Three-particle scattering from numerical lattice QCD

Scattering from the lattice: applications to phenomenology and beyond

Maxwell T. Hansen

May 14-18th, 2018

Outline

- [x] **Warm up and definitions**
  - [x] Basic set-up
  - [x] Finite-volume correlator
  - [x] Three non-interacting particles

- [ ] **Two particles in a box**
  - [ ] Alternative derivation
  - [ ] Truncation and application
  - [ ] Relating matrix elements

- [ ] **Three particles in a box**
  - [ ] 3-to-3 scattering
  - [ ] (Sketch of) derivation
  - [ ] An unexpected infinite-volume quantity
  - [ ] Relating energies to scattering

- [ ] **Testing the result**
  - [ ] Large-volume expansion
  - [ ] Effimov state in a box

- [ ] **Numerical explorations**
  - [ ] Truncation at low energies
  - [ ] 2-particle physics in 3-particle energies
  - [ ] Toy 3-particle resonance
  - [ ] Numerical Effimov state
  - [ ] Unphysical solutions

- [ ] **Looking forward**
  - [ ] 2-to-3 scattering and resonant subprocesses
  - [ ] Speculations

Review: ☑ Finite-volume calculations take $\int_{\mathbf{p}} \equiv \int \frac{d^3\mathbf{p}}{(2\pi)^3} \implies \frac{1}{L^3}\sum_{\mathbf{p}}$

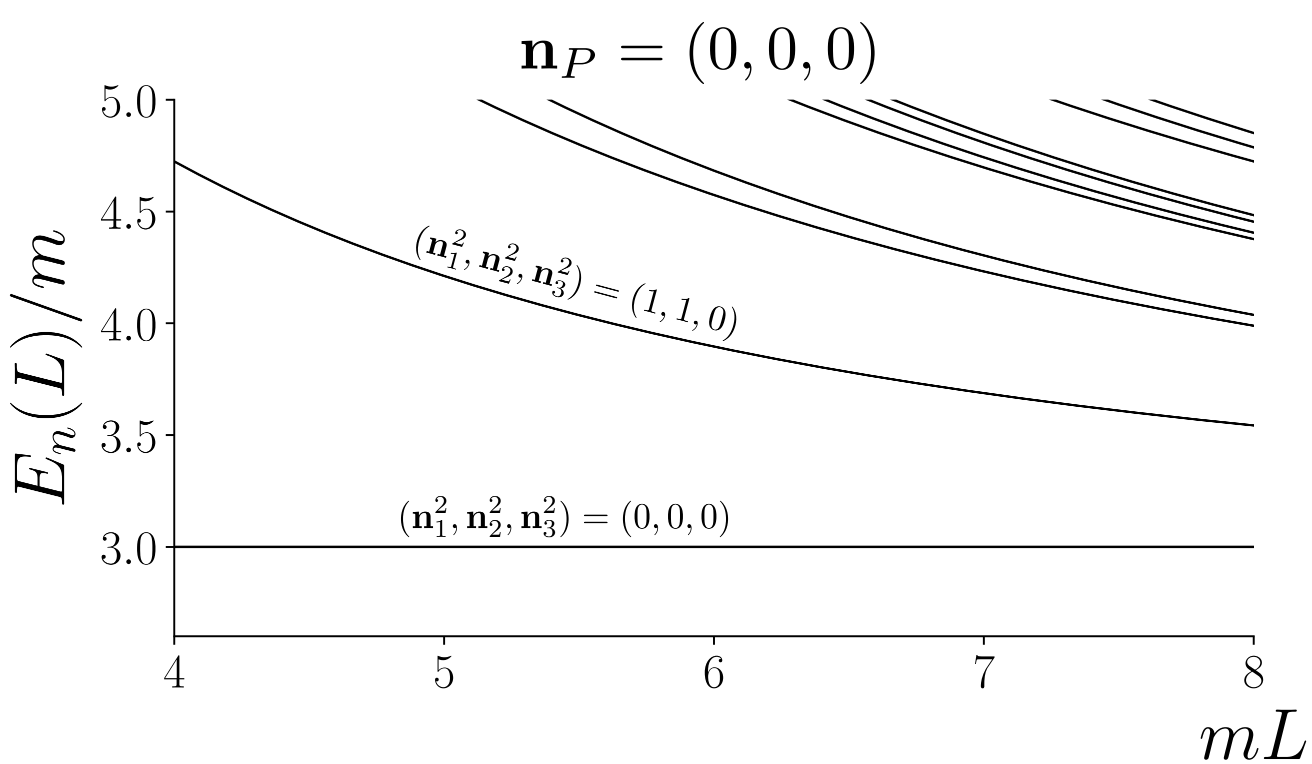
☑ Spectral decomposition… $C_L(P) = \sum_n 2E_n(L)L^3 \frac{i|\langle E_n, \mathbf{P}, L|\mathcal{O}^\dagger(0)|0\rangle|^2}{E^2 - E_n(L)^2 + i\epsilon}$

☑ Non-interacting correlator…

$$C_L(P) = C_\infty(P) + \frac{1}{6}\left[\frac{1}{L^6}\sum_{\mathbf{p},\mathbf{k}} - \int_{\mathbf{k},\mathbf{p}}\right] \frac{i|\langle \mathbf{P}, \mathbf{p}, \mathbf{k};\ \varphi\varphi\varphi|\mathcal{O}^\dagger(0)|0\rangle|^2}{2\omega_{\mathbf{k}}2\omega_{\mathbf{p}}2\omega_{\mathbf{pk}}[E - \omega_{\mathbf{k}} - \omega_{\mathbf{p}} - \omega_{\mathbf{pk}} + i\epsilon]} + \mathcal{O}(e^{-mL})$$

☑ Energies given by…

$$E_n(L) = \sqrt{\mathbf{p}_1^2 + m^2} + \sqrt{\mathbf{p}_1^2 + m^2} + \sqrt{\mathbf{p}_3^2 + m^2}$$



☑ Number of states $\propto L^6$

☑ Matrix elements related by…

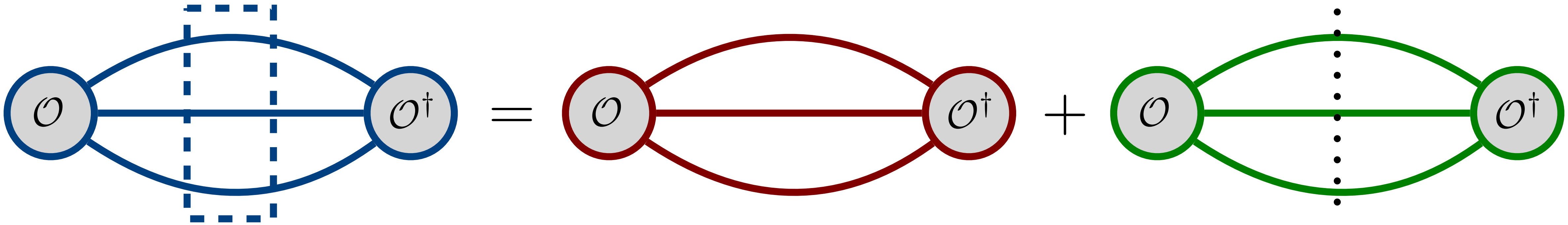
$$\frac{|\langle E_n, \mathbf{P}, L|\mathcal{O}^\dagger(0)|0\rangle|^2}{\left\langle |\langle \mathbf{P}, \mathbf{p}, \mathbf{k};\ \varphi\varphi\varphi|\mathcal{O}^\dagger(0)|0\rangle|^2 \right\rangle_{\Omega_n}} = \frac{\nu_n}{6L^9}\frac{1}{2\omega_{\mathbf{k}}2\omega_{\mathbf{p}}2\omega_{\mathbf{pk}}}$$

☑ States cluster due to NR degeneracy

$$E_n^{\rm NR}(L) = 3m + \frac{2\pi^2}{L^2}(n_1 + n_2 + n_3)$$

Three-particle cut identity

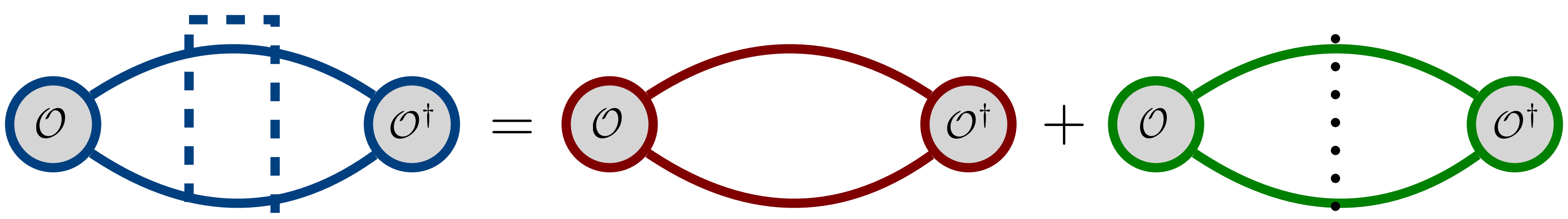
$$C_L(P) = C_\infty(P) + \frac{1}{6}\left[\frac{1}{L^6}\sum_{\mathbf{p},\mathbf{k}} - \int_{\mathbf{k},\mathbf{p}}\right] \frac{i|\widetilde{f}_{\mathrm{on}}(\mathbf{k},\hat{\mathbf{p}}^*)|^2}{2\omega_{\mathbf{k}}2\omega_{\mathbf{p}}2\omega_{\mathbf{pk}}[E-\omega_{\mathbf{k}}-\omega_{\mathbf{p}}-\omega_{\mathbf{pk}}+i\epsilon]} + \mathcal{O}(e^{-mL})$$

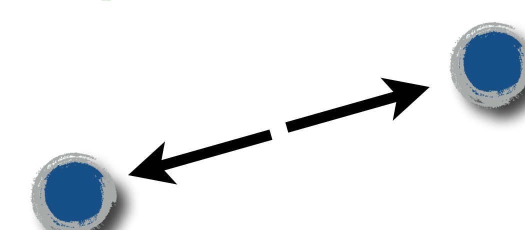


$$\frac{1}{L^3}\sum \qquad\qquad \int$$

Two-particle analog

$$C_L(P) = C_\infty(P) + \frac{1}{2}\left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int_{\mathbf{k}}\right] \frac{i|\widetilde{f}_{\mathrm{on}}(\hat{\mathbf{k}}^*)|^2}{2\omega_{\mathbf{k}}2\omega_{\mathbf{P}-\mathbf{p}}[E-\omega_{\mathbf{k}}-\omega_{\mathbf{P}-\mathbf{k}}+i\epsilon]} + \mathcal{O}(e^{-mL})$$



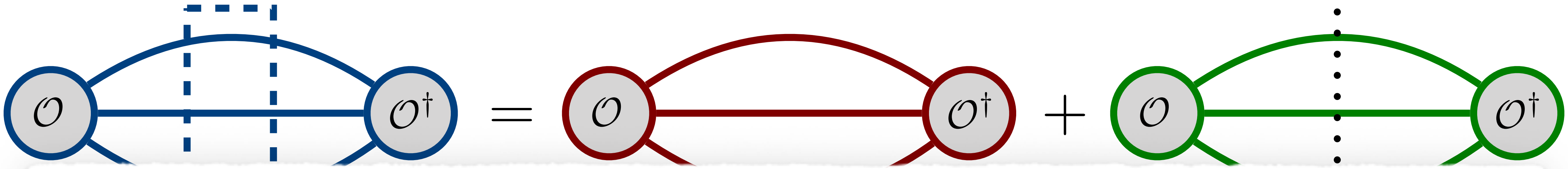


$$\mathbf{k}^* = \hat{\mathbf{k}}^* \sqrt{E^{*2}/4 - m^2}$$

$$E^* = \sqrt{E^2 - \mathbf{P}^2}$$

Three-particle cut identity

$$C_L(P) = C_\infty(P) + \frac{1}{6}\left[\frac{1}{L^6}\sum_{\mathbf{p},\mathbf{k}} - \int_{\mathbf{k},\mathbf{p}}\right] \frac{i|\widetilde{f}_{\rm on}(\mathbf{k},\hat{\mathbf{p}}^*)|^2}{2\omega_\mathbf{k} 2\omega_\mathbf{p} 2\omega_\mathbf{pk}[E-\omega_\mathbf{k}-\omega_\mathbf{p}-\omega_\mathbf{pk}+i\epsilon]} + \mathcal{O}(e^{-mL})$$



Stefan's comment....

$$\mathcal{O}(x) \equiv \frac{1}{6}\int_L d^4y \int_L d^4z \ f(y,z)\ \varphi(x)\varphi(x+y)\varphi(x+z)$$

This must be local in time!

Tw

This implies that the Fourier transform

C

$$\widetilde{f}(p,k) = \frac{1}{6}\sum_{\{k_1,k_2\}\in\{p,k,P-p-p\}} \int_L d^4z \int_L d^4y \ e^{-ik_1 z - ik_2 y} \ f(y,z)$$

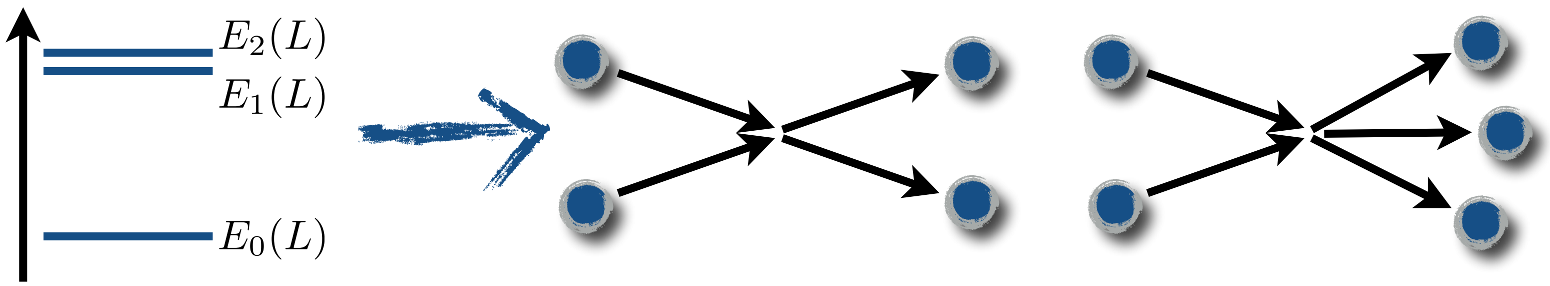
)

is independent of k^0 and p^0.

$$\mathbf{k}^* = \hat{\mathbf{k}}^* \sqrt{E^{*2}/4 - m^2}$$

$$E^* = \sqrt{E^2 - \mathbf{P}^2}$$

Finite-volume correlators (with interactions)



$$C_L(P) \equiv \int_L d^4x \; e^{-iPx} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(0) | 0 \rangle$$

Total 4-momentum

$$P = (E, \mathbf{P}) = (E, 2\pi \mathbf{n}_P / L)$$

c.m. frame energy: $E^{*2} = E^2 - \mathbf{P}^2$

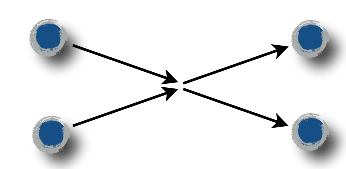
n-particle interpolator

e.g. $\pi(\mathbf{p})\pi(-\mathbf{p})$ **or** $\bar{q}\Gamma q$

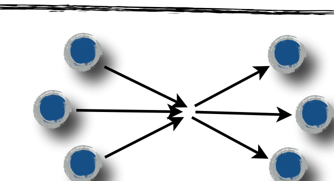
(only quantum numbers relevant)

Focus on a widow of energies to isolate particular on-shell states

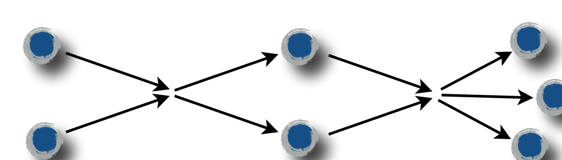
- **Two particles:** $\mathbb{Z}_2$ ✓ $0 < E^* < 4m$ $\quad$ $\mathbb{Z}_2$ ✗ $m < E^* < 3m$



- **Three particles** $\mathbb{Z}_2$ ✓ $m < E^* < 5m$



- **Two and three particles** $\mathbb{Z}_2$ ✗ $m < E^* < 4m$



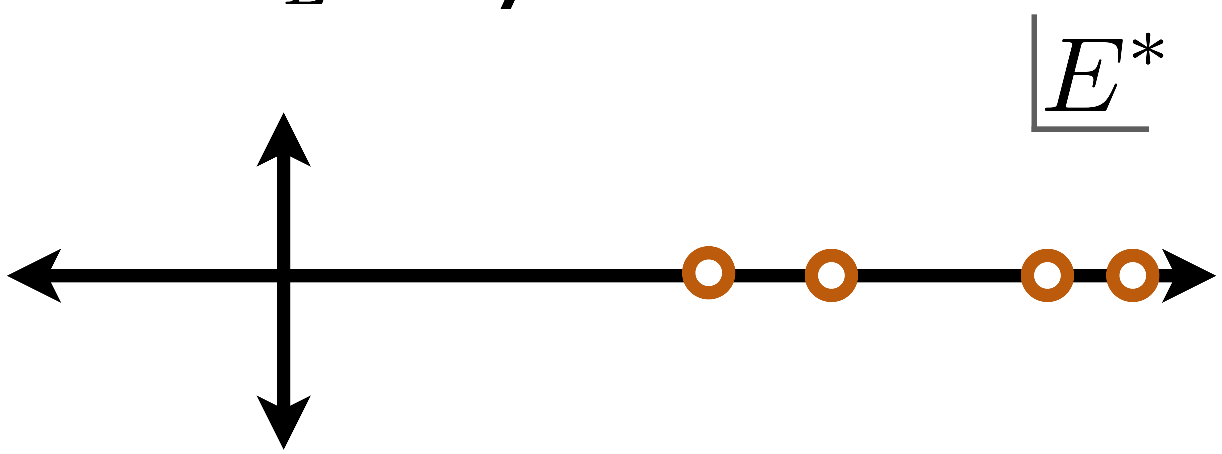
Of poles and branch cuts

$$C_L(P) \equiv \int_L d^4x \; e^{-iPx} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(0) | 0 \rangle$$

At fixed $L, \mathbf{P}$ poles in C_L give the finite-volume spectrum

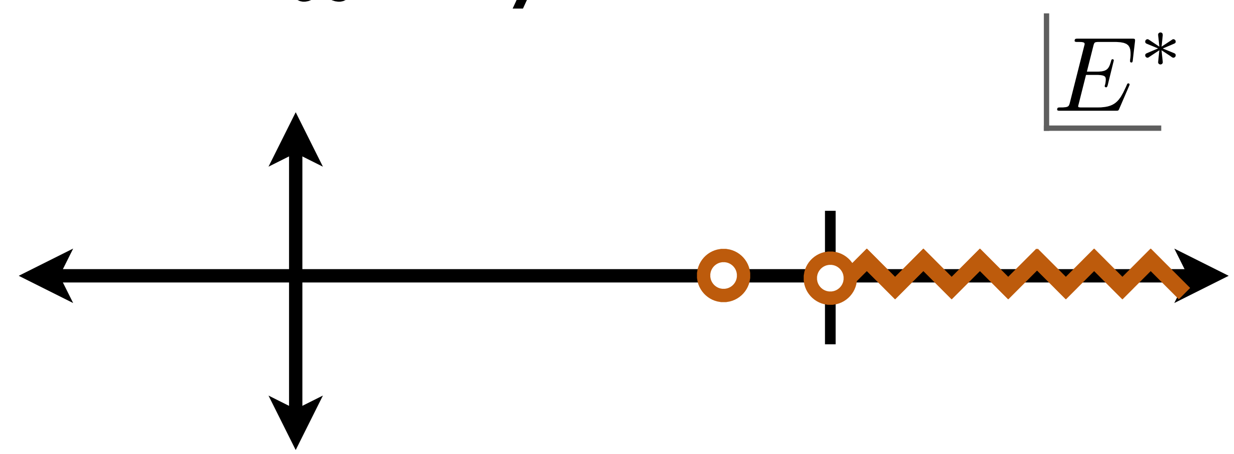
---

C_L analytic structure



- ☑ Real function (unitarity hidden)
- ☑ No cuts ⟶ only one sheet
- ☑ No resonance poles

C_∞ analytic structure

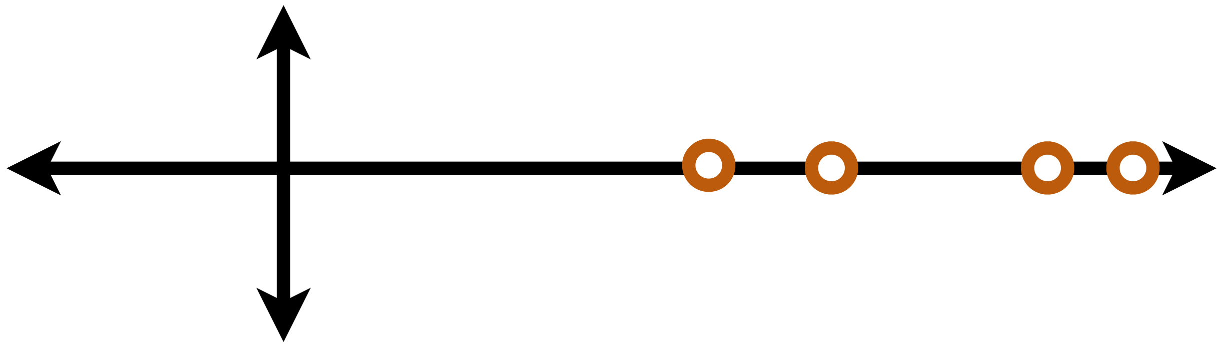


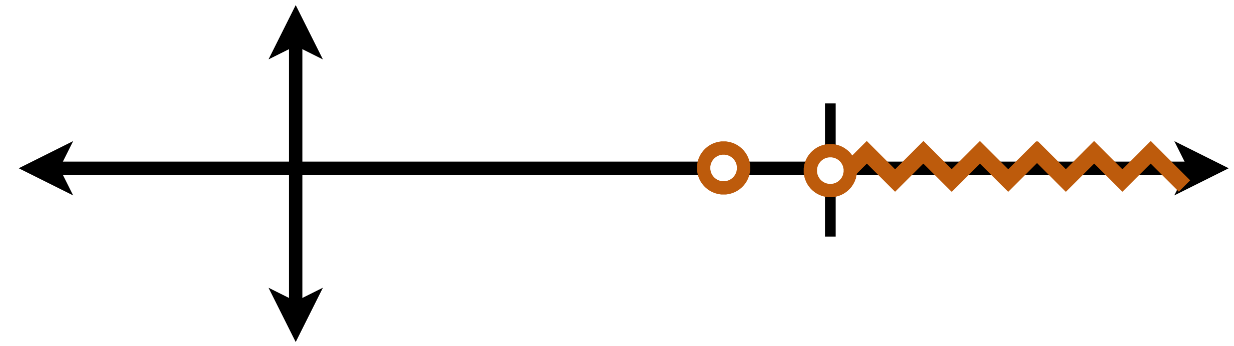
- ☑ Complex function (constrained by unitarity)
- ☑ Scattering states form cut
- ☑ Resonance poles on unphysical sheets

Want to relate $C_L \longleftrightarrow C_\infty$ ($\mathcal{M}_{n\to m}$ is just a specific choice of C_∞)

The idea is to “reach in” and correct the singularity structure

From poles to cuts: Toy example

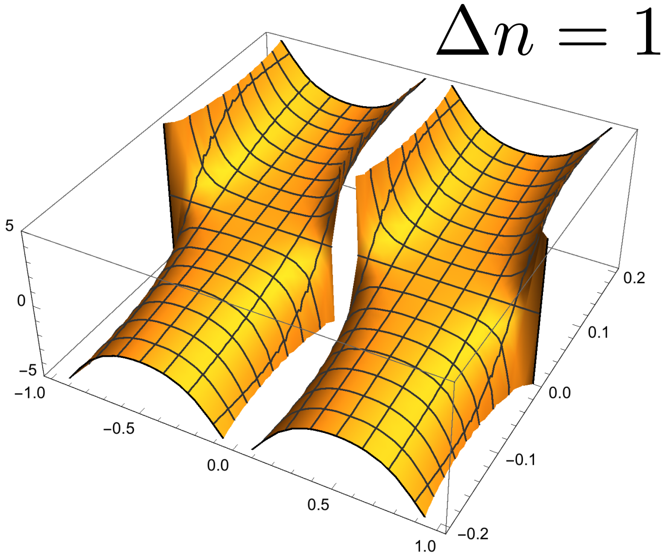




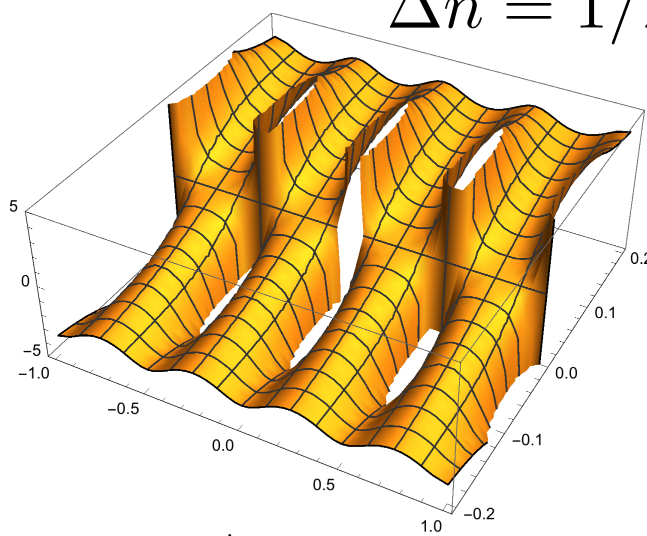
$$f(x,y,\Delta n) \equiv \sum_{n=-20,-20+\Delta n,\cdots}^{20} \Delta n \,\mathrm{Im}\frac{1}{n-x-iy}$$

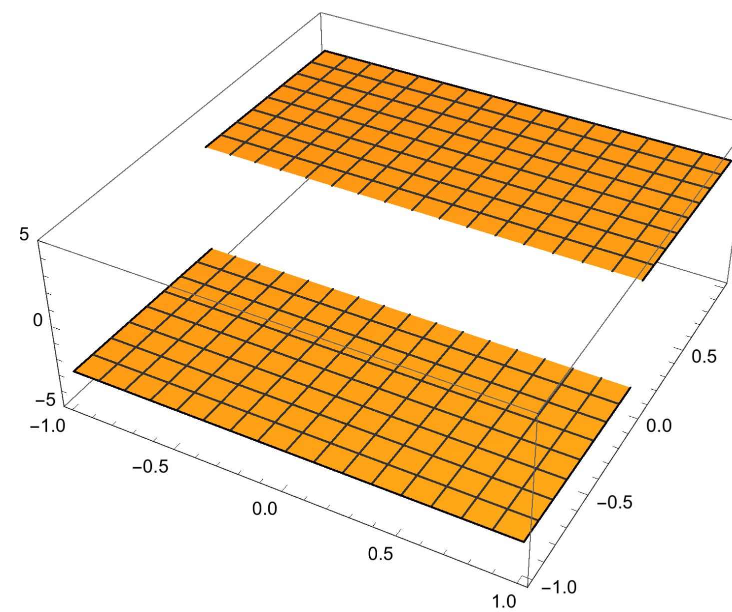
$$f_\infty(x,y) \equiv \mathrm{Im}\log\left[\frac{10-x-iy}{-10-x-iy}\right]$$

$\Delta n = 1$

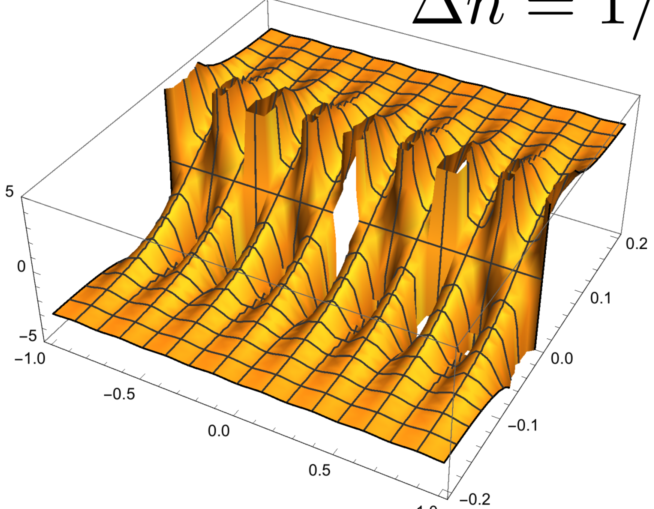


$\Delta n = 1/2$



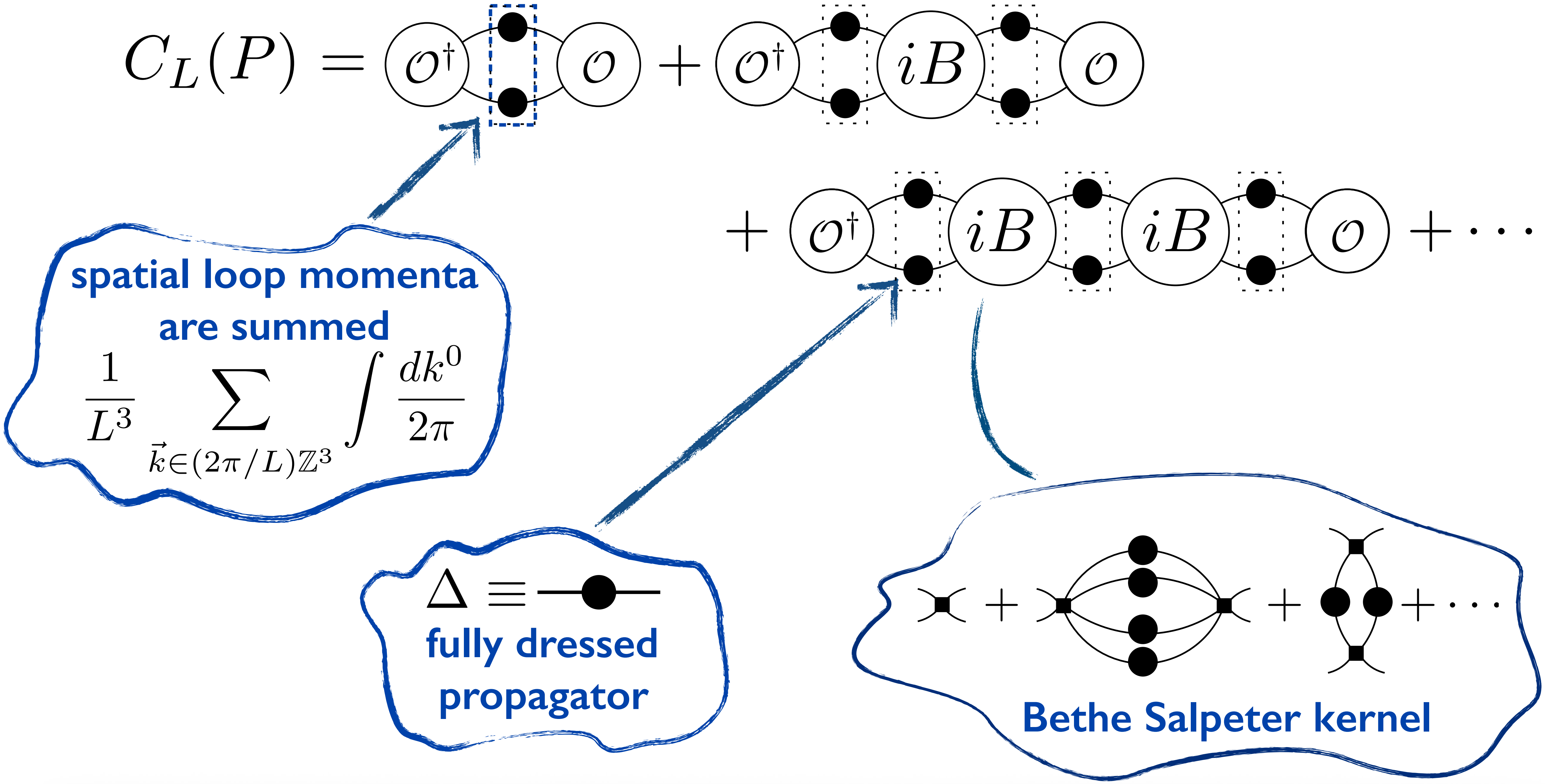


$\Delta n = 1/4$



Only in the infinite-volume limit can we generate the singularity that leads to a second sheet

Two identical particles: skeleton expansion

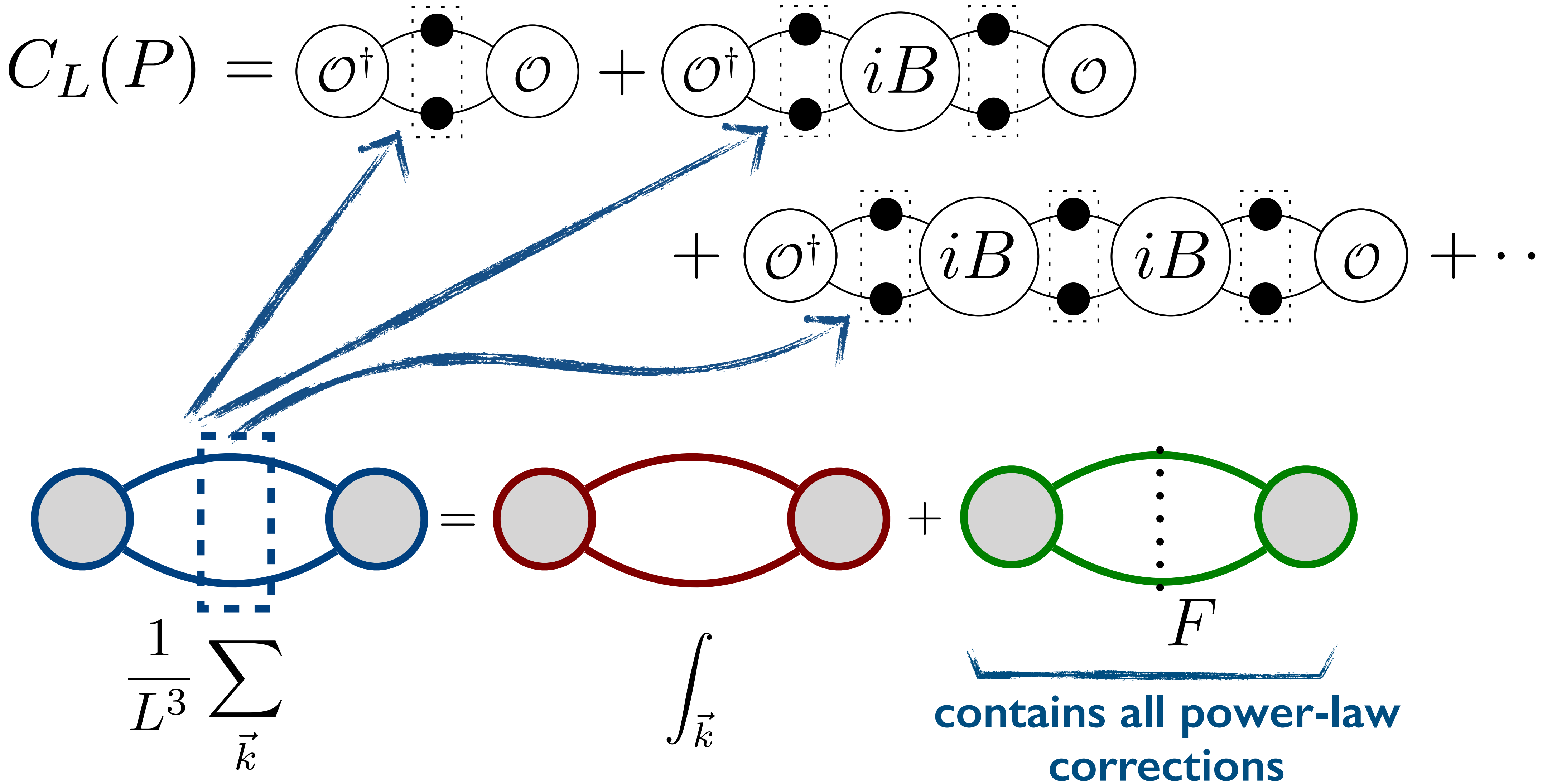


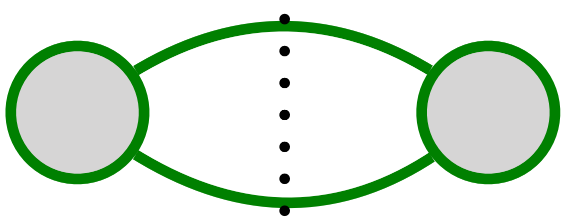
If $E^* < 4m$ **then**

$$K_L = K_\infty + \mathcal{O}(e^{-mL})$$
$$\Delta_L = \Delta_\infty + \mathcal{O}(e^{-mL})$$

Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)
Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

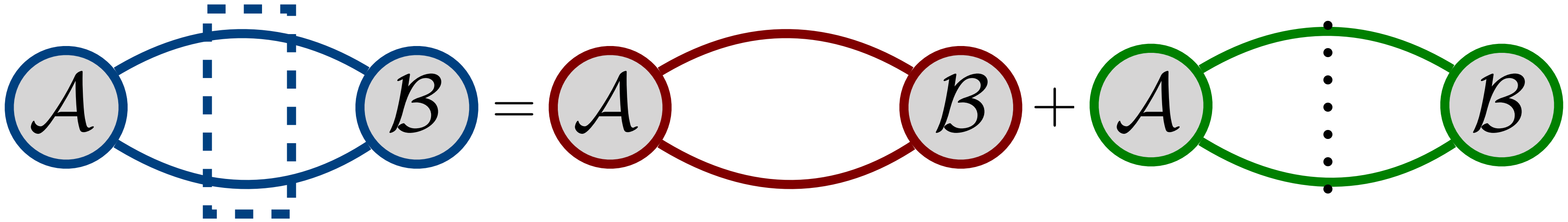
Two identical particles: loops and cuts

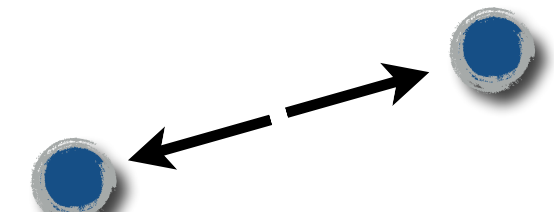


In  all four-momenta are projected on shell.

Physical, propagating states give dominate finite-volume effects.

Two-particle cut identity





$$\mathbf{k}^* = \hat{\mathbf{k}}^* \sqrt{E^{*2}/4 - m^2}$$

$$E^* = \sqrt{E^2 - \mathbf{P}^2}$$

$$C_L(P) = C_\infty(P) + \frac{1}{2}\left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int_{\mathbf{k}}\right] \frac{iA_{\text{on}}(\hat{\mathbf{k}}^*)B^*_{\text{on}}(\hat{\mathbf{k}}^*)}{2\omega_{\mathbf{k}} 2\omega_{\mathbf{P}-\mathbf{p}}[E - \omega_{\mathbf{k}} - \omega_{\mathbf{P}-\mathbf{k}} + i\epsilon]} + \mathcal{O}(e^{-mL})$$

decompose in spherical harmonics…

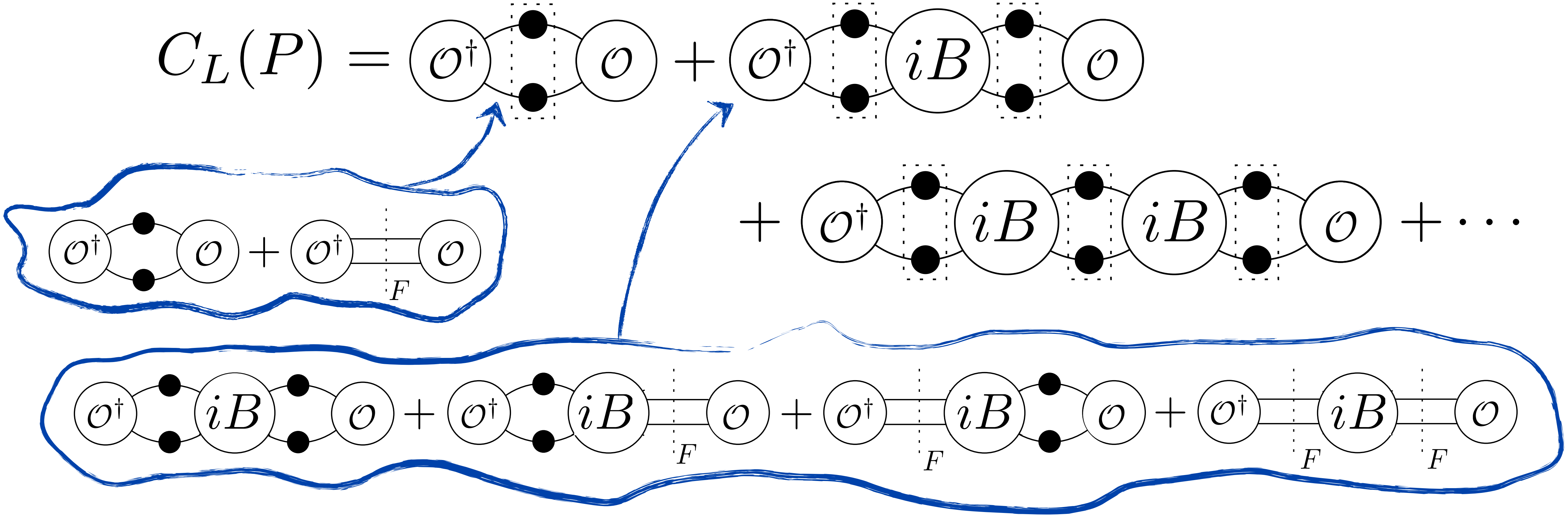
$$= C_\infty(P) + A_{\ell' m'} \frac{1}{2}\left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int_{\mathbf{k}}\right] \frac{i\mathcal{Y}_{\ell' m'}(\hat{\mathbf{k}}^*)\mathcal{Y}^*_{\ell m}(\hat{\mathbf{k}}^*)}{2\omega_{\mathbf{k}} 2\omega_{\mathbf{P}-\mathbf{p}}[E - \omega_{\mathbf{k}} - \omega_{\mathbf{P}-\mathbf{k}} + i\epsilon]} B^*_{\ell m} + \mathcal{O}(e^{-mL})$$

where…

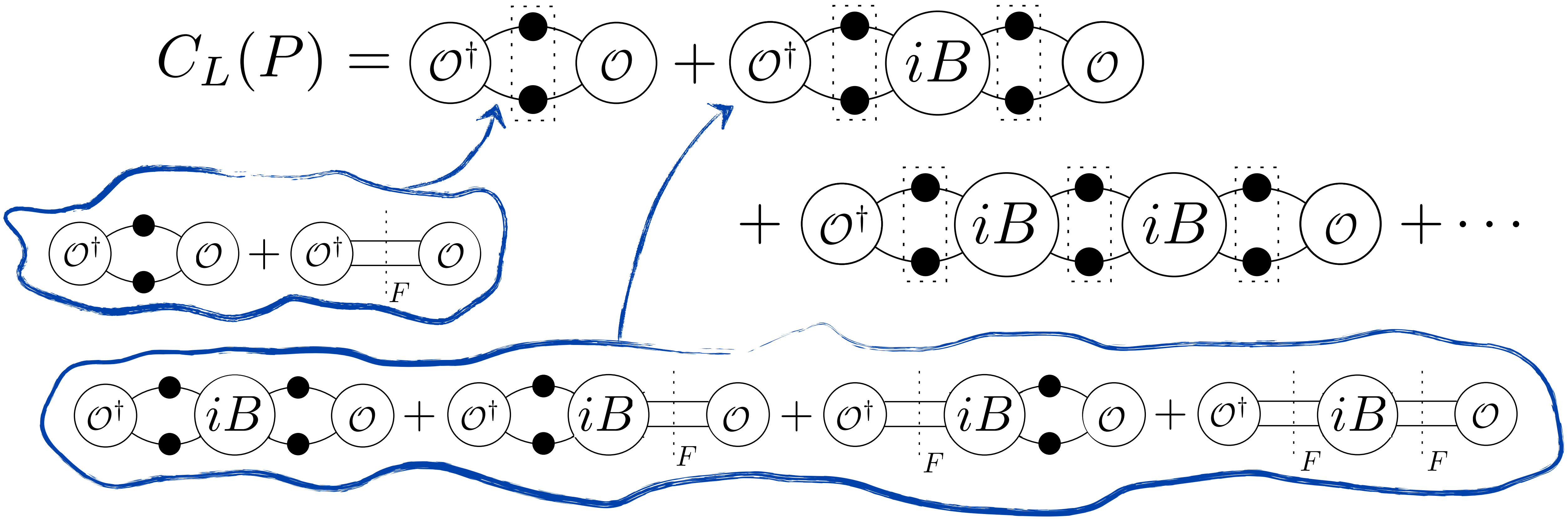
$$\mathcal{Y}_{\ell m}(\mathbf{k}^*) = \sqrt{4\pi}\left(\frac{k^*}{q^*}\right)^{\ell} Y_{\ell m}(\hat{\mathbf{k}}^*)\,, \qquad q^* = \sqrt{E^{*2}/4 - m^2}$$

Finally we define a matrix…

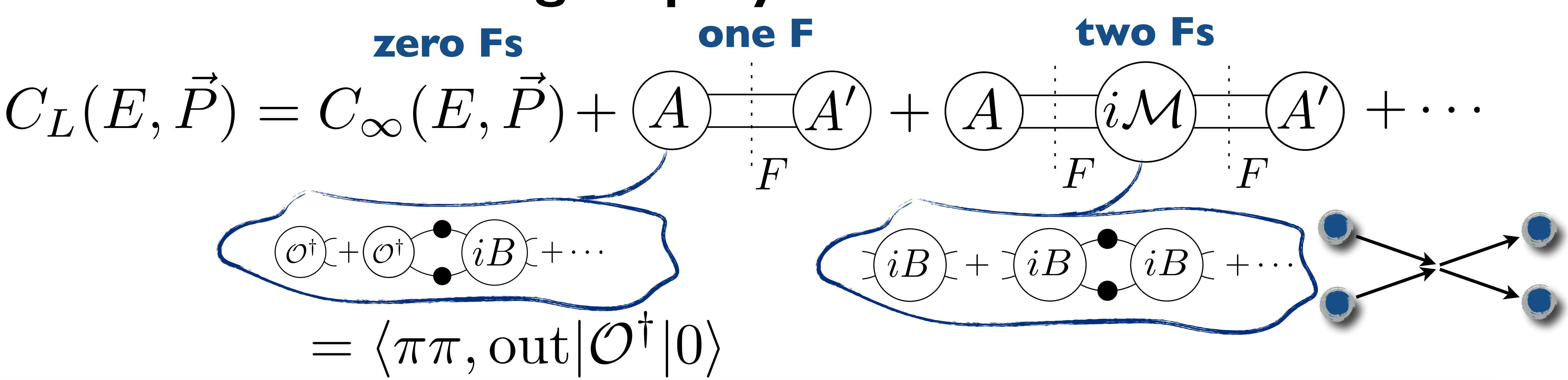
$$C_L(P) = C_\infty(P) + A_{\ell' m'} iF_{\ell' m';\ell m} B^*_{\ell m} + \mathcal{O}(e^{-mL})$$



$C_L(P) =$
$\mathcal{O}^\dagger$
$\mathcal{O}$
iB
F
$+ \cdots$



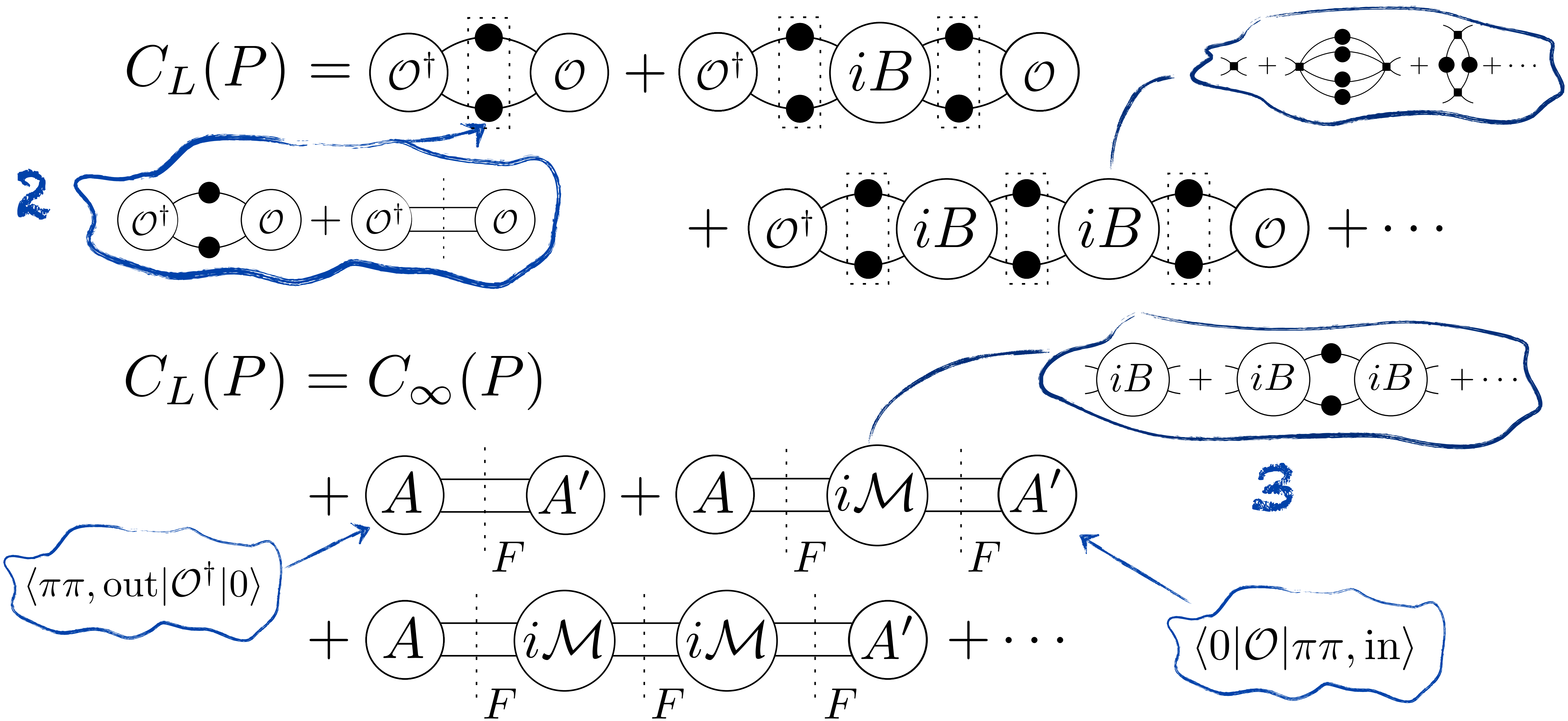
Now regroup by number of Fs



When we factorize diagrams and group infinite-volume parts…

physical observables emerge!

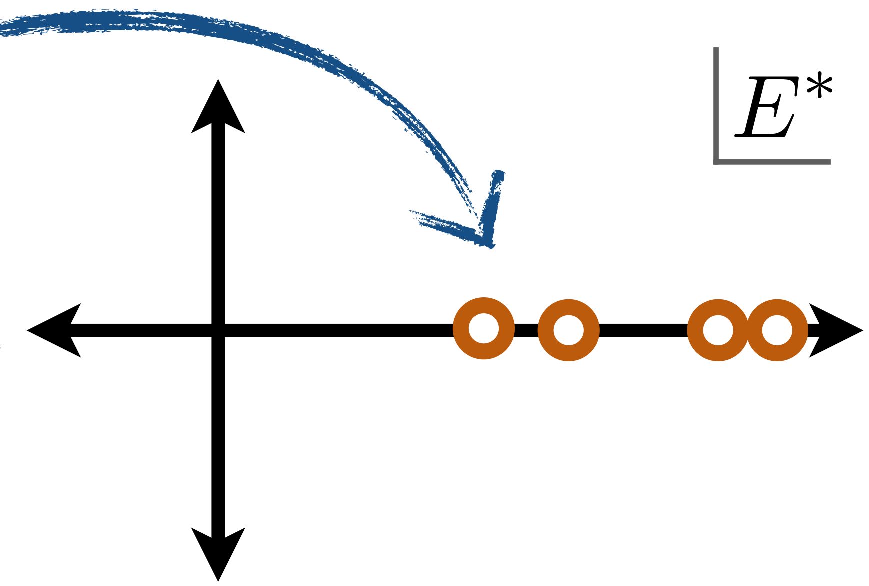
Two-particle review



We deduce…

poles are in here

$$C_L(P) = C_\infty(P) - A' F \frac{1}{1 + \mathcal{M}_{2\to 2} F} A$$

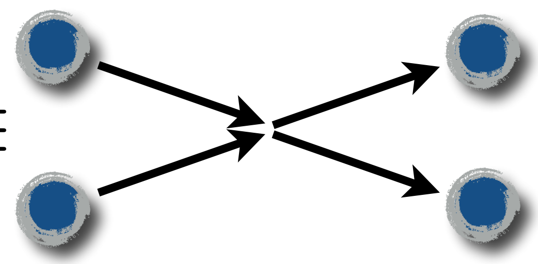


Two-particle result

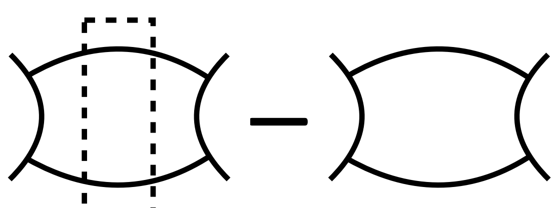
At fixed $(L, \vec{P})$, finite-volume energies are solutions to $\det[\mathcal{M}_{2\to2}^{-1} + F] = 0$

Rummukainen and Gottlieb, *Nucl. Phys.* B450, 397 (1995)
Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

Matrices defined using angular-momentum states

$\mathcal{M}_{2\to2} \equiv$  **diagonal matrix, parametrized by** $\delta_\ell(E^*)$

$F \equiv$ **non-diagonal matrix of known geometric functions**

$\equiv$  difference of two-particle loops in finite and infinite volume

depends on $L, E, \mathbf{P}$

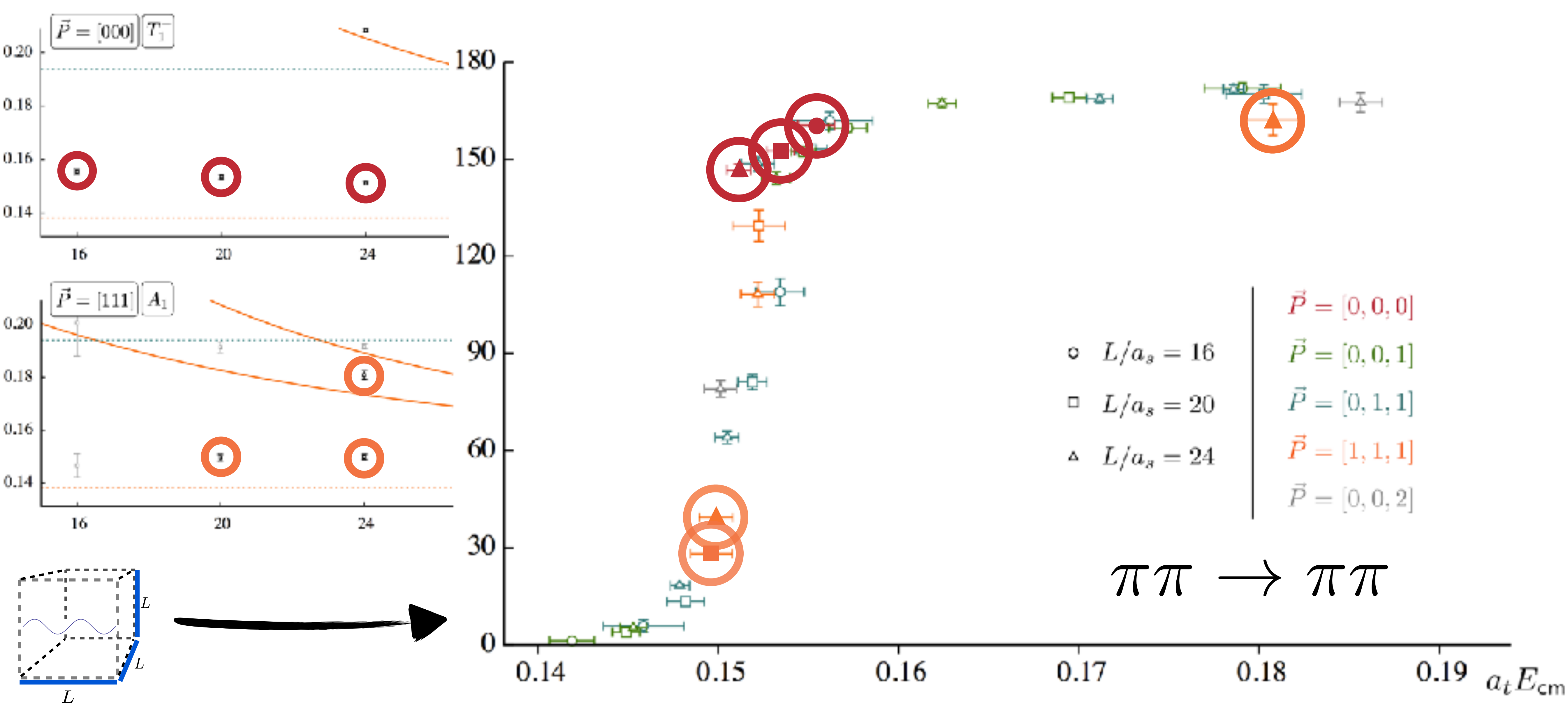
At low energies, lowest partial waves dominate $\mathcal{M}_{2\to2}$

$$\cot\delta(E_n^*) + \cot\phi(E_n, \vec{P}, L) = 0$$

scattering phase **known function**

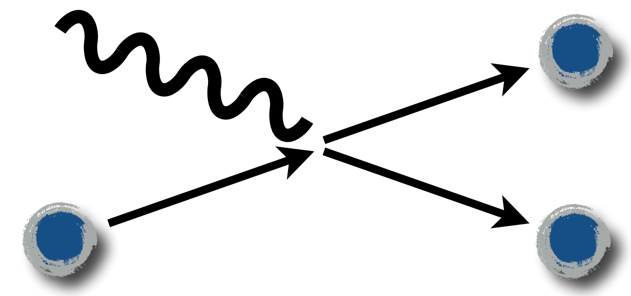
E.g. *p*-wave $\pi\pi \to \pi\pi$ scattering, $I^G(J^{PC}) = 1^+(1^{--})$

$$\cot \delta_{\ell=1}(E_n^*) + \cot \phi(E_n, \vec{P}, L) = 0$$



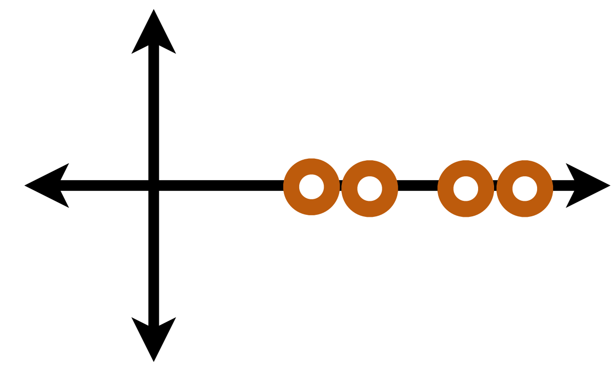
from Dudek, Edwards, Thomas in *Phys.Rev*. D87 (2013) 034505

Photoproduction $\langle \pi\pi, \text{out}|\mathcal{J}_\mu|\pi\rangle \equiv$



How can we get this from finite-volume observables?

Why did we expect $C_L(P)$ *to have poles?*



$$C_L(P) \equiv \int_L d^4x\, e^{-iPx} \langle 0|T\mathcal{O}(x)\mathcal{O}^\dagger(0)|0\rangle$$

Insert a complete set finite-volume of states

$$C_L(P) \xrightarrow[E\to E_n]{} -\frac{L^3 \langle 0|\mathcal{O}(0)|n,\vec{P},L\rangle\langle n,\vec{P},L|\mathcal{O}^\dagger(0)|0\rangle}{E-E_n}$$

Now compare this to our factorized result

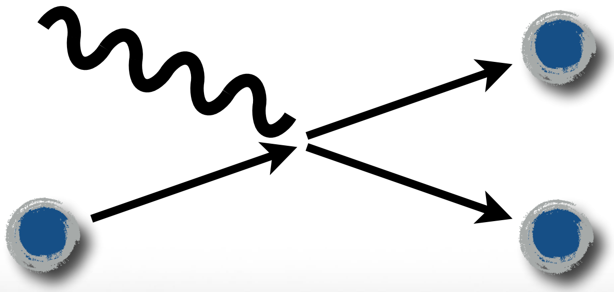
$$C_L(P) = C_\infty(P) - A' F \frac{1}{1+\mathcal{M}_{2\to 2}F} A$$

$\mathcal{R}$ **is the residue of this matrix**

$$\xrightarrow[E\to E_n]{} -\frac{\langle 0|\mathcal{O}(0)|\pi\pi,\text{in}\rangle \mathcal{R}(E_n,\vec{P},L)\langle \pi\pi,\text{out}|\mathcal{O}^\dagger(0)|0\rangle}{E-E_n}$$

Briceño, MTH, Walker-Loud (2015)

Photoproduction $\langle \pi\pi, \mathrm{out} | \mathcal{J}_\mu | \pi \rangle \equiv$



get this from the lattice

experimental observable

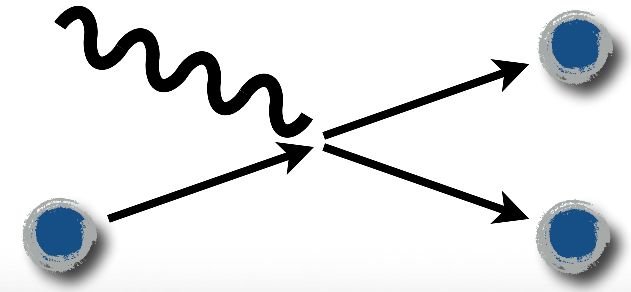
$$2\omega_\pi L^6 |\langle n, L | \mathcal{J}_\mu | \pi \rangle|^2 =$$

$$\langle \pi | \mathcal{J}_\mu | \pi\pi, \mathrm{in} \rangle \mathcal{R}(E_n, L) \langle \pi\pi, \mathrm{out} | \mathcal{J}_\mu | \pi \rangle$$

depends on scattering phase

Briceño, MTH, Walker-Loud (2015)

Photoproduction $\langle \pi\pi, \mathrm{out} | \mathcal{J}_\mu | \pi \rangle \equiv$



get this from the lattice

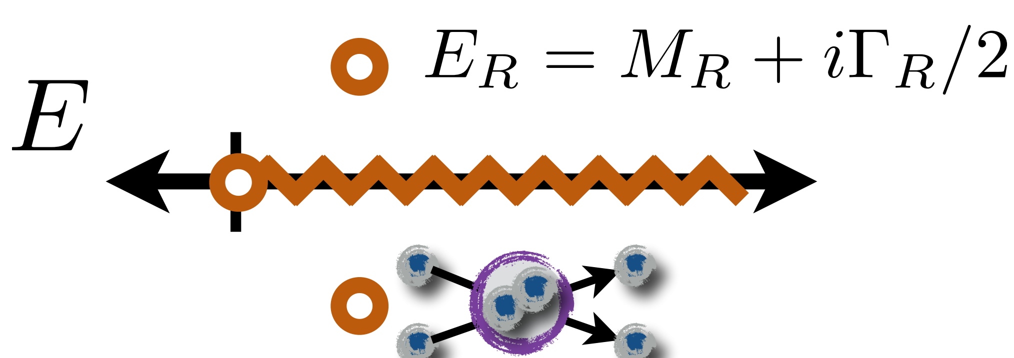
$$2\omega_\pi L^6 |\langle n, L | \mathcal{J}_\mu | \pi \rangle|^2 = \langle \pi | \mathcal{J}_\mu | \pi\pi, \mathrm{in} \rangle \mathcal{R}(E_n, L) \langle \pi\pi, \mathrm{out} | \mathcal{J}_\mu | \pi \rangle$$

experimental observable

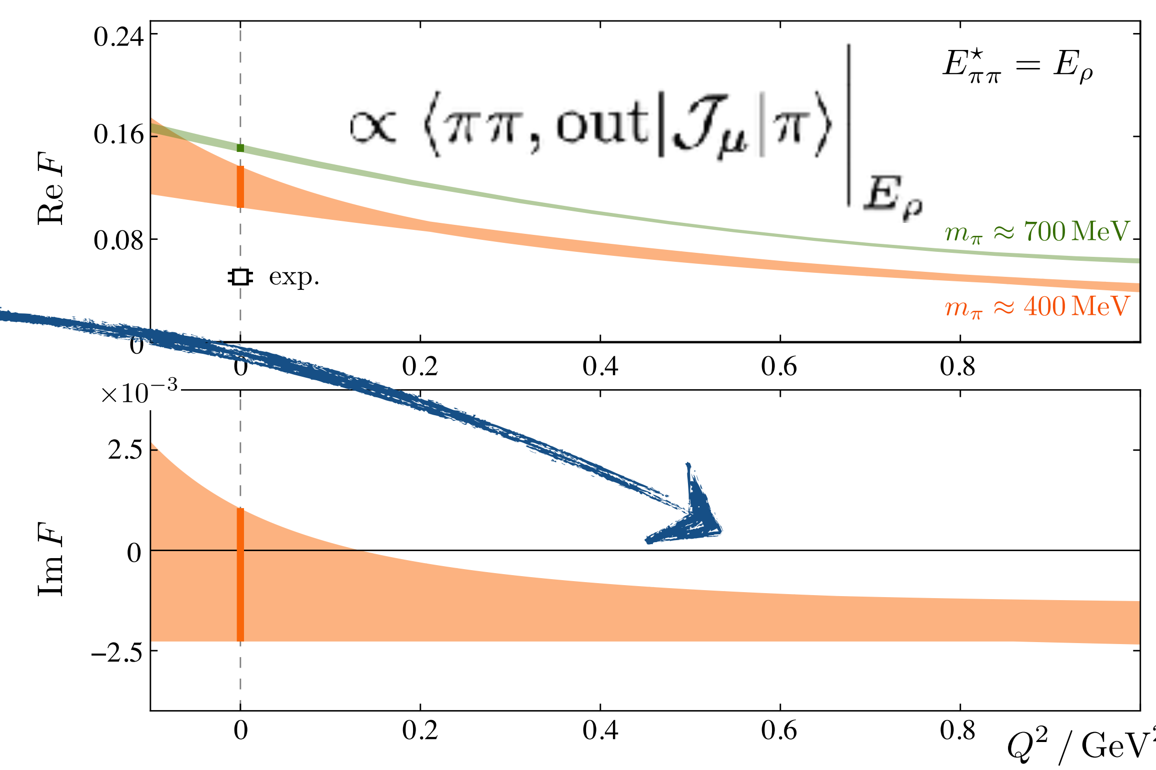
depends on scattering phase

Briceño, MTH, Walker-Loud (2015)

Transition factor defined by continuing the ππ state to the ρ pole is a complex number.

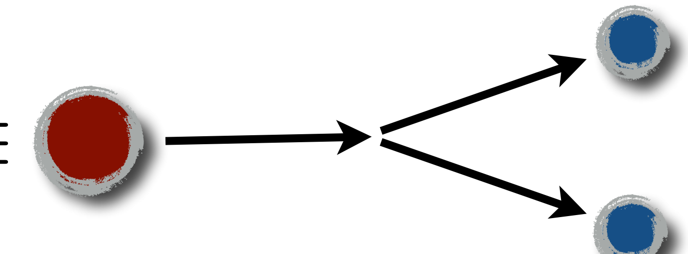


$\pi\gamma^* \to \rho$



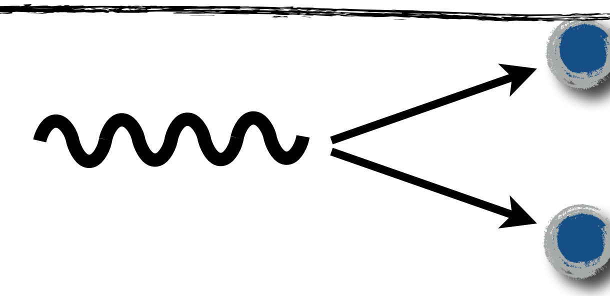
Briceño et. al., Phys. Rev. D93, 114508 (2016)

Same basic idea in many different contexts…

Kaon decay $\langle \pi\pi, \mathrm{out} | \mathcal{H} | K \rangle \equiv$ 

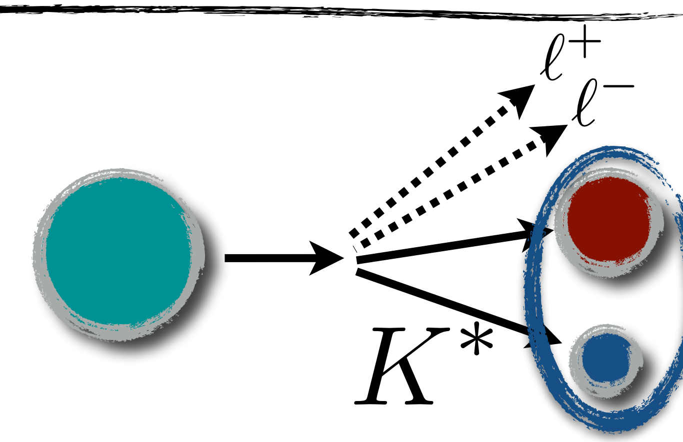
Lellouch, Lüscher (2001) ○ **Kim, Sachrajda, Sharpe** (2005) ○ **Christ, Kim, Yamazaki** (2005)

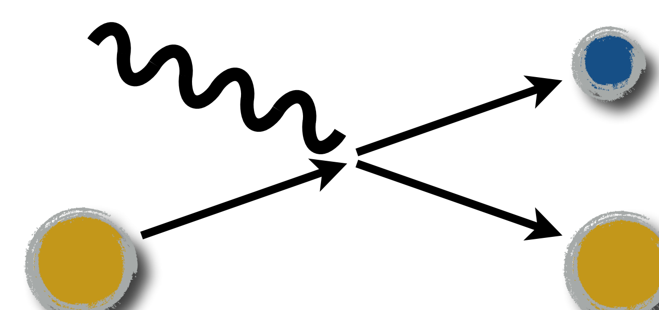
Implementation by RBC/UKQCD collaboration

Time-like form factors $\langle \pi\pi, \mathrm{out} | \mathcal{J}_\mu | 0 \rangle \equiv$ 

Meyer (2011)

Relevant for muon HVP contribution to muon g-2

Resonance transition amplitudes $\langle K\pi, \mathrm{out} | \mathcal{J}_{\alpha\beta} | B \rangle \equiv$ 

Particles with spin $\langle N\pi, \mathrm{out} | \mathcal{J}_\mu | N \rangle \equiv$ 

Agadjanov *et al.* (2014) ○ **Briceño, MTH, Walker-Loud** (2015) ○ **Briceño, MTH** (2016)

Outline

☑ Warm up and definitions
- ☑ Basic set-up
- ☑ Finite-volume correlator
- ☑ Three non-interacting particles

☑ Two particles in a box
- ☑ Alternative derivation
- ☑ Truncation and application
- ☑ Relating matrix elements

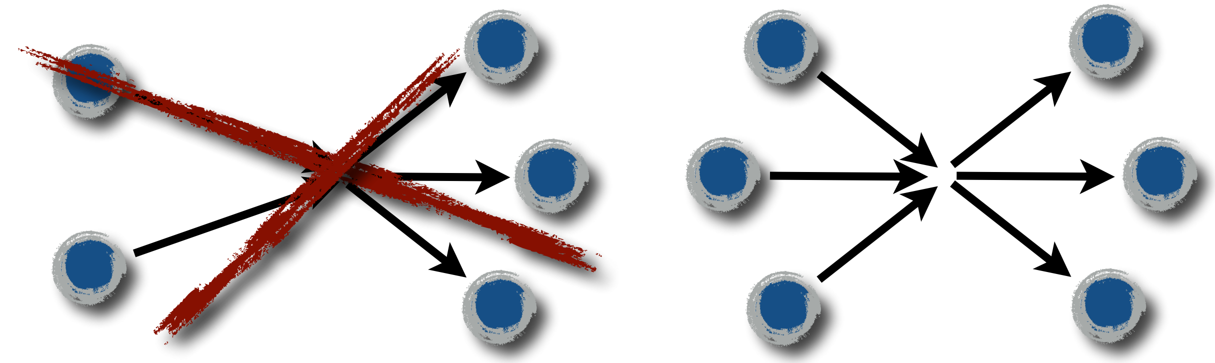
☐ Three particles in a box
- ☐ 3-to-3 scattering
- ☐ (Sketch of) derivation
- ☐ An unexpected infinite-volume quantity
- ☐ Relating energies to scattering

☐ Testing the result
- ☐ Large-volume expansion
- ☐ Effimov state in a box

☐ Numerical explorations
- ☐ Truncation at low energies
- ☐ 2-particle physics in 3-particle energies
- ☐ Toy 3-particle resonance
- ☐ Numerical Effimov state
- ☐ Unphysical solutions

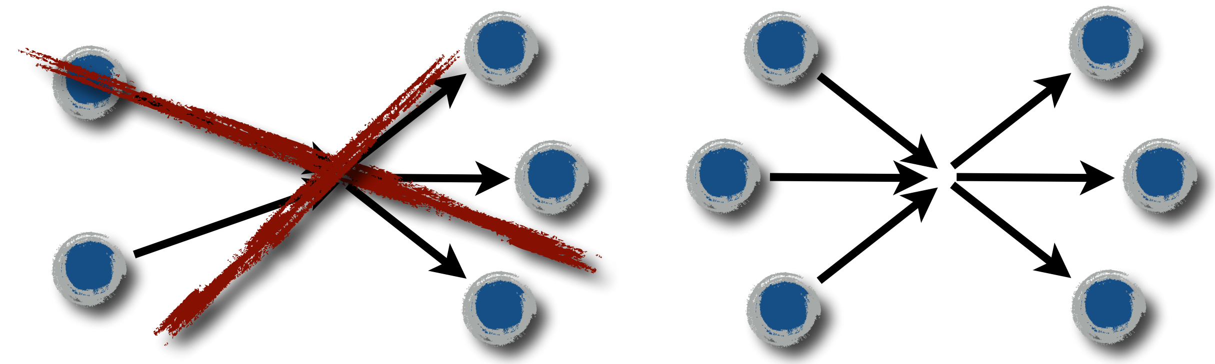
☐ Looking forward
- ☐ 2-to-3 scattering and resonant subprocesses
- ☐ Speculations

We focus here on identical scalar particles



For now we turn off **two-to-three scattering** using a symmetry

We focus here on identical scalar particles

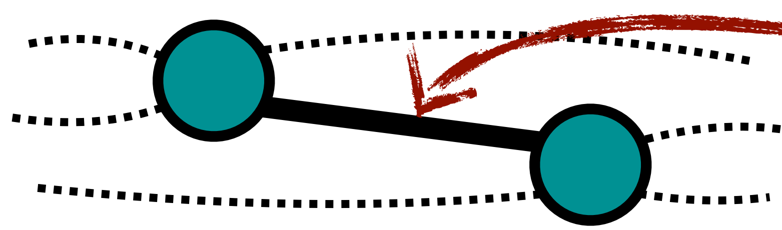


For now we turn off **two-to-three scattering** using a symmetry

Three-to-three amplitude has kinematic singularities

$$i\mathcal{M}_{3\to3} \equiv$$ fully connected correlator with six external legs amputated and projected on shell

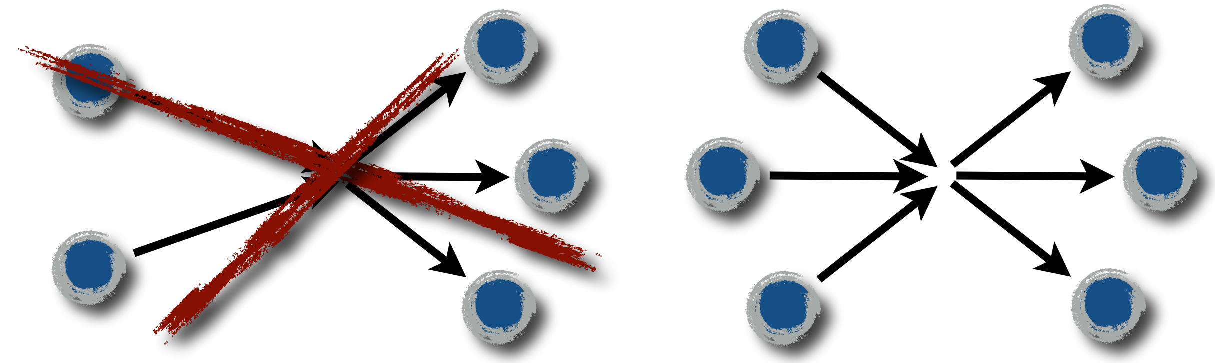
$=$



$+\cdots$

Certain external momenta put this on-shell!

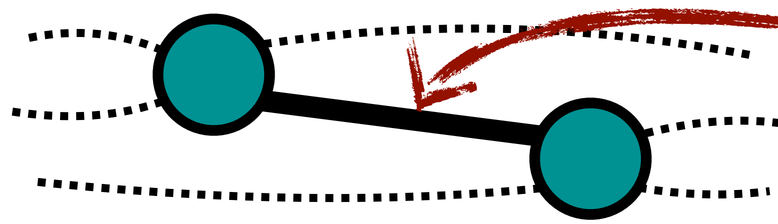
We focus here on identical scalar particles



For now we turn off two-to-three scattering using a symmetry

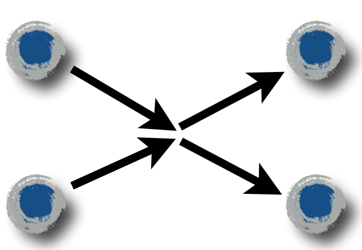
Three-to-three amplitude has kinematic singularities

$i\mathcal{M}_{3\to 3} \equiv$ fully connected correlator with six external legs amputated and projected on shell

$=$  $+ \cdots$

Certain external momenta put this on-shell!

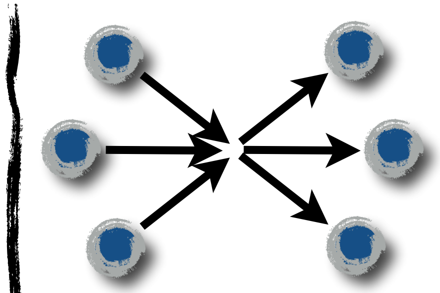
Three-to-three amplitude has more degrees of freedom



12 momentum components

-10 Poincaré generators

2 degrees of freedom



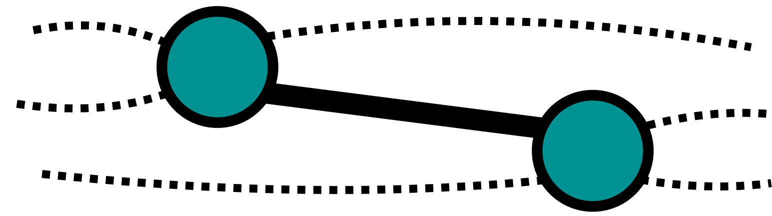
18 momentum components

-10 Poincaré generators

8 degrees of freedom

How can we extract a singular, eight-coordinate function using finite-volume energies?

Spectrum depends on a modified quantity with singularities removed

$$\mathcal{K}_{\mathrm{df},3} \not\ni$$ 

PV pole prescription

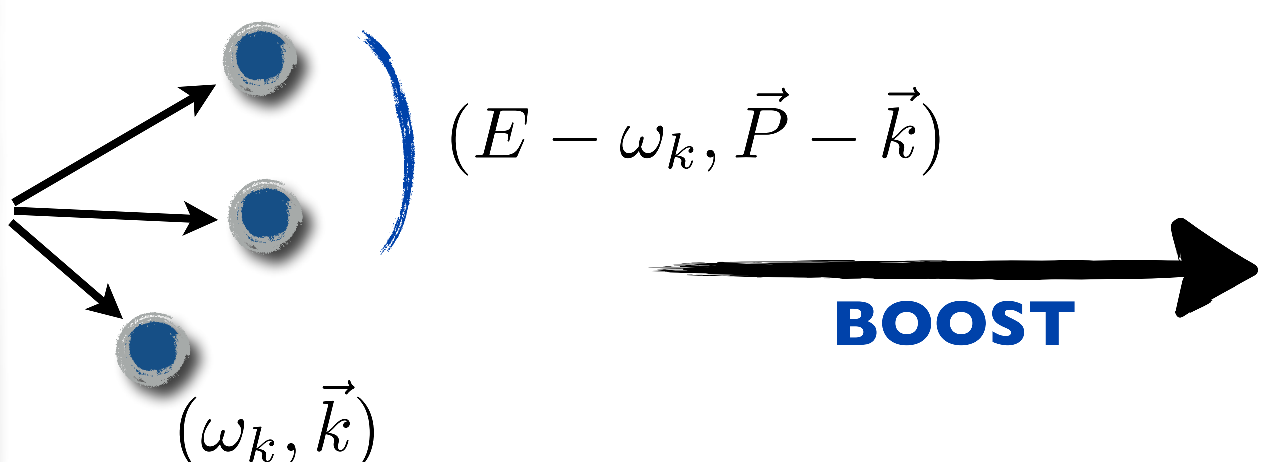
Same degrees of freedom as $\mathcal{M}_3$

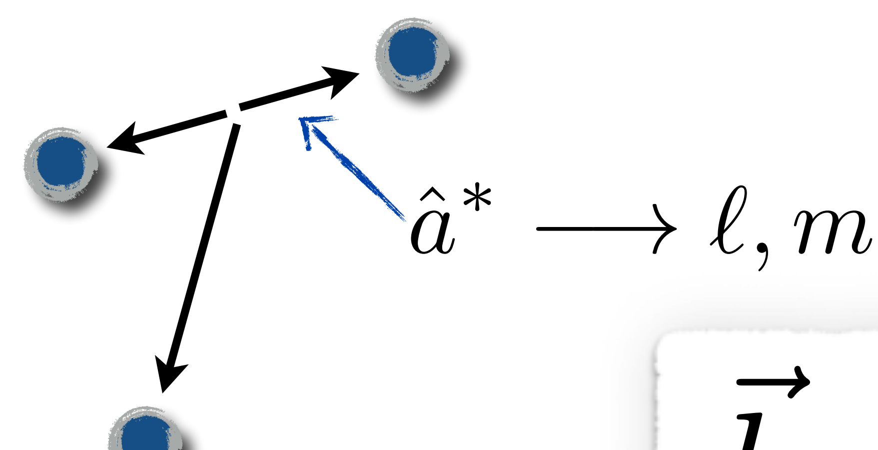
df stands for "divergence free"

Smooth, real function (easier to extract)

Relation to $\mathcal{M}_3$ is known (depends only on on-shell $\mathcal{M}_2$)

Degrees of freedom encoded in an extended matrix space

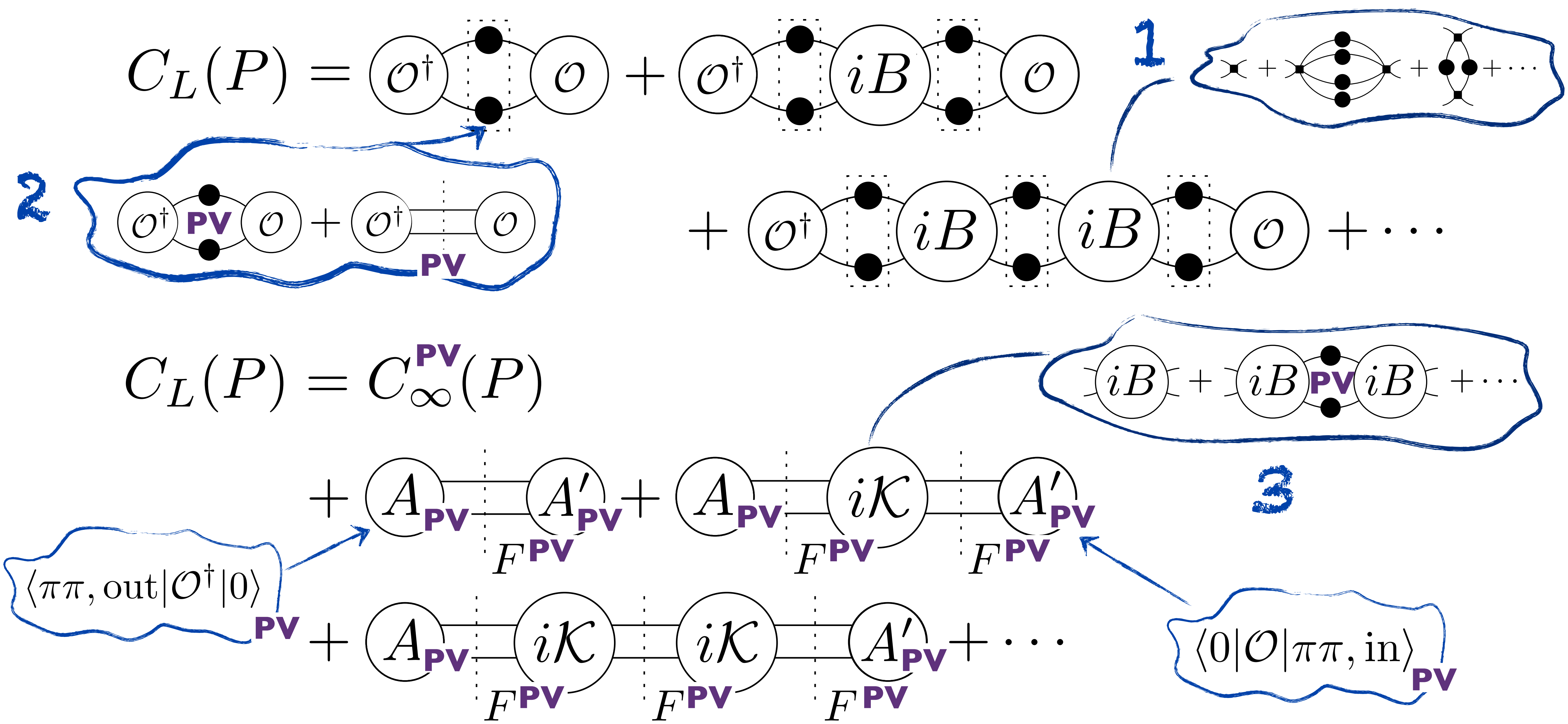




$$\vec{k}, \ell, m$$

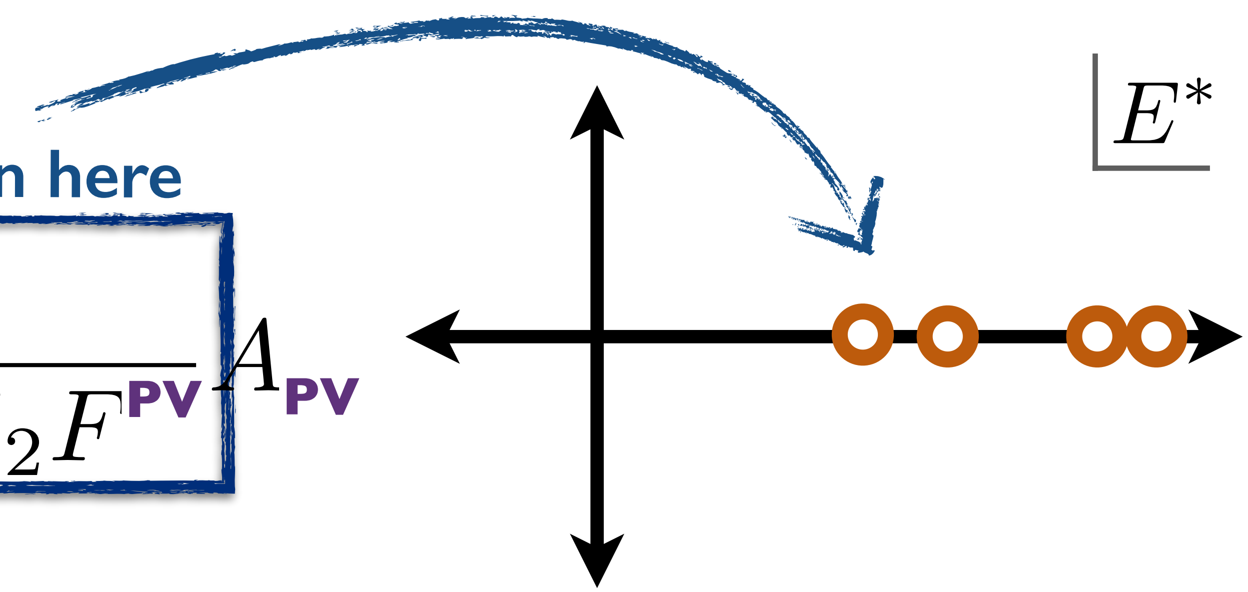
($\vec{k}$ is restricted to finite-volume momenta)

Aside: Changing pole prescriptions for two particles



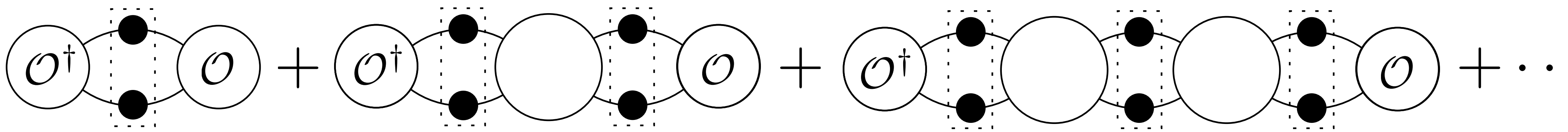
poles are in here

$$C_L(P) = C_\infty^{\mathsf{PV}}(P) - A'_{\mathsf{PV}} \boxed{F^{\mathsf{PV}} \frac{1}{1 + \mathcal{K}_2 F^{\mathsf{PV}}}} A_{\mathsf{PV}}$$

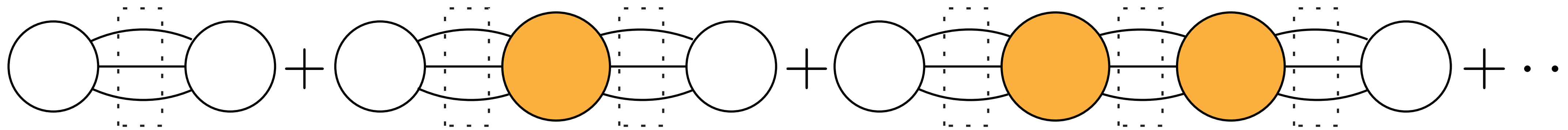


Back to three: new skeleton expansion

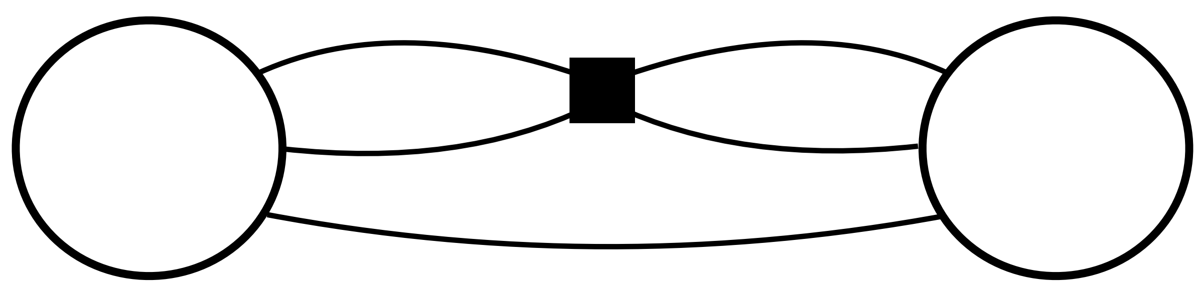
Recall for two particles we started with a "skeleton expansion"

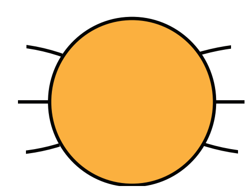
$C_L(P) =$ 

So now we need the same for three…

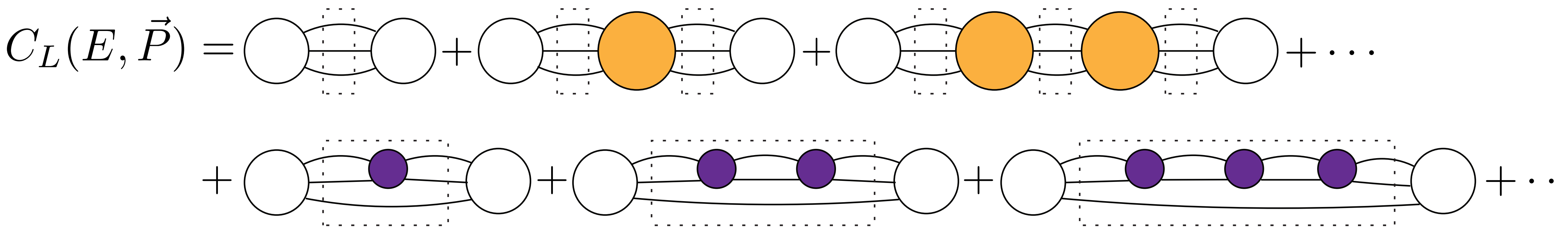
$C_L(E, \vec{P}) \overset{?}{=}$ 

No!… We must also accommodate diagrams like

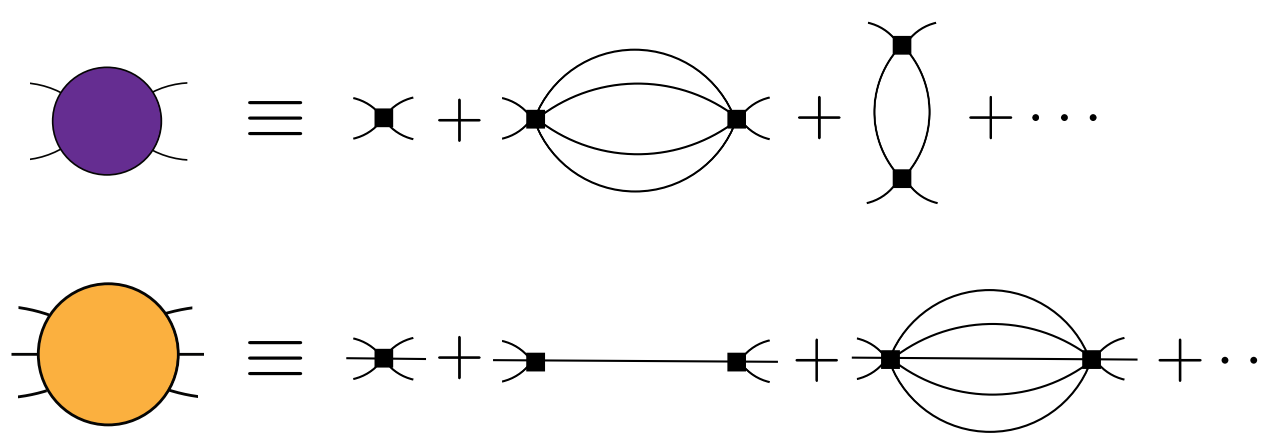


Disconnected diagrams in  **lead to singularities that invalidate the derivation**

Back to three: new skeleton expansion

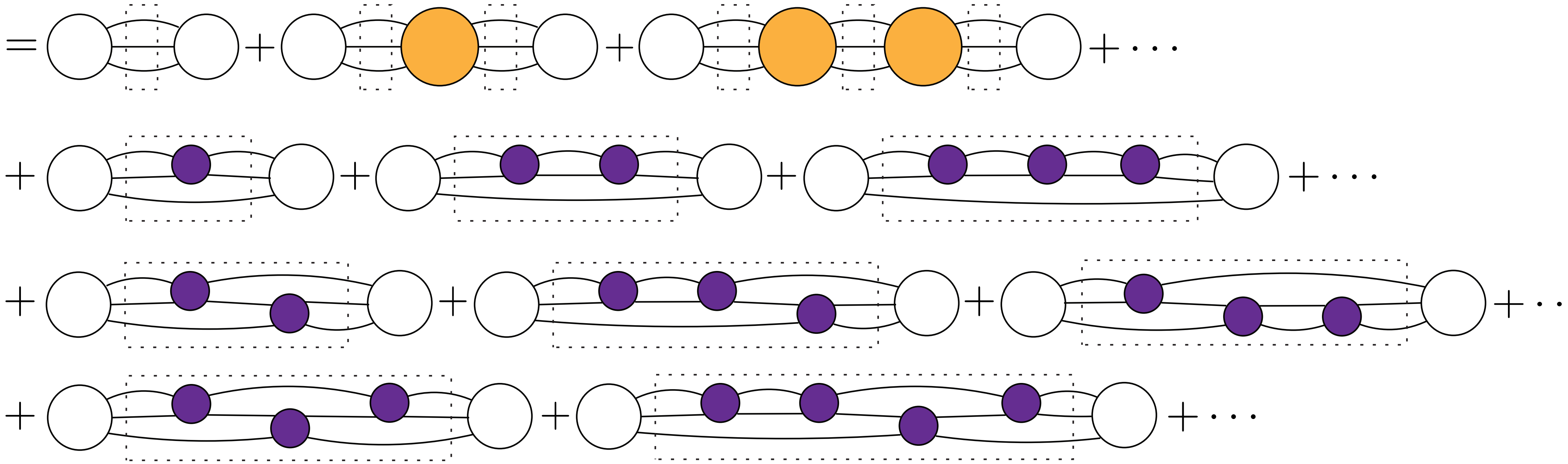


Kernel definitions:

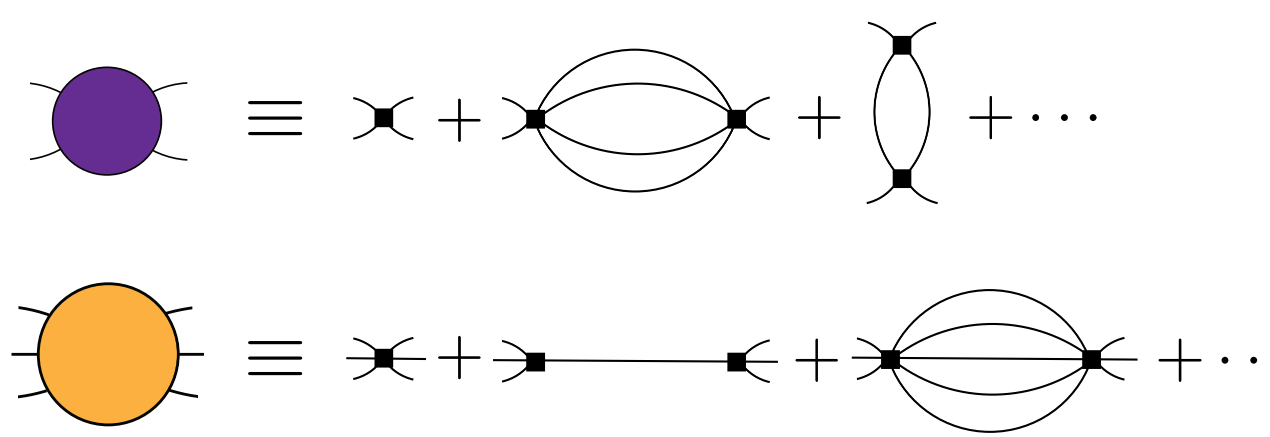


Back to three: new skeleton expansion

$C_L(E,\vec{P}) =$

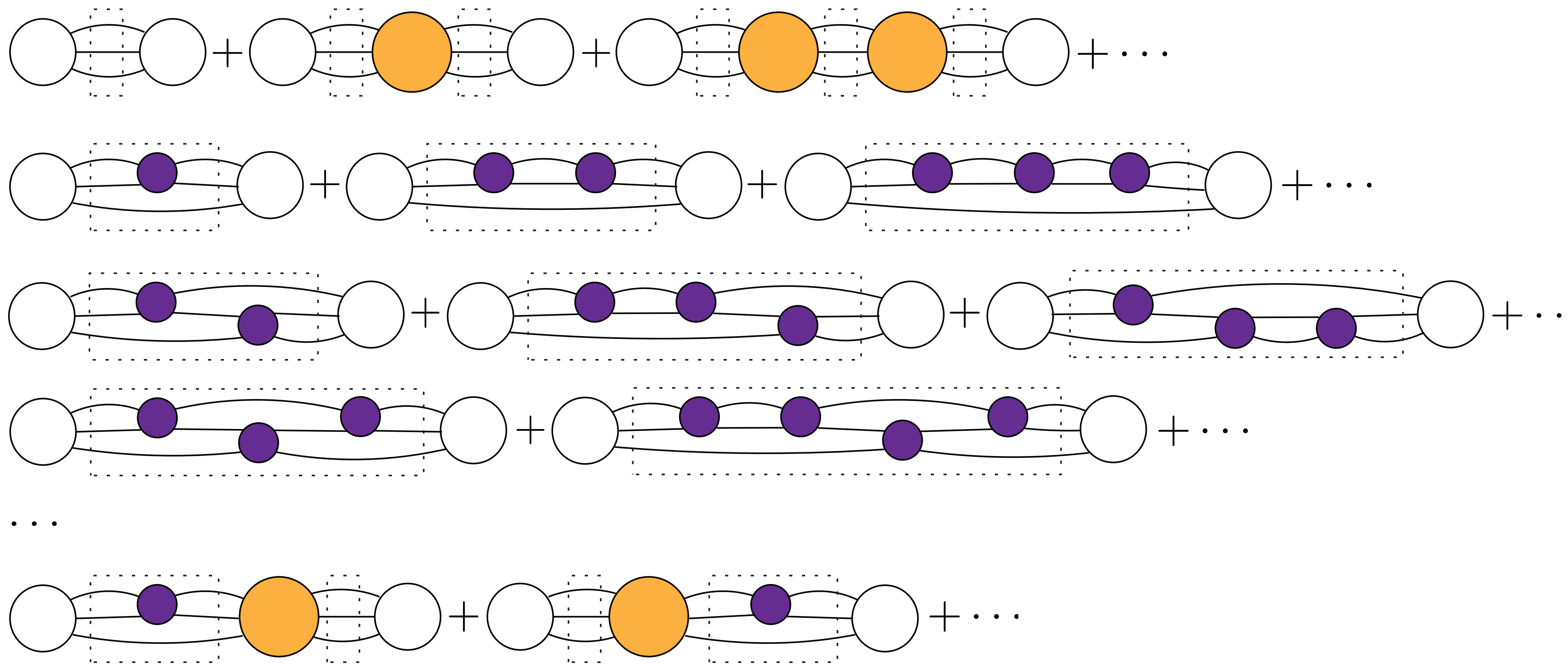


Kernel definitions:

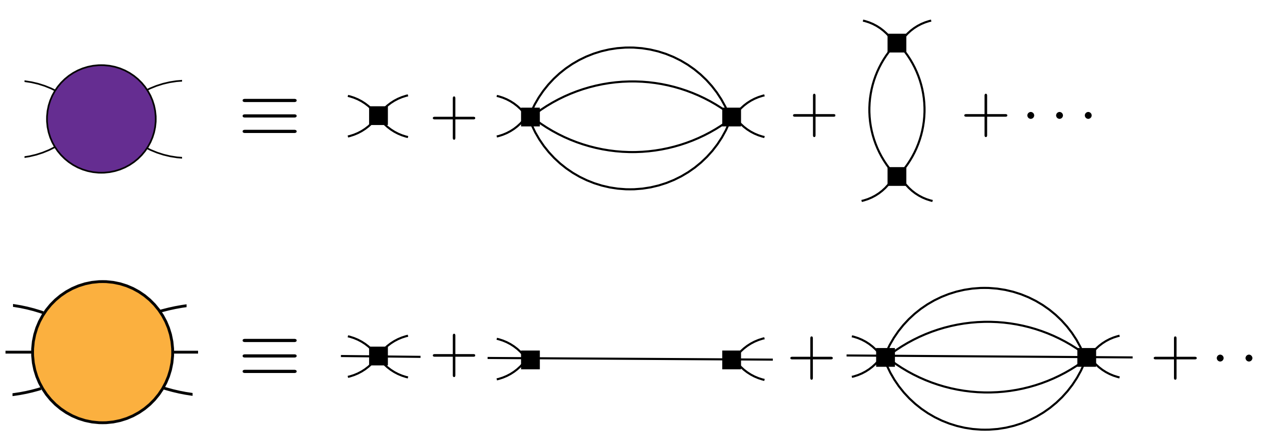


New skeleton expansion

$C_L(E,\vec{P}) =$



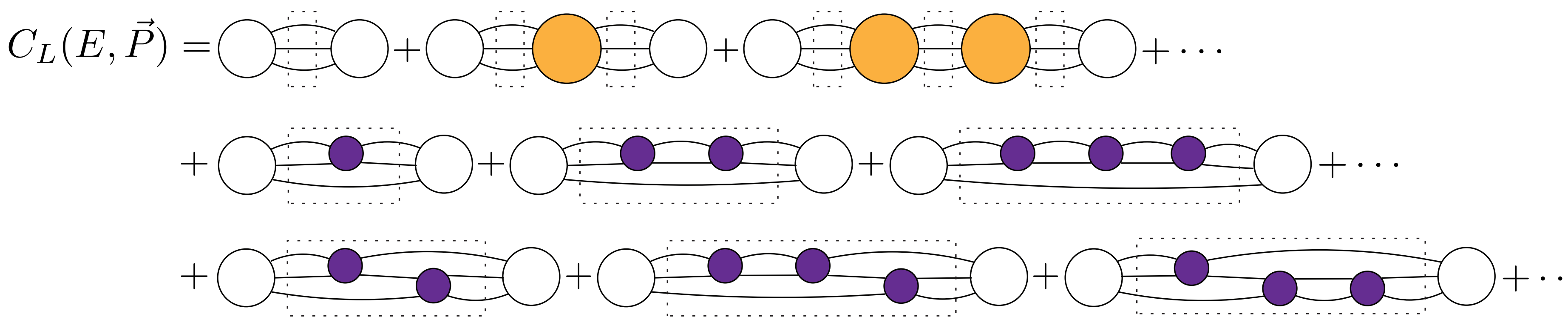
Kernel definitions:



- **All lines are fully dressed propagators**
- **Boxes represent sums over finite-volume momenta**
- **Kernels may contain fixed poles**

Basic approach

1. Work out the three particle skeleton expansion



2. Break diagrams into finite- and infinite-volume parts
3. Organize and sum terms to identify
   infinite-volume observables

Result

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) - A' F_3 \frac{1}{1 + \mathcal{K}_{\text{df},3} F_3} A$$

- Looks similar to the two-particle case
- All quantities defined with PV-pole prescription
- F_3 depends on finite-volume and two-to-two scattering

Quantization condition

At fixed $(L, \vec{P})$, finite-volume energies are solutions to

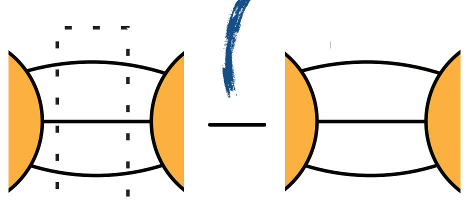
$$\det_{k,\ell,m}\left[\mathcal{K}_{\text{df},3}^{-1} + F_3\right] = 0$$

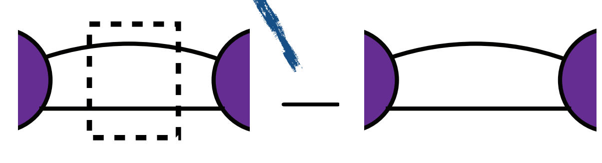
$F_3 \equiv$ **matrix that depends on geometric functions and** $\mathcal{M}_{2\to 2}$.

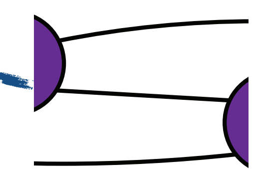
MTH and Sharpe (2014)

All of the complication is buried inside F_3

$$F_3 = \frac{F}{6\omega L^3} - \frac{F}{2\omega L^3}\frac{1}{1+\mathcal{M}_{2,L}G}\mathcal{M}_{2,L}F$$







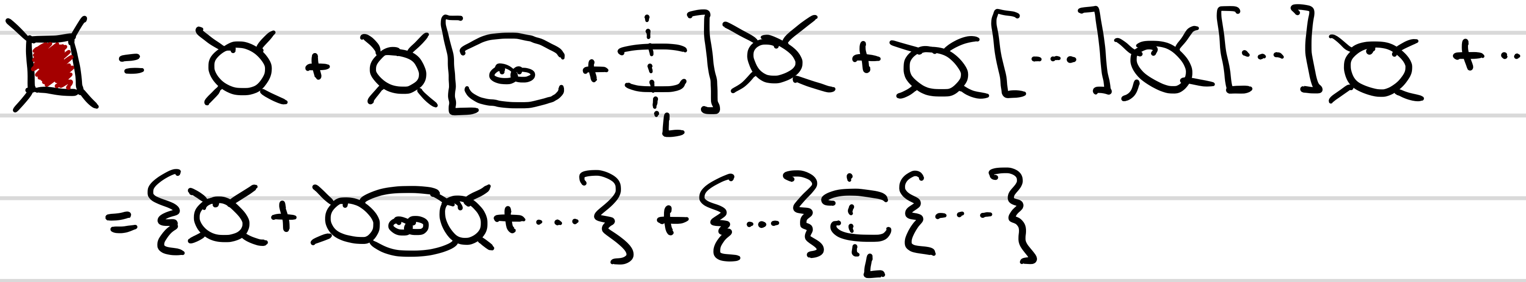
These are all matrices with indices

momentum of one particle $\vec{k} = \dfrac{2\pi\vec{n}}{L}$ $\otimes$ angular momentum of the other two ℓ, m

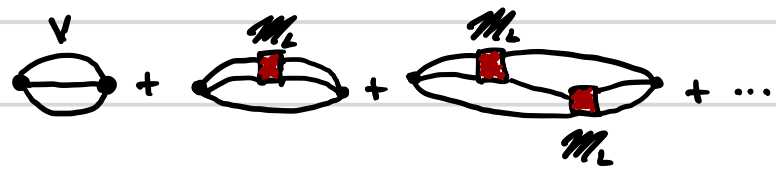
F and G are geometric functions

$\mathcal{M}_{2,L}$ **depends on F and** $\mathcal{M}_2$

More derivation details



$$= i\mathcal{M}\sum_{n=0}^{\infty}[iFi\mathcal{M}]^{-1} = i\mathcal{M}\frac{1}{1-iFi\mathcal{M}}$$

$C_L(E,\vec{P}) =$ 

$$= A'_{k'\ell'm'}\frac{iF_{k'\ell'm';k\ell m}}{6\omega_k L^3}A_{k\ell m} + A'\frac{iF}{2\omega L^3}i\mathcal{M}_L iFA + A'\frac{iF}{2\omega L^3}i\mathcal{M}_L iG\, i\mathcal{M}_L iFA +$$  $$+ \cdots$$

$$= A'\frac{iF}{6\omega L^3}A + A'\frac{iF}{2\omega L^3}i\mathcal{M}_L\sum_{n=0}^{\infty}[iG\,i\mathcal{M}_L]^n iFA + \cdots$$

$$= A'\underbrace{\left[\frac{iF}{2\omega L^3}\left(\frac{1}{3} + i\mathcal{M}_L\frac{1}{1-iG\,i\mathcal{M}_L}iF\right)\right]}_{iF_3}A + A' iF_3\sum_{n=0}^{\infty}[iK_{df}\,iF_3]^n A + C_\infty(E,\vec{P})$$

$$= C_\infty(E,\vec{P}) + A' iF_3(E,\vec{P},L)\frac{1}{1-iK_{df}(E^*)\,iF_3(E,\vec{P},L)}A$$

- So for fixed $\{\vec{P},L\}$, the F.V. energies are solns. to...

$$\det\left[K_{df}(E^*)^{-1} + F_3(E^*,\vec{P},L)\right] = 0$$

Quantization condition

At fixed $(L, \vec{P})$, finite-volume energies are solutions to

$$\det_{k,\ell,m}\left[\mathcal{K}_{\mathrm{df},3}^{-1} + F_3\right] = 0$$

$F_3 \equiv$ matrix that depends on geometric functions and $\mathcal{M}_{2\to2}$.

MTH and Sharpe (2014)

(1). Use two-particle q.c. to constrain $\mathcal{M}_2$ and determine $F_3(E, \vec{P}, L)$.

$$\det[\mathcal{M}_2^{-1} + F_2] = 0 \longrightarrow \mathcal{M}_2 \longrightarrow F_3(E, \vec{P}, L)$$

(2). Use decomposition + parametrization to express $\mathcal{K}_{\mathrm{df},3}(E^*)$ in terms of α_i.

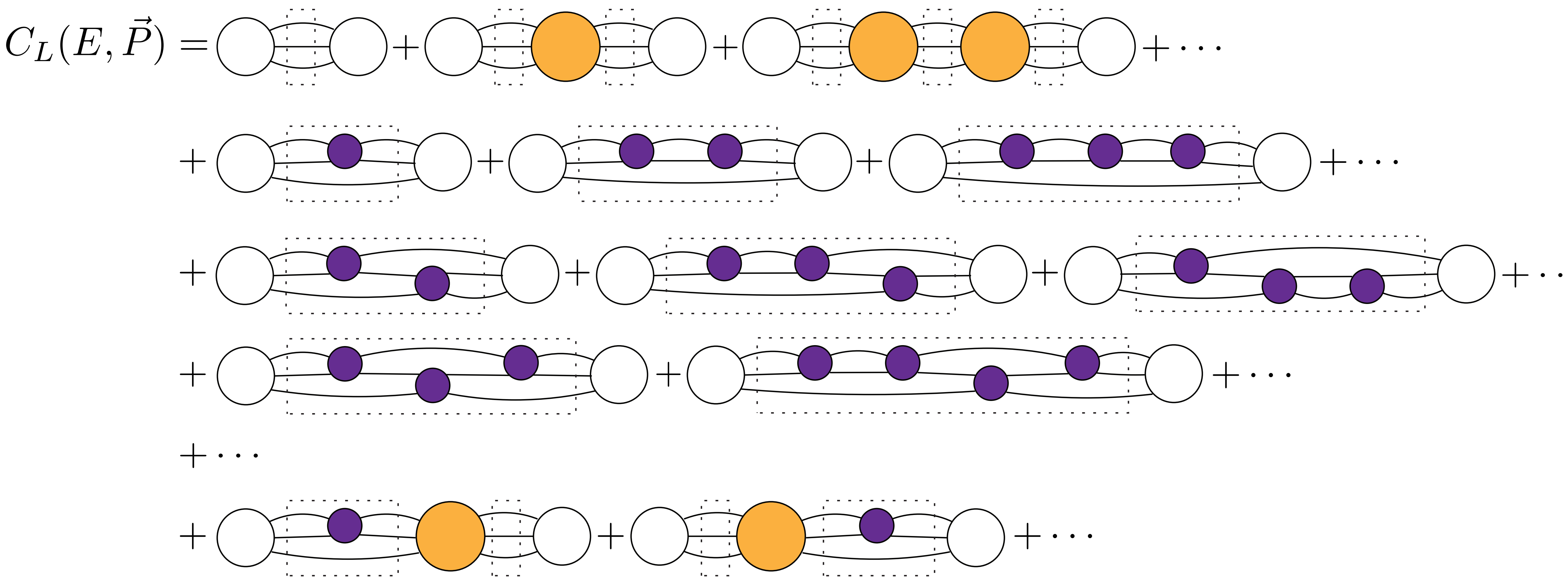
$$\mathcal{K}_{\mathrm{df},3}(E^*, \Omega_3', \Omega_3) \approx \mathcal{K}_{\mathrm{df},3}[\alpha_1, \cdots, \alpha_N] \longleftarrow$$ Recall, this is a real, smooth function

(3). Use three-particle q.c. with finite-volume energies to determine $\mathcal{K}_{\mathrm{df},3}(E^*)$.

$$\det[\mathcal{K}_{\mathrm{df},3}^{-1} + F_3] = 0 \longrightarrow \mathcal{K}_{\mathrm{df},3}(E^*)$$ ✓

Relating $\mathcal{K}_{\mathrm{df},3}$ to $\mathcal{M}_3$

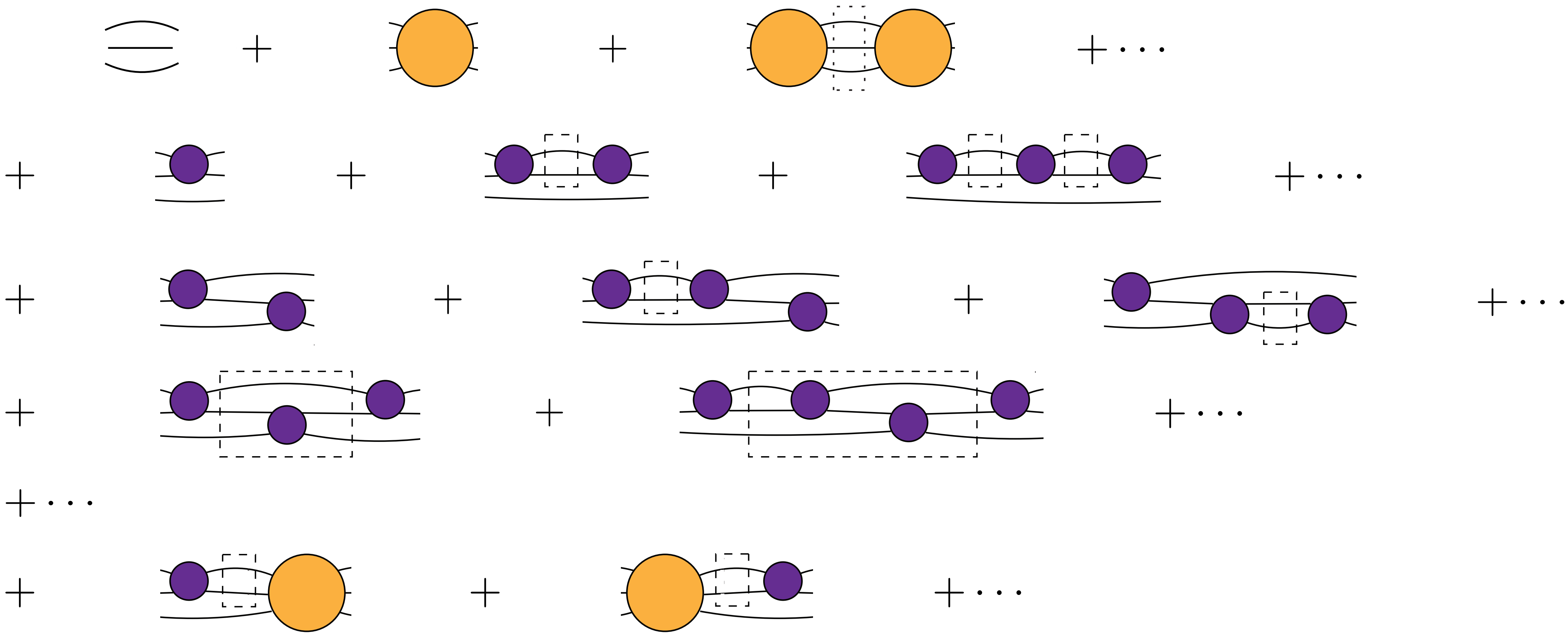
First we modify $C_L(E, \vec{P})$ to define $i\mathcal{M}_{L,3}$



Relating $\mathcal{K}_{\mathrm{df},3}$ to $\mathcal{M}_3$

First we modify $C_L(E, \vec{P})$ to define $i\mathcal{M}_{L,3}$

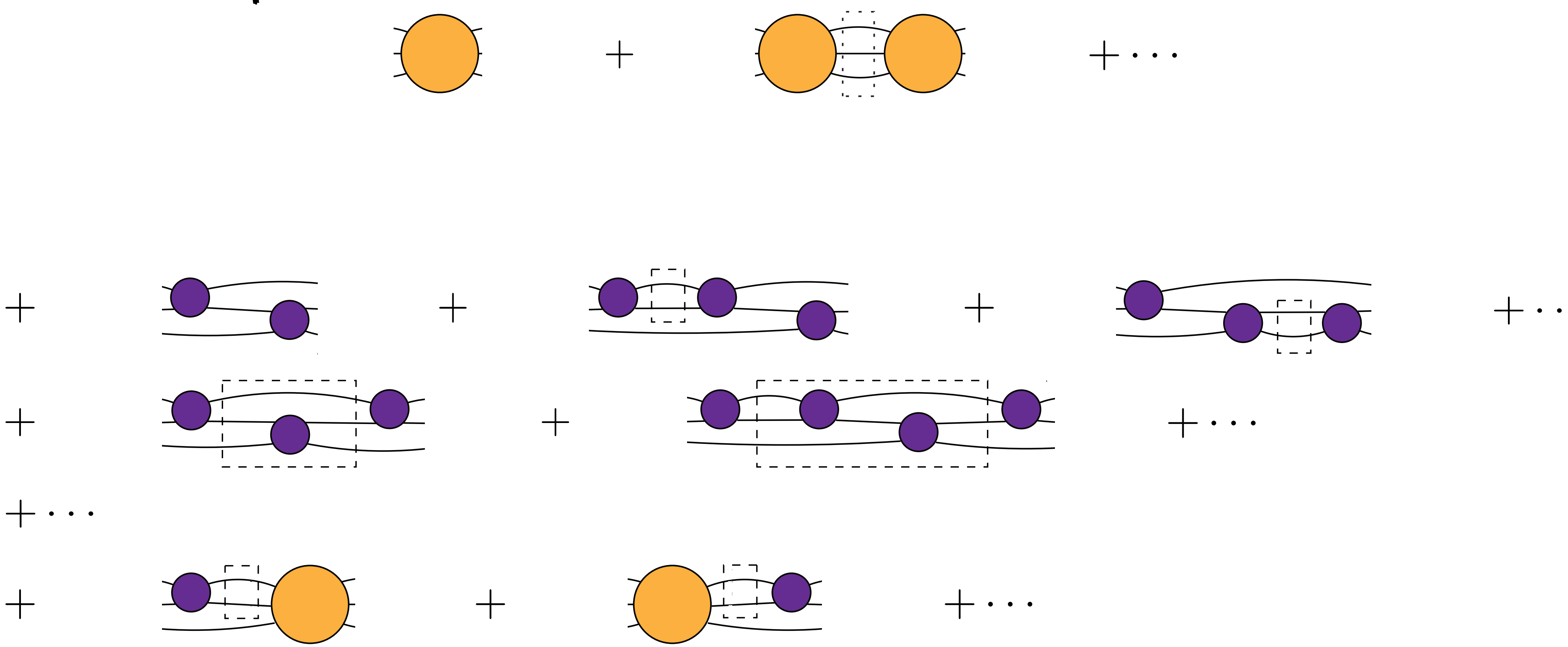
1. Amputate interpolating fields



Relating $\mathcal{K}_{\mathrm{df},3}$ to $\mathcal{M}_3$

First we modify $C_L(E,\vec{P})$ to define $i\mathcal{M}_{L,3}$

1. Amputate interpolating fields
2. Drop disconnected diagrams

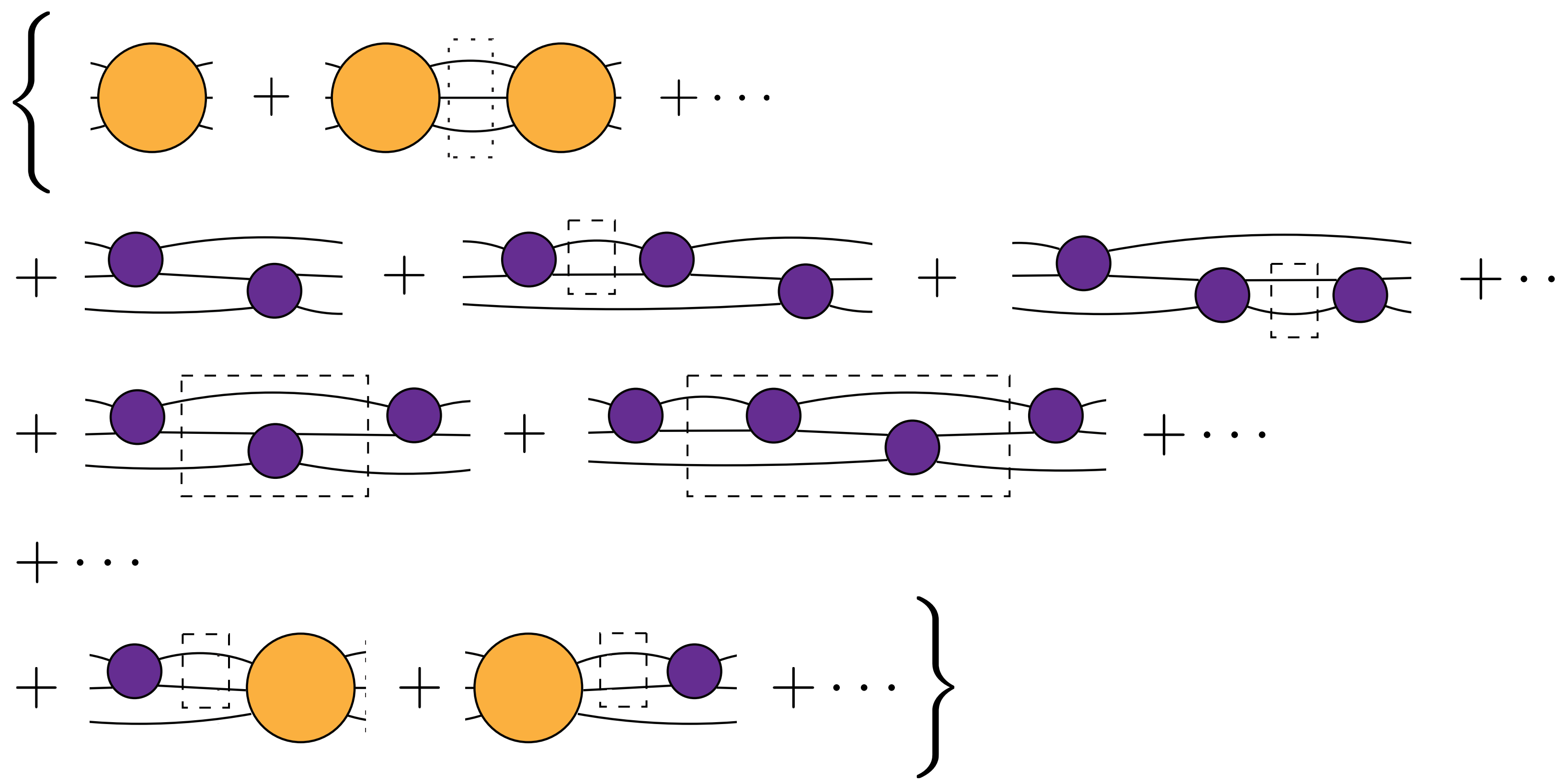


Relating $\mathcal{K}_{\mathrm{df},3}$ to $\mathcal{M}_3$

First we modify $C_L(E, \vec{P})$ to define $i\mathcal{M}_{L,3}$

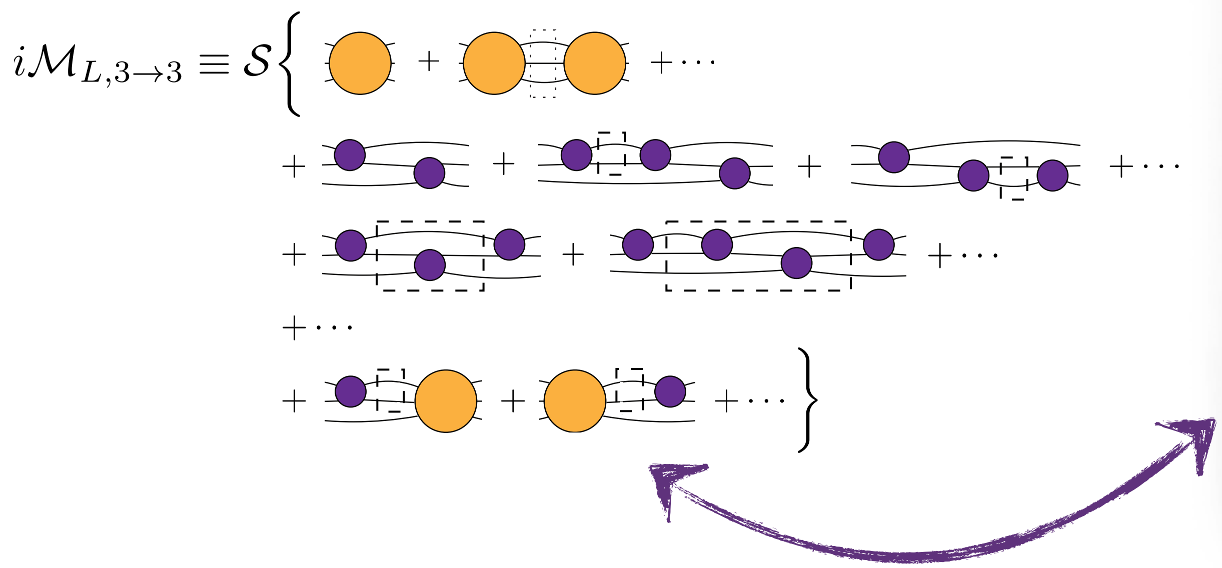
1. Amputate interpolating fields
2. Drop disconnected diagrams
3. Symmetrize

$i\mathcal{M}_{L,3\to3} \equiv \mathcal{S}$



Relating $\mathcal{K}_{\mathrm{df},3}$ to $\mathcal{M}_3$

Combined with our earlier analysis this gives a matrix equation



$$\mathcal{M}_{L,3} = \mathcal{S}\left[\mathcal{D}_L + \mathcal{L}_L \frac{1}{\mathcal{K}_{\mathrm{df},3}^{-1} + F_3} \mathcal{R}_L\right]$$

$$\mathcal{L}_L = \mathcal{X} F_3, \quad \mathcal{R}_L = F_3 \mathcal{X},$$

$$\mathcal{D}_L = -\mathcal{X}\left[F_3 - F_3\big|_{G\to0}\right]\mathcal{X}$$

with the "amputation matrix" $\mathcal{X} = \left(\frac{F}{2\omega L^3}\right)^{-1}$

With this analytic relation in hand we can…

(a) Set $E \to E + i\epsilon$, **(b) Send** $L \to \infty$, **(c) Send** $\epsilon \to 0^+$.

Leads to an integral equation for the scattering amplitude

$$\mathcal{M}_3(E^*) = \mathcal{I}\left[\mathcal{K}_{\mathrm{df},3}(E^*), \mathcal{M}_2\right]$$

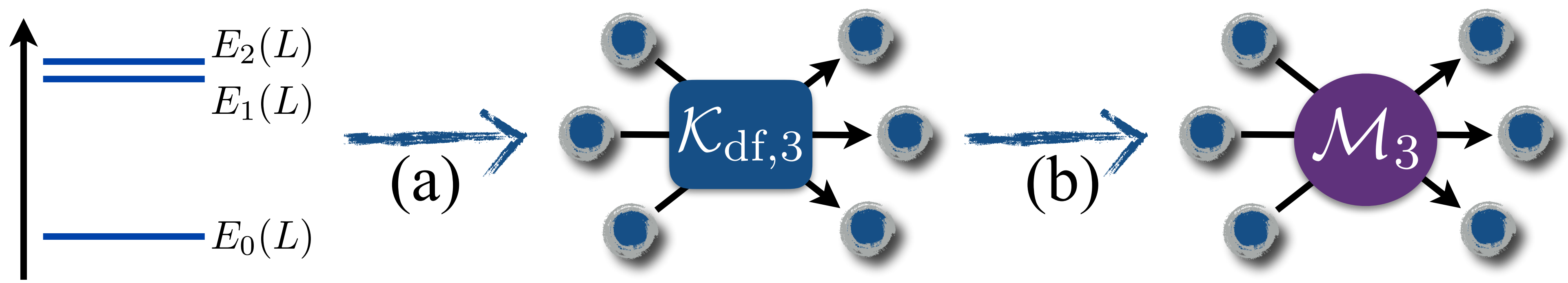
Fixed total energy, manifestly convergent, on-shell only, no reference to EFT, takes care of unitarity and singularities, useful independent of finite-volume physics?

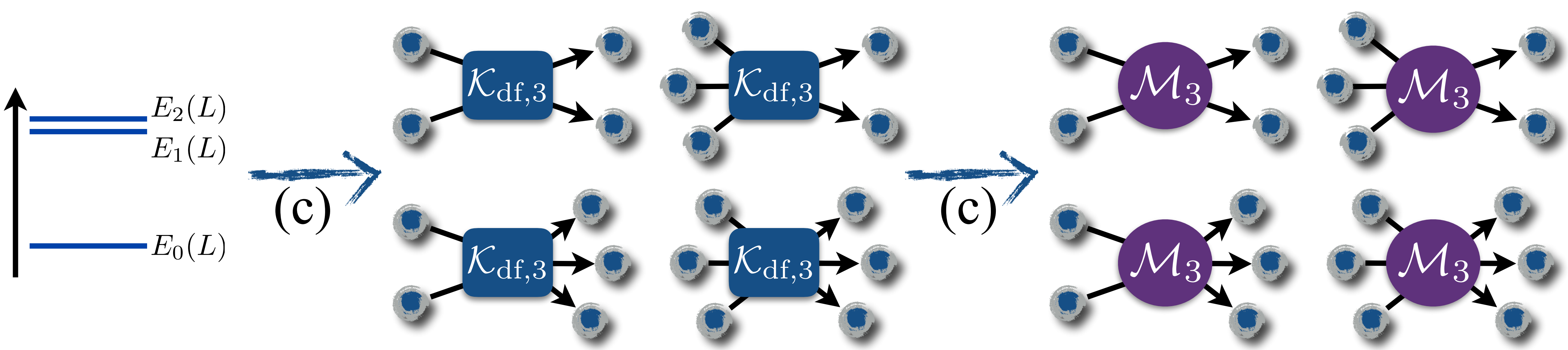
MTH and Sharpe (2015)

Current status

Model- & EFT-independent relation between

finite-volume energies and relativistic two-and-three particle scattering





(a),(b) ***MTH and Sharpe (2015),(2016)***

(c) ***Briceño, MTH, Sharpe (2017)***

Outline

☑ Warm up and definitions
- ☑ Basic set-up
- ☑ Finite-volume correlator
- ☑ Three non-interacting particles

☑ Two particles in a box
- ☑ Alternative derivation
- ☑ Truncation and application
- ☑ Relating matrix elements

☑ Three particles in a box
- ☑ 3-to-3 scattering
- ☑ (Sketch of) derivation
- ☑ An unexpected infinite-volume quantity
- ☑ Relating energies to scattering

☐ Testing the result
- ☐ Large-volume expansion
- ☐ Effimov state in a box

☐ Numerical explorations
- ☐ Truncation at low energies
- ☐ 2-particle physics in 3-particle energies
- ☐ Toy 3-particle resonance
- ☐ Numerical Effimov state
- ☐ Unphysical solutions

☐ Looking forward
- ☐ 2-to-3 scattering and resonant subprocesses
- ☐ Speculations

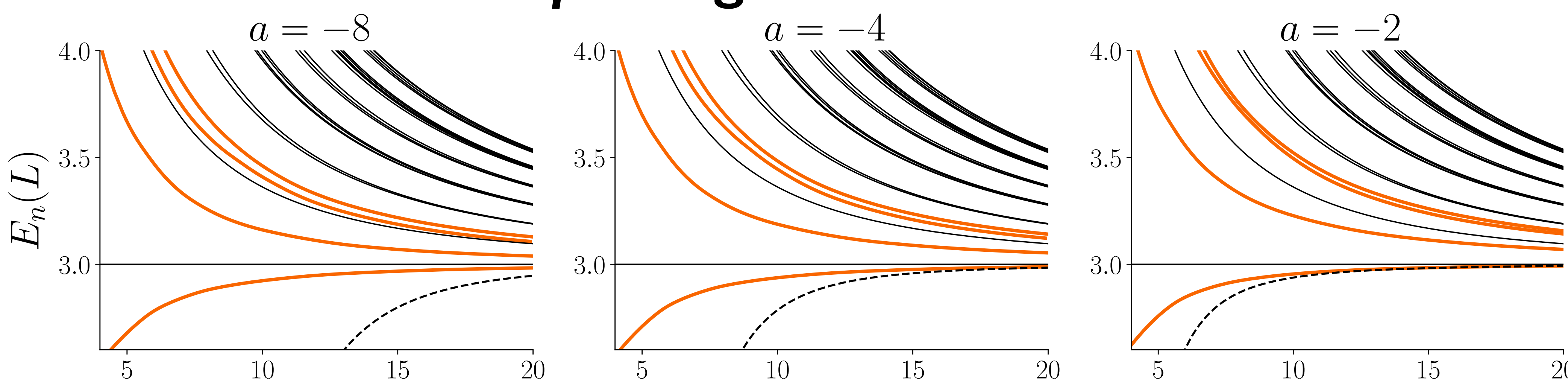
Teaser for tomorrow

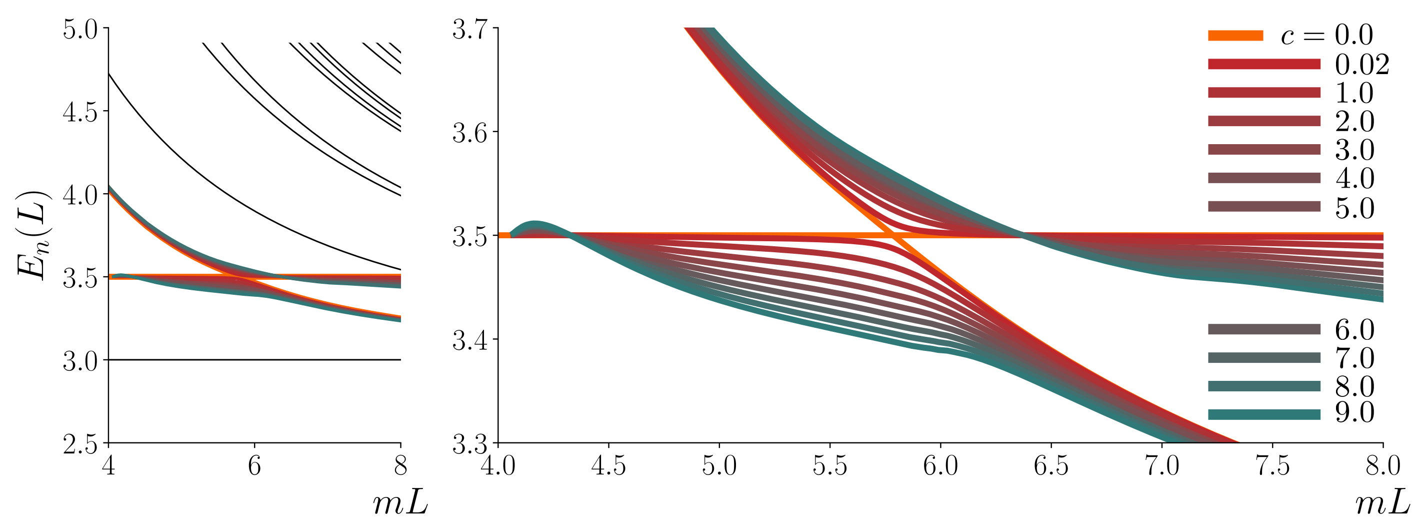
Reproducing (and extending) a 60 year old result

$$E = 3m + \frac{12\pi a}{mL^3}\left(1 + c_4\frac{a}{L} + \cdots\right) - \frac{\mathcal{M}_{\rm thr}}{48m^3L^6} + \cdots$$

Huang and Yang 1958

Exploring the solutions





Thanks!