2. Relativistic applications of Lellouch-Lüscher-type relations

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Scattering from the Lattice: applications to phenomenology and beyond,
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Cluster of Excellence

Generalization of quantization condition to relativistic QFT

Lüscher showed (1986,1990), to all order in perturbation theory, that the spectrum is still determined by a quantization condition of the type $\det[A - BM] = 0$.

For the A_1 and T_1 irreps (rest frame), neglecting scattering phase for $\ell \geq 4$ or $\ell \geq 3$ respectively,

$$\delta_\ell(k) + \phi(q) = n\pi, \qquad n \in \mathbb{Z}, \qquad q \equiv \frac{kL}{2\pi}.$$

Basis of current lattice applications (including moving frames) in

- ρ, K^* channels
- a_0, f_0 channels.

Partial wave expansion of the scattering amplitude

Let W be the center-of-mass energy: $W = 2\sqrt{k^2 + m_\pi^2}$.

Partial wave expansion of scattering amplitude for spinless particles:

$$\mathcal{M} = 16\pi W \sum_\ell (2\ell + 1) P_\ell(\cos\theta) t_\ell,$$

$$t_\ell(k) = \frac{1}{2ik}(\eta_\ell(k) e^{2i\delta_\ell(k)} - 1).$$

η_ℓ is the inelasticity, δ_ℓ the scattering phase.

One derivation of LL type relations

In particular, the differential

$$\Delta\delta_\ell(k) = -F_\ell(k,L)\frac{\Delta k}{k}$$

of the equation $\delta_\ell(k) + \phi(q) = n\pi$ still relates a change in the particle interaction and the spectrum on the torus.

Strategy is the same as in the non-relativistic case:

$$\begin{array}{ccc}
\delta_\ell & \overset{(L)}{\rightarrow} & E_n(L) \\
\downarrow \Delta V & & \downarrow \Delta V \\
\delta_\ell + \Delta\delta_\ell & \overset{(L)}{\rightarrow} & E_n(L) + \Delta E_n(L).
\end{array}$$

One derivation of LL type relations (II) [Lellouch, Lüscher 2000]

$K \to \pi\pi$ weak decay: the perturbing Hamiltonian is provided by the weak interaction:

$$V = \int d^3x \, \mathcal{L}_{\mathrm{w}}(x).$$

Since both initial and final states are both purely hadronic, energy and momentum must match for the transition to be at physical kinematics.

Box size tuned so that $E_n = m_K$ for one of the discrete two-pion states.

The degeneracy between the kaon and two-pion states is lifted by the weak interaction. To linear order:

$$E = m_K \pm |M|, \qquad M \equiv_L \langle \pi\pi | V | K \rangle; \quad \Delta k = \frac{m_K}{4k_\pi} |M|.$$

One derivation of LL type relations (III)

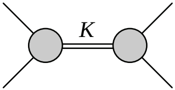
On the other hand, in infinite volume, the $\pi\pi$ phase shift is modified by the resonant production of a kaon. It is formally 2nd order in the weak interaction, but enhanced by the proximity to the kaon pole: $\frac{i}{p^2-m_K^2} = \frac{i}{2m_K\Delta p^0}$,
for $\boldsymbol{p}=0$ and $p^0 = M_K + \Delta p^0$. Evaluate this for $\Delta p^0 = \pm|M| \Rightarrow$

$$i\Delta\mathcal{M} = \langle(\pi\pi)_{\text{out}}|\mathcal{L}_{\text{w}}|K\rangle \frac{i}{2M_K(\pm|M|)} \langle K|\mathcal{L}_{\text{w}}|(\pi\pi)_{\text{in}}\rangle = (Ae^{i\delta_0}) \frac{i}{2M_K(\pm|M|)} (-Ae^{i\delta_0})$$

$$\Delta\mathcal{M} = \frac{i\,16\pi M_K}{k} e^{2i\delta} \Delta\delta_0 \Rightarrow \Delta\delta_0(k) = \mp\frac{k_\pi|A|^2}{32\pi m_K^2|M|} + \ldots$$

Inserting the expressions for Δk and for $\Delta\delta_0$ into , Lellouch & Lüscher obtain

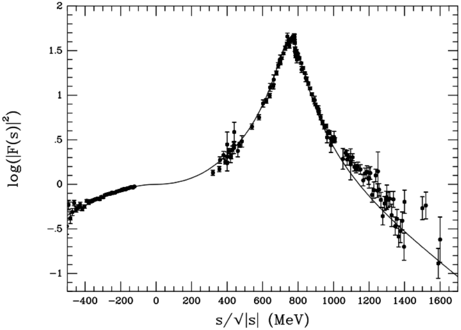
$$|A|^2 = 8\pi F_0(k,L)\left(\frac{m_K}{k_\pi}\right)^3 |M(L)|^2, \qquad \Gamma = \frac{k_\pi}{16\pi m_K^2}|A|^2.$$



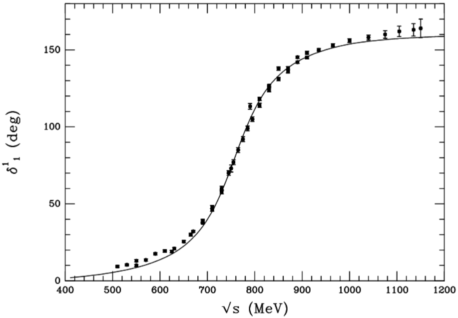
Pion form factor $F_\pi(q^2)$ (I)

Spacelike ($q^2 < 0$)	Timelike ($q^2 > 0$)
$\langle \pi^+ \vert \boldsymbol{j} \vert \pi^+ \rangle = (\boldsymbol{p} + \boldsymbol{p}')F_\pi(q^2)$	${}_{\text{out}}\langle \pi^+\pi^- \vert \boldsymbol{j} \vert 0 \rangle = (\boldsymbol{p}_+ - \boldsymbol{p}_-)F_\pi(q^2)$
$e^-\pi^+ \to e^-\pi^+$	$e^+e^- \to \pi^+\pi^-$

$|F_\pi(q^2)|^2$



phase of $F_\pi(q^2)$

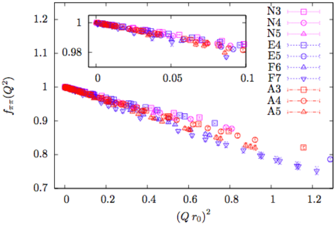


$\delta_1 = \pi^+\pi^-$-scattering phase (Watson thm)

[Figs. from Guerrero & Pich, hep-ph/9707347]

The pion form factor (II)

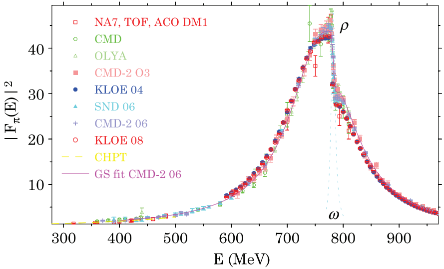
Spacelike pion form factor
Lattice data
[B.B. Brandt, A. Jüttner, H. Wittig 1301.3513]



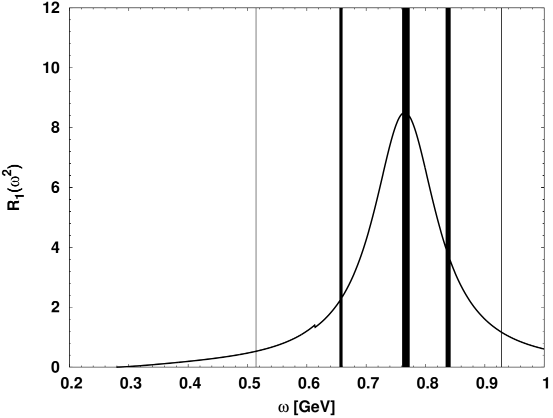
Timelike pion form factor:

$$q = p_+ + p_-, \quad k^2 = \frac{q^2}{4} - M_\pi^2$$

[plot from Jegerlehner & Nyffeler 0902.3360]



From discrete states on the torus to the timelike pion form factor



$$\rho(s) = \frac{1}{48\pi^2}\left(1-4m_\pi^2/s\right)^{3/2}|F_\pi(s)|^2, \qquad\qquad s<(4M_\pi)^2$$

$$|F_\pi(s)|^2 = \left(q\phi'(q)+k\tfrac{\partial\delta_1(k)}{\partial k}\right)\frac{3\pi s}{2k^5L^3}\left|{}_L\left\langle \pi\pi\left|\int \mathrm{d}\boldsymbol{x}\, j^z(\boldsymbol{x})\right|0\right\rangle\right|^2.$$

HM 1105.1892; figure: model for $F_\pi(s)$.

A derivation

Task: relate ${}_L\langle\pi\pi|\boldsymbol{j}|0\rangle$ to ${}_{\rm out}\langle\pi\pi|\boldsymbol{j}|0\rangle$ at a given c.m. energy E.

Idea: invoke a massive vector particle with mass $M = E$ which mixes with the two-pion state in the box.

$$V = -\int d^3x\, \boldsymbol{j}^a \cdot \boldsymbol{A}^a, \qquad \boldsymbol{A}(x) = \sum_{\boldsymbol{k}} \sum_{\sigma=1}^{3} \frac{\boldsymbol{e}^\sigma(\boldsymbol{k})}{\sqrt{2E_k L^3}} \left(a^b_{\boldsymbol{k},\sigma} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + a^{b\dagger}_{\boldsymbol{k},\sigma} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right)$$

In the box:

$$E^{\pm}_{\pi\pi} = M \pm |\mathcal{A}|, \quad \mathcal{A} = \frac{-e}{\sqrt{2M}} A_\psi, \quad L^{3/2} \langle \psi^a_\sigma | \hat{j}^b_{\sigma'}(\boldsymbol{x}) | 0 \rangle = \delta^{ab} \delta_{\sigma\sigma'} A_\psi .$$

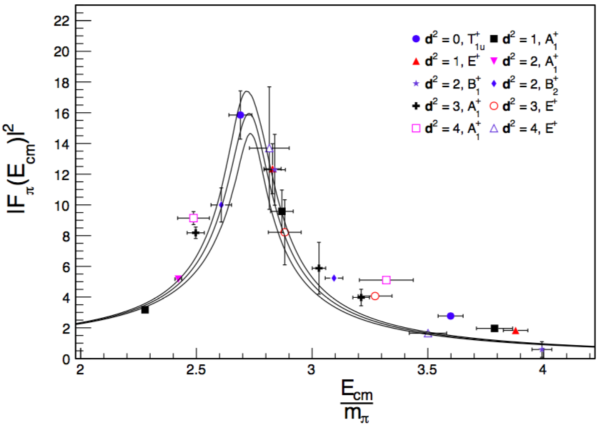
Infinite volume: the vector particle appears as a resonance in $\pi\pi$ scatt.,

$$\Delta\mathcal{M}(\pi\pi \to \pi\pi) = \langle (\pi_{\boldsymbol{p}'}\pi_{-\boldsymbol{p}'})^b, {\rm out} | \boldsymbol{j}^c | 0 \rangle \cdot \frac{e^2 \delta^{cd}}{q^2 - M^2} \cdot \langle 0 | \boldsymbol{j}^d | (\pi_{\boldsymbol{p}}\pi_{-\boldsymbol{p}})^a, {\rm in} \rangle .$$

Corresponding change in the scattering phase:

$$\Delta\delta_1 = \mp \frac{e^2}{12\pi M |\mathcal{A}|} |F_\pi|^2 \frac{k_\pi^3}{E_{\pi\pi}} \quad \Rightarrow \quad \text{insert into} \quad \Delta\delta_\ell(k) = -F_\ell(k,L) \frac{\Delta k}{k} .$$

Example of a lattice calculation of $F_\pi(q^2)$



From 1511.03251 J. Bulava, B. Hörz et al.; see also 1412.6319 X. Feng et al. and the recent 1710.03529 F. Erben et al.

$(g-2)_\mu$: a reminder

$$\boldsymbol{\mu} = g\,\mu_B \boldsymbol{s}, \qquad \mu_B = \frac{e}{2m_\mu}$$

- $g = 2$ in Dirac's theory
- $a_\mu \equiv (g-2)/2 = F_2(0) = \frac{\alpha}{2\pi}$ (Schwinger 1948)
- direct measurement (BNL): $a_\mu = (11659208.9 \pm 6.3) \cdot 10^{-10}$
- Standard Model prediction $a_\mu = (11659182.8 \pm 4.9) \cdot 10^{-10}$.
- $a_\mu^{\text{exp}} - a_\mu^{\text{th}} = (26.1 \pm 8.0) \cdot 10^{-10}$.

Numbers from 1105.3149 Hagiwara et al.

$(g-2)_\mu$: history and near future

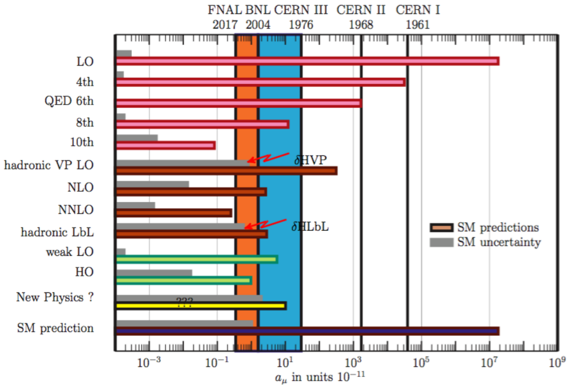
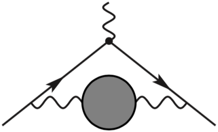
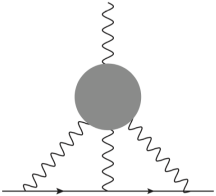


Fig. from Jegerlehner 1705.00263



Hadronic vacuum polarisation (HVP)



Hadronic light-by-light scattering (HLbL)

New experiments: $\times 4$ improvement in accuracy $\Longrightarrow$ theory effort needed:

- HVP target accuracy: $\lesssim 0.5\%$
- HLbL target accuracy: 10%.

HVP: definitions (Euclidean space)

- primary object on the lattice: $G_{\mu\nu}(x) = \langle j_\mu(x) j_\nu(0)\rangle$.

- polarization tensor: $\Pi_{\mu\nu}(Q) \equiv \int d^4x\, e^{iQ\cdot x} G_{\mu\nu}(x)$.

- O(4) invariance and current conservation $\partial_\mu j_\mu = 0$:

$$\Pi_{\mu\nu}(Q) = \left(Q_\mu Q_\nu - \delta_{\mu\nu} Q^2\right)\Pi(Q^2).$$

- Spectral representation: $\rho(s) = \frac{R(s)}{12\pi^2}$, $R(s) \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{4\pi\alpha(s)^2/(3s)}$,

$$\Pi(Q^2) - \Pi(0) = Q^2 \int_{4m_\pi^2}^{\infty} ds \frac{\rho(s)}{s(s+Q^2)}.$$

- $$a_\mu^{\text{hvp}} = 4\alpha^2 \int_0^\infty dQ^2\; K(Q^2; m_\mu^2)\; [\Pi(Q^2) - \Pi(0)]$$

- In the limit $m_\mu \to 0$: $\lim_{m_\mu \to 0} \frac{a_\mu^{\text{hvp}}}{m_\mu^2} = \frac{4}{3}\alpha^2 \Pi'(Q^2 = 0)$.

Lautrup, Peterman & de Rafael Phys.Rept 3 (1972) 193; Blum hep-lat/0212018

The time-momentum representation (TMR)

- mixed-representation Euclidean correlator: (natural on the lattice)

$$G_{\rm TMR}(x_0) = -\frac{1}{3}\sum_{k=1}^{3}\int d^3\boldsymbol{x}\; G_{kk}(x),$$

- the spectral representation:

$$G_{\rm TMR}(x_0) = \int_0^\infty d\omega\; \omega^2 \rho(\omega^2)\; e^{-\omega|x_0|}, \qquad x_0 \neq 0.$$

- Finally, the quantity $a_\mu^{\rm hvp}$ is given by

$$\begin{aligned} a_\mu^{\rm hvp} &= \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dx_0\; G(x_0)\, \tilde{f}(x_0), \\ \tilde{f}(x_0) &= \frac{2\pi^2}{m_\mu^2}\Big[-2 + 8\gamma_{\rm E} + \frac{4}{\hat{x}_0^2} + \hat{x}_0^2 - \frac{8}{\hat{x}_0} K_1(2\hat{x}_0) \\ &\qquad + 8\log(\hat{x}_0) + G^{2,1}_{1,3}\left(\hat{x}_0^2 \Big| {\textstyle \frac{3}{2} \atop 0,\,1,\,\frac{1}{2}}\right)\Big] \end{aligned}$$

where $\hat{x}_0 = m_\mu x_0$, $\gamma_{\rm E} = 0.577216..$ and $G^{m,n}_{p,q}$ is Meijer's function.

Bernecker & Meyer 1107.4388; Mainz-CLS 1705.01775.

TMR: a look at the integrand (light-quark conn. contribution)

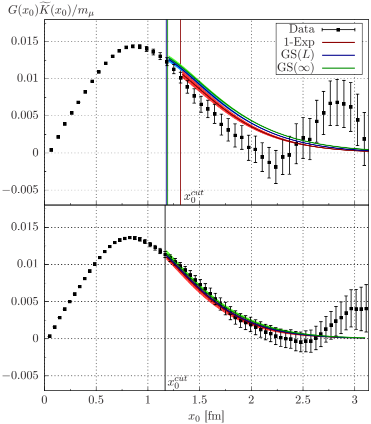
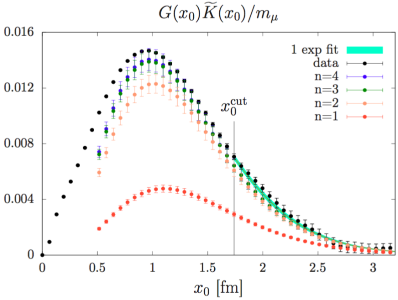


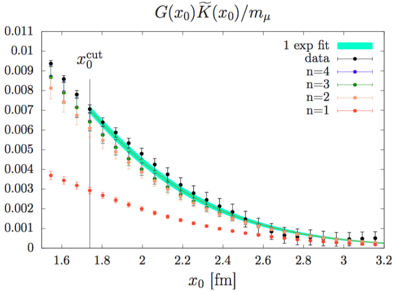
Illustration: $m_\pi = 190\,$MeV and $m_\pi = 270\,$MeV

- signal-to-noise deteriorates at long distances
- relative finite-volume effect becomes large at long distances
- how to control the long-distance part of the correlator?

$N_f = 2$ O(a) improved Wilson quarks: Mainz-CLS 1705.01775.

TMR on $N_f = 2+1$ CLS ensembles*





* Here, D200: $m_\pi = 200\,$MeV, $a = 0.064\,$fm.

- state-of-the-art spectroscopy including $\pi\pi$ interpolating operators
- benefit 1: if m has been computed with error δm, the relative error on e^{-mx_0} is $\delta m \cdot x_0 \Rightarrow$ only linear growth of the error.
- benefit 2: correct for finite-size effects from having discrete instead of continuum $\pi\pi$ states.

1107.4388; 1710.10072 (Mainz-CLS).

Spectroscopy: B. Hörz, J. Bulava et al. (1511.02351).

Summary

- Lellouch-Lüscher type relations have important applications in low-energy QCD phenomenology (e.g. flavor physics).

- In the ρ-meson channel, a dedicated calculation of the form factor in the timelike region helps remove finite-volume effects; perhaps also improve the statistical precision of lattice calculations of a_μ^{hvp}.

- The extension to moving frames and sometimes the use of dedicated boundary conditions plays an important role in practice.

Outlook

Can we go above the elastic scattering region?

- Extension to multi-two-particle channels exists.
- Three-particle channel available too. Many aspects to explore.
- Higher up in energy: even experiments perform inclusive measurements. Can we do the same on the lattice? $\rightsquigarrow$ Lecture 3.