{pmatrix}$$

{ *a* … *h* } are free parameters

$\pi\pi \to \pi\pi$ Argand diagram



$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering with m_π~391 MeV

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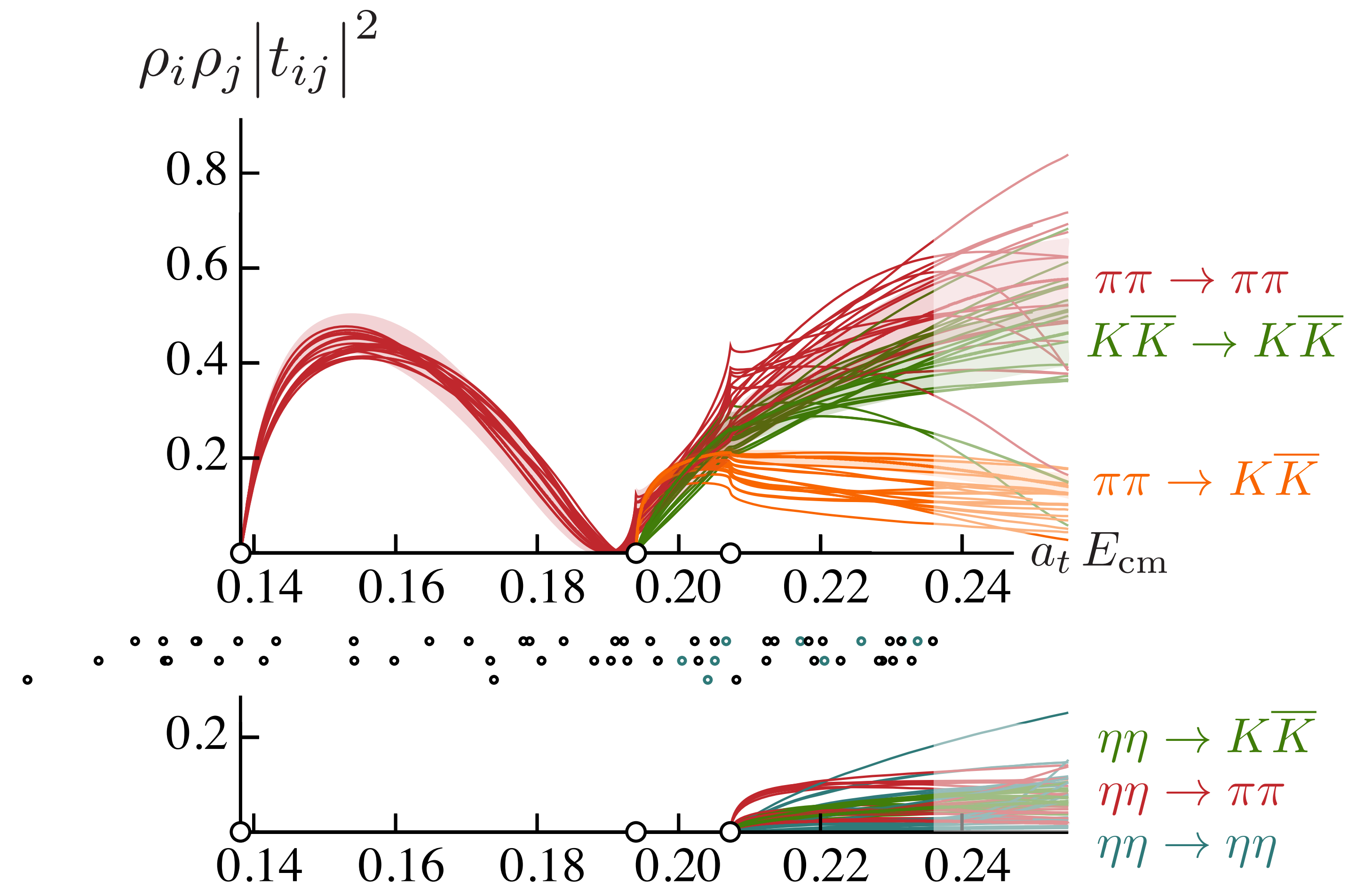
not obvious what amplitude parameterization likely to describe the spectra well — **try many …**

K^{-1} as matrix of polynomials,
K as matrix of polynomials,
K as pole plus matrix of polynomials,
simple versus Chew-Mandelstam phase-space …

keep choices that can describe spectra with good χ^2

variation with parameterization



$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering with m_π~391 MeV

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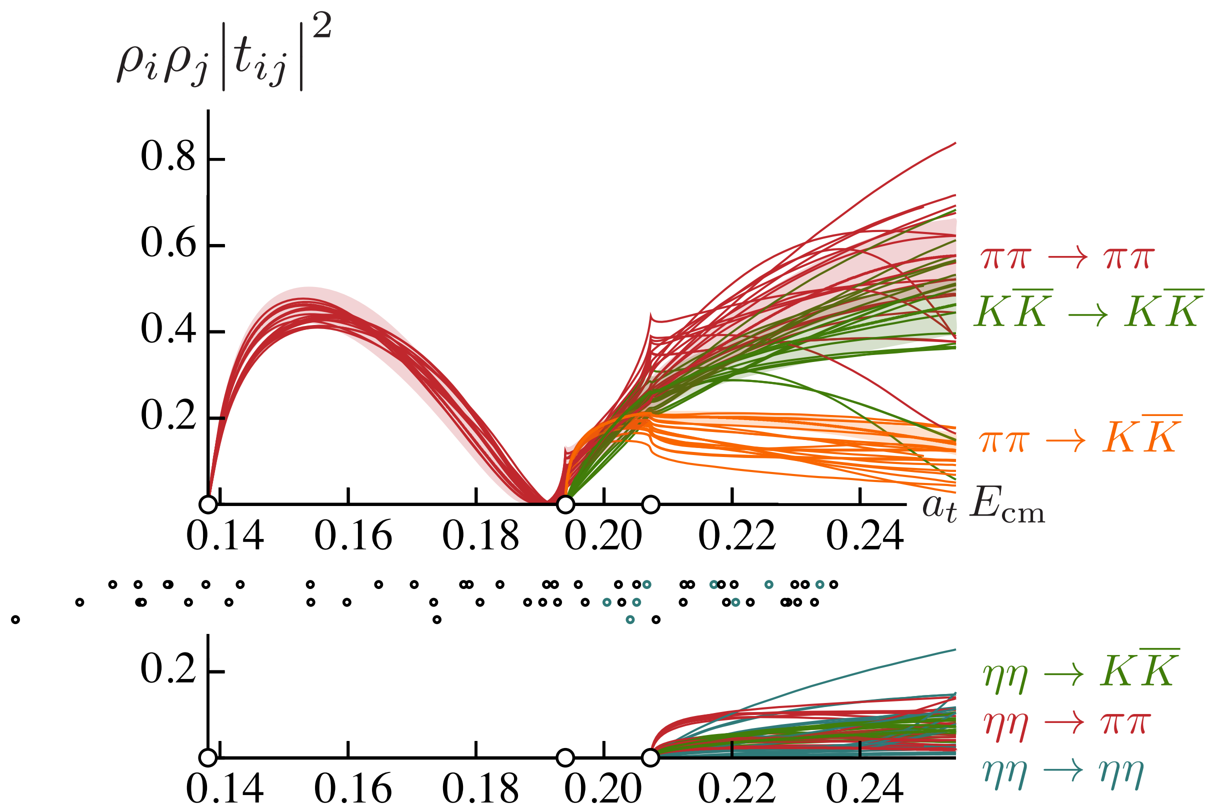
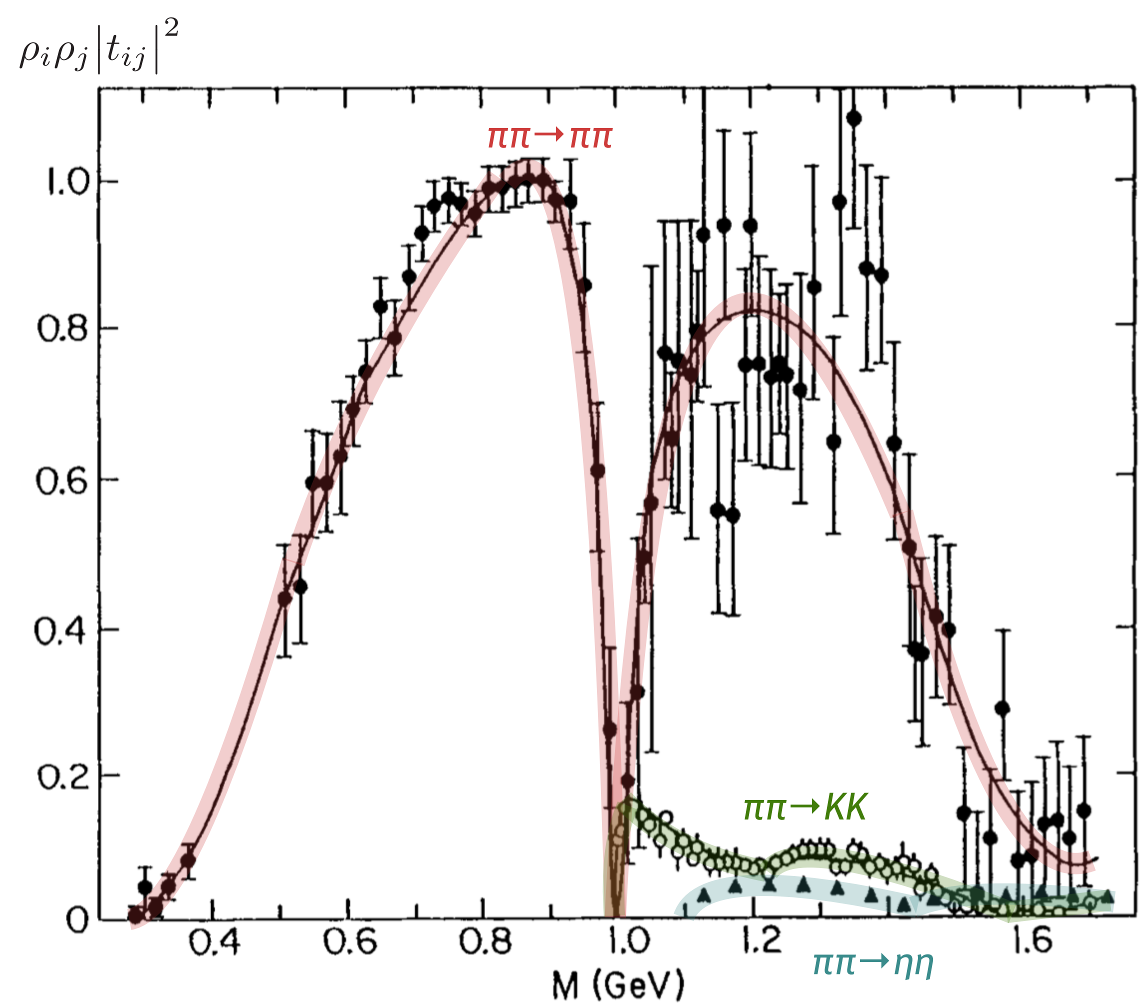
scattering amplitude 'prediction'



analogous experimental data



... but what do we do with this ?

... is this strange energy dependence due to resonances ?

$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering with m_π~391 MeV

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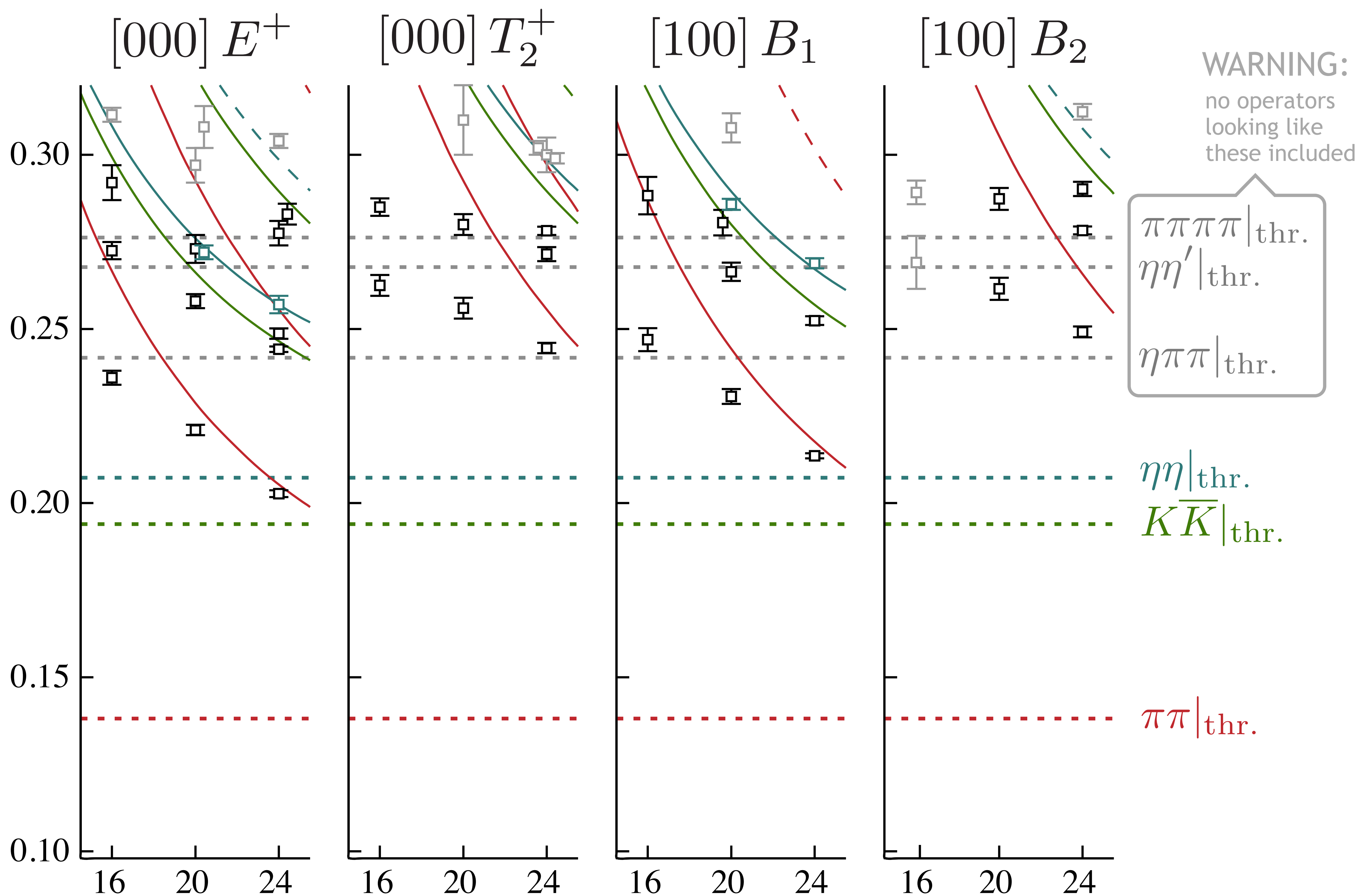
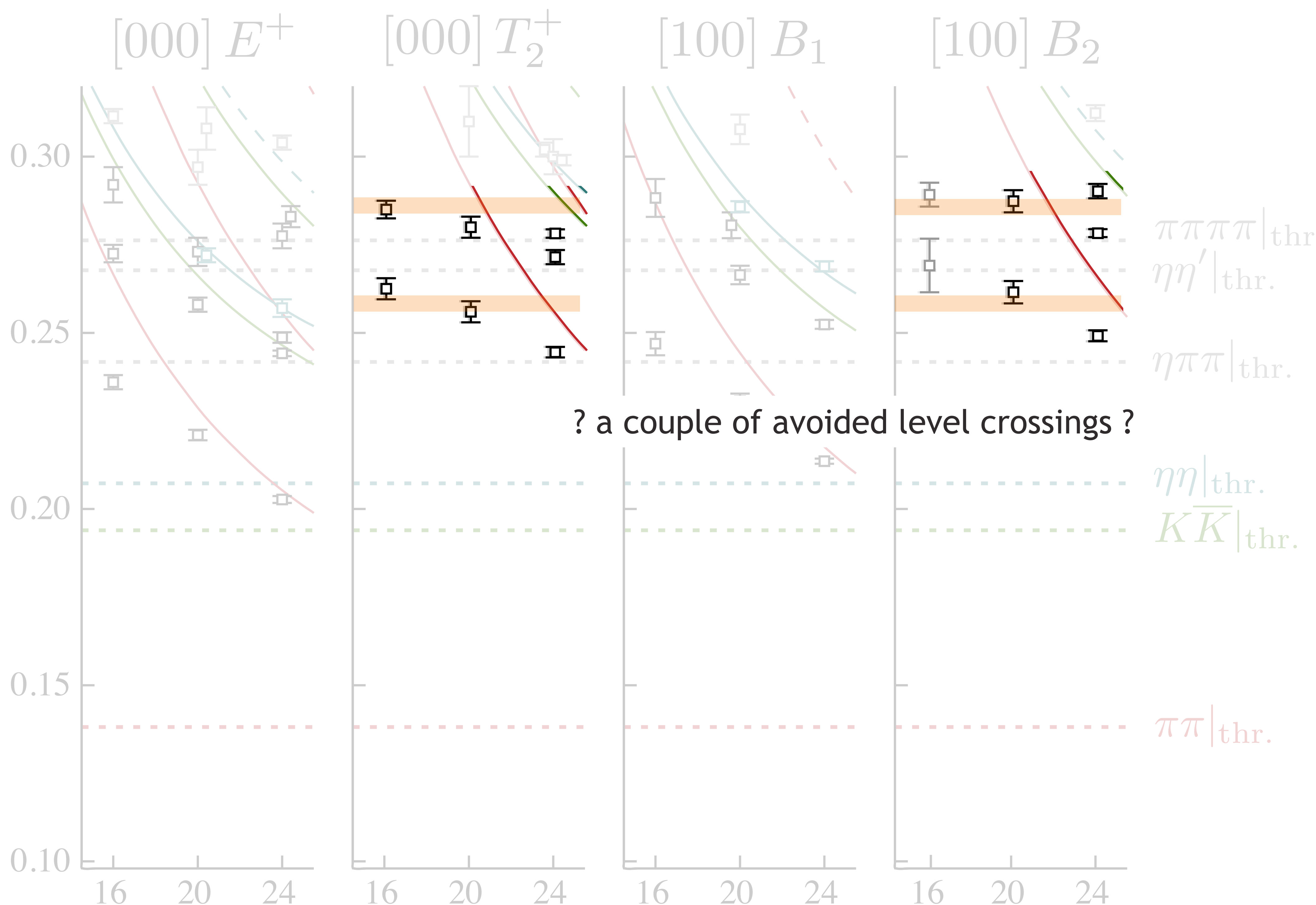
also computed spectra for irreps with lowest subduced spin J=2



also computed spectra for irreps with lowest subduced spin J=2



? a couple of avoided level crossings ?

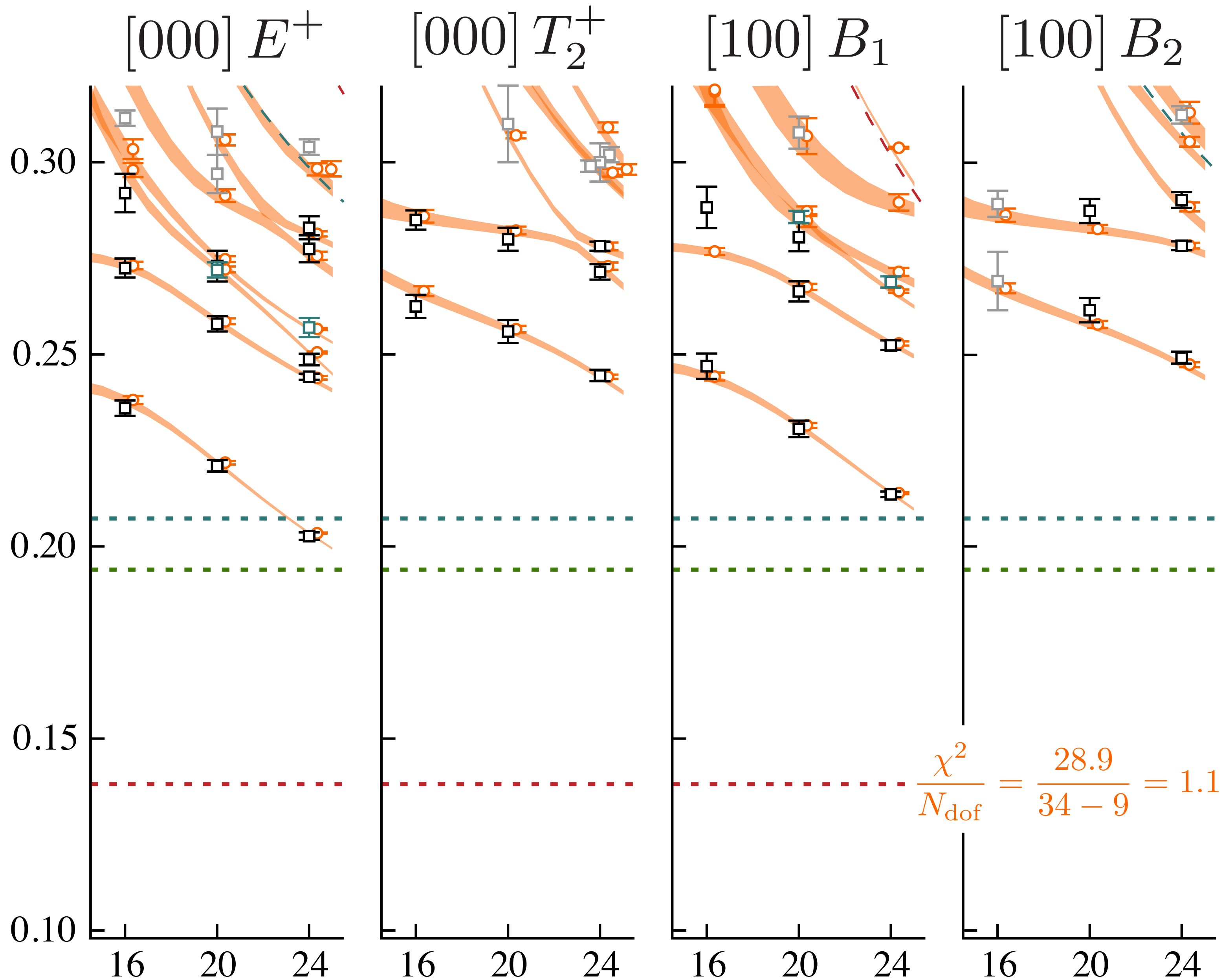
$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering with m_π~391 MeV

e.g. parameterize coupled *D*-wave ***t***-matrix with

$$K_{ij}(s) = \frac{g_i^{(1)} g_j^{(1)}}{m_1^2 - s} + \frac{g_i^{(2)} g_j^{(2)}}{m_2^2 - s} + \gamma_{ij} \qquad \gamma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma_{\eta\eta,\eta\eta} \end{pmatrix}$$

and the simple phase-space

best fit to lattice spectra



$$\frac{\chi^2}{N_{\rm dof}} = \frac{28.9}{34-9} = 1.15$$

$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering with m_π~391 MeV

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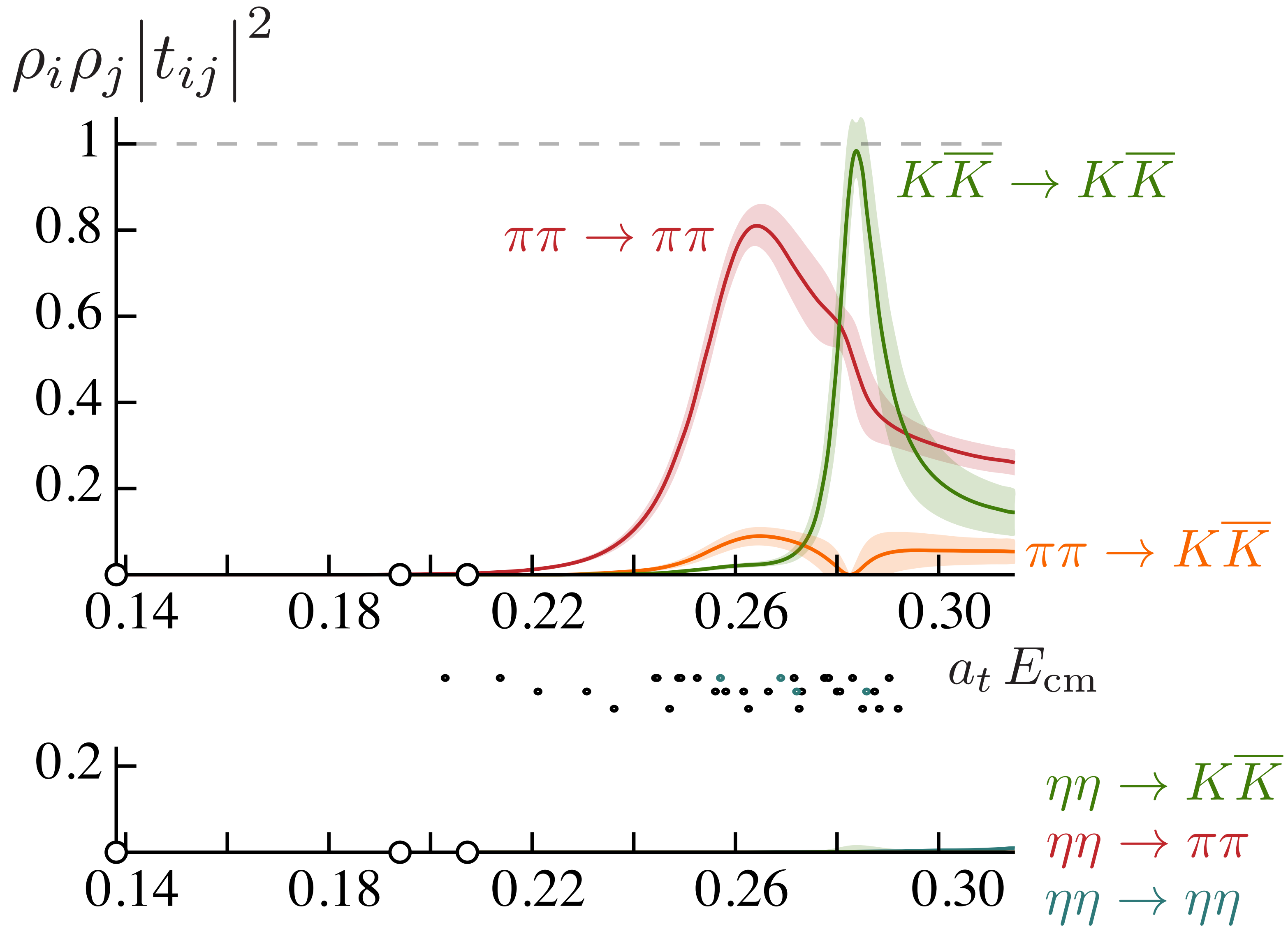
e.g. parameterize coupled *D*-wave ***t***-matrix with

$$K_{ij}(s) = \frac{g_i^{(1)} g_j^{(1)}}{m_1^2 - s} + \frac{g_i^{(2)} g_j^{(2)}}{m_2^2 - s} + \gamma_{ij} \qquad \gamma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma_{\eta\eta,\eta\eta} \end{pmatrix}$$

and the simple phase-space

D-wave amplitudes



$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering with m_π~391 MeV

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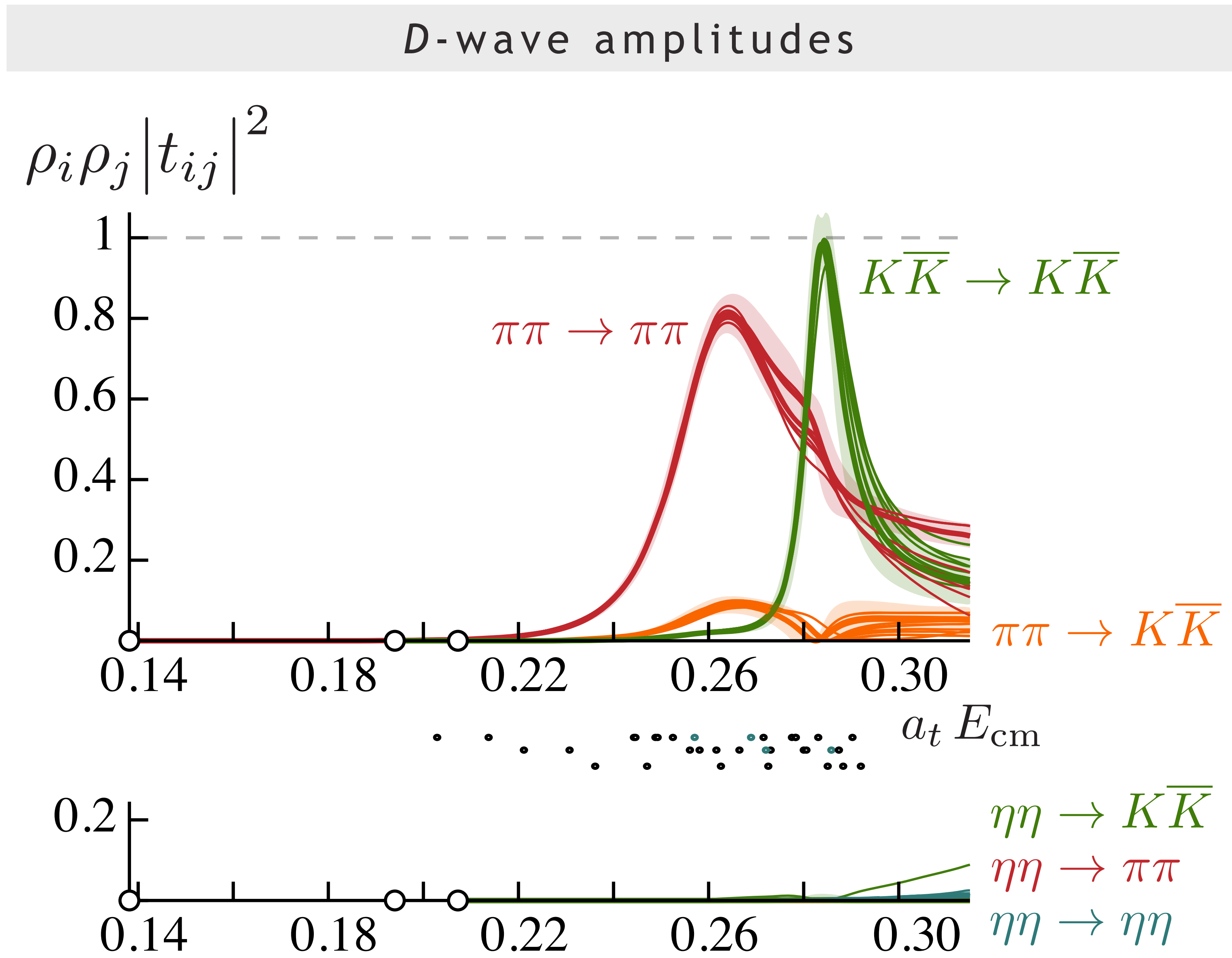
… and varying the particular choice of parameterization …



$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering with m_π~391 MeV

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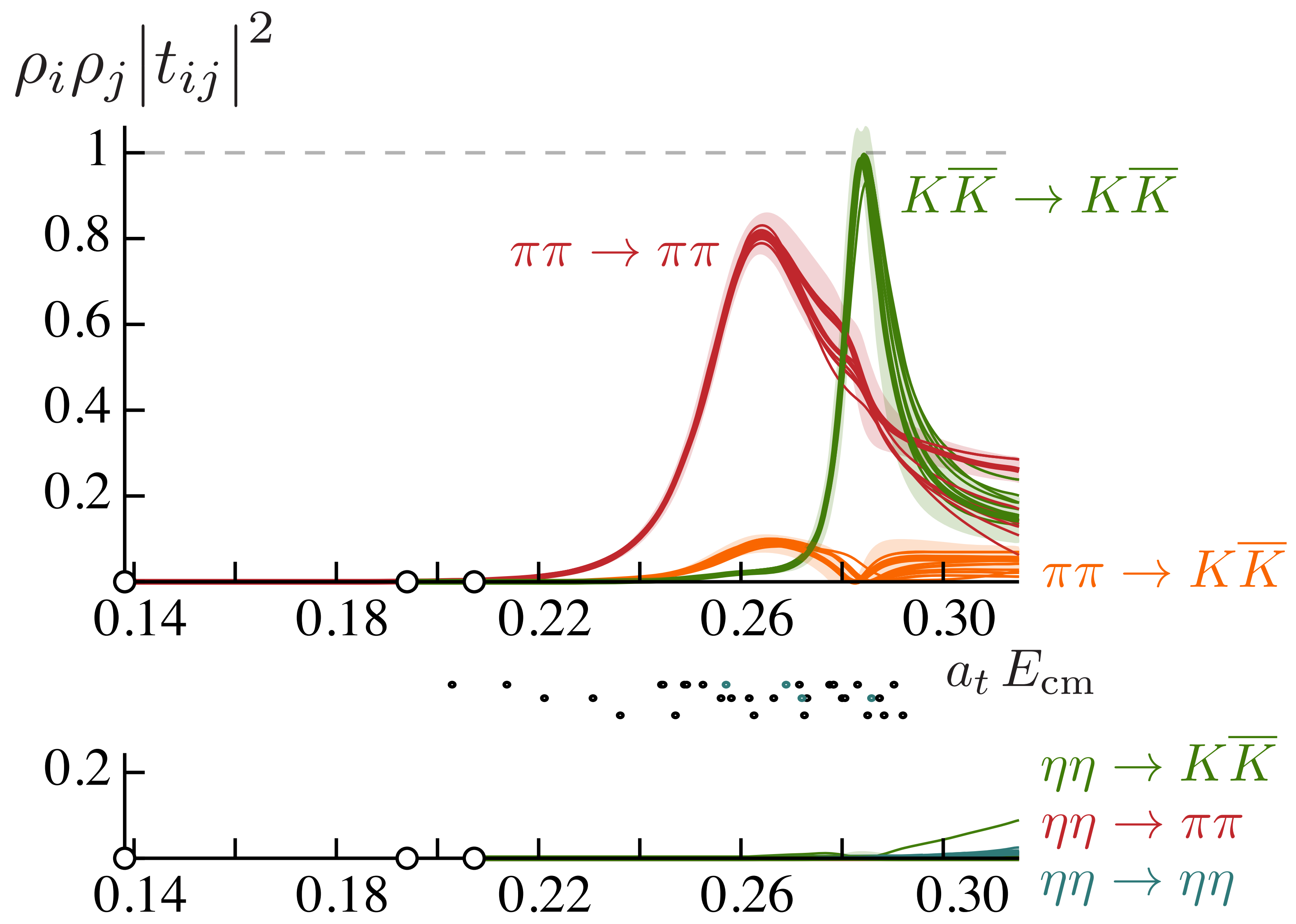
D-wave amplitudes



'looks like' two resonances

– lighter one has larger width, big coupling to $\pi\pi$

– heavier one has smaller width, big coupling to $K\bar{K}$

... there must be a more rigorous way to know the resonance content ?