Three-particle scattering from numerical lattice QCD

Scattering from the lattice: applications to phenomenology and beyond

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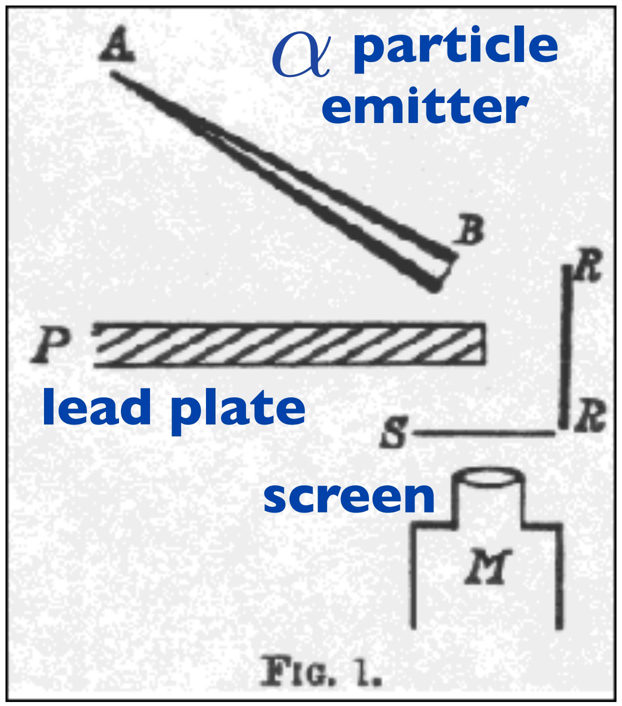
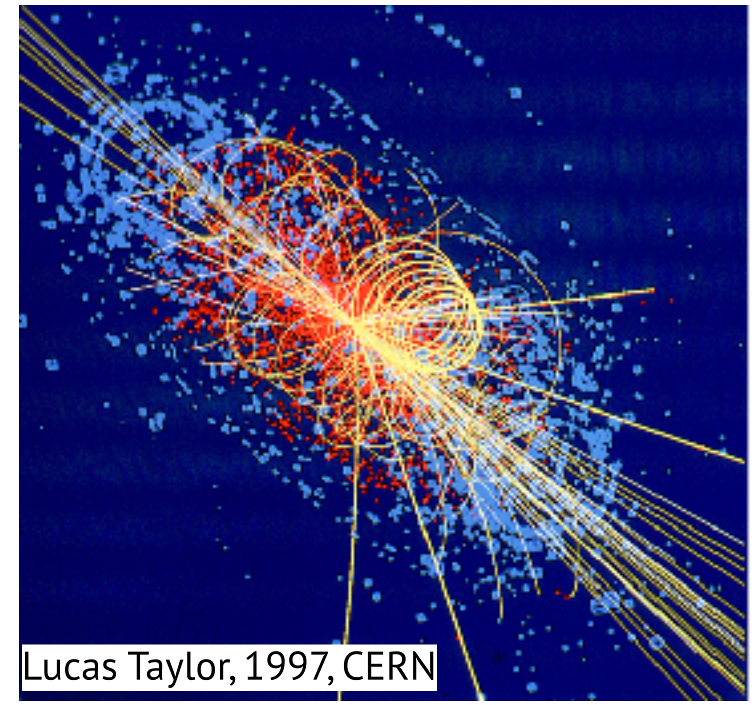
CERN

Motivation for scattering observables

Much of our knowledge of particle physics comes from
scattering experiments

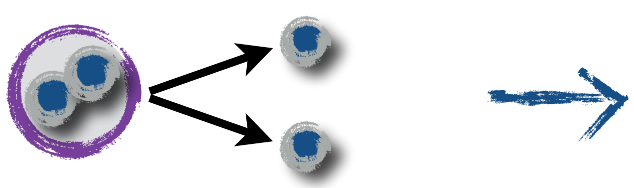


Rutherford 1911



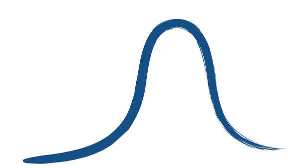
CMS 1997

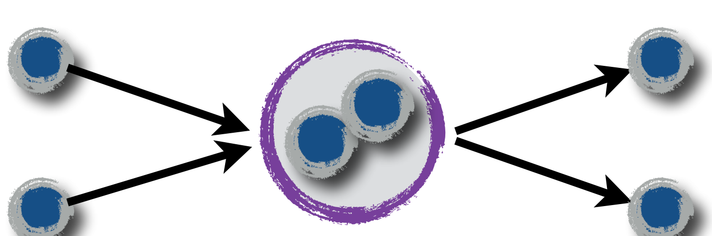
Short-lived states (resonances) seen via scattering



$$e^{-iHt}|\;\rangle \neq |\;\rangle e^{-iEt}$$

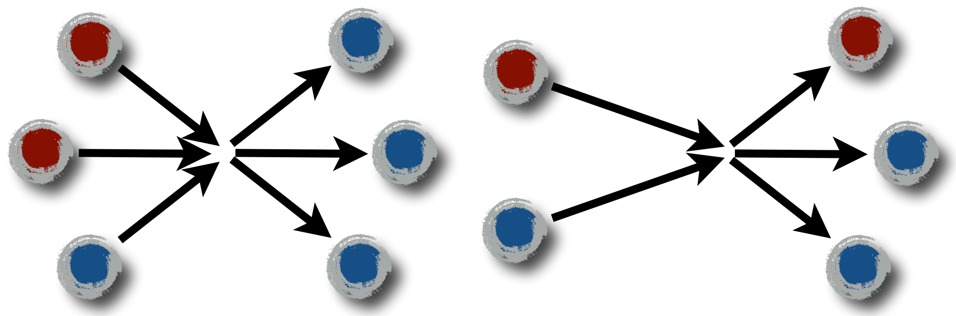
Not energy eigenstates, indeed not states in Hilbert space!





Instead, a resonance is a peak in a scattering rate.
Or, more generally, a pole in its analytic continuation.

The aim here is to derive a formalism for studying relativistic two- and three-particle systems from lattice QCD

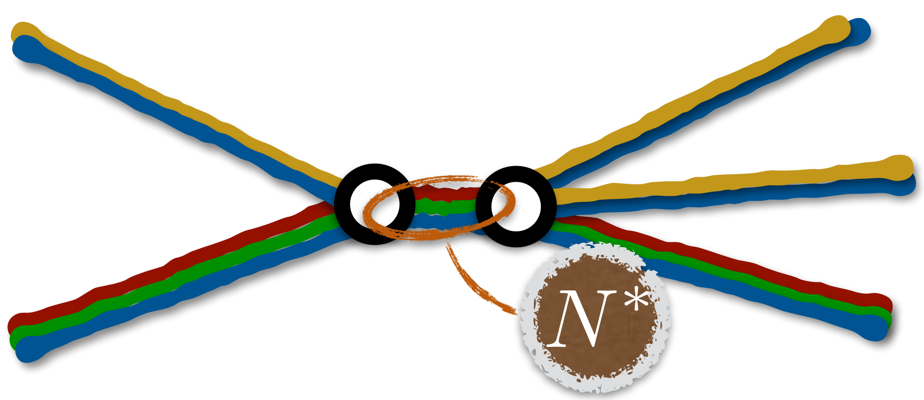


Potential applications…

Studying three-particle resonances

$\omega(782),\ a_1(1420) \to \pi\pi\pi$

$N(1440) \to N\pi, N\pi\pi$

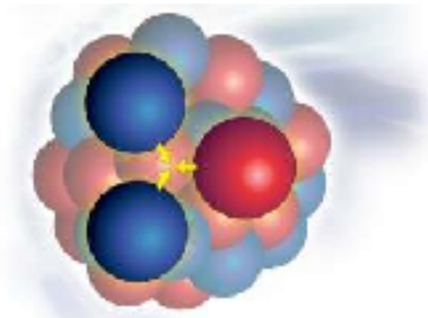


Calculating weak decays, form factors and transitions

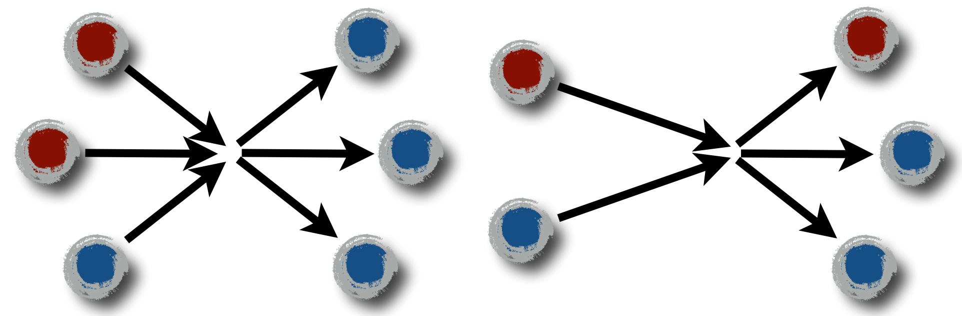
$K \to \pi\pi\pi \qquad N\gamma^* \to N\pi\pi$

Determining three-body interactions

NNN three-body forces needed as EFT input for studying larger nuclei and nuclear matter



The aim here is to derive a formalism for studying relativistic two- and three-particle systems from lattice QCD



Other motivations...

The Lüscher and Lellouch-Lüscher formalism only applies for

$\sqrt{s} = E_{\rm cm} <$ **multi-particle threshold**

$\pi K \to \pi K$		$\sqrt{s} = 2m_\pi + m_K$
$\pi\pi \to \pi\pi$	**only up to**	$\sqrt{s} = 4m_\pi$
$N\pi \to N\pi$		$\sqrt{s} = 2m_\pi + m_N$

3-particle formalism is a step on the way to ***n*-particle** formalism

Outline

- ☐ Warm up and definitions
- ☐ Two particles in a box
- ☐ Three particles in a box
- ☐ Testing the result
- ☐ Numerical explorations
- ☐ Looking forward

Outline

☐ Warm up and definitions
- ☐ Basic set-up
- ☐ Finite-volume correlator
- ☐ Three non-interacting particles

☐ Two particles in a box
- ☐ Alternative derivation
- ☐ Truncation and application
- ☐ Relating matrix elements

☐ Three particles in a box
- ☐ 3-to-3 scattering
- ☐ (Sketch of) derivation
- ☐ An unexpected infinite-volume quantity
- ☐ Relating energies to scattering

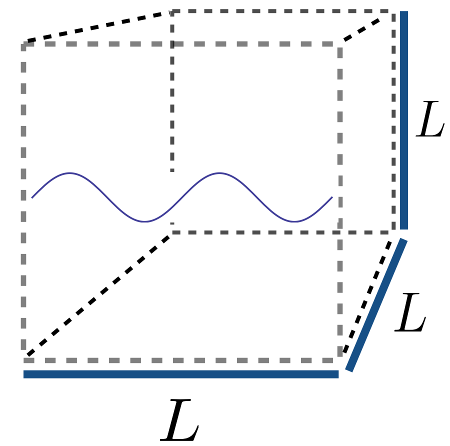
☐ Testing the result
- ☐ Large-volume expansion
- ☐ Effimov state in a box

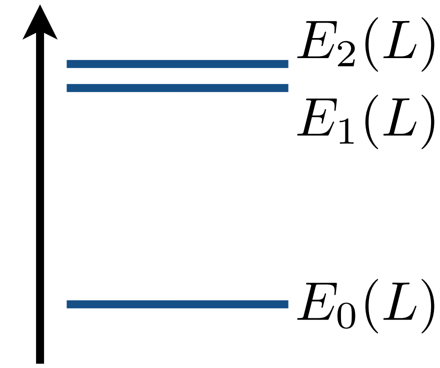
☐ Numerical explorations
- ☐ Truncation at low energies
- ☐ 2-particle physics in 3-particle energies
- ☐ Toy 3-particle resonance
- ☐ Numerical Effimov state
- ☐ Unphysical solutions

☐ Looking forward
- ☐ 2-to-3 scattering and resonant subprocesses
- ☐ Speculations

Finite-volume set-up

cubic, spatial volume (extent L)





periodic boundary conditions

$$q(\tau, \mathbf{x}) = q(\tau, \mathbf{x} + L\mathbf{e}_i) \quad \Bigg| \quad \pi(\tau, \mathbf{x}) = \pi(\tau, \mathbf{x} + L\mathbf{e}_i)$$

time direction **infinite**

L large enough to ignore e^{-mL}

Assume lattice effects are small or removed elsewhere

Work in continuum field theory throughout

Simplest quantization condition…

$$\pi(\tau, \mathbf{x}) = \int_L d^3\mathbf{x}\, e^{-i\mathbf{p}\cdot\mathbf{x}}\, \tilde{\pi}(\tau, \mathbf{p})$$

$$\pi(\tau, \mathbf{x} + L\mathbf{e}_i) = \int_L d^3\mathbf{x}\, e^{-i\mathbf{p}\cdot(\mathbf{x}+L\mathbf{e}_i)}\, \tilde{\pi}(\tau, \mathbf{p})$$

these must be equal

$$e^{-i\mathbf{p}\cdot L\mathbf{e}_i} = 1 \implies p_i L = 2\pi n_i$$

Quantization of momentum

$$\mathbf{p} = \frac{2\pi}{L}\mathbf{n}\,, \quad \mathbf{n} \in \mathbb{Z}^3$$

Fourier transform conventions (here in one dimension)

- Beginning in infinite volume, we need a convention to normalize the Fourier transform and its inverse

$$\mathcal{FT}^{-1}\Big[\mathcal{FT}[f]\Big] = f \quad \longrightarrow \quad \int \frac{dp}{N}\, e^{ipx} \left[\int \frac{dx'}{N'}\, e^{-ipx'} f(x') \right] = \frac{2\pi f(x)}{NN'}$$

$$\frac{2\pi}{NN'} = 1$$

simplify via...

$$\int dp\, e^{ip(x-x')} = 2\pi\delta(x-x')$$

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- Now we repeat the exercise in a finite volume

$$\frac{1}{N}\sum_n e^{ix(2\pi n/L)}\left[\int_0^L dx'\, e^{-ix'(2\pi n/L)} f(x')\right]$$

Rewrite this using $\sum_n e^{2\pi i z} = \sum_{n'} \delta(z+n')$

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$$\frac{1}{N}\sum_n e^{ix(2\pi n/L)}\Big[\int_0^L dx'\, e^{-ix'(2\pi n/L)} f(x')\Big] = \int_0^L dx' \sum_{n'} \frac{\delta\big[(x'-x)/L - n'\big]}{N} f(x') = \frac{L f(x)}{N}$$

$$= 1$$

Rewrite this using $\sum_n e^{2\pi i z} = \sum_{n'} \delta(z+n')$

argument can only vanish when $n' = 0$

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$$p = 2\pi n/L$$

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$$\lim_{L\to\infty} \frac{1}{L}\sum_p f(p) = \lim_{L\to\infty} \frac{1}{L}\sum_n f(2\pi n/L) = \lim_{L\to\infty} \frac{1}{L}\int dn\, f(2\pi n/L) = \lim_{L\to\infty} \frac{1}{L}\int dp \frac{L}{2\pi}\, f(p) = \int \frac{dp}{2\pi} f(p)$$

For a smooth function we can replace the sum with an integral

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$$\lim_{L\to\infty} \left[\frac{1}{L}\sum_p - \int \frac{dp}{2\pi} \right] f(p) = 0$$

Finite-volume correlator

Convenient to work with momentum space, finite-volume correlators

$$C_L(P) \equiv \int_L d^4x\; e^{-iPx} \langle 0|T\mathcal{O}(x)\mathcal{O}^\dagger(0)|0\rangle$$

Total 4-momentum

$$P = \big(E, \mathbf{P}\big) = \big(E, 2\pi\mathbf{n}_P/L\big)$$

c.m. frame energy: $E^{*2} = E^2 - \mathbf{P}^2$

3-particle interpolator

$$\mathcal{O}(x) \equiv \frac{1}{6}\int_L d^4y \int_L d^4z\; f(y,z)\; \varphi(x)\varphi(x+y)\varphi(x+z)$$

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To see why this is useful, perform a spectral decomposition: $\mathbb{I} = \sum_n |E_n, L\rangle\langle E_n, L|$

$$C_L(P) = \int_0^\infty dt \sum_n e^{i(E - E_n + i\epsilon)t} \langle 0 | \widetilde{\mathcal{O}}(0, \mathbf{P}) | E_n, \mathbf{P}, L \rangle \langle E_n, \mathbf{P}, L | \mathcal{O}^\dagger(0) | 0 \rangle + (t < 0)$$

Use $H \to H - i\epsilon$

Separate the two time orderings

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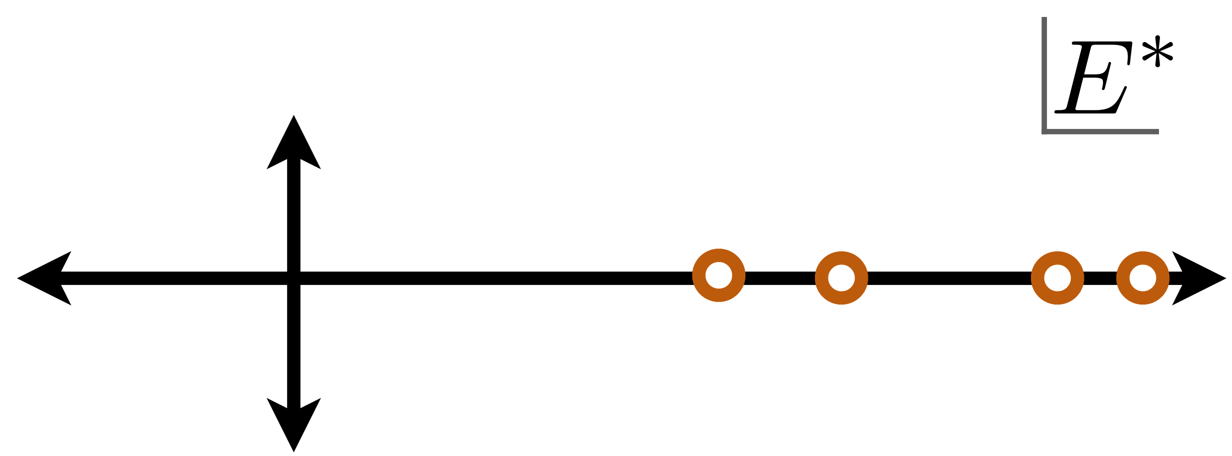
$$= L^3 \sum_n \frac{e^{i(E - E_n + i\epsilon)t}}{i(E - E_n + i\epsilon)} \Bigg|_0^\infty |\langle E_n, \mathbf{P}, L | \mathcal{O}^\dagger(0) | 0 \rangle|^2 + (t < 0)$$

$$C_L(P) = \sum_n 2E_n(L) L^3 \frac{i |\langle E_n, \mathbf{P}, L | \mathcal{O}^\dagger(0) | 0 \rangle|^2}{E^2 - E_n(L)^2 + i\epsilon}$$



Pole locations give the finite-volume spectrum

Diagrammatic evaluation (non-interacting)

We can also calculate the correlator explicitly by contracting fields

$$C_L(P) = \int_L d^4x \ e^{-iPx} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(0) | 0 \rangle = \sum_n 2E_n(L) L^3 \frac{i |\langle E_n, \mathbf{P}, L | \mathcal{O}^\dagger(0) | 0 \rangle|^2}{E^2 - E_n(L)^2 + i\epsilon}$$

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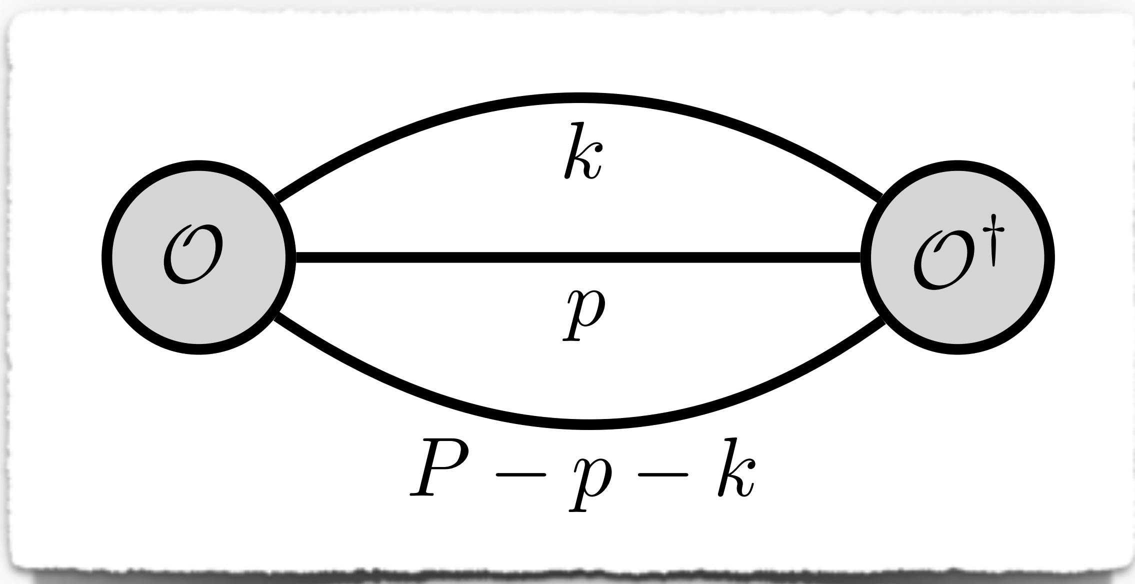

All loop momenta are summed rather than integrated: $\displaystyle \int \frac{d^4p}{(2\pi)^4} \Longrightarrow \int \frac{dp^0}{2\pi} \frac{1}{L^3} \sum_{\mathbf{p}}$

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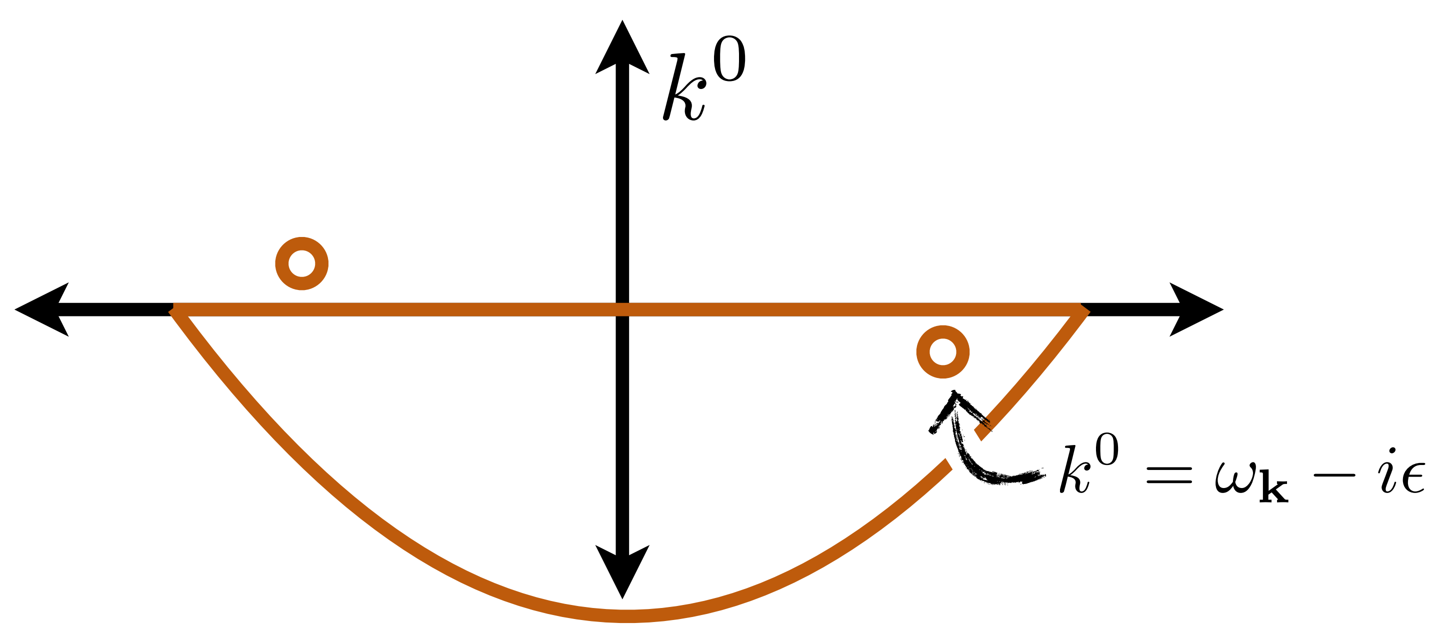
$$= \frac{1}{6} \frac{1}{L^6} \sum_{\mathbf{p},\mathbf{k}} \frac{i |\widetilde{f}(k,p)|^2}{2\omega_{\mathbf{k}} 2\omega_{\mathbf{p}} [(P-p-k)^2 - m^2 + i\epsilon]} + \mathcal{R}_L(E)$$



$\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$

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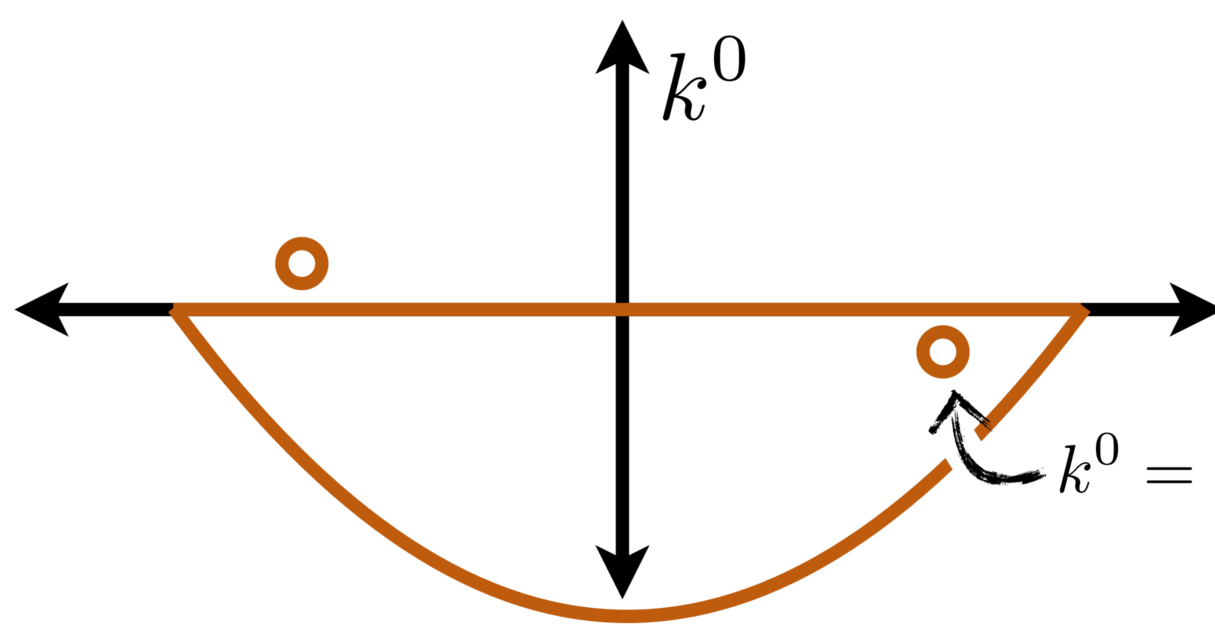
from other poles



$k^0 = \omega_{\mathbf{k}} - i\epsilon$ | $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$

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via... $$\frac{1}{A^2 - B^2} = \frac{1}{2B(A-B)} - \frac{1}{2B(A+B)}$$

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where $\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$ **and** $\omega_{\mathbf{pk}} = \sqrt{(\mathbf{P} - \mathbf{p} - \mathbf{k})^2 + m^2}$

Diagrammatic evaluation (non-interacting)

We can also calculate the correlator explicitly by contracting fields

$$C_L(P) = \int_L d^4x \; e^{-iPx} \langle 0|T\mathcal{O}(x)\mathcal{O}^\dagger(0)|0\rangle = \sum_n 2E_n(L)L^3 \frac{i|\langle E_n, \mathbf{P}, L|\mathcal{O}^\dagger(0)|0\rangle|^2}{E^2 - E_n(L)^2 + i\epsilon}$$

3-particle interpolator

$$\mathcal{O}(x) \equiv \frac{1}{6}\int_L d^4y \int_L d^4z \; f(y,z) \; \varphi(x)\varphi(x+y)\varphi(x+z)$$

$$C_L(P) = \frac{1}{6}\int_{p^0} \frac{1}{L^3}\sum_{\mathbf{p}} \int_{k^0} \frac{1}{L^3}\sum_{\mathbf{k}} \frac{i^3|\widetilde{f}(k,p)|^2}{[k^2 - m^2 + i\epsilon][p^2 - m^2 + i\epsilon][(P-p-k)^2 - m^2 + i\epsilon]}$$

$$= \frac{1}{6}\frac{1}{L^6}\sum_{\mathbf{p},\mathbf{k}} \frac{i|\widetilde{f}(k,p)|^2}{2\omega_{\mathbf{k}}2\omega_{\mathbf{p}}[(P-p-k)^2 - m^2 + i\epsilon]} + \mathcal{R}_L(E)$$

$$= \frac{1}{6}\frac{1}{L^6}\sum_{\mathbf{p},\mathbf{k}} \frac{i|\widetilde{f}(k,p)|^2}{2\omega_{\mathbf{k}}2\omega_{\mathbf{p}}2\omega_{\mathbf{pk}}[E - \omega_{\mathbf{k}} - \omega_{\mathbf{p}} - \omega_{\mathbf{pk}} + i\epsilon]} + \mathcal{R}'_L(P)$$

Aside…

$$\mathcal{R}'_L(P) = \frac{1}{L^6}\sum_{\mathbf{k},\mathbf{p}} \mathcal{S}(\mathbf{p},\mathbf{k})$$

smooth function

Diagrammatic evaluation (non-interacting)

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Aside…

$$\mathcal{R}'_L(P) = \frac{1}{L^6} \sum_{\mathbf{k},\mathbf{p}} \mathcal{S}(\mathbf{p},\mathbf{k}) = \int_{\mathbf{p},\mathbf{k}} \mathcal{S}(\mathbf{p},\mathbf{k}) + \mathcal{O}(e^{-mL})$$

$$\approx \mathcal{R}'_\infty(P)$$

smooth function

Diagrammatic evaluation (non-interacting)

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Some lessons from non-interacting particles

$$C_L(P) = \sum_n 2E_n(L)L^3 \frac{i|\langle E_n, \mathbf{P}, L|\mathcal{O}^\dagger(0)|0\rangle|^2}{E^2 - E_n(L)^2 + i\epsilon}$$

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- ☑ The number of solutions scales as L^6

  As $L \to \infty$ (at fixed iε) the sum-integral difference term must vanish

- ☑ The finite-volume energies are given by

$$E_n(L) = \sqrt{\mathbf{p}_1^2 + m^2} + \sqrt{\mathbf{p}_1^2 + m^2} + \sqrt{\mathbf{p}_3^2 + m^2}$$

where $\mathbf{p}_i = \frac{2\pi}{L}\mathbf{n}_i\,,\quad \mathbf{n}_i \in \mathbb{Z}^3$ and $\sum_i \mathbf{p}_i = \mathbf{P}$

- ☑ The residues at the poles must also match…

Details on $\widetilde{f}(k,p)$ ...

$\widetilde{f}(k,p)$ is the symmetrized Fourier transform of $f(x,y)$

$$\widetilde{f}(p,k) = \frac{1}{6} \sum_{\{k_1,k_2\} \in \{p,k,P-p-p\}} \int_L d^4z \int_L d^4y \; e^{-ik_1 z - ik_2 y} \; f(y,z)$$

recall...

$$\mathcal{O}(x) \equiv \frac{1}{6} \int_L d^4y \int_L d^4z \; f(y,z) \; \varphi(x)\varphi(x+y)\varphi(x+z)$$

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The "on-shell" version of this is an infinite-volume matrix element

$$\widetilde{f}(k,p)\Big|_{k^0=\omega_{\mathbf{k}},\; p^0=\omega_{\mathbf{p}},\; E-p^0-k^0=\omega_{\mathbf{pk}}} = \langle 0|\mathcal{O}(0)|\mathbf{P},\mathbf{k},\mathbf{p};\varphi\,\varphi\,\varphi\rangle$$

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The first two constraints are straightforward

This one is trickier

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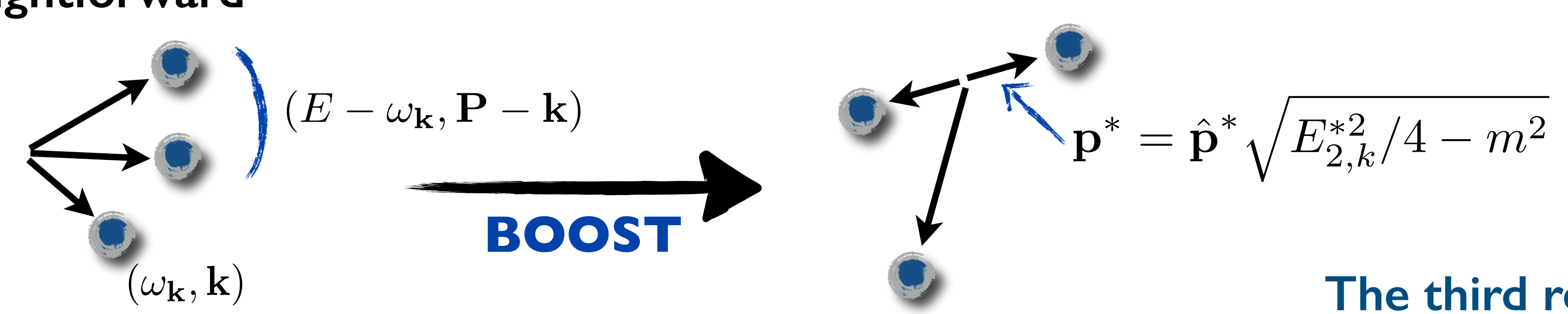

The third requirement restricts the three-momentum magnitude

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The "on-shell" version of this is an infinite-volume matrix element

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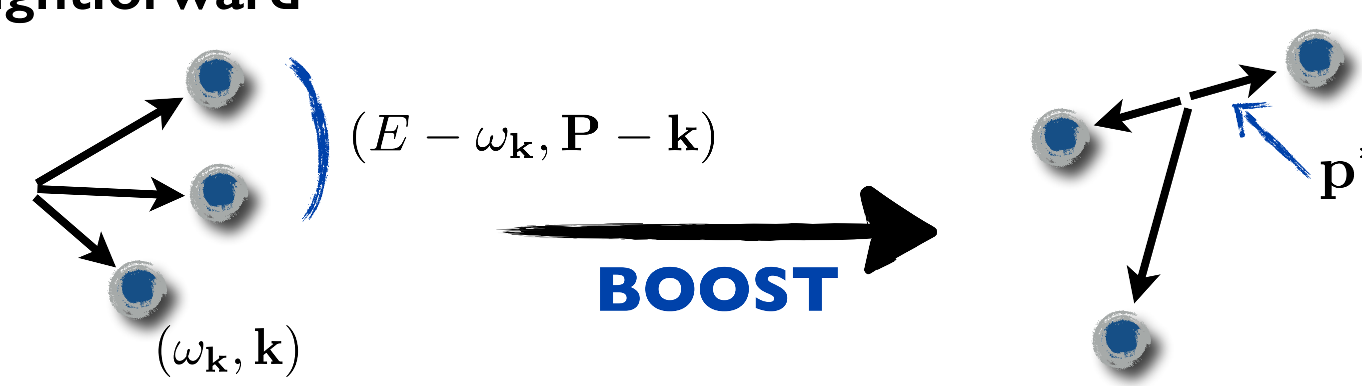
The first two constraints are straightforward

This one is trickier



$$\mathbf{p}^* = \hat{\mathbf{p}}^* \sqrt{E_{2,k}^{*2}/4 - m^2}$$

The third requirement restricts the three-momentum magnitude

Putting $\widetilde{f}(k,p)$ on shell

Now return to our finite-volume correlator

$$C_L(P) = C_\infty(P) + \frac{1}{6}\left[\frac{1}{L^6}\sum_{\mathbf{p},\mathbf{k}} - \int_{\mathbf{k},\mathbf{p}}\right] \frac{i|\widetilde{f}(k,p)|^2}{2\omega_{\mathbf{k}}2\omega_{\mathbf{p}}2\omega_{\mathbf{pk}}[E-\omega_{\mathbf{k}}-\omega_{\mathbf{p}}-\omega_{\mathbf{pk}}+i\epsilon]} + \mathcal{O}(e^{-mL})$$

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Note that this can be (trivially) rewritten as

$$C_L(P) = C_\infty(P) + \frac{1}{6}\left[\frac{1}{L^6}\sum_{\mathbf{p},\mathbf{k}} - \int_{\mathbf{k},\mathbf{p}}\right] \frac{i|\widetilde{f}_{\text{on}}(\mathbf{k},\hat{\mathbf{p}}^*)|^2}{2\omega_{\mathbf{k}}2\omega_{\mathbf{p}}2\omega_{\mathbf{pk}}[E-\omega_{\mathbf{k}}-\omega_{\mathbf{p}}-\omega_{\mathbf{pk}}+i\epsilon]} + \mathcal{R}''_L(P)$$

where $$\mathcal{R}''_L(P) = \frac{1}{6}\left[\frac{1}{L^6}\sum_{\mathbf{p},\mathbf{k}} - \int_{\mathbf{k},\mathbf{p}}\right] \frac{i\Big(|\widetilde{f}(k,p)|^2 - |\widetilde{f}_{\text{on}}(\mathbf{k},\hat{\mathbf{p}}^*)|^2\Big)}{2\omega_{\mathbf{k}}2\omega_{\mathbf{p}}2\omega_{\mathbf{pk}}[E-\omega_{\mathbf{k}}-\omega_{\mathbf{p}}-\omega_{\mathbf{pk}}+i\epsilon]}$$

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this further reduces to…

smooth

$$\mathcal{R}''_L(P) = \frac{1}{6}\left[\frac{1}{L^6}\sum_{\mathbf{p},\mathbf{k}} - \int_{\mathbf{k},\mathbf{p}}\right] \frac{i\mathcal{S}(\mathbf{p},\mathbf{k})[E-\omega_{\mathbf{k}}-\omega_{\mathbf{p}}-\omega_{\mathbf{pk}}+i\epsilon]}{2\omega_{\mathbf{k}}2\omega_{\mathbf{p}}2\omega_{\mathbf{pk}}[E-\omega_{\mathbf{k}}-\omega_{\mathbf{p}}-\omega_{\mathbf{pk}}+i\epsilon]}$$

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Now return to our finite-volume correlator

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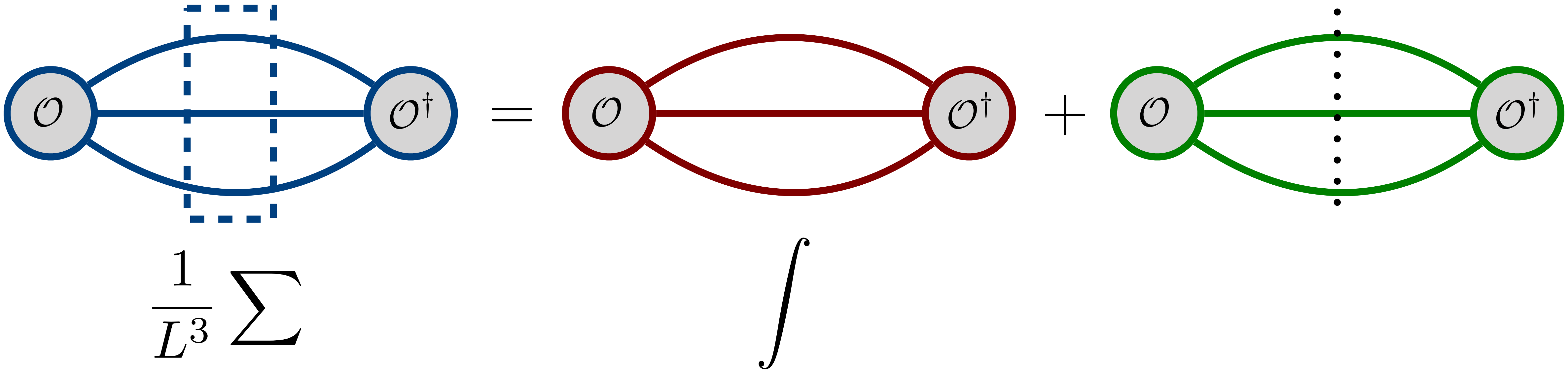
Three-particle cut identity

$$C_L(P) = C_\infty(P) + \frac{1}{6}\left[\frac{1}{L^6}\sum_{\mathbf{p},\mathbf{k}} - \int_{\mathbf{k},\mathbf{p}}\right] \frac{i|\widetilde{f}_{\text{on}}(\mathbf{k},\hat{\mathbf{p}}^*)|^2}{2\omega_{\mathbf{k}}2\omega_{\mathbf{p}}2\omega_{\mathbf{pk}}[E-\omega_{\mathbf{k}}-\omega_{\mathbf{p}}-\omega_{\mathbf{pk}}+i\epsilon]} + \mathcal{O}(e^{-mL})$$



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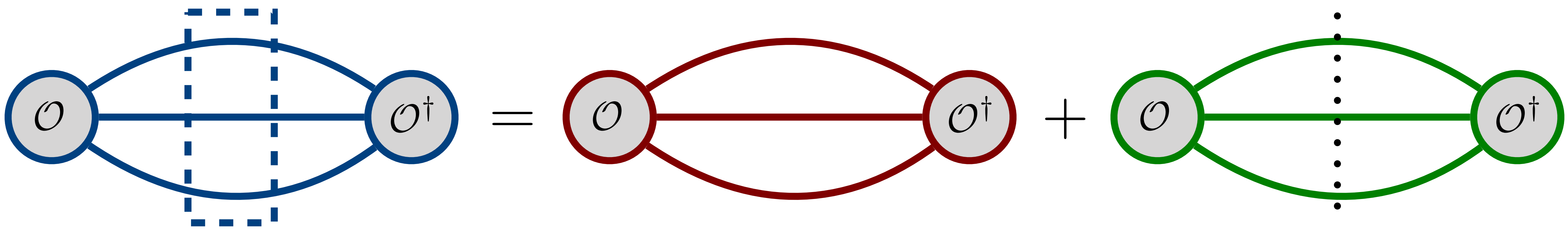
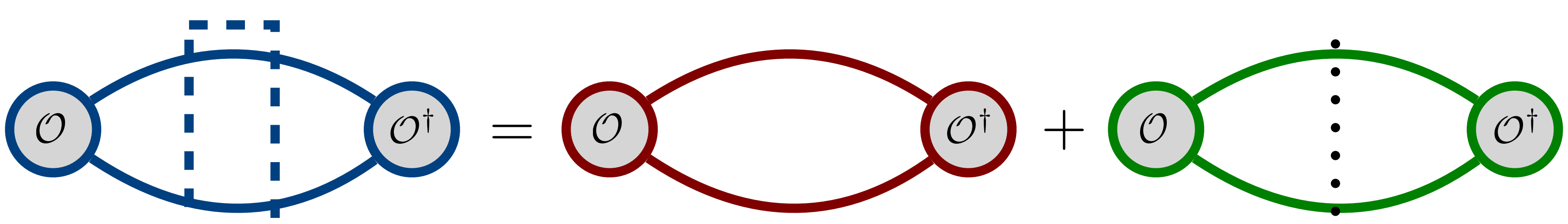
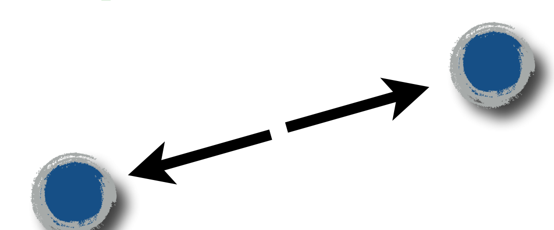


$$\frac{1}{L^3}\sum \qquad\qquad \int$$

Two-particle analog

$$C_L(P) = C_\infty(P) + \frac{1}{2}\left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int_{\mathbf{k}}\right] \frac{i|\widetilde{f}_{\rm on}(\hat{\mathbf{k}}^*)|^2}{2\omega_{\mathbf{k}}2\omega_{\mathbf{P}-\mathbf{p}}[E-\omega_{\mathbf{k}}-\omega_{\mathbf{P}-\mathbf{k}}+i\epsilon]} + \mathcal{O}(e^{-mL})$$





$$\mathbf{k}^* = \hat{\mathbf{k}}^*\sqrt{E^{*2}/4 - m^2}$$

$$E^* = \sqrt{E^2 - \mathbf{P}^2}$$

Final lesson from non-interacting particles

$$C_L(P) = \sum_n 2E_n(L)L^3 \frac{i|\langle E_n, \mathbf{P}, L|\mathcal{O}^\dagger(0)|0\rangle|^2}{E^2 - E_n(L)^2 + i\epsilon}$$

$$C_L(P) = C_\infty(P) + \frac{1}{6}\left[\frac{1}{L^6}\sum_{\mathbf{p},\mathbf{k}} - \int_{\mathbf{k},\mathbf{p}}\right] \frac{i|\langle \mathbf{P}, \mathbf{p}, \mathbf{k};\, \varphi\varphi\varphi|\mathcal{O}^\dagger(0)|0\rangle|^2}{2\omega_{\mathbf{k}} 2\omega_{\mathbf{p}} 2\omega_{\mathbf{pk}}[E - \omega_{\mathbf{k}} - \omega_{\mathbf{p}} - \omega_{\mathbf{pk}} + i\epsilon]} + \mathcal{O}(e^{-mL})$$

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☑ Matching the residues at the poles gives a **relation on matrix elements**

Final lesson from non-interacting particles

$$C_L(P) = \sum_n 2E_n(L) L^3 \frac{i|\langle E_n, \mathbf{P}, L|\mathcal{O}^\dagger(0)|0\rangle|^2}{E^2 - E_n(L)^2 + i\epsilon}$$

$$C_L(P) = C_\infty(P) + \frac{1}{6}\left[\frac{1}{L^6}\sum_{\mathbf{p},\mathbf{k}} - \int_{\mathbf{k},\mathbf{p}}\right] \frac{i|\langle \mathbf{P}, \mathbf{p}, \mathbf{k};\, \varphi\varphi\varphi|\mathcal{O}^\dagger(0)|0\rangle|^2}{2\omega_{\mathbf{k}} 2\omega_{\mathbf{p}} 2\omega_{\mathbf{pk}}[E - \omega_{\mathbf{k}} - \omega_{\mathbf{p}} - \omega_{\mathbf{pk}} + i\epsilon]} + \mathcal{O}(e^{-mL})$$

☑ Matching the residues at the poles gives a **relation on matrix elements**

Zoom in on the pole…

in the spectral decomposition

$$C_L(P)\Big|_{E=E_m(L)+\delta} = i\frac{L^3|\langle E_m, \mathbf{P}, L|\mathcal{O}^\dagger(0)|0\rangle|^2}{\delta} + \mathcal{O}(\delta^0)$$

Final lesson from non-interacting particles

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in the spectral decomposition

in the QFT result

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$$C_L(P) = \sum_n 2E_n(L) L^3 \frac{i|\langle E_n, \mathbf{P}, L|\mathcal{O}^\dagger(0)|0\rangle|^2}{E^2 - E_n(L)^2 + i\epsilon}$$

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☑ Matching the residues at the poles gives a **relation on matrix elements**

Zoom in on the pole…

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in the QFT result

$$C_L(P)\Big|_{E=E_m(L)+\delta} = i\frac{L^3|\langle E_m, \mathbf{P}, L|\mathcal{O}^\dagger(0)|0\rangle|^2}{\delta} + \mathcal{O}(\delta^0) = \frac{1}{6}\frac{1}{L^6}\sum_{\mathbf{p},\mathbf{k}\in\Omega_m} i\frac{|\langle \mathbf{P}, \mathbf{p}, \mathbf{k};\, \varphi\varphi\varphi|\mathcal{O}^\dagger(0)|0\rangle|^2}{\delta} + \mathcal{O}(\delta^0)$$

$$|\langle E_n, \mathbf{P}, L|\mathcal{O}^\dagger(0)|0\rangle|^2 = \frac{\nu_n}{6L^9}\frac{1}{2\omega_{\mathbf{k}} 2\omega_{\mathbf{p}} 2\omega_{\mathbf{pk}}}\Big\langle |\langle \mathbf{P}, \mathbf{p}, \mathbf{k};\, \varphi\varphi\varphi|\mathcal{O}^\dagger(0)|0\rangle|^2 \Big\rangle_{\Omega_n}$$

Non-interacting three-particle Lellouch-Lüscher factor

Plotting the non-interacting spectrum

$$\frac{E_n(L)}{m} = \sqrt{1 + \frac{4\pi^2 n_1}{(mL)^2}} + \sqrt{1 + \frac{4\pi^2 n_2}{(mL)^2}} + \sqrt{1 + \frac{4\pi^2 n_3}{(mL)^2}}$$

$n_1 = \mathbf{n}_1^2\,,\ n_2 = \mathbf{n}_2^2\,,\ n_3 = (\mathbf{n}_P - \mathbf{n}_1 - \mathbf{n}_2)^2$

Note: $mL = m_\pi L$ (not milliliters)

Plotting the non-interacting spectrum

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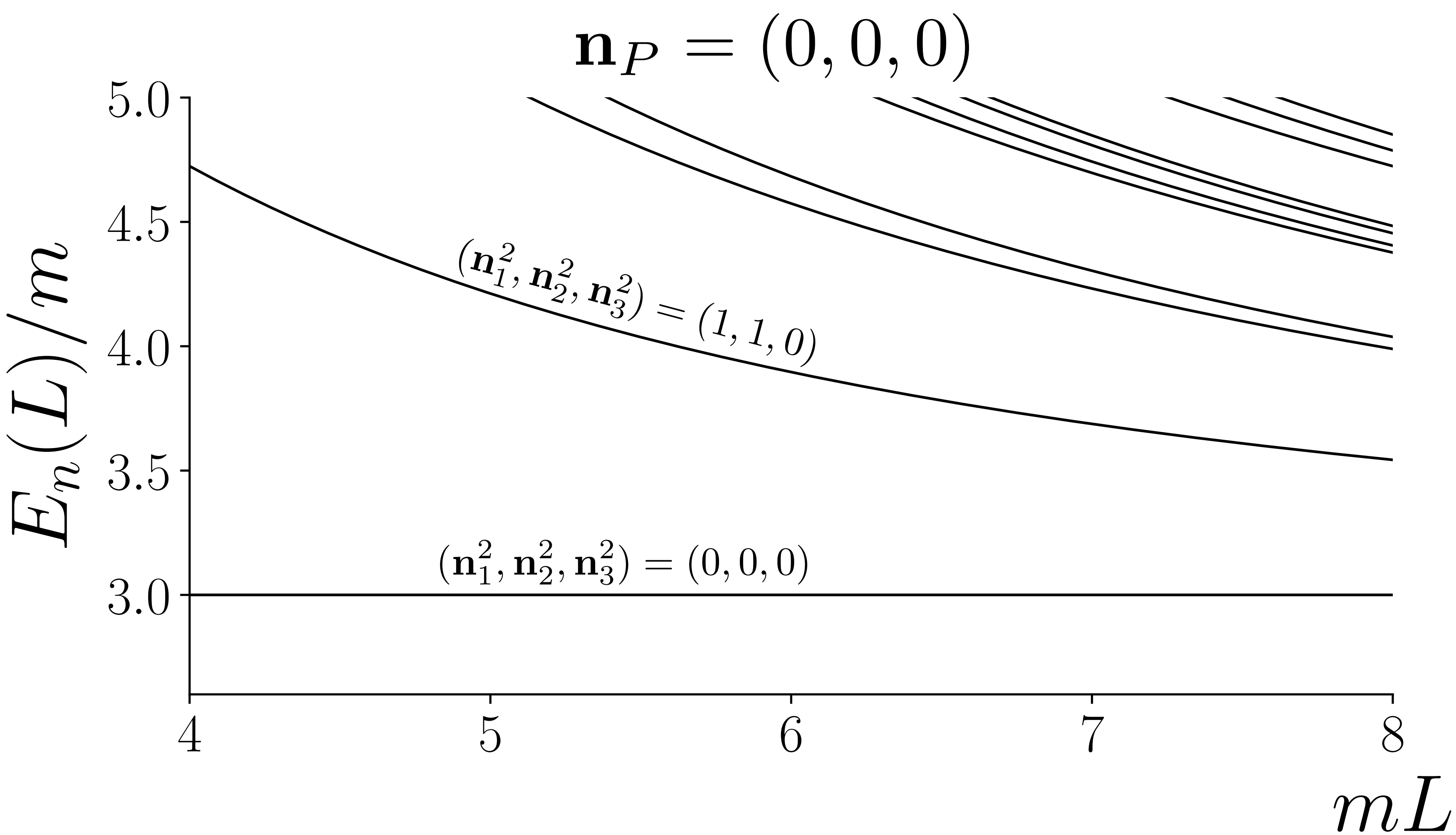

Plotting the non-interacting spectrum

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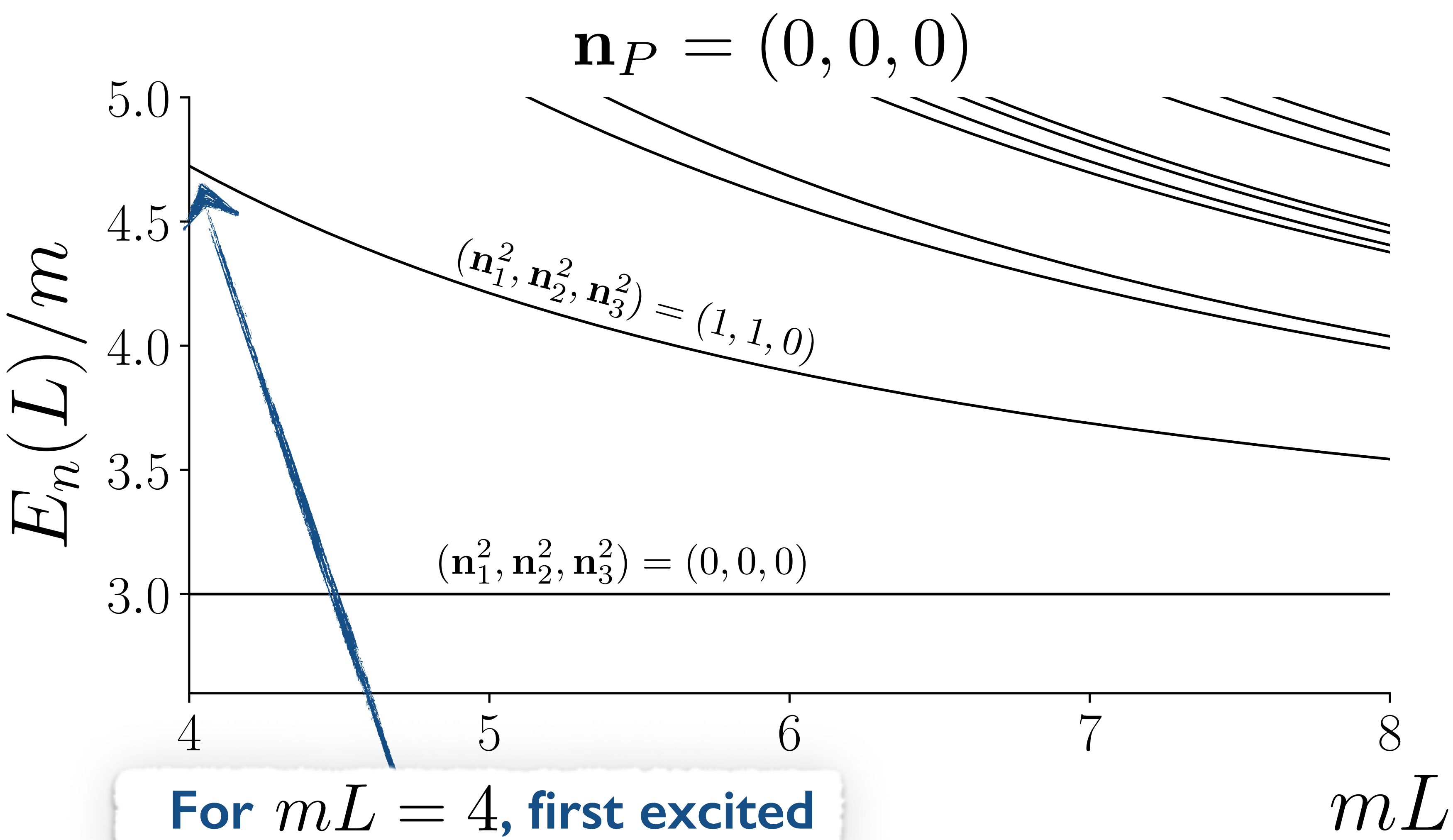

For $mL = 4$, first excited state is already relativistic

$$\frac{p^2}{m^2} = \left(\frac{2\pi}{mL}\right)^2 \approx 2.46$$

Plotting the non-interacting spectrum

$$\frac{E_n(L)}{m} = \sqrt{1+\frac{4\pi^2 n_1}{(mL)^2}} + \sqrt{1+\frac{4\pi^2 n_2}{(mL)^2}} + \sqrt{1+\frac{4\pi^2 n_3}{(mL)^2}}$$

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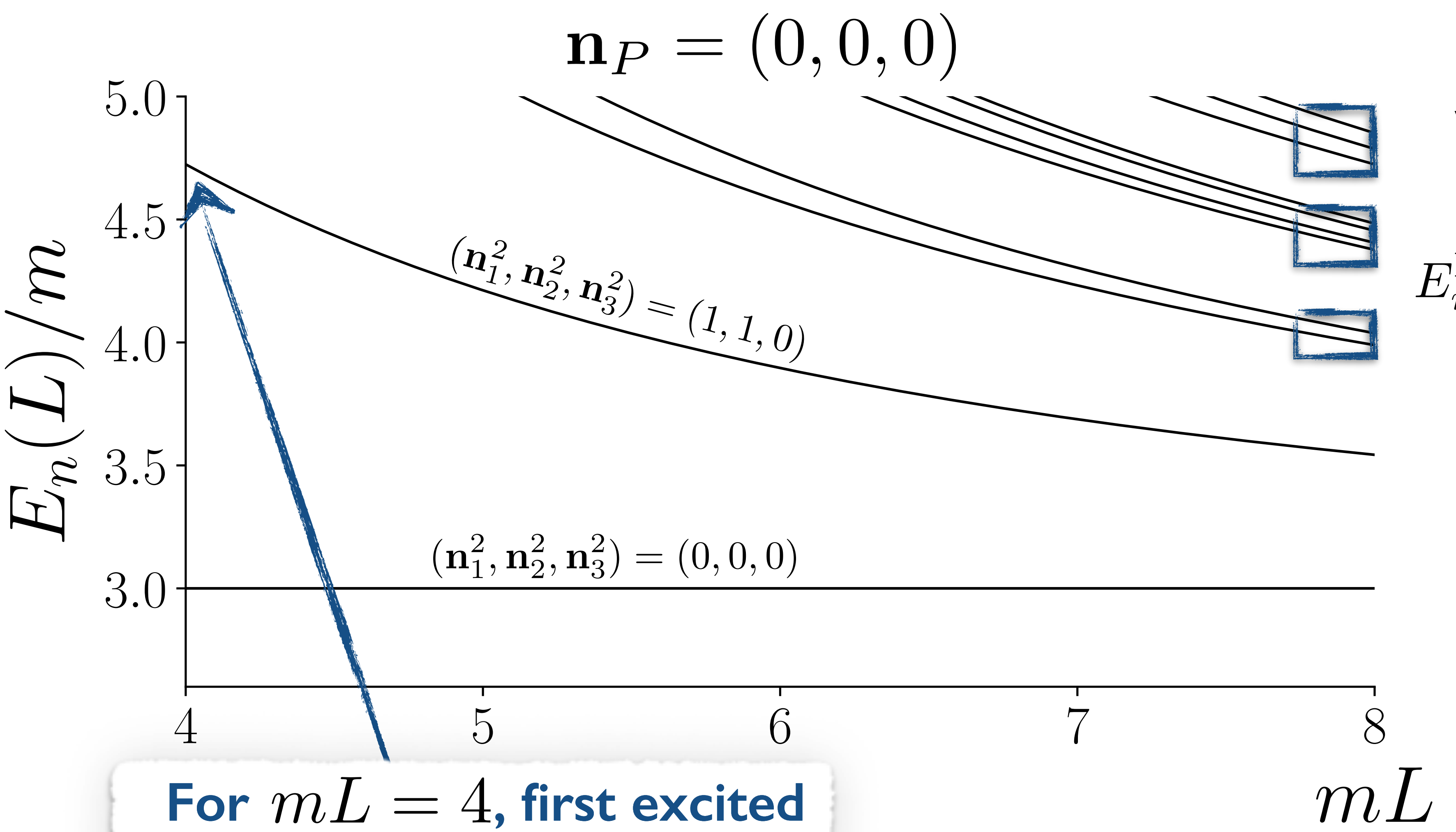


Why are these states clustered?

Accidental NR degeneracy!

$$E_n^{\rm NR}(L) = 3m + \frac{2\pi^2}{L^2}(n_1+n_2+n_3)$$

For $mL = 4$, first excited state is already relativistic

$$\frac{p^2}{m^2} = \left(\frac{2\pi}{mL}\right)^2 \approx 2.46$$

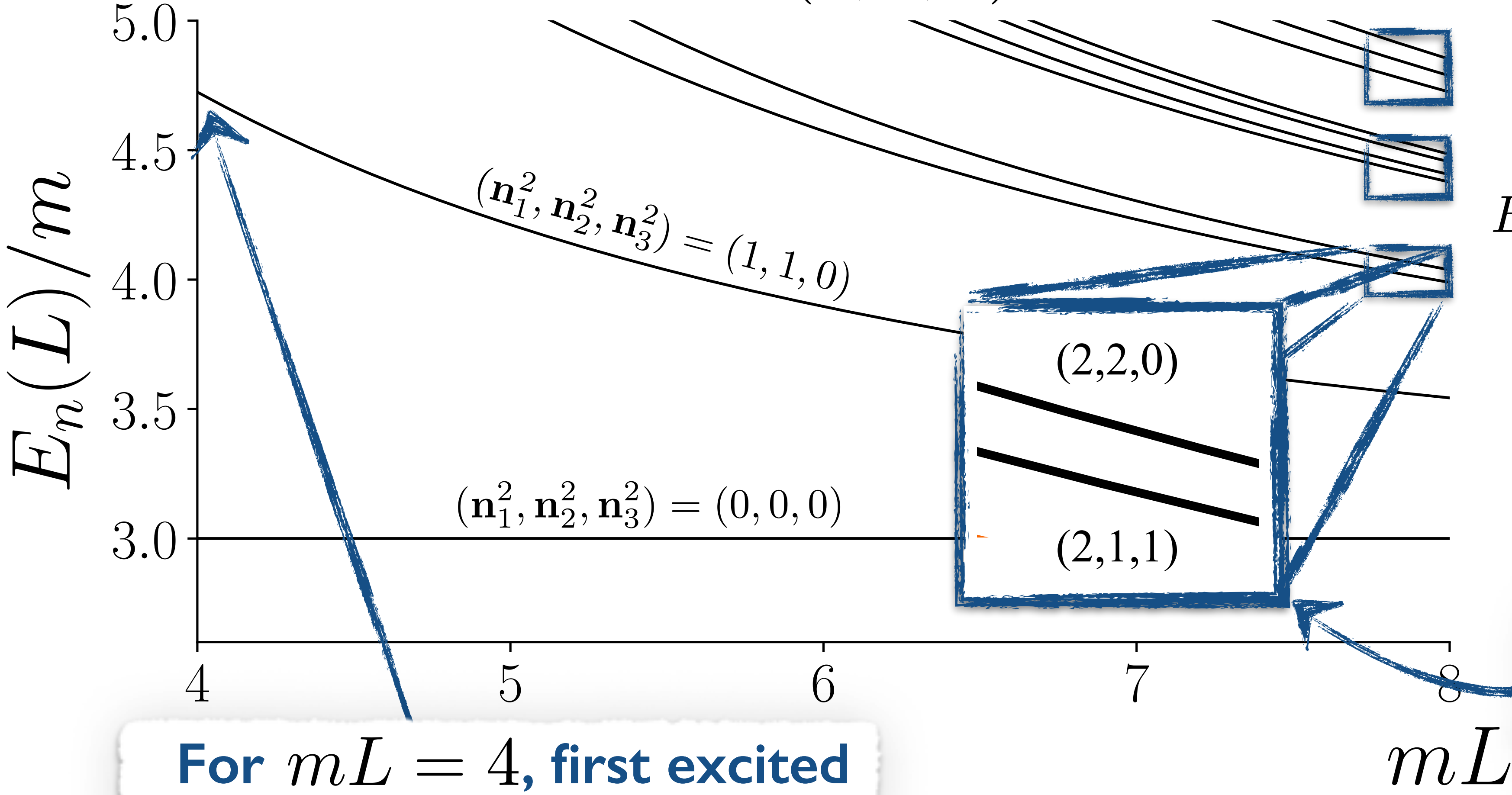
Plotting the non-interacting spectrum

$$\frac{E_n(L)}{m} = \sqrt{1 + \frac{4\pi^2 n_1}{(mL)^2}} + \sqrt{1 + \frac{4\pi^2 n_2}{(mL)^2}} + \sqrt{1 + \frac{4\pi^2 n_3}{(mL)^2}}$$

$n_1 = \mathbf{n}_1^2\,,\ n_2 = \mathbf{n}_2^2\,,\ n_3 = (\mathbf{n}_P - \mathbf{n}_1 - \mathbf{n}_2)^2$

Note: $mL = m_\pi L$ (not milliliters)

$\mathbf{n}_P = (0,0,0)$



Why are these states clustered?

Accidental NR degeneracy!

$$E_n^{\rm NR}(L) = 3m + \frac{2\pi^2}{L^2}(n_1 + n_2 + n_3)$$

These two states are degenerate in the NR theory

For $mL = 4$, first excited state is already relativistic

$$\frac{p^2}{m^2} = \left(\frac{2\pi}{mL}\right)^2 \approx 2.46$$

Review: ☑ Finite-volume calculations take $\int_{\mathbf{p}} \equiv \int \frac{d^3\mathbf{p}}{(2\pi)^3} \implies \frac{1}{L^3}\sum_{\mathbf{p}}$

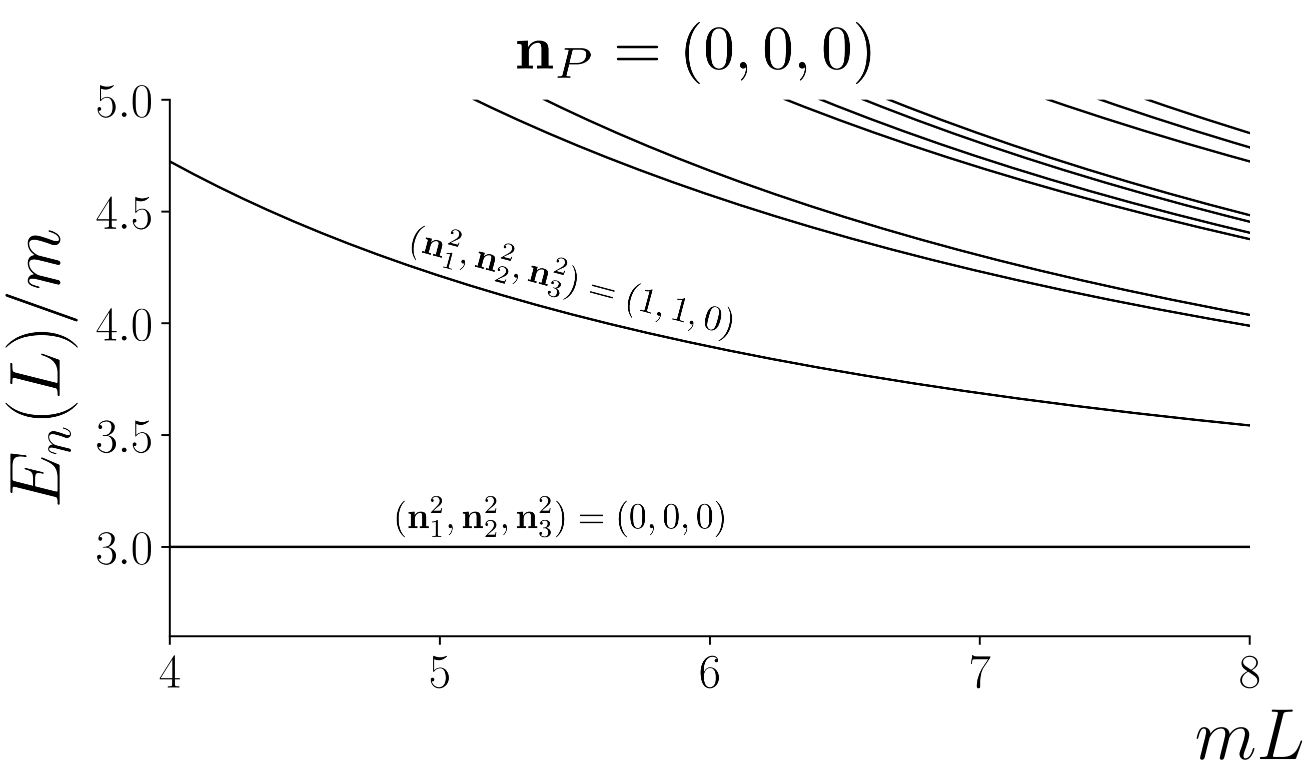
☑ Spectral decomposition… $C_L(P) = \sum_n 2E_n(L)L^3 \frac{i|\langle E_n, \mathbf{P}, L|\mathcal{O}^\dagger(0)|0\rangle|^2}{E^2 - E_n(L)^2 + i\epsilon}$

☑ Non-interacting correlator…

$$C_L(P) = C_\infty(P) + \frac{1}{6}\left[\frac{1}{L^6}\sum_{\mathbf{p},\mathbf{k}} - \int_{\mathbf{k},\mathbf{p}}\right] \frac{i|\langle \mathbf{P}, \mathbf{p}, \mathbf{k};\, \varphi\varphi\varphi|\mathcal{O}^\dagger(0)|0\rangle|^2}{2\omega_{\mathbf{k}}2\omega_{\mathbf{p}}2\omega_{\mathbf{pk}}[E - \omega_{\mathbf{k}} - \omega_{\mathbf{p}} - \omega_{\mathbf{pk}} + i\epsilon]} + \mathcal{O}(e^{-mL})$$

☑ Energies given by…

$$E_n(L) = \sqrt{\mathbf{p}_1^2 + m^2} + \sqrt{\mathbf{p}_1^2 + m^2} + \sqrt{\mathbf{p}_3^2 + m^2}$$



☑ Number of states $\propto L^6$

☑ Matrix elements related by…

$$\frac{|\langle E_n, \mathbf{P}, L|\mathcal{O}^\dagger(0)|0\rangle|^2}{\left\langle |\langle \mathbf{P}, \mathbf{p}, \mathbf{k};\, \varphi\varphi\varphi|\mathcal{O}^\dagger(0)|0\rangle|^2 \right\rangle_{\Omega_n}} = \frac{\nu_n}{6L^9}\frac{1}{2\omega_{\mathbf{k}}2\omega_{\mathbf{p}}2\omega_{\mathbf{pk}}}$$

☑ States cluster due to NR degeneracy

$$E_n^{\mathrm{NR}}(L) = 3m + \frac{2\pi^2}{L^2}(n_1 + n_2 + n_3)$$

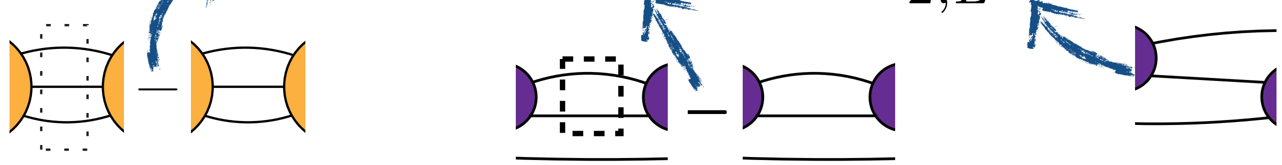
Outline

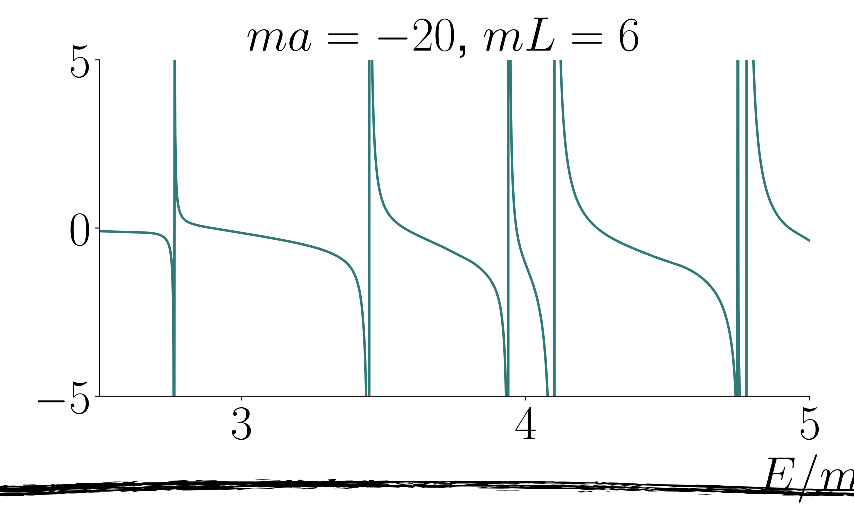
- [x] **Warm up and definitions**
  - [x] Basic set-up
  - [x] Finite-volume correlator
  - [x] Three non-interacting particles
- [ ] **Two particles in a box**
  - [ ] Alternative derivation
  - [ ] Truncation and application
  - [ ] Relating matrix elements
- [ ] **Three particles in a box**
  - [ ] 3-to-3 scattering
  - [ ] (Sketch of) derivation
  - [ ] An unexpected infinite-volume quantity
  - [ ] Relating energies to scattering
- [ ] **Testing the result**
  - [ ] Large-volume expansion
  - [ ] Effimov state in a box
- [ ] **Numerical explorations**
  - [ ] Truncation at low energies
  - [ ] 2-particle physics in 3-particle energies
  - [ ] Toy 3-particle resonance
  - [ ] Numerical Effimov state
  - [ ] Unphysical solutions
- [ ] **Looking forward**
  - [ ] 2-to-3 scattering and resonant subprocesses
  - [ ] Speculations

Teaser for the coming lectures

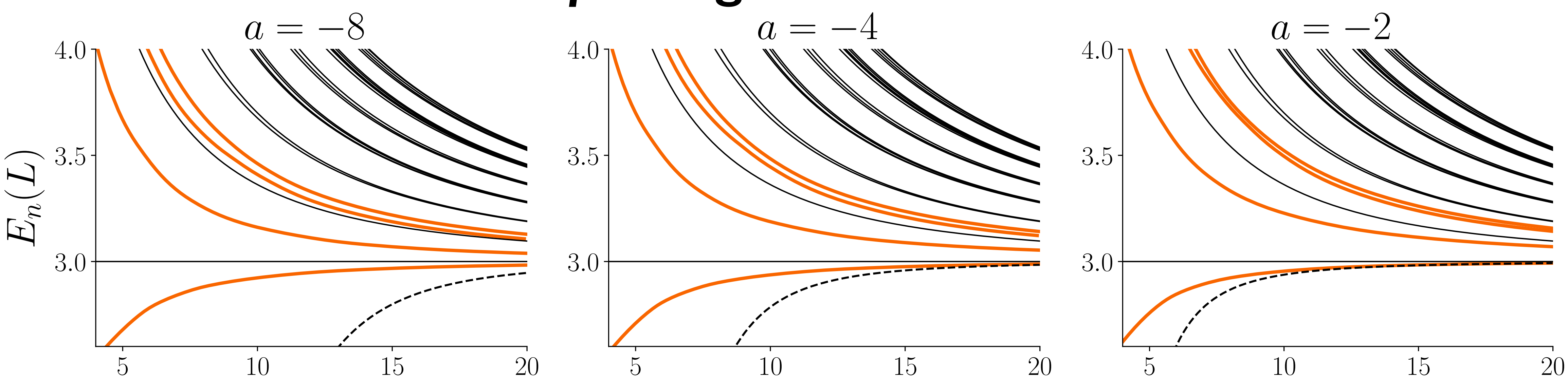
A new quantization condition

$$F_3 = \frac{F}{6\omega L^3} - \frac{F}{2\omega L^3}\frac{1}{1+\mathcal{M}_{2,L}G}\mathcal{M}_{2,L}F$$

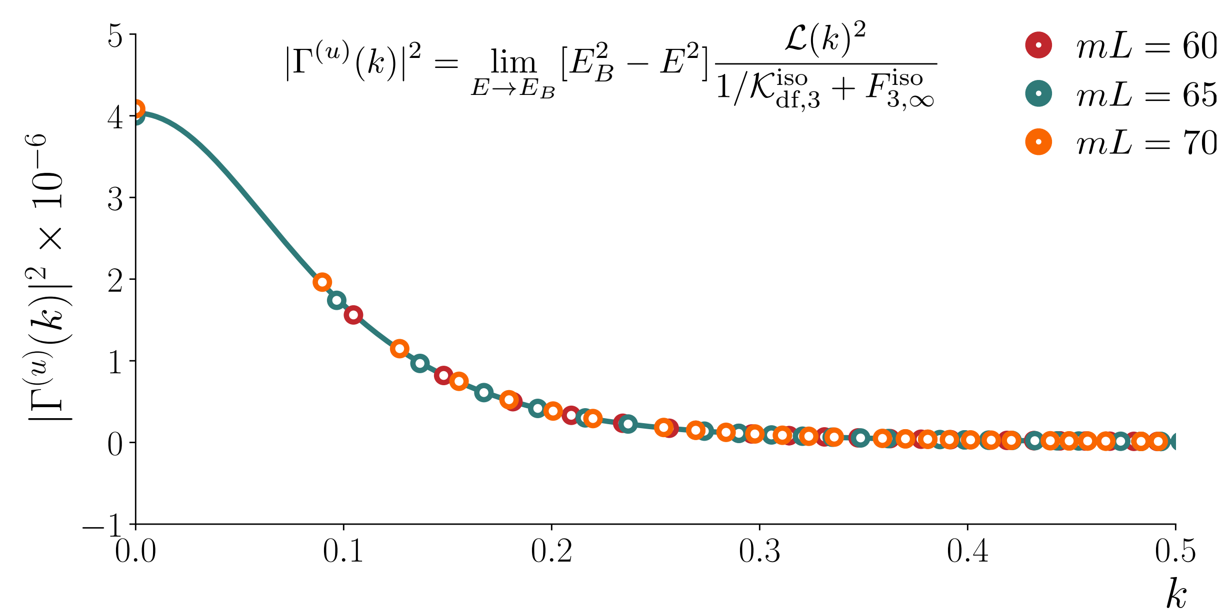




Exploring the solutions



Efimov bound states in and outside of the box



Thanks for listening!