



Three-particle scattering from numerical lattice QCD

Scattering from the lattice: applications to phenomenology and beyond

Maxwell T. Hansen

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Outline

Warm up and definitions

- Basic set-up
 - Finite-volume correlator
 - Three non-interacting particles
-

Two particles in a box

- Alternative derivation
- Truncation and application
- Relating matrix elements

Three particles in a box

- 3-to-3 scattering
 - (Sketch of) derivation
 - An unexpected infinite-volume quantity
 - Relating energies to scattering
-

Testing the result

- Large-volume expansion
- Effimov state in a box

Numerical explorations

- Truncation at low energies
- 2-particle physics in 3-particle energies
- Toy 3-particle resonance
- Numerical Effimov state
- Unphysical solutions

Looking forward

- 2-to-3 scattering and resonant subprocesses
- Speculations

Review: Finite-volume calculations take $\int_{\mathbf{p}} \equiv \int \frac{d^3\mathbf{p}}{(2\pi)^3} \implies \frac{1}{L^3} \sum_{\mathbf{p}}$

Spectral decomposition... $C_L(P) = \sum_n 2E_n(L) L^3 \frac{i|\langle E_n, \mathbf{P}, L | \mathcal{O}^\dagger(0) | 0 \rangle|^2}{E^2 - E_n(L)^2 + i\epsilon}$

Non-interacting correlator...

$$C_L(P) = C_\infty(P) + \frac{1}{6} \left[\frac{1}{L^6} \sum_{\mathbf{p}, \mathbf{k}} - \int_{\mathbf{k}, \mathbf{p}} \right] \frac{i|\langle \mathbf{P}, \mathbf{p}, \mathbf{k}; \varphi\varphi\varphi | \mathcal{O}^\dagger(0) | 0 \rangle|^2}{2\omega_{\mathbf{k}} 2\omega_{\mathbf{p}} 2\omega_{\mathbf{p}\mathbf{k}} [E - \omega_{\mathbf{k}} - \omega_{\mathbf{p}} - \omega_{\mathbf{p}\mathbf{k}} + i\epsilon]} + \mathcal{O}(e^{-mL})$$

Energies given by...

$$E_n(L) = \sqrt{\mathbf{p}_1^2 + m^2} + \sqrt{\mathbf{p}_2^2 + m^2} + \sqrt{\mathbf{p}_3^2 + m^2}$$

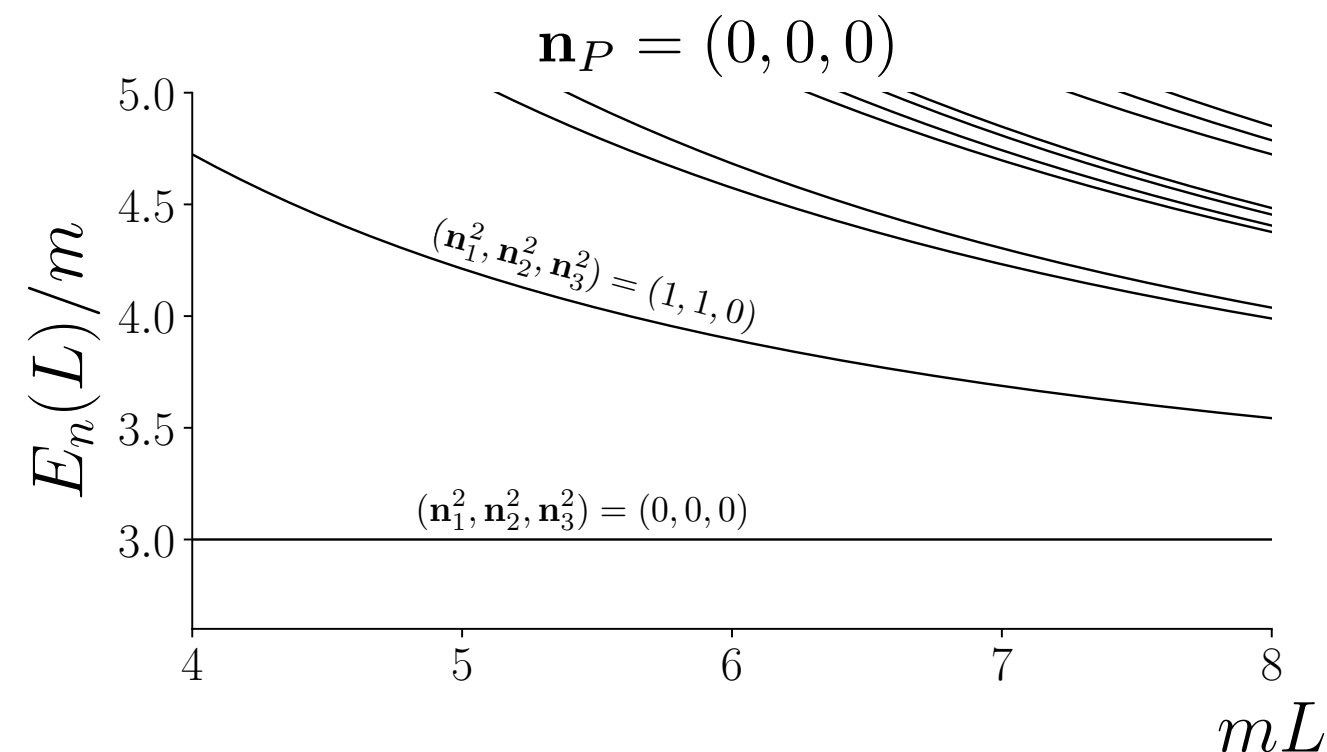
Number of states $\propto L^6$

Matrix elements related by...

$$\frac{|\langle E_n, \mathbf{P}, L | \mathcal{O}^\dagger(0) | 0 \rangle|^2}{\langle |\langle \mathbf{P}, \mathbf{p}, \mathbf{k}; \varphi\varphi\varphi | \mathcal{O}^\dagger(0) | 0 \rangle|^2 \rangle_{\Omega_n}} = \frac{\nu_n}{6L^9} \frac{1}{2\omega_{\mathbf{k}} 2\omega_{\mathbf{p}} 2\omega_{\mathbf{p}\mathbf{k}}}$$

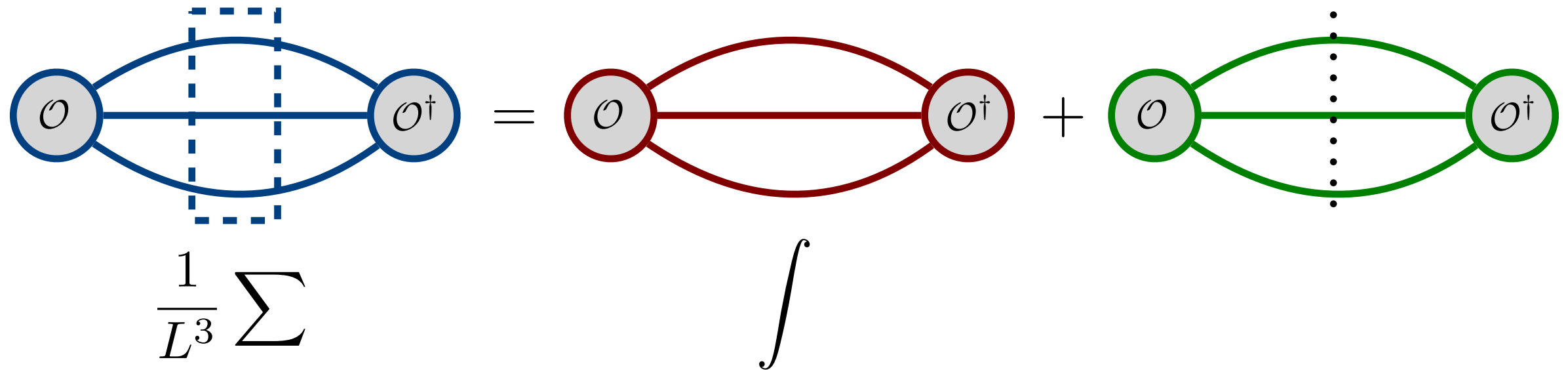
States cluster due to NR degeneracy

$$E_n^{\text{NR}}(L) = 3m + \frac{2\pi^2}{L^2} (n_1 + n_2 + n_3)$$



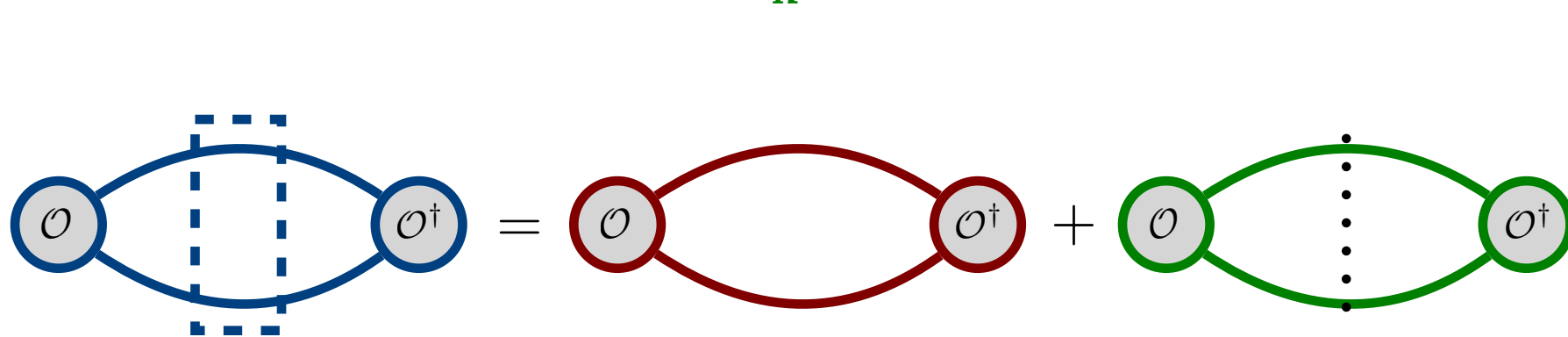
Three-particle cut identity

$$C_L(P) = C_\infty(P) + \frac{1}{6} \left[\frac{1}{L^6} \sum_{\mathbf{p}, \mathbf{k}} - \int_{\mathbf{k}, \mathbf{p}} \right] \frac{i |\tilde{f}_{\text{on}}(\mathbf{k}, \hat{\mathbf{p}}^*)|^2}{2\omega_{\mathbf{k}} 2\omega_{\mathbf{p}} 2\omega_{\mathbf{p}\mathbf{k}} [E - \omega_{\mathbf{k}} - \omega_{\mathbf{p}} - \omega_{\mathbf{p}\mathbf{k}} + i\epsilon]} + \mathcal{O}(e^{-mL})$$



Two-particle analog

$$C_L(P) = C_\infty(P) + \frac{1}{2} \left[\frac{1}{L^3} \sum_{\mathbf{k}} - \int_{\mathbf{k}} \right] \frac{i |\tilde{f}_{\text{on}}(\hat{\mathbf{k}}^*)|^2}{2\omega_{\mathbf{k}} 2\omega_{\mathbf{P}-\mathbf{k}} [E - \omega_{\mathbf{k}} - \omega_{\mathbf{P}-\mathbf{k}} + i\epsilon]} + \mathcal{O}(e^{-mL})$$



$$\mathbf{k}^* = \hat{\mathbf{k}}^* \sqrt{E^{*2}/4 - m^2}$$

$$E^* = \sqrt{E^2 - \mathbf{P}^2}$$

Three-particle cut identity

$$C_L(P) = C_\infty(P) + \frac{1}{6} \left[\frac{1}{L^6} \sum_{\mathbf{p}, \mathbf{k}} - \int_{\mathbf{k}, \mathbf{p}} \right] \frac{i |\tilde{f}_{\text{on}}(\mathbf{k}, \hat{\mathbf{p}}^*)|^2}{2\omega_{\mathbf{k}} 2\omega_{\mathbf{p}} 2\omega_{\mathbf{p}\mathbf{k}} [E - \omega_{\mathbf{k}} - \omega_{\mathbf{p}} - \omega_{\mathbf{p}\mathbf{k}} + i\epsilon]} + \mathcal{O}(e^{-mL})$$



Stefan's comment....

$$\mathcal{O}(x) \equiv \frac{1}{6} \int_L d^4y \int_L d^4z f(y, z) \varphi(x) \varphi(x+y) \varphi(x+z)$$

This must be local in time!

This implies that the Fourier transform

$$\tilde{f}(p, k) = \frac{1}{6} \sum_{\{k_1, k_2\} \in \{p, k, P-p-k\}} \int_L d^4z \int_L d^4y e^{-ik_1 z - ik_2 y} f(y, z)$$

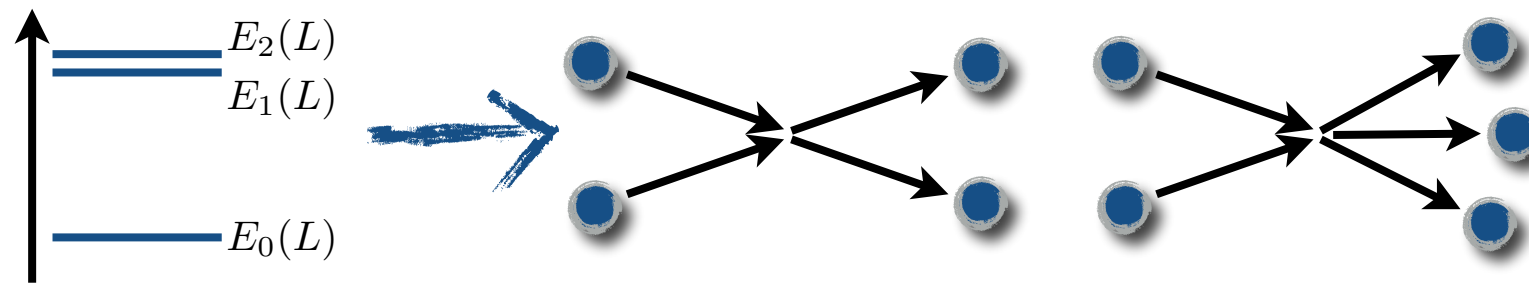
is independent of k^0 and p^0 .



$$\mathbf{k}^* = \hat{\mathbf{k}}^* \sqrt{E^{*2}/4 - m^2}$$

$$E^* = \sqrt{E^2 - \mathbf{P}^2}$$

Finite-volume correlators (with interactions)



$$C_L(P) \equiv \int_L d^4x e^{-iPx} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(0) | 0 \rangle$$

Total 4-momentum

$$P = (E, \mathbf{P}) = (E, 2\pi \mathbf{n}_P / L)$$

c.m. frame energy: $E^{*2} = E^2 - \mathbf{P}^2$

n-particle interpolator

e.g. $\pi(\mathbf{p})\pi(-\mathbf{p})$ or $\bar{q}\Gamma q$

(only quantum numbers relevant)

Focus on a window of energies to isolate particular on-shell states

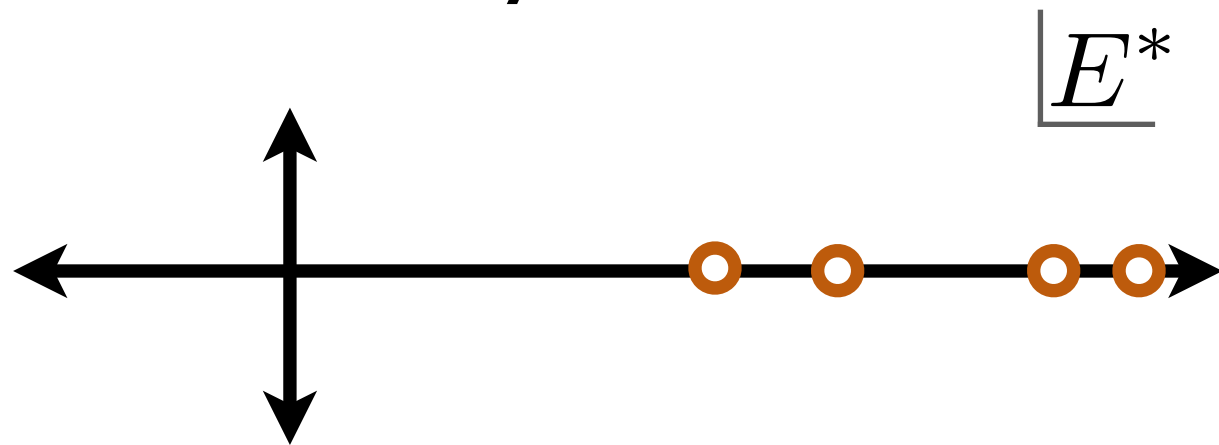
Two particles:	\mathbb{Z}_2 ✓	$0 < E^* < 4m$	\mathbb{Z}_2 ✗	$m < E^* < 3m$	
Three particles	\mathbb{Z}_2 ✓	$m < E^* < 5m$			
Two and three particles	\mathbb{Z}_2 ✗	$m < E^* < 4m$			

Of poles and branch cuts

$$C_L(P) \equiv \int_L d^4x e^{-iPx} \langle 0|T\mathcal{O}(x)\mathcal{O}^\dagger(0)|0\rangle$$

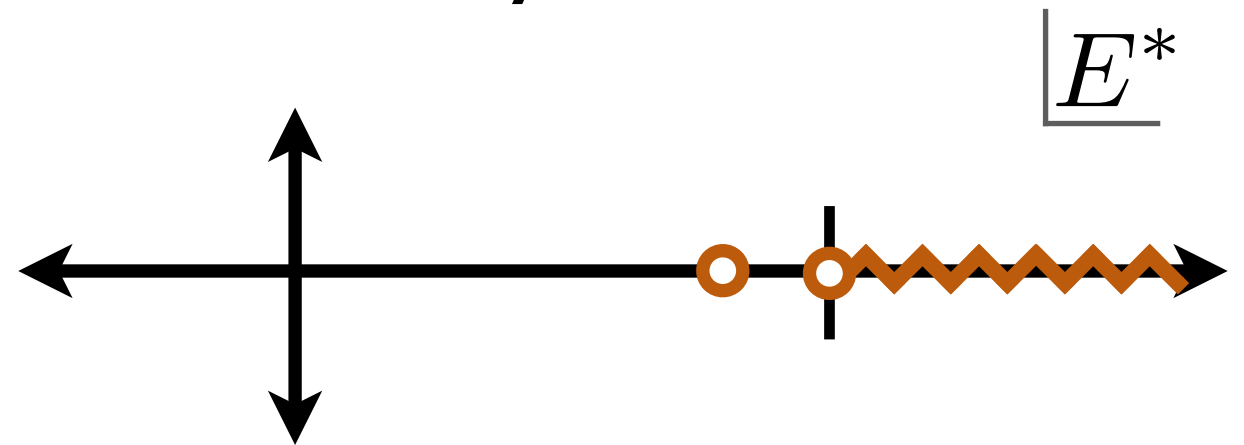
At fixed L , P poles in C_L give **the finite-volume spectrum**

C_L analytic structure



- ✓ Real function (unitarity hidden)
- ✓ No cuts \rightarrow only one sheet
- ✓ No resonance poles

C_∞ analytic structure

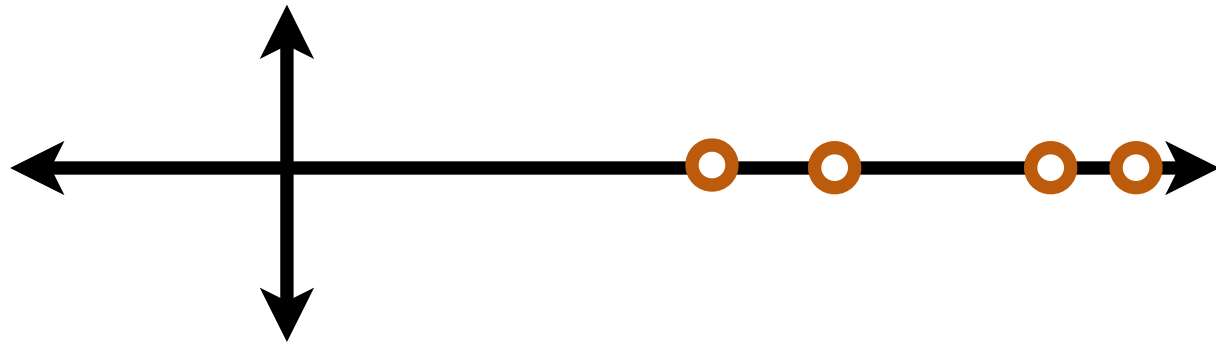


- ✓ Complex function (constrained by unitarity)
- ✓ Scattering states form cut
- ✓ Resonance poles on unphysical sheets

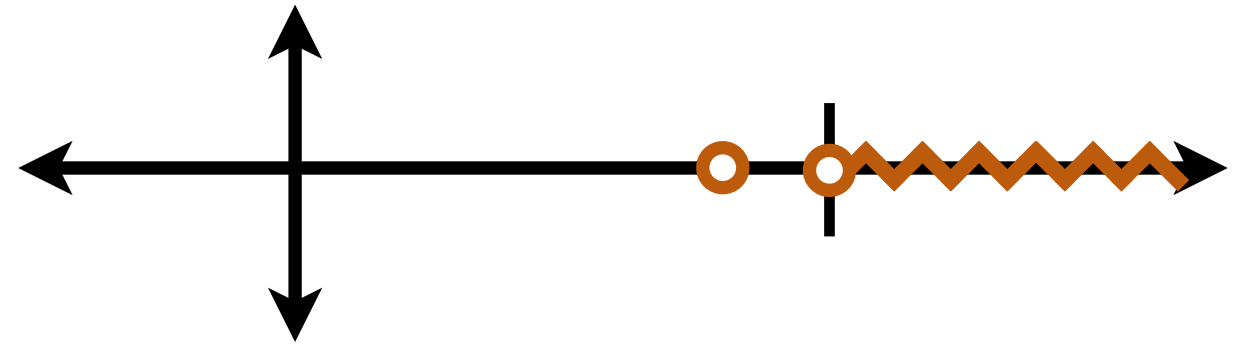
Want to relate $C_L \leftrightarrow C_\infty$ ($\mathcal{M}_{n \rightarrow m}$ is just a specific choice of C_∞)

The idea is to “reach in” and correct the singularity structure

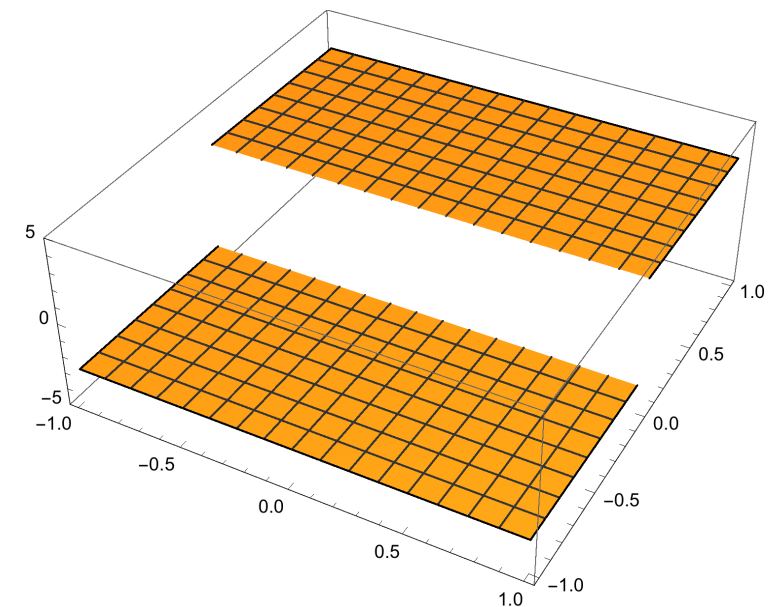
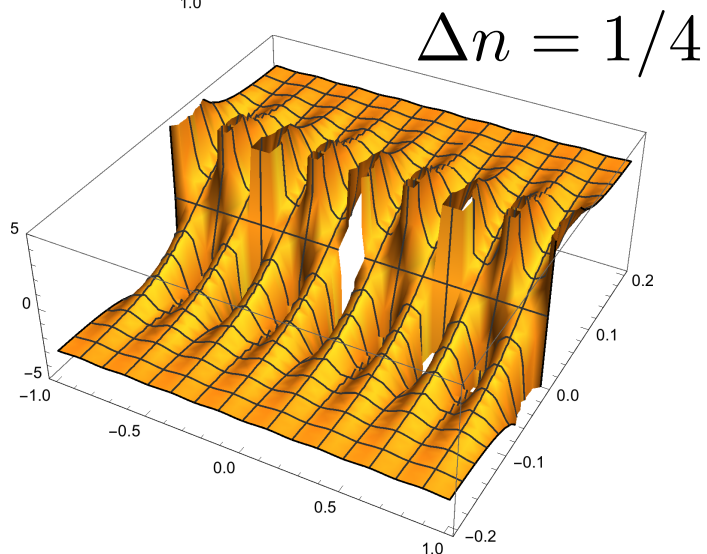
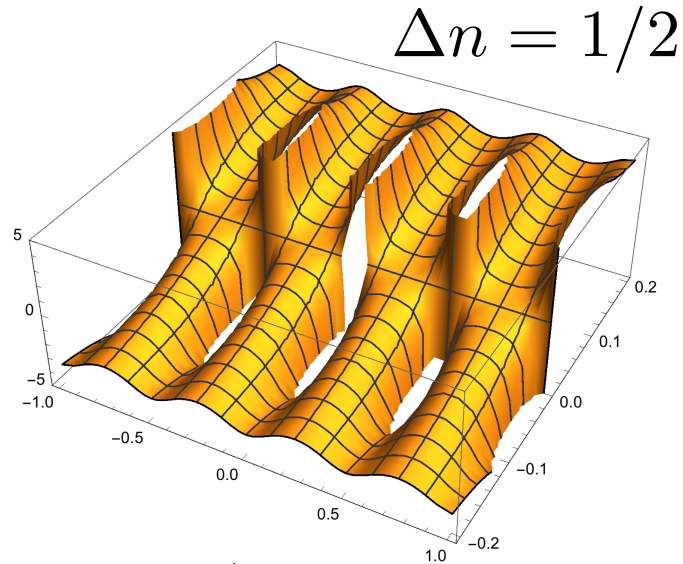
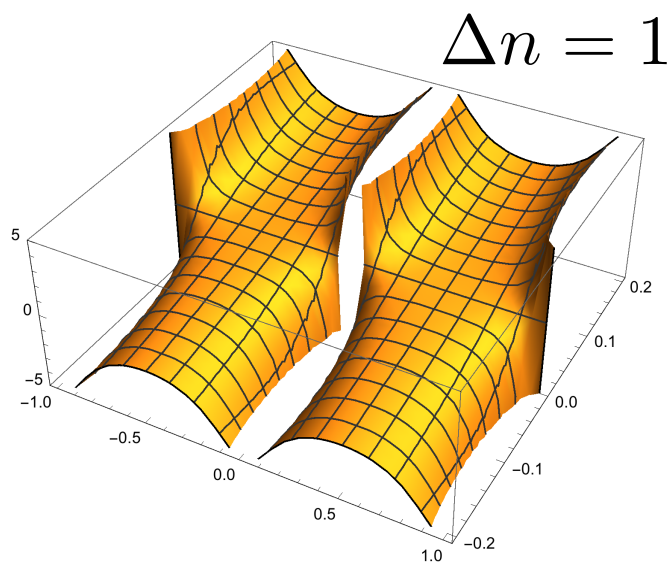
From poles to cuts: Toy example



$$f(x, y, \Delta n) \equiv \sum_{n=-20, -20+\Delta n, \dots}^{20} \Delta n \operatorname{Im} \frac{1}{n - x - iy}$$



$$f_{\infty}(x, y) \equiv \operatorname{Im} \log \left[\frac{10 - x - iy}{-10 - x - iy} \right]$$



**Only in the infinite-volume limit
can we generate the singularity
that leads to a second sheet**

Two identical particles: skeleton expansion

$$C_L(P) = \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \dots$$

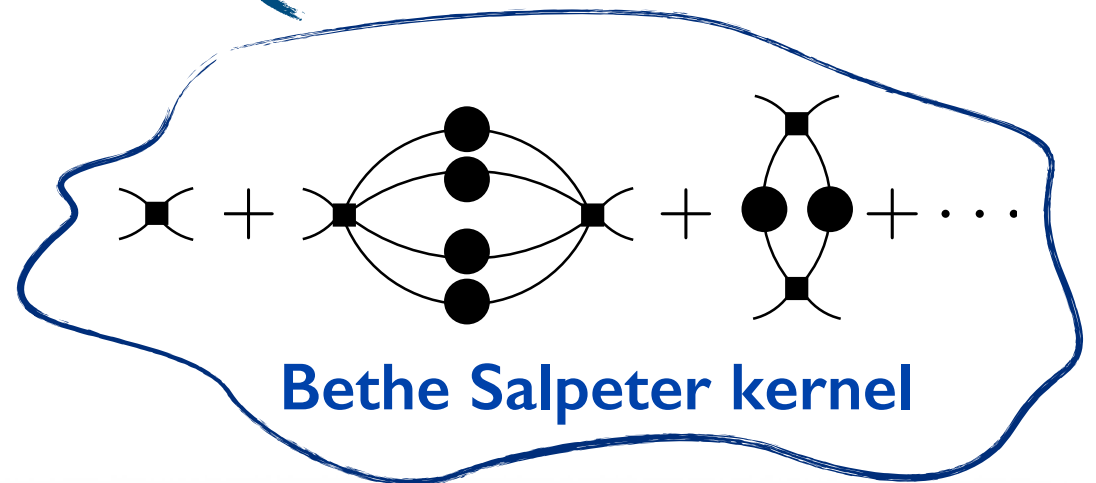
The diagrams show a sequence of terms in the skeleton expansion. Each term consists of a chain of circles representing operators. The first term is \mathcal{O}^\dagger followed by a dashed box containing two vertices connected by a line, followed by \mathcal{O} . The second term is \mathcal{O}^\dagger followed by a dashed box containing two vertices, then a circle labeled iB , then another dashed box with two vertices, and finally \mathcal{O} . The third term is \mathcal{O}^\dagger followed by a dashed box with two vertices, then a circle labeled iB , then another dashed box with two vertices, then a second circle labeled iB , then a third dashed box with two vertices, and finally \mathcal{O} . Ellipses indicate further terms in the series.

spatial loop momenta
are summed

$$\frac{1}{L^3} \sum_{\vec{k} \in (2\pi/L)\mathbb{Z}^3} \int \frac{dk^0}{2\pi}$$

$\Delta \equiv \text{diagram}$
fully dressed
propagator

The diagram shows a horizontal line with a central black dot, representing a fully dressed propagator.



if $E^* < 4m$ **then**

$$K_L = K_\infty + \mathcal{O}(e^{-mL})$$

$$\Delta_L = \Delta_\infty + \mathcal{O}(e^{-mL})$$

Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)

Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

Two identical particles: loops and cuts

$$C_L(P) = \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \dots$$

The equation shows a series of diagrams representing the expansion of $C_L(P)$. Each diagram consists of two external particles, \mathcal{O}^\dagger and \mathcal{O} , connected by a loop structure. The first diagram is a simple loop with two internal vertices. The second diagram includes a propagator iB between the two internal vertices. The third diagram includes two iB propagators. Blue arrows point from the first two diagrams to the corresponding terms in the expansion above.

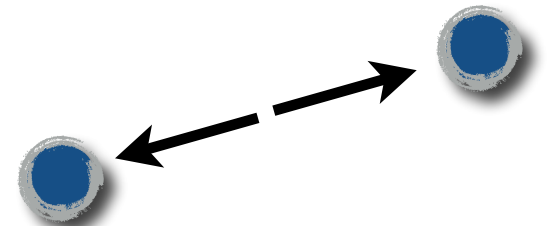
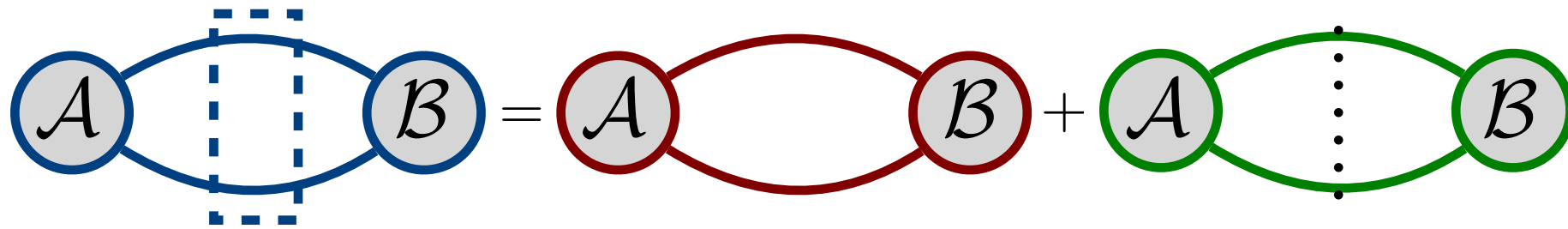
$$\frac{1}{L^3} \sum_{\vec{k}} \text{diagram}_1 = \int_{\vec{k}} \text{diagram}_2 + \underbrace{\text{diagram}_3}_F$$

The diagram on the left shows two grey particles connected by a blue loop with a dashed box around the internal vertices. The diagram on the right shows the same two grey particles connected by a red loop. The diagram on the far right shows the same two grey particles connected by a green loop with a vertical ellipsis between the internal vertices, labeled F . A blue bracket under F is labeled "contains all power-law corrections".

In  all four-momenta are projected on shell.

Physical, propagating states give dominate finite-volume effects.

Two-particle cut identity



$$\mathbf{k}^* = \hat{\mathbf{k}}^* \sqrt{E^{*2}/4 - m^2}$$

$$E^* = \sqrt{E^2 - \mathbf{P}^2}$$

$$C_L(P) = C_\infty(P) + \frac{1}{2} \left[\frac{1}{L^3} \sum_{\mathbf{k}} - \int_{\mathbf{k}} \right] \frac{iA_{\text{on}}(\hat{\mathbf{k}}^*) B_{\text{on}}^*(\hat{\mathbf{k}}^*)}{2\omega_{\mathbf{k}} 2\omega_{\mathbf{P}-\mathbf{k}} [E - \omega_{\mathbf{k}} - \omega_{\mathbf{P}-\mathbf{k}} + i\epsilon]} + \mathcal{O}(e^{-mL})$$

decompose in spherical harmonics...

$$= C_\infty(P) + A_{\ell'm'} \frac{1}{2} \left[\frac{1}{L^3} \sum_{\mathbf{k}} - \int_{\mathbf{k}} \right] \frac{i\mathcal{Y}_{\ell'm'}(\hat{\mathbf{k}}^*) \mathcal{Y}_{\ell m}^*(\hat{\mathbf{k}}^*)}{2\omega_{\mathbf{k}} 2\omega_{\mathbf{P}-\mathbf{k}} [E - \omega_{\mathbf{k}} - \omega_{\mathbf{P}-\mathbf{k}} + i\epsilon]} B_{\ell m}^* + \mathcal{O}(e^{-mL})$$

where...

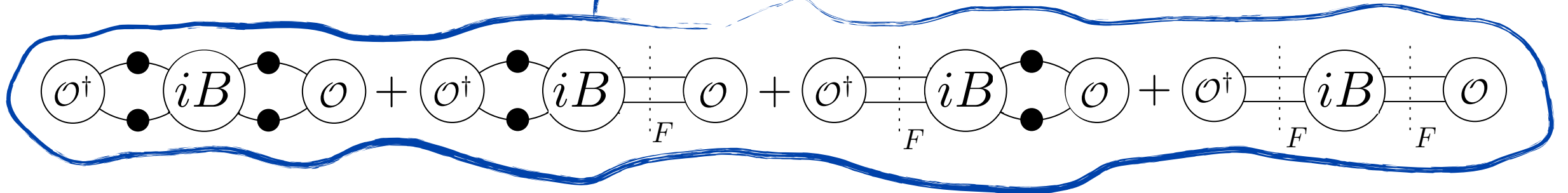
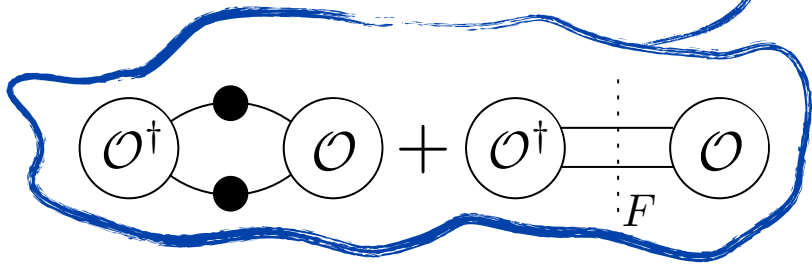
$$\mathcal{Y}_{\ell m}(\mathbf{k}^*) = \sqrt{4\pi} \left(\frac{k^*}{q^*} \right)^\ell Y_{\ell m}(\hat{\mathbf{k}}^*), \quad q^* = \sqrt{E^{*2}/4 - m^2}$$

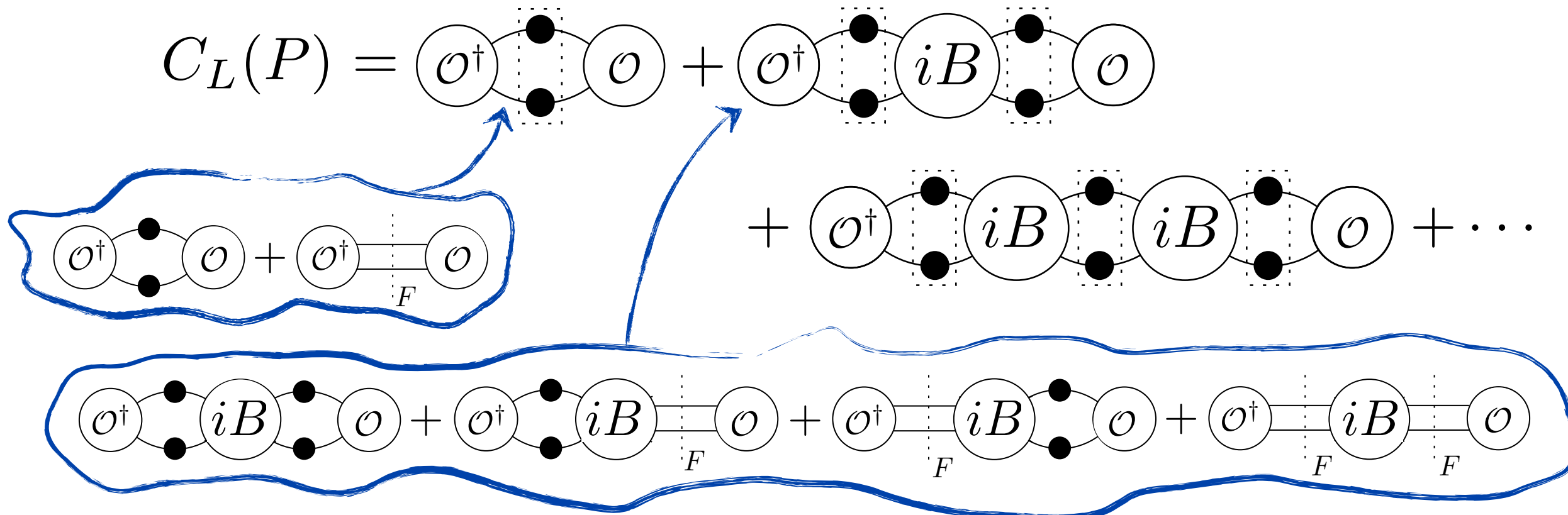
Finally we define a matrix...

$$C_L(P) = C_\infty(P) + A_{\ell'm'} iF_{\ell'm';\ell m} B_{\ell m}^* + \mathcal{O}(e^{-mL})$$

$$C_L(P) = \begin{array}{c} \circlearrowleft \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} \circlearrowright \mathcal{O} \\ + \circlearrowleft \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} \circlearrowright iB \begin{array}{c} \bullet \\ \bullet \end{array} \circlearrowright \mathcal{O} \end{array}$$

$$+ \begin{array}{c} \circlearrowleft \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} \circlearrowright iB \begin{array}{c} \bullet \\ \bullet \end{array} \circlearrowright iB \begin{array}{c} \bullet \\ \bullet \end{array} \circlearrowright \mathcal{O} \\ + \dots \end{array}$$

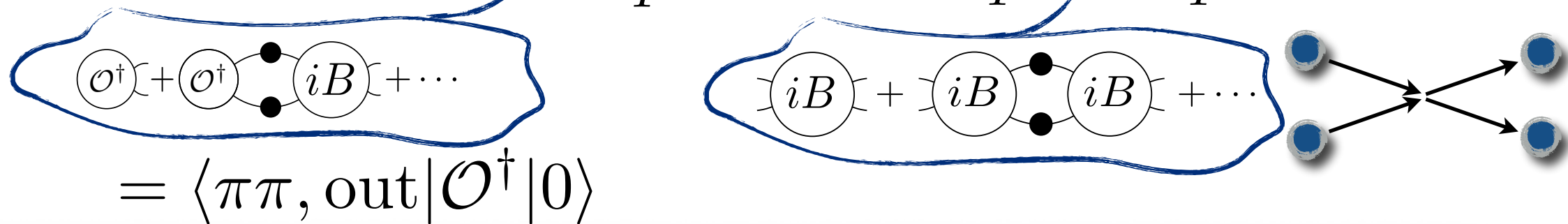




Now regroup by number of Fs

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + \text{diagram}_1 + \text{diagram}_2 + \dots$$

zero Fs
one F
two Fs



When we factorize diagrams and group infinite-volume parts...
physical observables emerge!

Two-particle review

$$C_L(P) = \text{diagram 1} + \text{diagram 2}$$

Diagram 1: \mathcal{O}^\dagger (circle) with two external lines (dots) and two internal lines (dots) connecting to a central vertex (circle with two dots), which then connects to \mathcal{O} (circle).

Diagram 2: \mathcal{O}^\dagger (circle) with two external lines (dots) and two internal lines (dots) connecting to a central vertex (circle with two dots), which then connects to iB (circle with two dots), which then connects to another central vertex (circle with two dots), which finally connects to \mathcal{O} (circle).

2

$$\text{diagram 1} + \text{diagram 2}$$

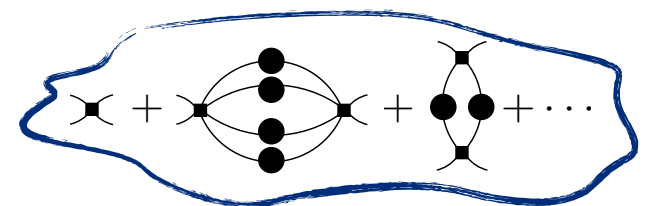
Diagram 1: \mathcal{O}^\dagger (circle) with two external lines (dots) and two internal lines (dots) connecting to \mathcal{O} (circle).

Diagram 2: \mathcal{O}^\dagger (circle) with two external lines (dots) and two internal lines (dots) connecting to \mathcal{O} (circle).

$$+ \text{diagram 3} + \text{diagram 4} + \dots$$

Diagram 3: \mathcal{O}^\dagger (circle) with two external lines (dots) and two internal lines (dots) connecting to iB (circle with two dots), which then connects to another iB (circle with two dots), which finally connects to \mathcal{O} (circle).

Diagram 4: \mathcal{O}^\dagger (circle) with two external lines (dots) and two internal lines (dots) connecting to iB (circle with two dots), which then connects to another iB (circle with two dots), which then connects to a third iB (circle with two dots), which finally connects to \mathcal{O} (circle).



$$C_L(P) = C_\infty(P)$$

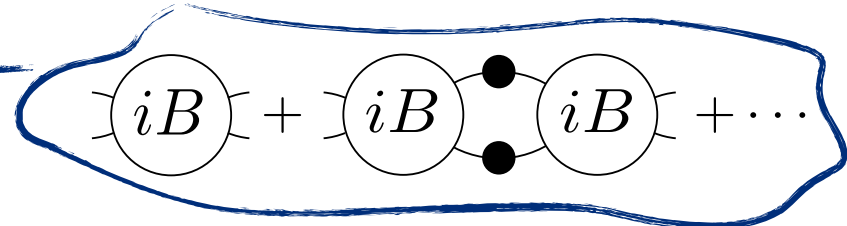
$$+ \text{diagram 5} + \text{diagram 6} + \dots$$

Diagram 5: A (circle) with a vertical dashed line labeled F connecting to A' (circle).

Diagram 6: A (circle) with a vertical dashed line labeled F connecting to $i\mathcal{M}$ (circle with two dots), which then connects to another vertical dashed line labeled F connecting to A' (circle).

Diagram 7: A (circle) with a vertical dashed line labeled F connecting to $i\mathcal{M}$ (circle with two dots), which then connects to another $i\mathcal{M}$ (circle with two dots), which then connects to a third vertical dashed line labeled F connecting to A' (circle).

$\langle \pi\pi, \text{out} | \mathcal{O}^\dagger | 0 \rangle$

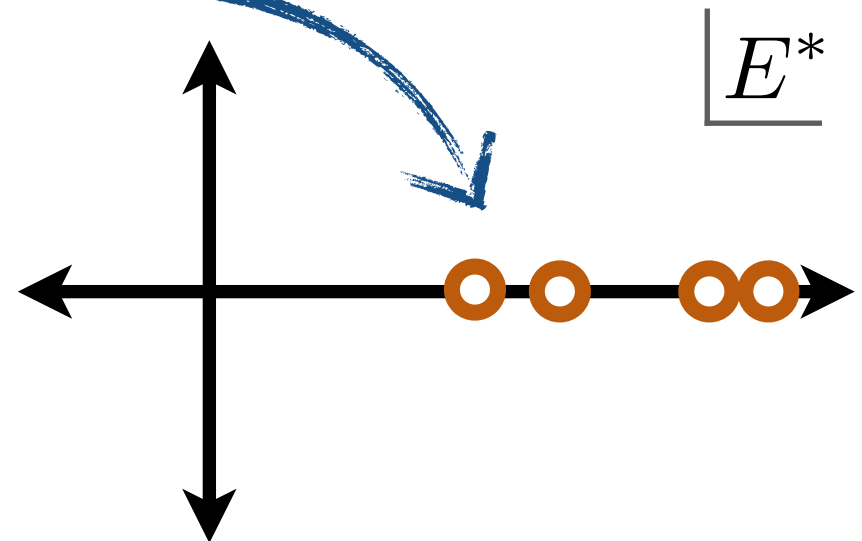


$\langle 0 | \mathcal{O} | \pi\pi, \text{in} \rangle$

We deduce...

$$C_L(P) = C_\infty(P) - A' F \frac{1}{1 + \mathcal{M}_{2 \rightarrow 2}} F A$$

The fraction $\frac{1}{1 + \mathcal{M}_{2 \rightarrow 2}}$ is enclosed in a blue hand-drawn box.



Two-particle result

At fixed (L, \vec{P}) , finite-volume energies are solutions to $\det[\mathcal{M}_{2 \rightarrow 2}^{-1} + F] = 0$

Rummukainen and Gottlieb, *Nucl. Phys.* B450, 397 (1995)
 Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

Matrices defined using angular-momentum states

$\mathcal{M}_{2 \rightarrow 2} \equiv$  diagonal matrix, parametrized by $\delta_\ell(E^*)$

$F \equiv$ non-diagonal matrix of known geometric functions

\equiv  difference of two-particle loops in finite and infinite volume depends on L, E, \mathbf{P}

At low energies, lowest partial waves dominate $\mathcal{M}_{2 \rightarrow 2}$

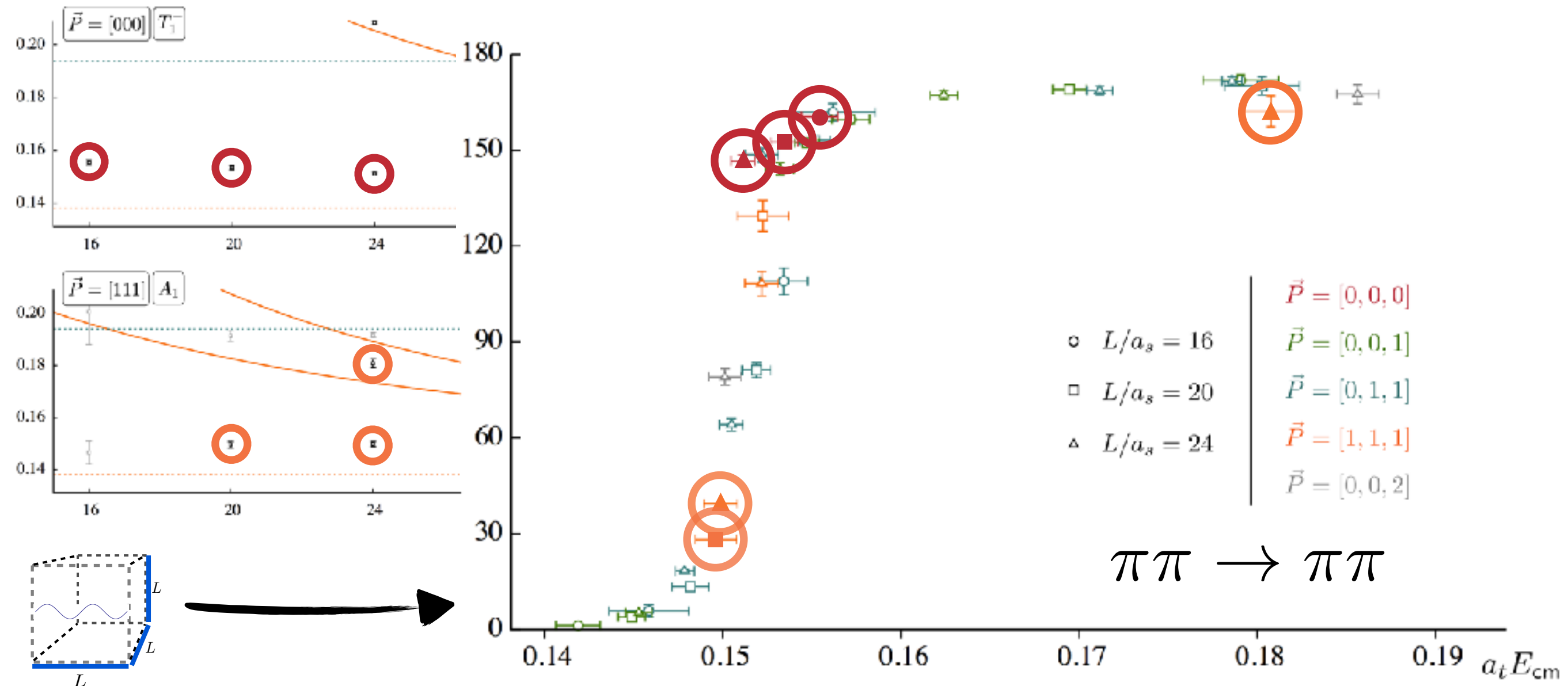
$$\cot \delta(E_n^*) + \cot \phi(E_n, \vec{P}, L) = 0$$

scattering phase

known function

E.g. p -wave $\pi\pi \rightarrow \pi\pi$ scattering, $I^G(J^{PC}) = 1^+(1^{--})$

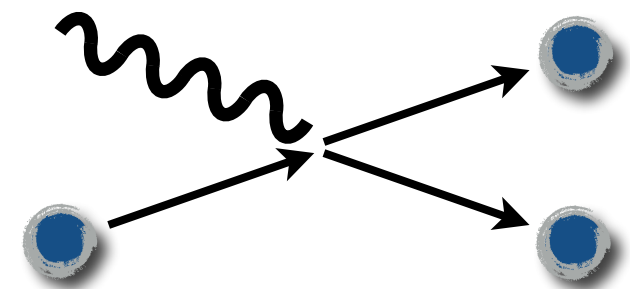
$$\cot \delta_{\ell=1}(E_n^*) + \cot \phi(E_n, \vec{P}, L) = 0$$



from Dudek, Edwards, Thomas in *Phys.Rev.* D87 (2013) 034505

Photoproduction

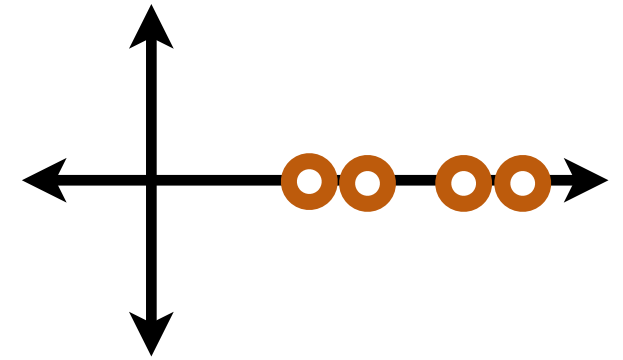
$$\langle \pi\pi, \text{out} | \mathcal{J}_\mu | \pi \rangle \equiv$$



How can we get this from finite-volume observables?

Why did we expect $C_L(P)$ to have poles?

$$C_L(P) \equiv \int_L d^4x e^{-iPx} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(0) | 0 \rangle$$



Insert a complete set finite-volume of states

$$C_L(P) \xrightarrow{E \rightarrow E_n} \frac{L^3 \langle 0 | \mathcal{O}(0) | n, \vec{P}, L \rangle \langle n, \vec{P}, L | \mathcal{O}^\dagger(0) | 0 \rangle}{E - E_n}$$

Now compare this to our factorized result

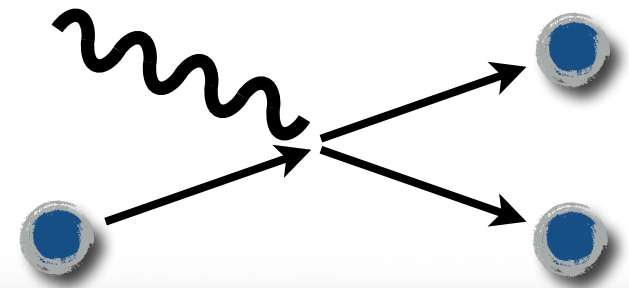
$$C_L(P) = C_\infty(P) - A' F \frac{1}{1 + \mathcal{M}_{2 \rightarrow 2} F} A$$

\mathcal{R} is the residue of this matrix

$$\xrightarrow{E \rightarrow E_n} \frac{\langle 0 | \mathcal{O}(0) | \pi\pi, \text{in} \rangle \mathcal{R}(E_n, \vec{P}, L) \langle \pi\pi, \text{out} | \mathcal{O}^\dagger(0) | 0 \rangle}{E - E_n}$$

Photoproduction

$$\langle \pi\pi, \text{out} | \mathcal{J}_\mu | \pi \rangle \equiv$$



get this from the lattice

$$2\omega_\pi L^6 |\langle n, L | \mathcal{J}_\mu | \pi \rangle|^2 =$$

**experimental
observable**

$$\langle \pi | \mathcal{J}_\mu | \pi\pi, \text{in} \rangle \mathcal{R}(E_n, L) \langle \pi\pi, \text{out} | \mathcal{J}_\mu | \pi \rangle$$

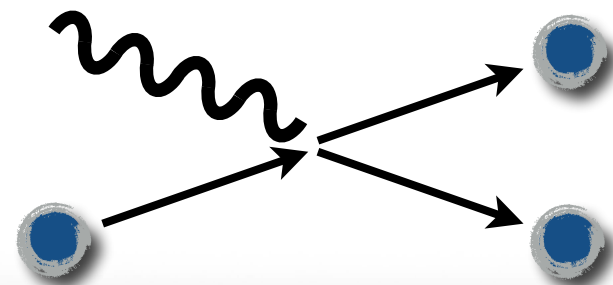
depends on scattering phase



Briceño, MTH, Walker-Loud (2015)

Photoproduction

$$\langle \pi\pi, \text{out} | \mathcal{J}_\mu | \pi \rangle \equiv$$



get this from the lattice

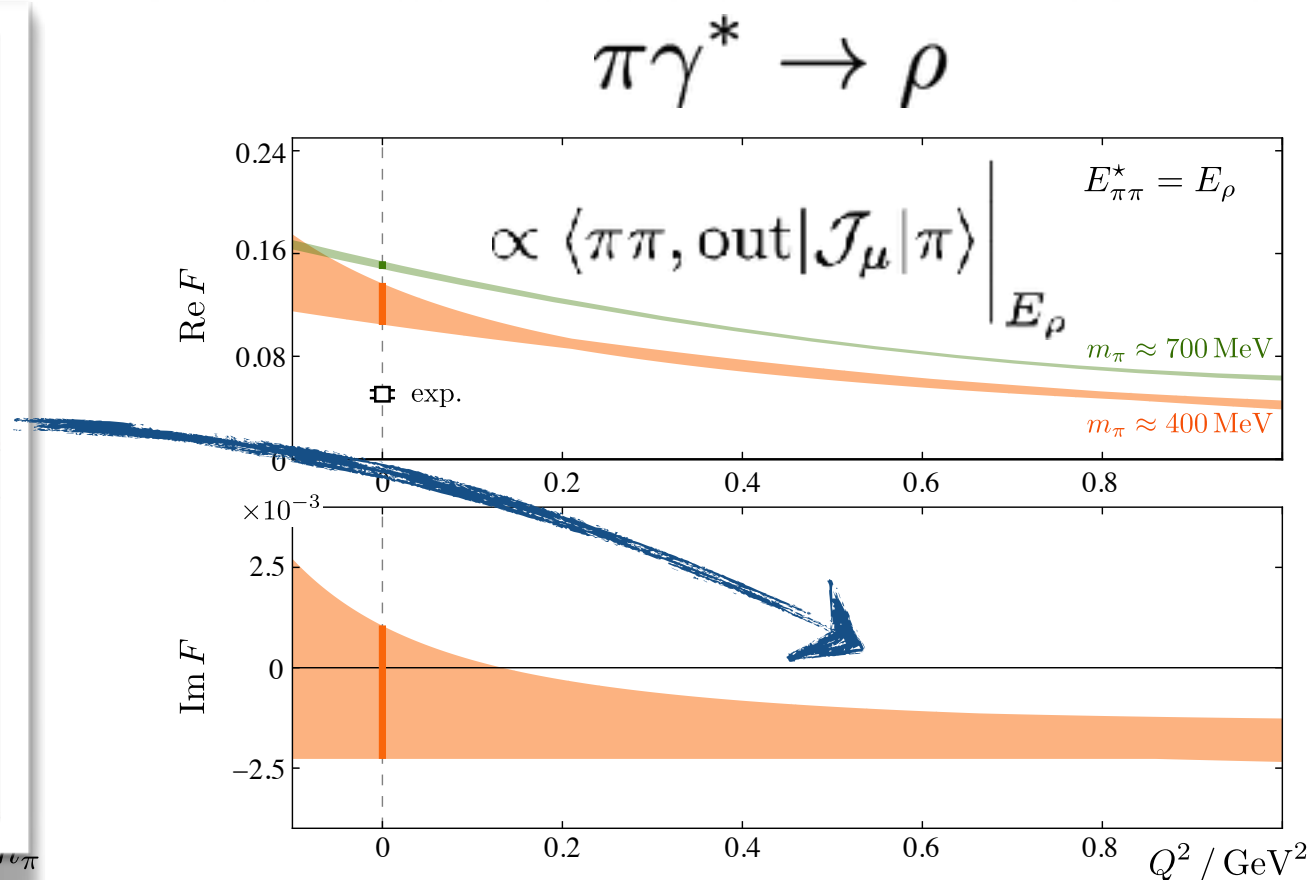
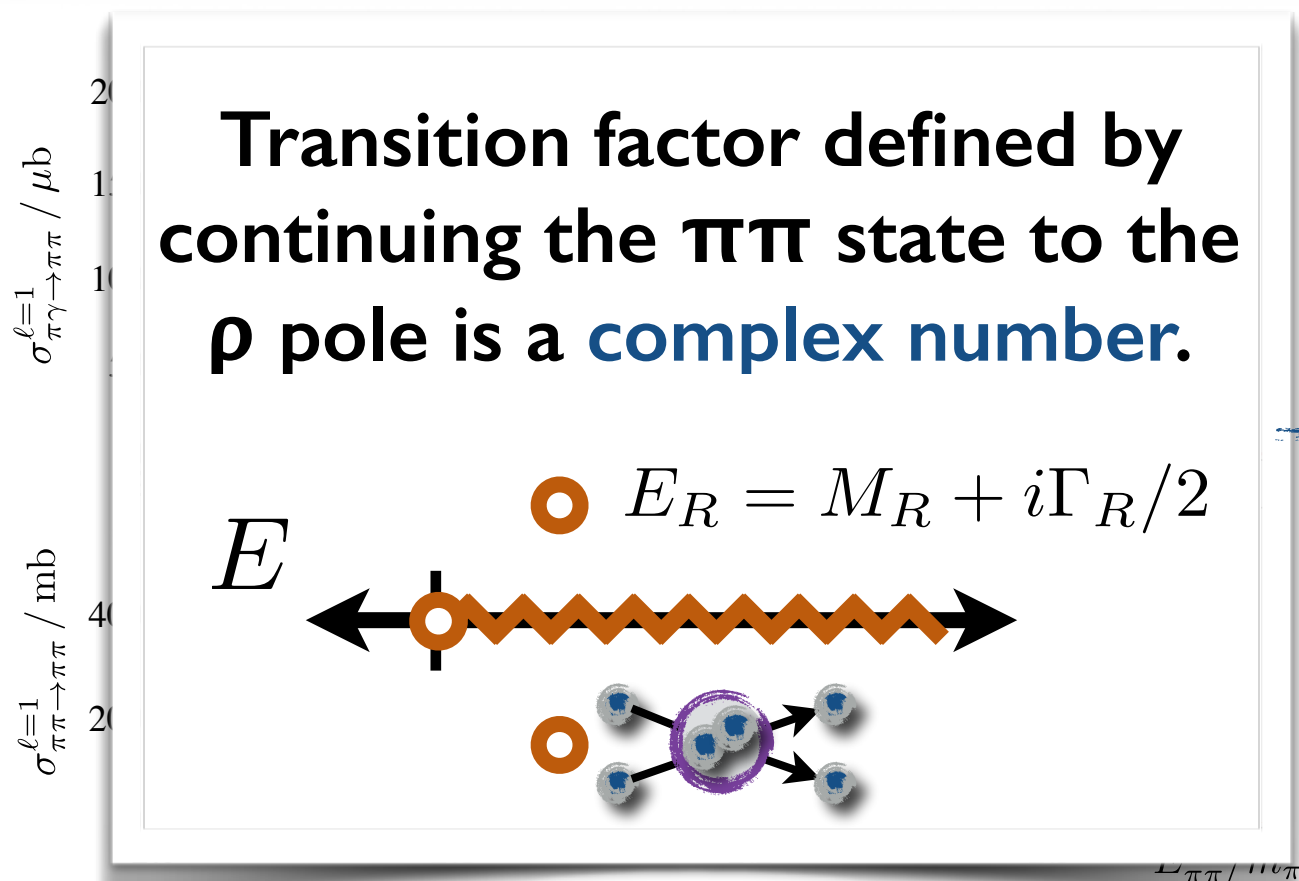
$$2\omega_\pi L^6 |\langle n, L | \mathcal{J}_\mu | \pi \rangle|^2 =$$

experimental observable

$$\langle \pi | \mathcal{J}_\mu | \pi\pi, \text{in} \rangle \mathcal{R}(E_n, L) \langle \pi\pi, \text{out} | \mathcal{J}_\mu | \pi \rangle$$

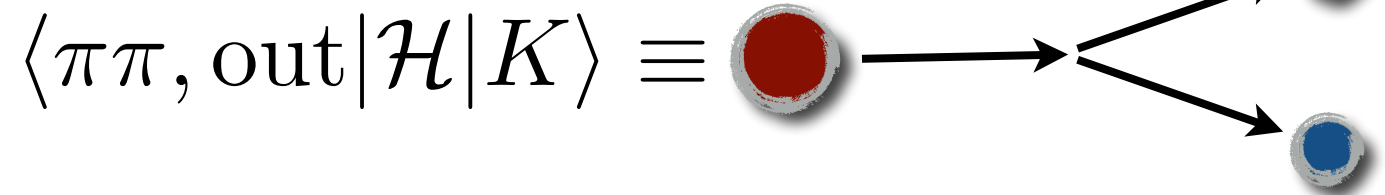
depends on scattering phase

Briceño, MTH, Walker-Loud (2015)



Same basic idea in many different contexts...

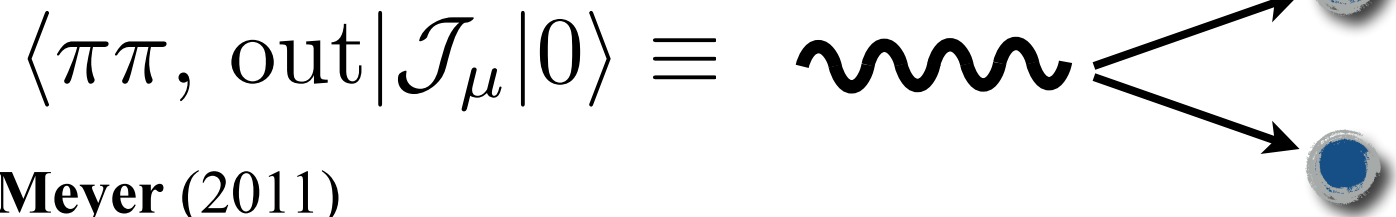
Kaon decay



Lellouch, Lüscher (2001) Kim, Sachrajda, Sharpe (2005) Christ, Kim, Yamazaki (2005)

Implementation by RBC/UKQCD collaboration

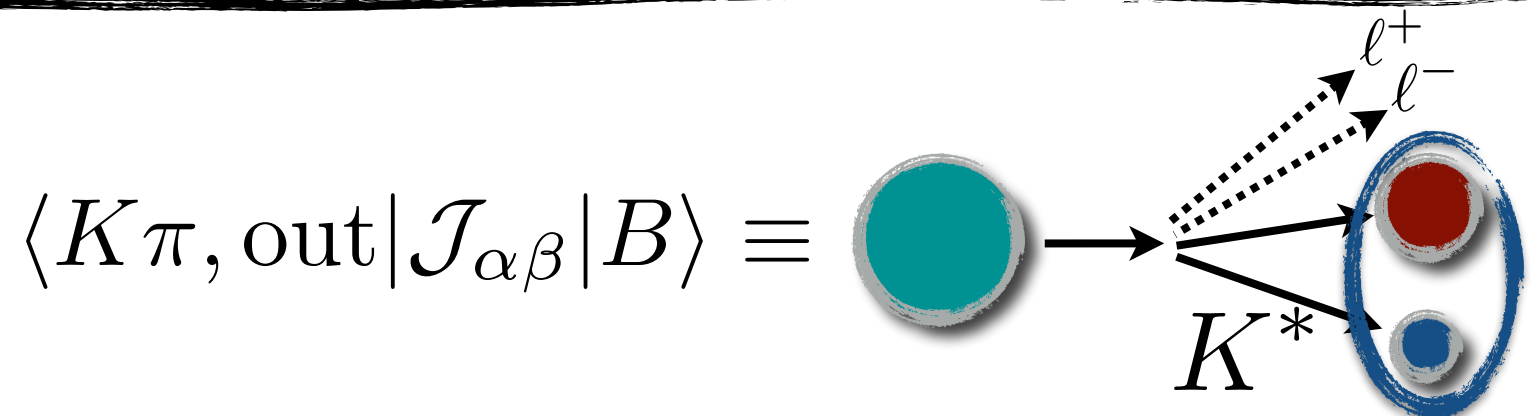
Time-like form factors



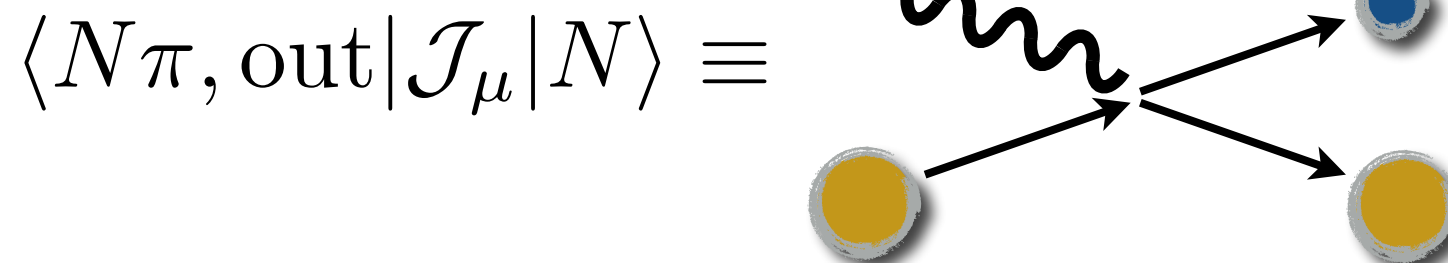
Meyer (2011)

Relevant for muon HVP contribution to muon g-2

Resonance transition amplitudes



Particles with spin



Agadjanov *et al.* (2014) Briceño, MTH, Walker-Loud (2015) Briceño, MTH (2016)



Outline

Warm up and definitions

- Basic set-up
 - Finite-volume correlator
 - Three non-interacting particles
-

Two particles in a box

- Alternative derivation
- Truncation and application
- Relating matrix elements

Three particles in a box

- 3-to-3 scattering
 - (Sketch of) derivation
 - An unexpected infinite-volume quantity
 - Relating energies to scattering
-

Testing the result

- Large-volume expansion
- Effimov state in a box

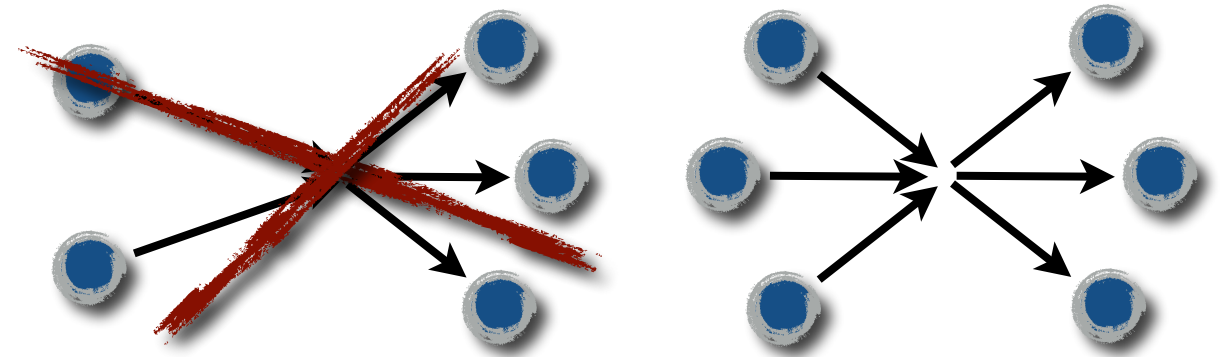
Numerical explorations

- Truncation at low energies
- 2-particle physics in 3-particle energies
- Toy 3-particle resonance
- Numerical Effimov state
- Unphysical solutions

Looking forward

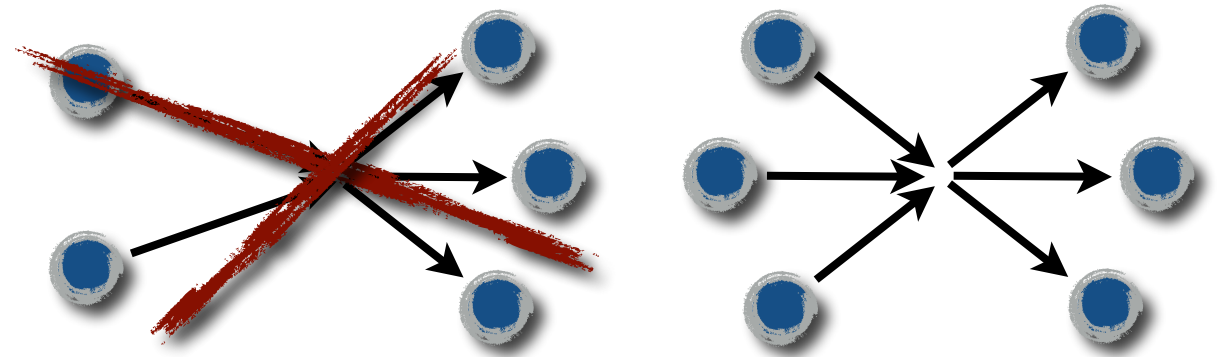
- 2-to-3 scattering and resonant subprocesses
- Speculations

We focus here on
identical scalar particles



For now we turn off two-to-three scattering using a symmetry

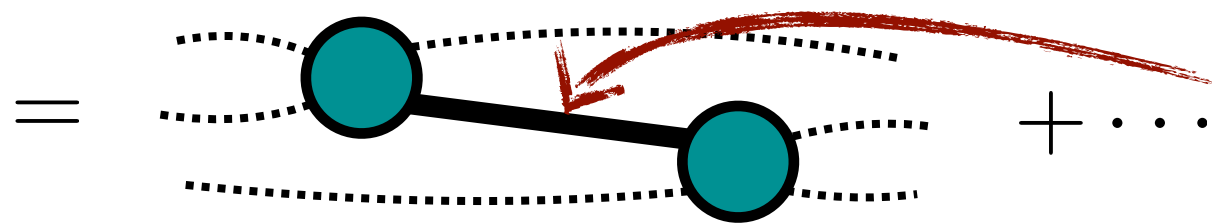
We focus here on identical scalar particles



For now we turn off two-to-three scattering using a symmetry

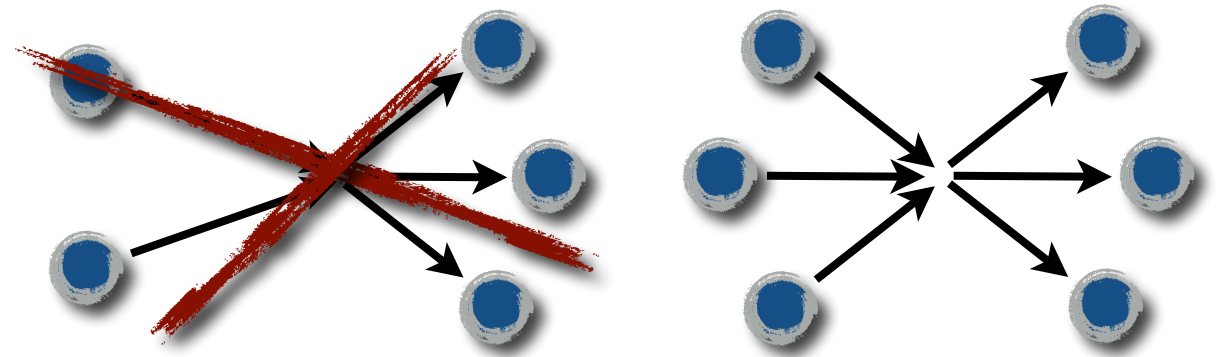
Three-to-three amplitude has kinematic singularities

$i\mathcal{M}_{3\rightarrow 3} \equiv$ fully connected correlator with six external legs amputated and projected on shell



Certain external momenta put this on-shell!

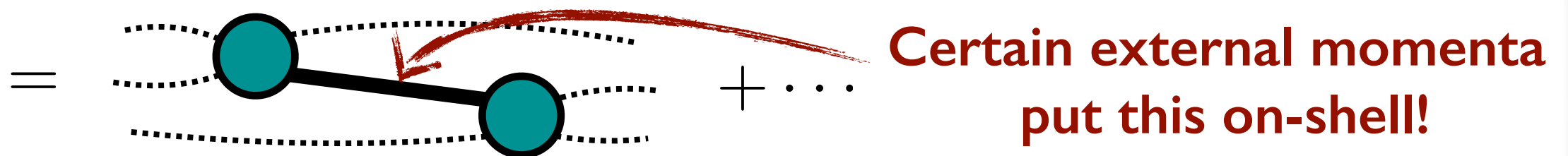
We focus here on identical scalar particles



For now we turn off two-to-three scattering using a symmetry

Three-to-three amplitude has kinematic singularities

$i\mathcal{M}_{3\rightarrow 3} \equiv$ fully connected correlator with six external legs amputated and projected on shell



Three-to-three amplitude has more degrees of freedom



2 degrees of freedom



8 degrees of freedom

How can we extract a singular, eight-coordinate function using finite-volume energies?

Spectrum depends on a modified quantity with singularities removed



PV pole prescription

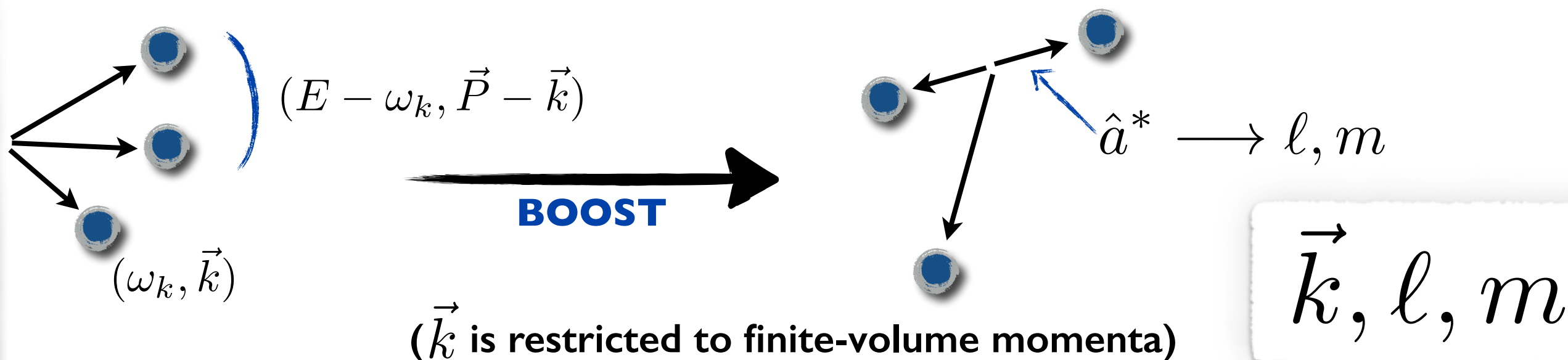
df stands for “divergence free”

Same degrees of freedom as \mathcal{M}_3

Smooth, real function (easier to extract)

Relation to \mathcal{M}_3 is known (depends only on on-shell \mathcal{M}_2)

Degrees of freedom encoded in an extended matrix space



Aside: Changing pole prescriptions for two particles

$$C_L(P) = \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \dots$$

Diagram 1: \mathcal{O}^\dagger (circle) connected to a pair of black dots (circles) which are connected to \mathcal{O} (circle). A dashed box encloses the pair of dots.

Diagram 2: \mathcal{O}^\dagger (circle) connected to a pair of black dots (circles) which are connected to iB (circle), which is then connected to another pair of black dots (circles) connected to \mathcal{O} (circle). Dashed boxes enclose both pairs of dots.

Diagram 3: A cloud containing a series of diagrams: \mathcal{O}^\dagger connected to a pair of dots, which connect to \mathcal{O} ; \mathcal{O}^\dagger connected to a pair of dots, which connect to a pair of dots, which connect to \mathcal{O} ; and so on.

2

$$\text{diagram 1} + \text{diagram 2} \quad \text{PV}$$

Diagram 1: \mathcal{O}^\dagger (circle) connected to a pair of black dots (circles) which are connected to \mathcal{O} (circle). A dashed box encloses the pair of dots. The label "PV" is written in purple below the diagram.

Diagram 2: \mathcal{O}^\dagger (circle) connected to a pair of black dots (circles) which are connected to \mathcal{O} (circle). A dashed box encloses the pair of dots. The label "PV" is written in purple below the diagram.

$$C_L(P) = C_\infty^{\text{PV}}(P)$$

3

$$\text{diagram 1} + \text{diagram 2} + \dots$$

Diagram 1: A_{PV} (circle) connected to A'_{PV} (circle) via a horizontal line labeled F_{PV} below it. A dashed box encloses the A_{PV} circle. The label "PV" is written in purple below the diagram.

Diagram 2: A_{PV} (circle) connected to A'_{PV} (circle) via a horizontal line labeled F_{PV} below it. In the middle of this line is a circle labeled $i\mathcal{K}$. A dashed box encloses the A_{PV} circle. The label "PV" is written in purple below the diagram.

Diagram 3: A cloud containing a series of diagrams: iB (circle); iB (circle) connected to iB (circle) via a horizontal line labeled F_{PV} below it; iB (circle) connected to iB (circle) via a horizontal line labeled F_{PV} below it, with a pair of black dots (circles) in the middle; and so on.

Diagram 4: A_{PV} (circle) connected to A'_{PV} (circle) via a horizontal line labeled F_{PV} below it. In the middle of this line are two circles labeled $i\mathcal{K}$. A dashed box encloses the A_{PV} circle. The label "PV" is written in purple below the diagram.

Diagram 5: A_{PV} (circle) connected to A'_{PV} (circle) via a horizontal line labeled F_{PV} below it. In the middle of this line are three circles labeled $i\mathcal{K}$. A dashed box encloses the A_{PV} circle. The label "PV" is written in purple below the diagram.

Diagram 6: A cloud containing a series of diagrams: iB (circle); iB (circle) connected to iB (circle) via a horizontal line labeled F_{PV} below it; iB (circle) connected to iB (circle) via a horizontal line labeled F_{PV} below it, with a pair of black dots (circles) in the middle; and so on.

Diagram 7: A_{PV} (circle) connected to A'_{PV} (circle) via a horizontal line labeled F_{PV} below it. In the middle of this line are four circles labeled $i\mathcal{K}$. A dashed box encloses the A_{PV} circle. The label "PV" is written in purple below the diagram.

Diagram 8: A_{PV} (circle) connected to A'_{PV} (circle) via a horizontal line labeled F_{PV} below it. In the middle of this line are five circles labeled $i\mathcal{K}$. A dashed box encloses the A_{PV} circle. The label "PV" is written in purple below the diagram.

$$C_L(P) = C_\infty^{\text{PV}}(P) - A'_{\text{PV}} \left[F_{\text{PV}} \frac{1}{1 + \mathcal{K}_2 F_{\text{PV}}} \right] A_{\text{PV}}$$

poles are in here

Diagram: A complex plane with a horizontal axis labeled E^* and a vertical axis. Three orange circles representing poles are located on the positive real axis. A blue arrow points from the diagram to the term in brackets in the equation above.

Back to three: new skeleton expansion

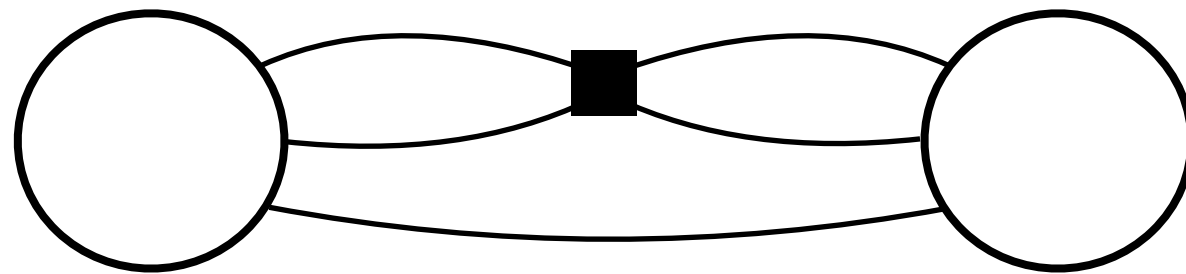
Recall for two particles we started with a “skeleton expansion”

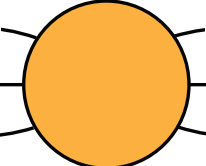
$$C_L(P) = \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \dots$$

So now we need the same for three...

$$C_L(E, \vec{P}) \stackrel{?}{=} \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \dots$$

No!... We must also accommodate diagrams like



Disconnected diagrams in  lead to singularities that invalidate the derivation

Back to three: new skeleton expansion

$$C_L(E, \vec{P}) = \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

$$+ \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots$$

The diagrams in the expansion are:

- Diagram 1: Two white circles connected by two arcs, enclosed in a dashed box.
- Diagram 2: A white circle, a dashed box containing two arcs, an orange circle, another dashed box with two arcs, and a final white circle.
- Diagram 3: A white circle, a dashed box with two arcs, an orange circle, a dashed box with two arcs, another orange circle, a dashed box with two arcs, and a final white circle.
- Diagram 4: A white circle, a dashed box containing a purple circle and two arcs, and a final white circle.
- Diagram 5: A white circle, a dashed box containing two purple circles and two arcs, and a final white circle.
- Diagram 6: A white circle, a dashed box containing three purple circles and two arcs, and a final white circle.

Kernel definitions:

$$\text{Purple circle} \equiv \text{Diagram A} + \text{Diagram B} + \text{Diagram C} + \dots$$

The diagrams in the kernel definition for the purple circle are:

- Diagram A: A vertex with four external lines.
- Diagram B: A vertex with four external lines and two internal arcs.
- Diagram C: A vertex with four external lines and two internal arcs forming a lens shape.

$$\text{Orange circle} \equiv \text{Diagram D} + \text{Diagram E} + \text{Diagram F} + \dots$$

The diagrams in the kernel definition for the orange circle are:

- Diagram D: A vertex with four external lines.
- Diagram E: A vertex with four external lines and a single internal line.
- Diagram F: A vertex with four external lines and two internal arcs.

Back to three: new skeleton expansion

$$\begin{aligned}
 C_L(E, \vec{P}) = & \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots \\
 & + \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots \\
 & + \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \dots \\
 & + \text{Diagram 10} + \text{Diagram 11} + \dots
 \end{aligned}$$

The diagrams in the expansion are:

- Diagram 1: Two white circles connected by two arcs, enclosed in a dashed box.
- Diagram 2: Two white circles connected by two arcs, with an orange circle in the middle, enclosed in a dashed box.
- Diagram 3: Two white circles connected by two arcs, with two orange circles in the middle, enclosed in a dashed box.
- Diagram 4: Two white circles connected by two arcs, with one purple circle in the middle, enclosed in a dashed box.
- Diagram 5: Two white circles connected by two arcs, with two purple circles in the middle, enclosed in a dashed box.
- Diagram 6: Two white circles connected by two arcs, with three purple circles in the middle, enclosed in a dashed box.
- Diagram 7: Two white circles connected by two arcs, with two purple circles in the middle, enclosed in a dashed box.
- Diagram 8: Two white circles connected by two arcs, with three purple circles in the middle, enclosed in a dashed box.
- Diagram 9: Two white circles connected by two arcs, with four purple circles in the middle, enclosed in a dashed box.
- Diagram 10: Two white circles connected by two arcs, with three purple circles in the middle, enclosed in a dashed box.
- Diagram 11: Two white circles connected by two arcs, with four purple circles in the middle, enclosed in a dashed box.

Kernel definitions:

$$\text{Purple circle} \equiv \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

The diagrams in the kernel definition for the purple circle are:

- Diagram 1: A purple circle with four external lines.
- Diagram 2: A diagram with two vertices connected by two arcs, each vertex having two external lines.
- Diagram 3: A diagram with two vertices connected by two arcs, each vertex having two external lines, with a loop on each vertex.

$$\text{Orange circle} \equiv \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

The diagrams in the kernel definition for the orange circle are:

- Diagram 1: An orange circle with four external lines.
- Diagram 2: A diagram with two vertices connected by a straight line, each vertex having two external lines.
- Diagram 3: A diagram with two vertices connected by two arcs, each vertex having two external lines.

New skeleton expansion

$$\begin{aligned}
 C_L(E, \vec{P}) = & \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots \\
 & + \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots \\
 & + \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \dots \\
 & + \text{Diagram 10} + \text{Diagram 11} + \dots \\
 & + \dots \\
 & + \text{Diagram 12} + \text{Diagram 13} + \dots
 \end{aligned}$$

The diagrams in the expansion are as follows:

- Diagram 1: Two white circles connected by two lines, with a dashed box around the lines.
- Diagram 2: Two white circles connected by two lines, with an orange circle in the middle. Dashed boxes are around the lines and the orange circle.
- Diagram 3: Two white circles connected by two lines, with two orange circles in the middle. Dashed boxes are around the lines and each orange circle.
- Diagram 4: Two white circles connected by two lines, with one purple circle in the middle. Dashed boxes are around the lines and the purple circle.
- Diagram 5: Two white circles connected by two lines, with two purple circles in the middle. Dashed boxes are around the lines and each purple circle.
- Diagram 6: Two white circles connected by two lines, with three purple circles in the middle. Dashed boxes are around the lines and each purple circle.
- Diagram 7: Two white circles connected by two lines, with two purple circles in the middle. Dashed boxes are around the lines and each purple circle.
- Diagram 8: Two white circles connected by two lines, with three purple circles in the middle. Dashed boxes are around the lines and each purple circle.
- Diagram 9: Two white circles connected by two lines, with four purple circles in the middle. Dashed boxes are around the lines and each purple circle.
- Diagram 10: Two white circles connected by two lines, with three purple circles in the middle. Dashed boxes are around the lines and each purple circle.
- Diagram 11: Two white circles connected by two lines, with four purple circles in the middle. Dashed boxes are around the lines and each purple circle.
- Diagram 12: Two white circles connected by two lines, with one purple circle and one orange circle in the middle. Dashed boxes are around the lines, the purple circle, and the orange circle.
- Diagram 13: Two white circles connected by two lines, with one orange circle and one purple circle in the middle. Dashed boxes are around the lines, the orange circle, and the purple circle.

Kernel definitions:

$$\begin{aligned}
 \text{Purple circle} & \equiv \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots \\
 \text{Orange circle} & \equiv \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots
 \end{aligned}$$

The diagrams in the kernel definitions are as follows:

- Diagram 1 (Purple): A purple circle with four external lines.
- Diagram 2 (Purple): A loop diagram with two external lines.
- Diagram 3 (Purple): A tadpole diagram with two external lines.
- Diagram 4 (Orange): A diagram with two external lines and a central orange circle.
- Diagram 5 (Orange): A diagram with two external lines and a central orange circle connected to a purple circle.
- Diagram 6 (Orange): A loop diagram with two external lines and a central orange circle.

- All lines are fully dressed propagators
- Boxes represent sums over finite-volume momenta
- Kernels may contain fixed poles

Basic approach

1. Work out the three particle skeleton expansion

$$C_L(E, \vec{P}) = \begin{array}{l} \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots \\ \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots \\ \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \dots \end{array}$$

2. Break diagrams into finite- and infinite-volume parts

3. Organize and sum terms to identify
infinite-volume observables

Result

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) - A' F_3 \frac{1}{1 + \mathcal{K}_{\text{df},3} F_3} A$$

- Looks similar to the two-particle case
- All quantities defined with PV-pole prescription
- F_3 depends on finite-volume and two-to-two scattering

Quantization condition

At fixed (L, \vec{P}) , finite-volume energies are solutions to

$$\det_{k,\ell,m} \left[\mathcal{K}_{\text{df},3}^{-1} + F_3 \right] = 0$$

$F_3 \equiv$ matrix that depends on **geometric functions** and $\mathcal{M}_{2 \rightarrow 2}$.
MTH and Sharpe (2014)

All of the complication is buried inside F_3

$$F_3 = \frac{F}{6\omega L^3} - \frac{F}{2\omega L^3} \frac{1}{1 + \mathcal{M}_{2,L} G} \mathcal{M}_{2,L} F$$

These are all matrices with indices

momentum of
one particle

$$\vec{k} = \frac{2\pi\vec{n}}{L}$$



angular momentum
of the other two

$$\ell, m$$

F and **G** are geometric functions

$\mathcal{M}_{2,L}$ depends on **F** and \mathcal{M}_2

More derivation details

$$\text{[Diagram with red square]} = \text{[Diagram]} + \text{[Diagram]} \left[\text{[Diagram]} + \text{[Diagram]} \right] \text{[Diagram]} + \text{[Diagram]} \left[\dots \right] \text{[Diagram]} \left[\dots \right] \text{[Diagram]} + \dots$$

$$= \left\{ \text{[Diagram]} + \text{[Diagram]} \text{[Diagram]} + \dots \right\} + \left\{ \dots \right\} \text{[Diagram]} \left\{ \dots \right\}$$

$$= i\mathcal{M} \sum_{n=0}^{\infty} [iF_i\mathcal{M}]^n = i\mathcal{M} \frac{1}{1 - iF_i\mathcal{M}}$$

$$C_L(E, \vec{P}) = \text{[Diagram]} + \text{[Diagram]} + \text{[Diagram]} + \dots$$

$$= A'_{k'k} \frac{iF_{k'k}}{6\omega_L^3} A_{k\alpha} + A' \frac{iF}{2\omega_L^3} i\mathcal{M}_L iFA + A' \frac{iF}{2\omega_L^3} i\mathcal{M}_L iG i\mathcal{M}_L iFA + \text{[Diagram]} + \dots$$

$$= A' \frac{iF}{6\omega_L^3} A + A' \frac{iF}{2\omega_L^3} i\mathcal{M}_L \sum_{n=0}^{\infty} [iG i\mathcal{M}_L]^n iFA + \dots$$

$$= A' \underbrace{\left[\frac{iF}{2\omega_L^3} \left(\frac{1}{3} + i\mathcal{M}_L \frac{1}{1 - iG i\mathcal{M}_L} iF \right) \right]}_{iF_3} A + A' iF_3 \sum_{n=0}^{\infty} [iK_{\alpha\beta} iF_3]^n A + C_{\infty}(E, \vec{P})$$

$$= C_{\infty}(E, \vec{P}) + A' iF_3(E, \vec{P}, L) \frac{1}{1 - iK_{\alpha\beta}(E^*) iF_3(E, \vec{P}, L)} A$$

- So for fixed $\{\vec{P}, L\}$, the F.v. energies are solns. to...

$$\det [K_{\alpha\beta}(E^*)^{-1} + F_3(E^*, \vec{P}, L)] = 0$$

Quantization condition

At fixed (L, \vec{P}) , finite-volume energies are solutions to

$$\det_{k,\ell,m} \left[\mathcal{K}_{\text{df},3}^{-1} + F_3 \right] = 0$$

$F_3 \equiv$ matrix that depends on geometric functions and $\mathcal{M}_{2 \rightarrow 2}$.

MTH and Sharpe (2014)

(1). Use two-particle q.c. to constrain \mathcal{M}_2 and determine $F_3(E, \vec{P}, L)$.

$$\det[\mathcal{M}_2^{-1} + F_2] = 0 \longrightarrow \mathcal{M}_2 \longrightarrow F_3(E, \vec{P}, L)$$

(2). Use decomposition + parametrization to express $\mathcal{K}_{\text{df},3}(E^*)$ in terms of α_i .

$$\mathcal{K}_{\text{df},3}(E^*, \Omega'_3, \Omega_3) \approx \mathcal{K}_{\text{df},3}[\alpha_1, \dots, \alpha_N] \longleftarrow \text{Recall, this is a real, smooth function}$$

(3). Use three-particle q.c. with finite-volume energies to determine $\mathcal{K}_{\text{df},3}(E^*)$.

$$\det[\mathcal{K}_{\text{df},3}^{-1} + F_3] = 0 \longrightarrow \mathcal{K}_{\text{df},3}(E^*) \checkmark$$

Relating $\mathcal{K}_{\text{df},3}$ to \mathcal{M}_3

First we modify $C_L(E, \vec{P})$ to define $i\mathcal{M}_{L,3}$

$$\begin{aligned} C_L(E, \vec{P}) = & \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots \\ & + \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots \\ & + \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \dots \\ & + \text{Diagram 10} + \text{Diagram 11} + \dots \\ & + \dots \\ & + \text{Diagram 12} + \text{Diagram 13} + \dots \end{aligned}$$

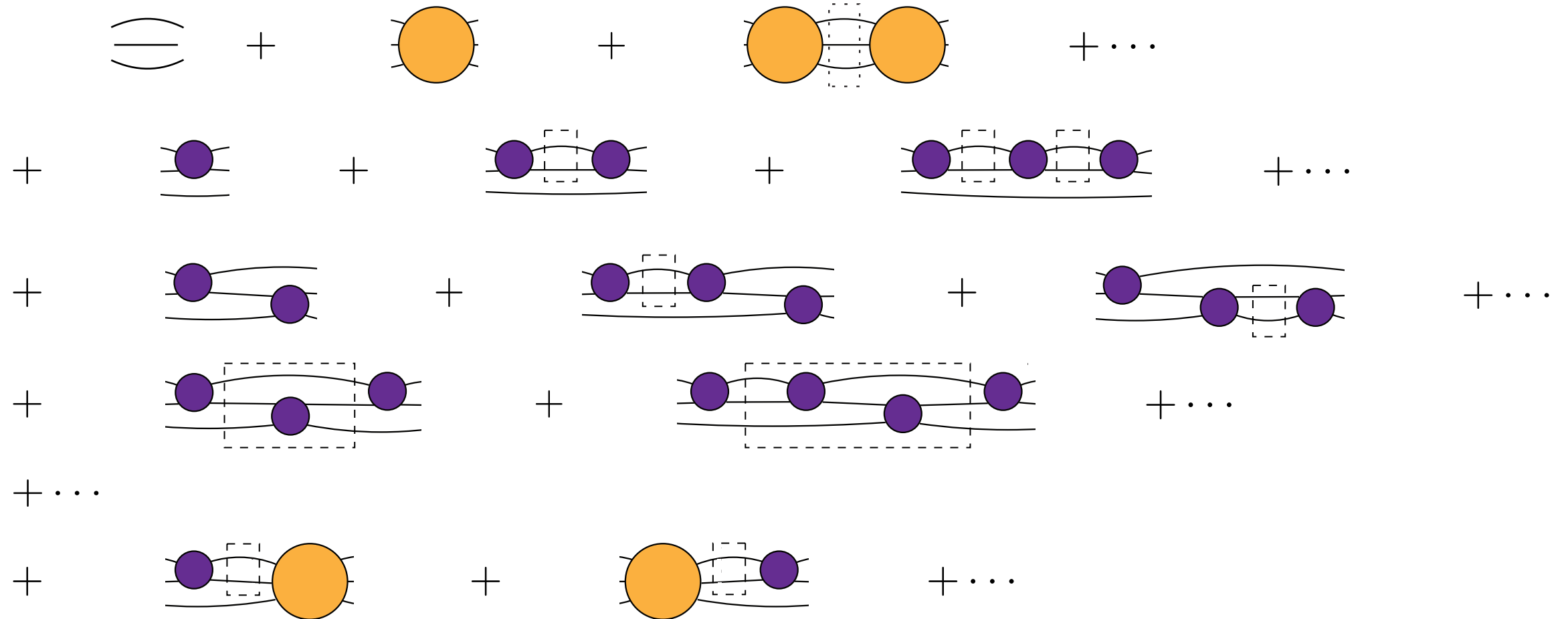
The diagrams in the equation represent terms in a series expansion of $C_L(E, \vec{P})$. Each diagram consists of two external white circles connected by two lines. The internal structure is as follows:

- Diagram 1:** Two white circles connected by two lines.
- Diagram 2:** Two white circles connected by two lines, with a single orange circle in the middle.
- Diagram 3:** Two white circles connected by two lines, with two orange circles in the middle.
- Diagram 4:** Two white circles connected by two lines, with a single purple circle in the middle.
- Diagram 5:** Two white circles connected by two lines, with two purple circles in the middle.
- Diagram 6:** Two white circles connected by two lines, with three purple circles in the middle.
- Diagram 7:** Two white circles connected by two lines, with two purple circles in the middle, and a dashed box around the first purple circle.
- Diagram 8:** Two white circles connected by two lines, with three purple circles in the middle, and a dashed box around the first two purple circles.
- Diagram 9:** Two white circles connected by two lines, with four purple circles in the middle, and a dashed box around the first three purple circles.
- Diagram 10:** Two white circles connected by two lines, with three purple circles in the middle, and a dashed box around the first two purple circles.
- Diagram 11:** Two white circles connected by two lines, with four purple circles in the middle, and a dashed box around the first three purple circles.
- Diagram 12:** Two white circles connected by two lines, with a purple circle in the middle, followed by an orange circle, and then another white circle.
- Diagram 13:** Two white circles connected by two lines, with an orange circle in the middle, followed by a purple circle, and then another white circle.

Relating $\mathcal{K}_{\text{df},3}$ to \mathcal{M}_3

First we modify $C_L(E, \vec{P})$ to define $i\mathcal{M}_{L,3}$

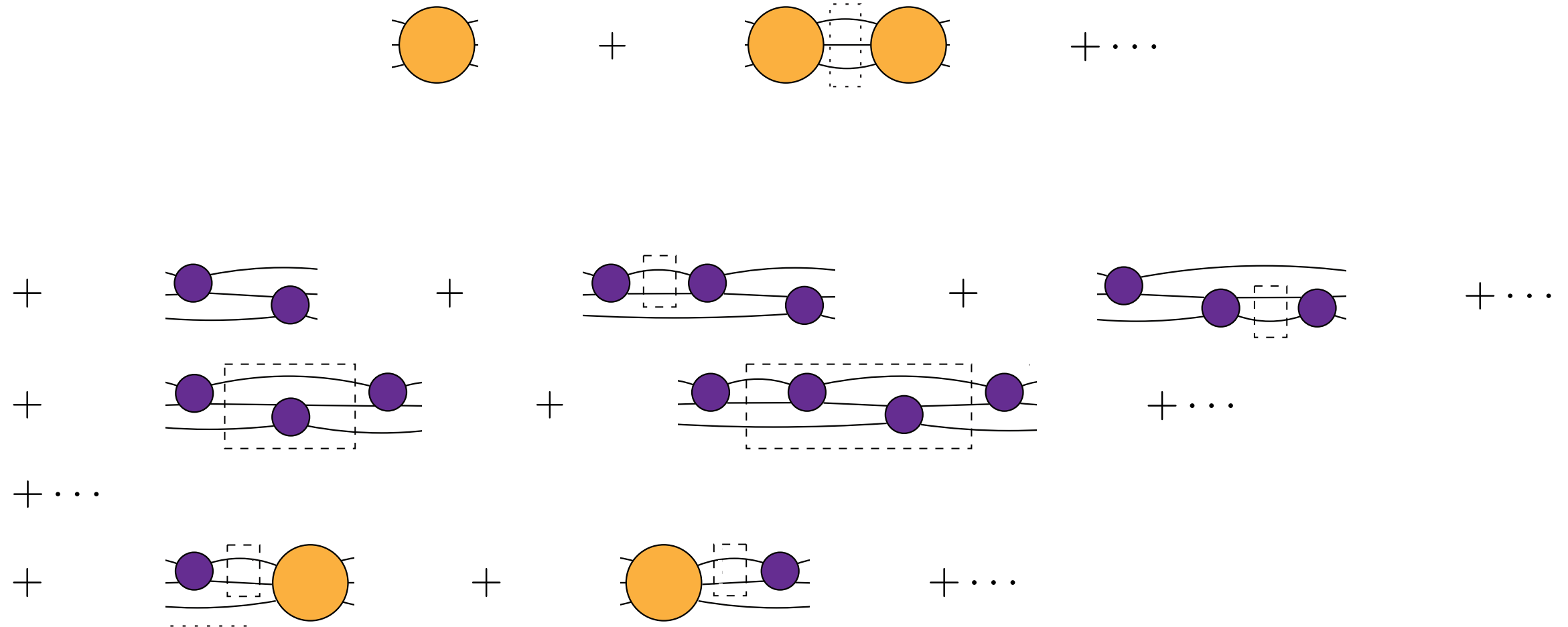
1. Amputate interpolating fields



Relating $\mathcal{K}_{\text{df},3}$ to \mathcal{M}_3

First we modify $C_L(E, \vec{P})$ to define $i\mathcal{M}_{L,3}$

1. Amputate interpolating fields
2. Drop disconnected diagrams



Relating $\mathcal{K}_{\text{df},3}$ to \mathcal{M}_3

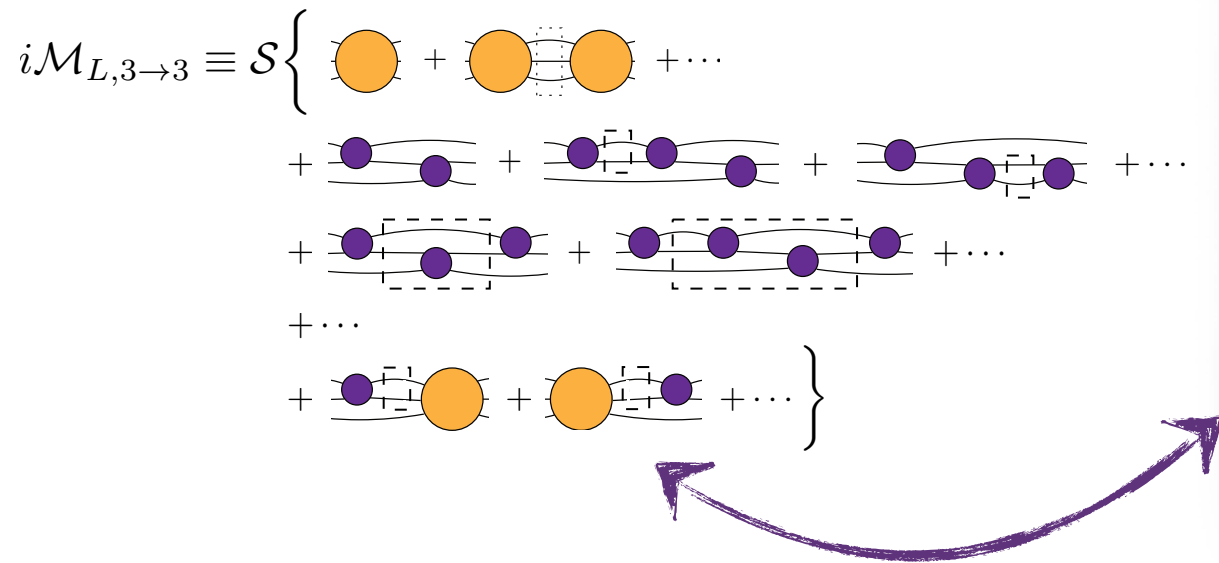
First we modify $C_L(E, \vec{P})$ to define $i\mathcal{M}_{L,3}$

1. Amputate interpolating fields
2. Drop disconnected diagrams
3. Symmetrize

$$i\mathcal{M}_{L,3 \rightarrow 3} \equiv \mathcal{S} \left\{ \begin{array}{l} \text{Orange circle} + \text{Orange circle} \text{---} \text{Orange circle} + \dots \\ + \text{Purple circle} \text{---} \text{Purple circle} + \text{Purple circle} \text{---} \text{Purple circle} \text{---} \text{Purple circle} + \dots \\ + \text{Purple circle} \text{---} \text{Purple circle} \text{---} \text{Purple circle} + \text{Purple circle} \text{---} \text{Purple circle} \text{---} \text{Purple circle} \text{---} \text{Purple circle} + \dots \\ + \dots \\ + \text{Purple circle} \text{---} \text{Orange circle} + \text{Orange circle} \text{---} \text{Purple circle} + \dots \end{array} \right\}$$

Relating $\mathcal{K}_{\text{df},3}$ to \mathcal{M}_3

Combined with our earlier analysis
this gives a matrix equation



$$\mathcal{M}_{L,3} = \mathcal{S} \left[\mathcal{D}_L + \mathcal{L}_L \frac{1}{\mathcal{K}_{\text{df},3}^{-1} + F_3} \mathcal{R}_L \right]$$

$$\mathcal{L}_L = \mathcal{X} F_3, \quad \mathcal{R}_L = F_3 \mathcal{X},$$

$$\mathcal{D}_L = -\mathcal{X} [F_3 - F_3|_{G \rightarrow 0}] \mathcal{X}$$

with the “amputation matrix” $\mathcal{X} = \left(\frac{F}{2\omega L^3} \right)^{-1}$

With this analytic relation in hand we can...

(a) Set $E \rightarrow E + i\epsilon$, (b) Send $L \rightarrow \infty$, (c) Send $\epsilon \rightarrow 0^+$.

Leads to an integral equation for the scattering amplitude

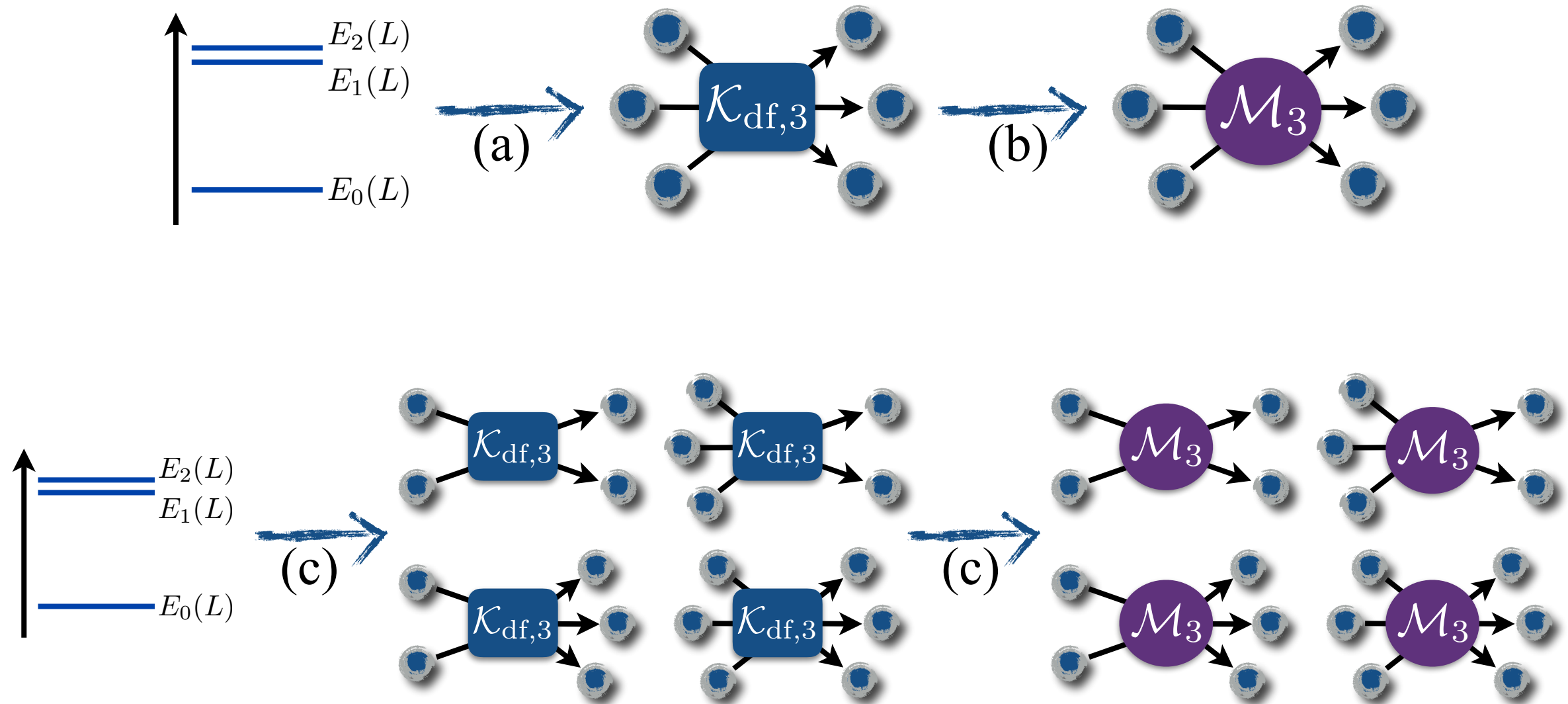
$$\mathcal{M}_3(E^*) = \mathcal{I} [\mathcal{K}_{\text{df},3}(E^*), \mathcal{M}_2]$$

Fixed total energy, manifestly convergent, on-shell only, no reference to EFT,
takes care of unitarity and singularities, useful independent of finite-volume physics?

MTH and Sharpe (2015)

Current status

Model- & EFT-independent relation between finite-volume energies and relativistic two-and-three particle scattering



(a),(b) *MTH and Sharpe (2015),(2016)*

(c) *Briceño, MTH, Sharpe (2017)*



Outline

Warm up and definitions

- Basic set-up
 - Finite-volume correlator
 - Three non-interacting particles
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Two particles in a box

- Alternative derivation
- Truncation and application
- Relating matrix elements

Three particles in a box

- 3-to-3 scattering
 - (Sketch of) derivation
 - An unexpected infinite-volume quantity
 - Relating energies to scattering
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Testing the result

- Large-volume expansion
- Effimov state in a box

Numerical explorations

- Truncation at low energies
- 2-particle physics in 3-particle energies
- Toy 3-particle resonance
- Numerical Effimov state
- Unphysical solutions

Looking forward

- 2-to-3 scattering and resonant subprocesses
- Speculations

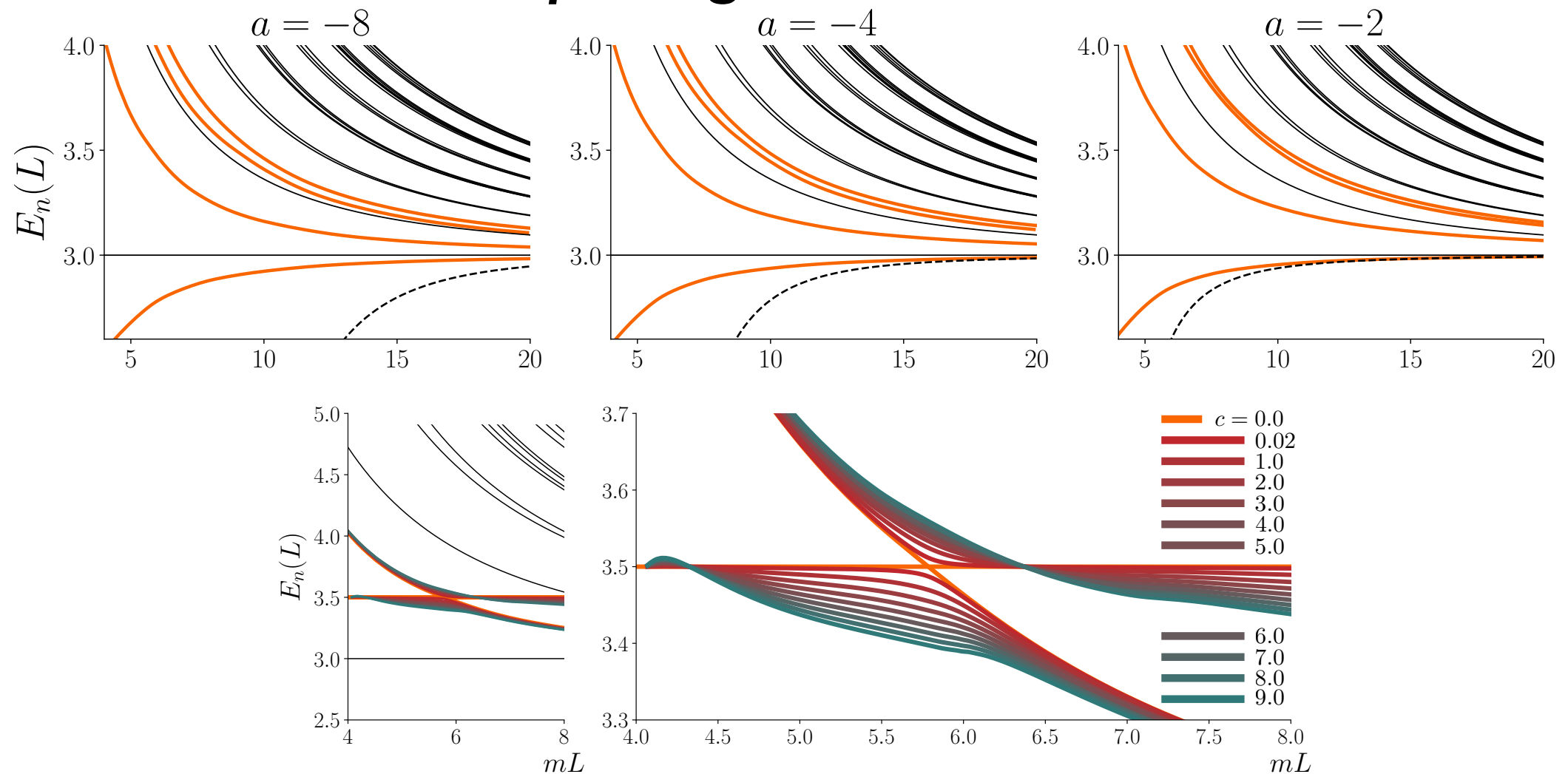
Teaser for tomorrow

Reproducing (and extending) a 60 year old result

$$E = 3m + \frac{12\pi a}{mL^3} \left(1 + c_4 \frac{a}{L} + \dots \right) - \frac{\mathcal{M}_{\text{thr}}}{48m^3 L^6} + \dots$$

Huang and Yang 1958

Exploring the solutions



Thanks!