



Three-particle scattering from numerical lattice QCD

Scattering from the lattice: applications to phenomenology and beyond

Maxwell T. Hansen

May 14-18th, 2018



Review:

☑ Alternative two-particle derivation (Kim, Sachrajda, Sharpe)

$$C_L(P) = \begin{array}{c} \textcircled{\mathcal{O}^\dagger} \textcircled{\mathcal{O}} \\ \text{---} \end{array} + \begin{array}{c} \textcircled{\mathcal{O}^\dagger} \textcircled{iB} \textcircled{\mathcal{O}} \\ \text{---} \end{array} \\ + \begin{array}{c} \textcircled{\mathcal{O}^\dagger} \textcircled{iB} \textcircled{iB} \textcircled{\mathcal{O}} \\ \text{---} \end{array} + \dots$$

$$C_L(P) = C_\infty(P) - A'F \frac{1}{1 + \mathcal{M}_{2 \rightarrow 2}F} A$$

☑ Lellouch-Lüscher via pole matching

$$\frac{L^3 \langle 0 | \mathcal{O}(0) | n, \vec{P}, L \rangle \langle n, \vec{P}, L | \mathcal{O}^\dagger(0) | 0 \rangle}{E - E_n}$$

$$\frac{\langle 0 | \mathcal{O}(0) | \pi\pi, \text{in} \rangle \mathcal{R}(E_n, \vec{P}, L) \langle \pi\pi, \text{out} | \mathcal{O}^\dagger(0) | 0 \rangle}{E - E_n}$$

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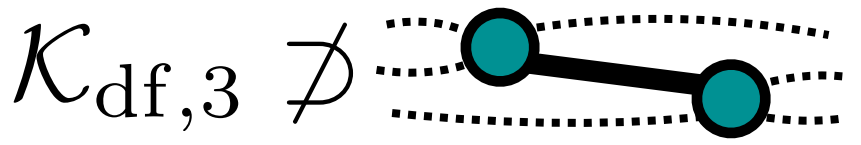
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- A real, smooth, three-particle quantity...



- Three-particle quantization condition

$$C_L(E, \vec{P}) = \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \text{Diagram 4} \\ \text{Diagram 5} \\ \text{Diagram 6} \\ \text{Diagram 7} \\ \text{Diagram 8} \\ \text{Diagram 9} \\ \text{Diagram 10} \\ \text{Diagram 11} \\ \text{Diagram 12} \\ \text{Diagram 13} \\ \text{Diagram 14} \\ \text{Diagram 15} \\ \text{Diagram 16} \\ \text{Diagram 17} \\ \text{Diagram 18} \\ \text{Diagram 19} \\ \text{Diagram 20} \\ \text{Diagram 21} \\ \text{Diagram 22} \\ \text{Diagram 23} \\ \text{Diagram 24} \\ \text{Diagram 25} \\ \text{Diagram 26} \\ \text{Diagram 27} \\ \text{Diagram 28} \\ \text{Diagram 29} \\ \text{Diagram 30} \\ \text{Diagram 31} \\ \text{Diagram 32} \\ \text{Diagram 33} \\ \text{Diagram 34} \\ \text{Diagram 35} \\ \text{Diagram 36} \\ \text{Diagram 37} \\ \text{Diagram 38} \\ \text{Diagram 39} \\ \text{Diagram 40} \\ \text{Diagram 41} \\ \text{Diagram 42} \\ \text{Diagram 43} \\ \text{Diagram 44} \\ \text{Diagram 45} \\ \text{Diagram 46} \\ \text{Diagram 47} \\ \text{Diagram 48} \\ \text{Diagram 49} \\ \text{Diagram 50} \\ \text{Diagram 51} \\ \text{Diagram 52} \\ \text{Diagram 53} \\ \text{Diagram 54} \\ \text{Diagram 55} \\ \text{Diagram 56} \\ \text{Diagram 57} \\ \text{Diagram 58} \\ \text{Diagram 59} \\ \text{Diagram 60} \\ \text{Diagram 61} \\ \text{Diagram 62} \\ \text{Diagram 63} \\ \text{Diagram 64} \\ \text{Diagram 65} \\ \text{Diagram 66} \\ \text{Diagram 67} \\ \text{Diagram 68} \\ \text{Diagram 69} \\ \text{Diagram 70} \\ \text{Diagram 71} \\ \text{Diagram 72} \\ \text{Diagram 73} \\ \text{Diagram 74} \\ \text{Diagram 75} \\ \text{Diagram 76} \\ \text{Diagram 77} \\ \text{Diagram 78} \\ \text{Diagram 79} \\ \text{Diagram 80} \\ \text{Diagram 81} \\ \text{Diagram 82} \\ \text{Diagram 83} \\ \text{Diagram 84} \\ \text{Diagram 85} \\ \text{Diagram 86} \\ \text{Diagram 87} \\ \text{Diagram 88} \\ \text{Diagram 89} \\ \text{Diagram 90} \\ \text{Diagram 91} \\ \text{Diagram 92} \\ \text{Diagram 93} \\ \text{Diagram 94} \\ \text{Diagram 95} \\ \text{Diagram 96} \\ \text{Diagram 97} \\ \text{Diagram 98} \\ \text{Diagram 99} \\ \text{Diagram 100} \end{array} + \dots$$

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) - A'F_3 \frac{1}{1 + \mathcal{K}_{\text{df},3}F_3} A$$

- Road to physics:

- I. Use q.c. + energy levels to determine $\mathcal{K}_{\text{df},3}$
- II. Use known integral equation to relate $\mathcal{K}_{\text{df},3}$ to \mathcal{M}_3



Outline

Warm up and definitions

- Basic set-up
 - Finite-volume correlator
 - Three non-interacting particles
-

Two particles in a box

- Alternative derivation
- Truncation and application
- Relating matrix elements

Three particles in a box

- 3-to-3 scattering
 - (Sketch of) derivation
 - An unexpected infinite-volume quantity
 - Relating energies to scattering
-

Testing the result

- Know issues
- Large-volume expansion
- Effimov state in a box

Other methods

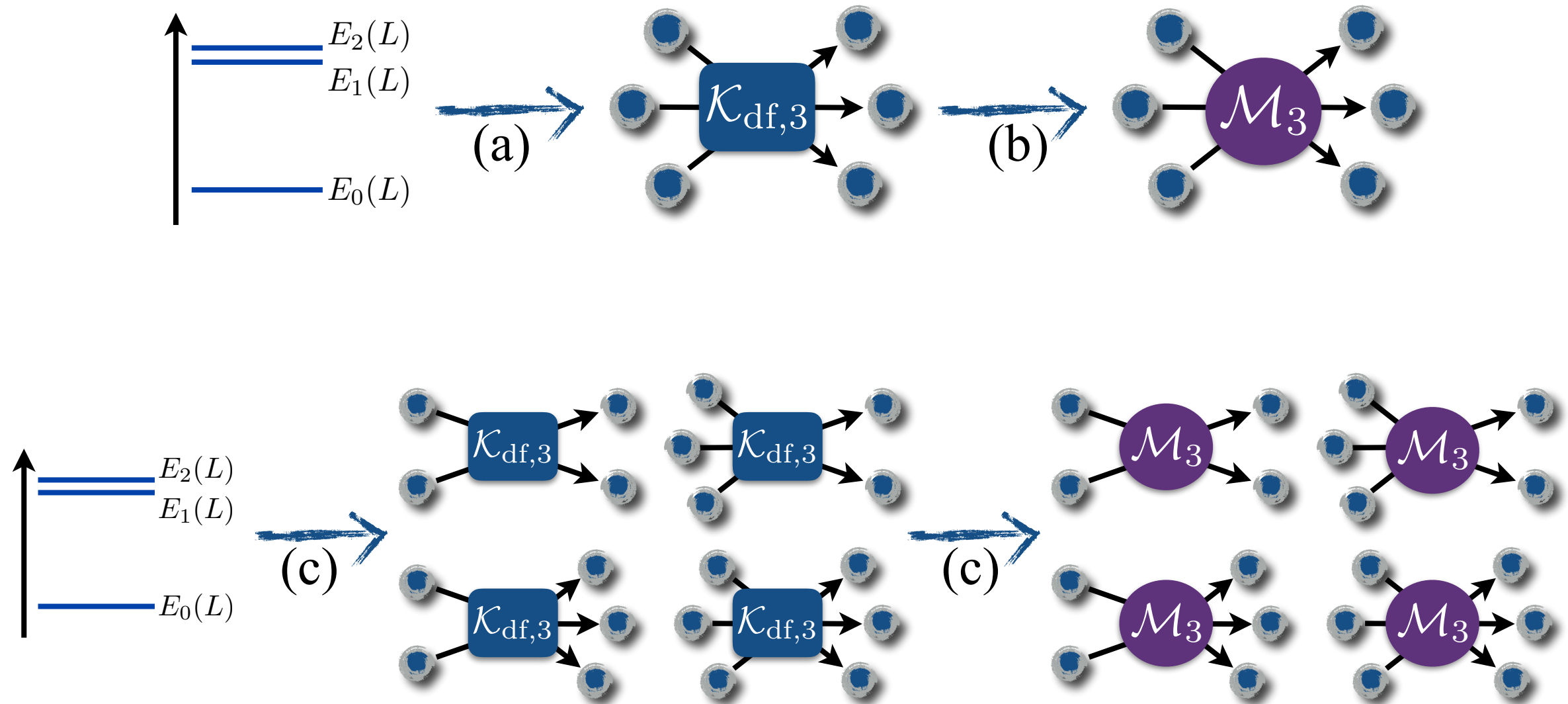
Numerical explorations

- Truncation at low energies
- Toy solutions for various systems
- Unphysical solutions

Looking forward

Current status

Model- & EFT-independent relation between finite-volume energies and relativistic two-and-three particle scattering



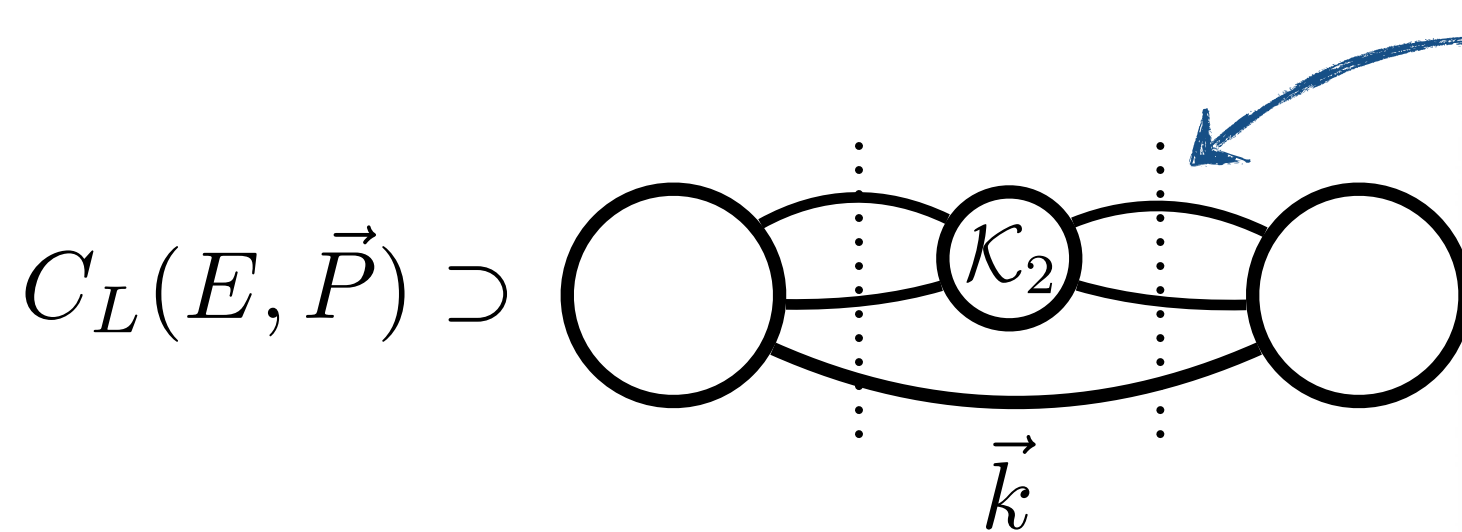
(a),(b) *MTH and Sharpe (2015),(2016)*

(c) *Briceño, MTH, Sharpe (2017)*

Smooth cutoff function

$\mathcal{K}_{\text{df},3}$ and F_3 depend on a smooth cutoff function

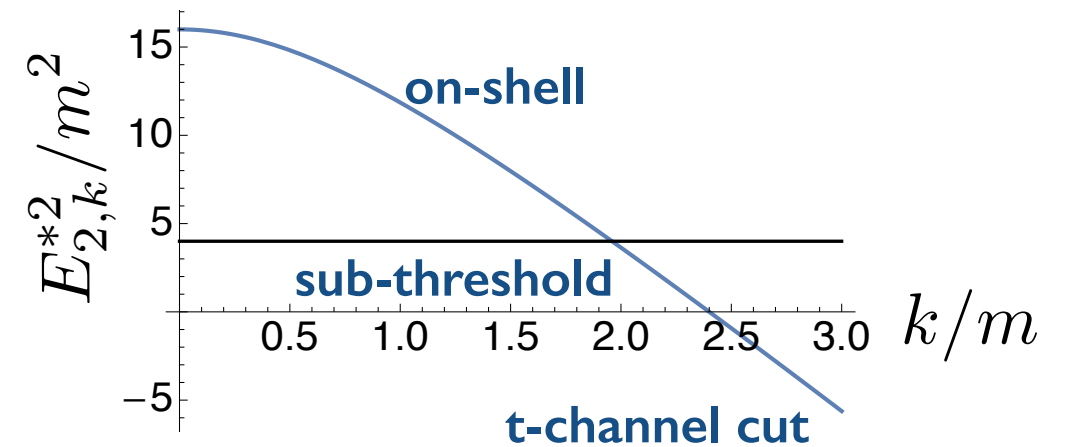
To see why, consider one of the contributions to C_L ...



How do we define this on-shell cut?

Energy of top two particles is:

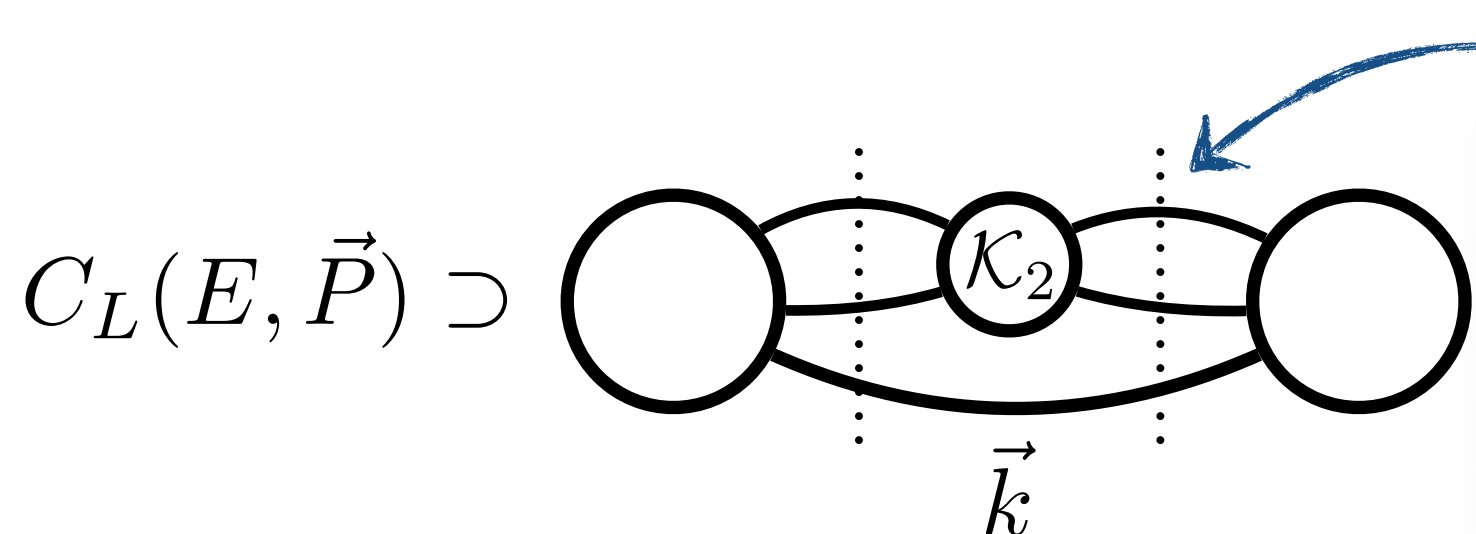
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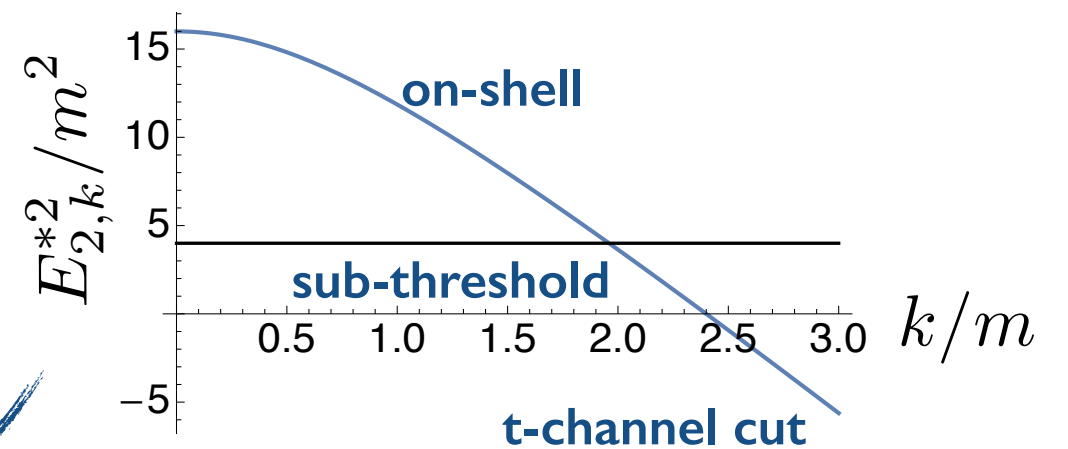
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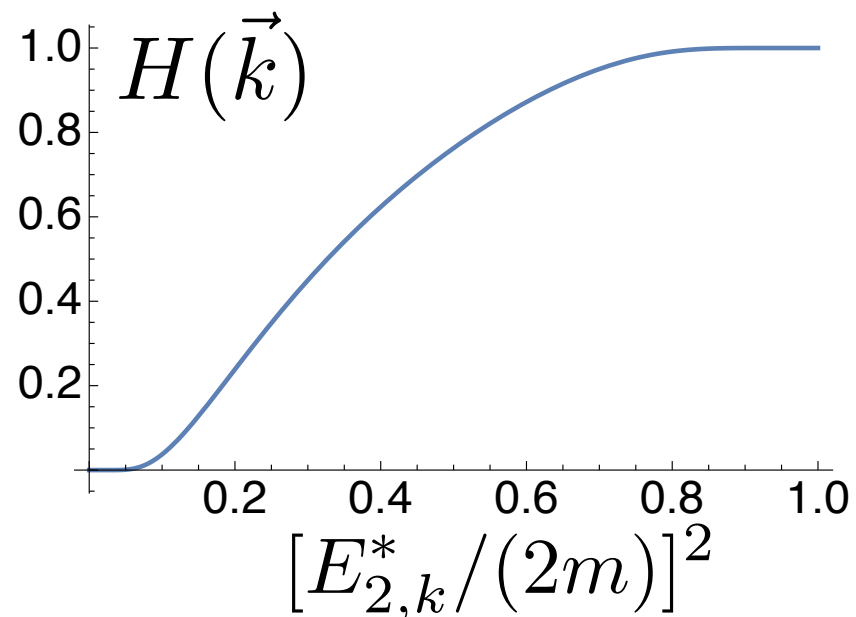
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To keep on-shell states and avoid spurious off-shell contributions...

Cuts are defined with



🤔 Is this really necessary? 🤔

We choose to sum subdiagrams into \mathcal{K}_2

On-shell \mathcal{K}_2 gives important volume effects
No important effects far below threshold

Must connect the two regions

Important limitation

Current formalism requires no poles in \mathcal{K}_2 ... **Derivation assumes**

$$\frac{1}{L^3} \sum_{\vec{k}} \text{Diagram} = \int_{\vec{k}} \text{Diagram}$$

The diagram shows two large circles connected by two arcs. The upper arc has two vertices, each with a fermion line labeled f . A central circle labeled \mathcal{K}_2 is connected to these two vertices. The lower arc is a simple line connecting the two large circles. The momentum \vec{k} is indicated below the central circle.

Given that we are seeking an EFT-independent mapping...
Is it intuitive that \mathcal{K}_2 poles need special treatment?

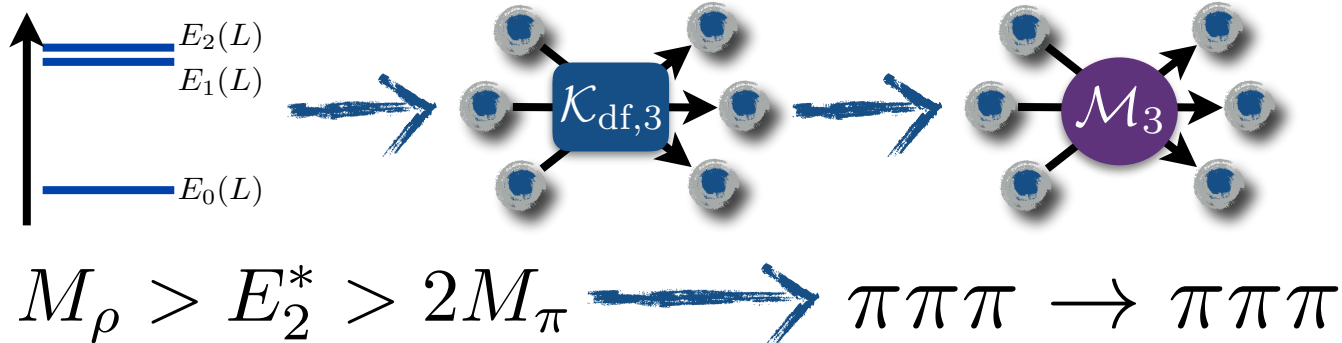
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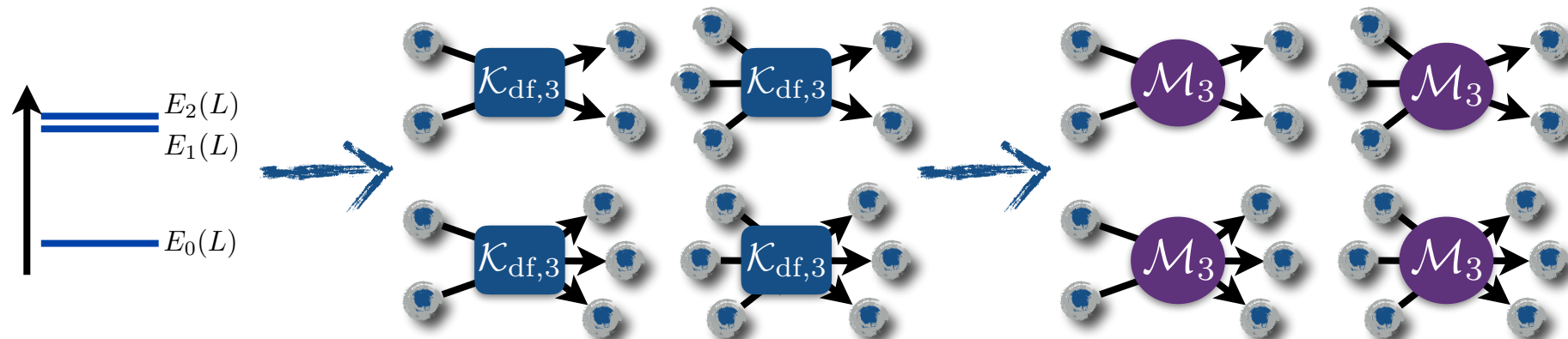
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Update
We now have a complete derivation of formalism that includes K_2 poles.
Briceño, MTH, Sharpe (underway)

Need to bridge the gap

$$E_2^* > 2M_\pi \gg M_\rho \longrightarrow \rho\pi \rightarrow \pi\pi\pi$$



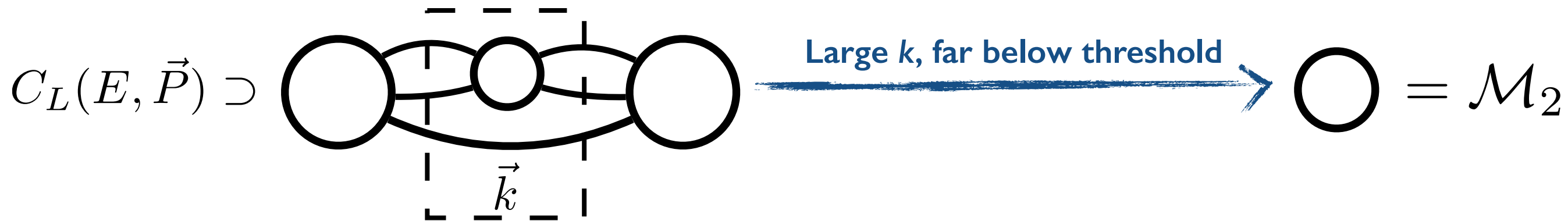
The most technical detail of all...

Far below threshold there is no ambiguity about which two-to-two scattering quantity appears in C_L



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Reason: $\frac{1}{L^3} \sum_{\vec{k}} \frac{1}{(2\omega_k)^2 (E_{\text{sub}} - 2\omega_k)} = \int_{\vec{k}} \frac{1}{(2\omega_k)^2 (E_{\text{sub}} - 2\omega_k)} = \text{Analytic Continuation} \left[\int_{\vec{k}} \frac{1}{(2\omega_k)^2 (E - 2\omega_k + i\epsilon)} \right]$

This means that our subthreshold \mathcal{K}_2 is non-standard

$$\mathcal{K}_2^{-1} \propto p^* \cot \delta(p^*) + [1 - H(\vec{k})] \kappa(p^*)$$

K matrix above threshold, smooth at threshold, interpolates to the amplitude below threshold



Why are you telling me this?



It is important because our formalism breaks down when there are poles in this definition of \mathcal{K}_2 .

Testing the formalism (Weak interactions)

☑ Expansion is well known for small a/L

$$E = 3m + \frac{12\pi a}{mL^3} \left(1 + c_4 \frac{a}{L} + c_5 \frac{a^2}{L^2} \right) + \mathcal{O}(1/L^6)$$

Huang and Yang (1957); Beane, Detmold, Savage, (2007); Tan(2007)

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$$\det[F_3(E, L)^{-1} + \mathcal{K}_{\text{df},3}(E)] = 0$$

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To enhance an eigenvalue of $\mathcal{H} = \tilde{\mathcal{K}}_2^{-1} + \tilde{F} + \tilde{G}$

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🔧 The leading order follows from

$$\mathcal{H}_{00} = -\frac{1}{64\pi m^2 a} + \frac{1}{16L^3 m^3 \Delta E} + \frac{1}{8m^3 L^3 \Delta E} + \dots$$

Testing the formalism (Weak interactions)

We reproduce known results through l/L^5 and derive a relation at l/L^6

Note: Relativistic effects enter at l/L^6 , same order as three-to-three

$$E = 3m + \frac{12\pi a}{mL^3} \left(1 + c_4 \frac{a}{L} + \dots \right) - \frac{\mathcal{M}_{\text{thr}}}{48m^3 L^6} + \dots$$

MTH and Sharpe (2017)

Terms through l/L^5 are from tuning the eigenvalue of $\mathcal{H} = \tilde{\mathcal{K}}_2^{-1} + \tilde{F} + \tilde{G}$

Testing the formalism (Weak interactions)

We reproduce known results through $1/L^5$ and derive a relation at $1/L^6$

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☑ We checked this in $\lambda\phi^4$ through $\mathcal{O}(\lambda^4)$

MTH and Sharpe (2016), Sharpe (2017)

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Meißner, Rios and Rusetsky, *Phys. Rev. Lett.* 114, 091602 (2015) + erratum

The infinite-volume boundstate energy, $E_B \equiv 3m - \frac{\kappa^2}{m}$
is shifted in finite volume by an amount

$$\Delta E(L) = c |A|^2 \frac{\kappa^2}{m} \frac{1}{(\kappa L)^{3/2}} e^{-2\kappa L/\sqrt{3}} + \dots$$

(Note: In the original image, a blue arrow points from the text below to the $|A|^2$ term, and a blue bracket groups the constant c and the exponential term.)

$c = -96.351 \dots$
geometric constant from
Efimov wavefunction

normalization correction factor
(close to one)

Assumes two-body potential, unitary limit, P=0, s-wave only

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Our formalism gives a general relation between scattering amplitudes and energy levels. So we substitute...

$$\mathcal{M}_3 \sim -\frac{\Gamma \bar{\Gamma}}{E^2 - E_B^2} \quad \mathcal{M}_2 = -\frac{16\pi E_2^*}{ip^*}$$

and study the lowest three-particle finite-volume level

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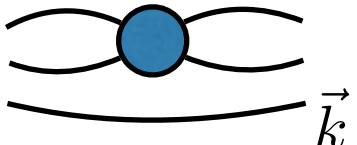
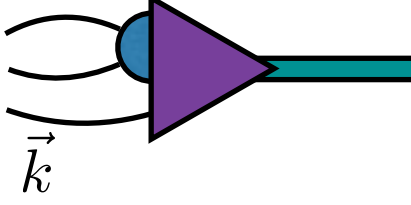
and study the lowest three-particle finite-volume level

We aim to reproduce the exponent, leading power and overall constant using our relativistic formalism

Reproducing the result...

1. Show that the relativistic quantization predicts (at leading order in $1/L$)

$$\Delta E(L) = -\frac{1}{2E_B} \left[\frac{1}{L^3} \sum_{\vec{k}} - \int_{\vec{k}} \right] \frac{\bar{\Gamma}^{(u)}(k) \Gamma^{(u)}(k)}{2\omega_k \mathcal{M}_2(k)}$$


s-wave scattering amplitude
usymmetrized residue factor


2. Derive the functional forms of the infinite-volume quantities

$$\Gamma^{(u)}(k) = \frac{3^{3/8} \pi^{1/4}}{4} A \sqrt{-c} \mathcal{M}_2(k) \quad \mathcal{M}_2(k) = \frac{32\pi m}{\kappa} \left[1 + \frac{3k^2}{4\kappa^2} \right]^{-1/2}$$

follows from matching to
Effimov wavefunction

unitary amplitude with spectator
“stealing” some momentum

3. Evaluate the sum-integral difference with Poisson summation

$$\begin{aligned} \Delta E(L) &= c|A|^2 \frac{3^{3/4} \pi^{3/2}}{3\kappa} 6 \int_{\vec{k}} e^{iL\hat{x}\cdot\vec{k}} \frac{1}{2\omega_k} \left[1 + \frac{3k^2}{4\kappa^2} \right]^{-1/2} \\ &= c|A|^2 \frac{\kappa^2}{m} \frac{1}{(\kappa L)^{3/2}} e^{-2\kappa L/\sqrt{3}} + \dots \end{aligned}$$

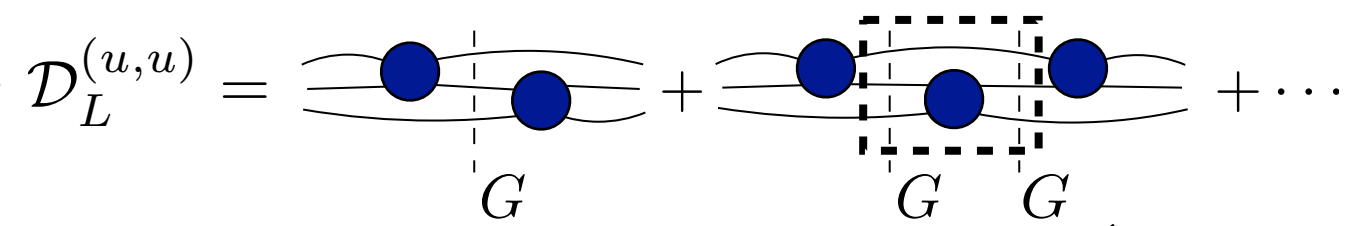
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We use the second form of the finite-volume correlator

$$\mathcal{M}_{3,L} = \mathcal{S} \left[\mathcal{D}_L^{(u,u)} + \mathcal{L}_L^{(u)} \mathcal{K}_{\text{df},3} \frac{1}{1 + F_3 \mathcal{K}_{\text{df},3}} \mathcal{R}_L^{(u)} \right]$$

$\mathcal{D}_L^{(u,u)} =$  $+ \dots$ here we use that $\mathcal{M}_{2,L} = \mathcal{M}_2$ to the order we work

$= \left[\frac{1}{L^3} \sum_{\vec{k}} - \int_{\vec{k}} \right] \left[\text{diagram with 2 particles} + \text{diagram with 3 particles} + \dots \right] \frac{1}{\mathcal{M}_2(\vec{k})} \left[\text{diagram with 2 particles} + \dots \right] + \dots$

$$\mathcal{D}_L^{(u,u)} = \mathcal{D}_\infty^{(u,u)} + \left[\frac{1}{L^3} \sum_{\vec{k}} - \int_{\vec{k}} \right] \mathcal{D}_\infty^{(u,u)} \frac{1}{2\omega_k \mathcal{M}_2(\vec{k})} \mathcal{D}_L^{(u,u)}$$

Reproducing the result (more details)...

1. Show that the relativistic quantization predicts (at leading order in $1/L$)

$$\Delta E(L) = -\frac{1}{2E_B} \left[\frac{1}{L^3} \sum_{\vec{k}} - \int_{\vec{k}} \right] \frac{\bar{\Gamma}^{(u)}(k) \Gamma^{(u)}(k)}{2\omega_k \mathcal{M}_2(k)}$$

We use the second form of the finite-volume correlator

$$\mathcal{M}_{3,L} = \mathcal{S} \left[\mathcal{D}_L^{(u,u)} + \mathcal{L}_L^{(u)} \mathcal{K}_{\text{df},3} \frac{1}{1 + F_3 \mathcal{K}_{\text{df},3}} \mathcal{R}_L^{(u)} \right]$$

Expansion and analysis of all terms shows that the same relation holds for the full (unsymmetrized) three-to-three scattering amplitude

$$\mathcal{M}_{3,L}^{(u,u)} = \mathcal{M}_3^{(u,u)} + \left[\frac{1}{L^3} \sum_{\vec{k}} - \int_{\vec{k}} \right] \mathcal{M}_3^{(u,u)} \frac{1}{2\omega_k \mathcal{M}_2(\vec{k})} \mathcal{M}_{3,L}^{(u,u)}$$

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Substituting pole ansatz and solving gives the claimed result

$$\frac{1}{E^2 - [E_B + \Delta E(L)]^2} = \frac{1}{E^2 - E_B^2} + \left[\frac{1}{L^3} \sum_{\vec{k}} - \int_{\vec{k}} \right] \frac{1}{E^2 - E_B^2} \frac{\bar{\Gamma}^{(u)}(k) \Gamma^{(u)}(k)}{2\omega_k \mathcal{M}_2(\vec{k})} \frac{1}{E^2 - [E_B + \Delta E(L)]^2}$$

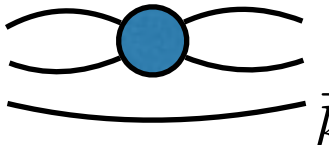
finite-volume pole infinite-volume pole

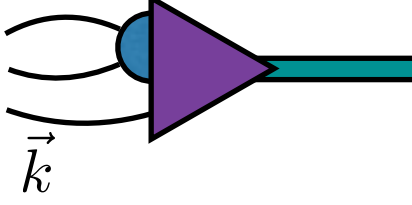
Reproducing the result (more details)...

1. Show that the relativistic quantization predicts (at leading order in $1/L$)

✓

$$\Delta E(L) = -\frac{1}{2E_B} \left[\frac{1}{L^3} \sum_{\vec{k}} - \int_{\vec{k}} \right] \frac{\bar{\Gamma}^{(u)}(k) \Gamma^{(u)}(k)}{2\omega_k \mathcal{M}_2(k)}$$


s-wave scattering amplitude


usymmetrized residue factor

2. Derive the functional forms of the infinite-volume quantities

$$\Gamma^{(u)}(k) = \frac{3^{3/8} \pi^{1/4}}{4} A \sqrt{-c} \mathcal{M}_2(k) \quad \mathcal{M}_2(k) = \frac{32\pi m}{\kappa} \left[1 + \frac{3k^2}{4\kappa^2} \right]^{-1/2}$$

follows from matching to
Effimov wavefunction

unitary amplitude with spectator
“stealing” some momentum

3. Evaluate the sum-integral difference with Poisson summation

$$\begin{aligned} \Delta E(L) &= c|A|^2 \frac{3^{3/4} \pi^{3/2}}{3\kappa} 6 \int_{\vec{k}} e^{iL\hat{x}\cdot\vec{k}} \frac{1}{2\omega_k} \left[1 + \frac{3k^2}{4\kappa^2} \right]^{-1/2} \\ &= c|A|^2 \frac{\kappa^2}{m} \frac{1}{(\kappa L)^{3/2}} e^{-2\kappa L/\sqrt{3}} + \dots \end{aligned}$$

Reproducing the result (more details)...

2. Derive the functional forms of the infinite-volume quantities

$$\mathcal{M}_2(k) = -\frac{16\pi E_2^*}{ip^*} \xrightarrow[\text{leading order in binding momentum}]{\text{subthreshold,}} \mathcal{M}_2(k) = \frac{32\pi m}{\kappa} \left[1 + \frac{3k^2}{4\kappa^2}\right]^{-1/2}$$

$p^* = \sqrt{[(E - \omega_k)^2 - \vec{k}^2]/4 - m^2}$

To derive the residue factor we match to the non-relativistic wavefunction

$$\left[-\frac{1}{2m} \sum_i \frac{\partial^2}{\partial \mathbf{r}_i^2} + \sum_{ij} V(\mathbf{r}_i - \mathbf{r}_j) \right] \psi(\mathbf{r}_1, \mathbf{r}_2) = -\frac{\kappa^2}{m} \psi(\mathbf{r}_1, \mathbf{r}_2)$$

this can be re-expressed using the Faddeev equation

$$\psi = \phi_1 + \phi_2 + \phi_3 \quad \left[-\frac{1}{2m} \sum_i \frac{\partial^2}{\partial \mathbf{r}_i^2} + \frac{\kappa^2}{m} \right] \phi_3(\mathbf{r}_1, \mathbf{r}_2) = V(\mathbf{r}_1 - \mathbf{r}_2) \psi(\mathbf{r}_1, \mathbf{r}_2)$$

We have found that the unsymmetrized residue factor is given by

$$\Gamma^{(u)}(k) = \lim_{\text{on shell}} 4\sqrt{3}m^2 \left(-\frac{\kappa^2}{m} - H_0 \right) \tilde{\phi}_3$$

Substituting the known wave function and expanding about the leading singularity, we find

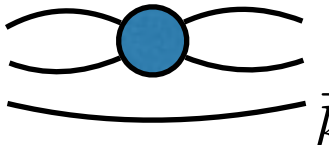
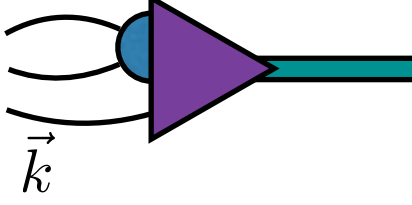
$$\Gamma^{(u)}(k) = \frac{3^{3/8} \pi^{1/4}}{4} A \sqrt{-c} \mathcal{M}_2(k)$$

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1. Show that the relativistic quantization predicts (at leading order in $1/L$)

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3. Evaluate the sum-integral difference with Poisson summation

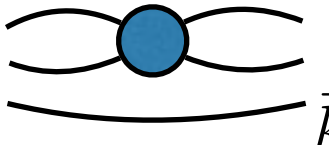
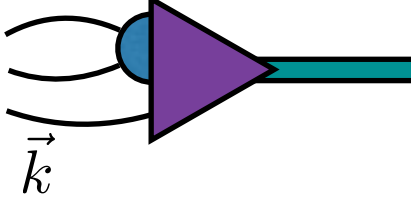
$$\begin{aligned} \Delta E(L) &= c|A|^2 \frac{3^{3/4} \pi^{3/2}}{3\kappa} 6 \int_{\vec{k}} e^{iL\hat{x}\cdot\vec{k}} \frac{1}{2\omega_k} \left[1 + \frac{3k^2}{4\kappa^2} \right]^{-1/2} \\ &= c|A|^2 \frac{\kappa^2}{m} \frac{1}{(\kappa L)^{3/2}} e^{-2\kappa L/\sqrt{3}} + \dots \end{aligned}$$

Reproducing the result (more details)...

1. Show that the relativistic quantization predicts (at leading order in $1/L$)

✓

$$\Delta E(L) = -\frac{1}{2E_B} \left[\frac{1}{L^3} \sum_{\vec{k}} - \int_{\vec{k}} \right] \frac{\bar{\Gamma}^{(u)}(k) \Gamma^{(u)}(k)}{2\omega_k \mathcal{M}_2(k)}$$


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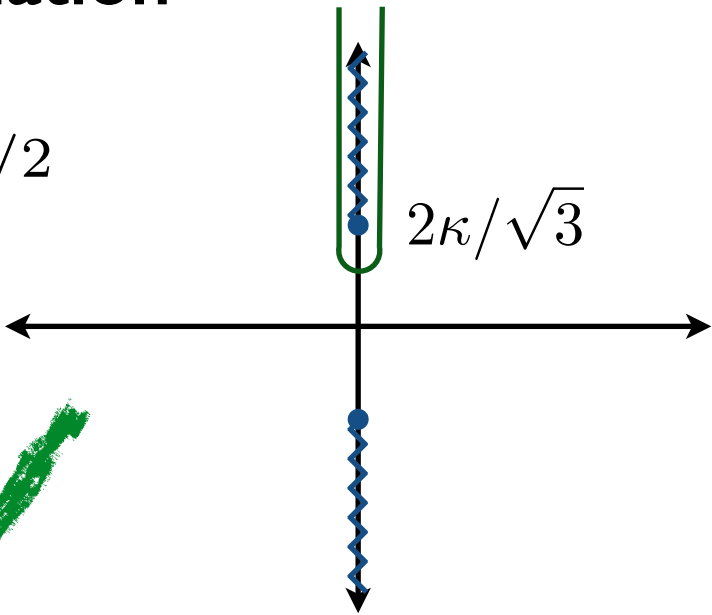
unitary amplitude with spectator
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3. Evaluate the sum-integral difference with Poisson summation

singularity from two-to-two amplitude

$$\Delta E(L) = c|A|^2 \frac{3^{3/4} \pi^{3/2}}{3\kappa} 6 \int_{\vec{k}} e^{iL\hat{x}\cdot\vec{k}} \frac{1}{2\omega_k} \left[1 + \frac{3k^2}{4\kappa^2} \right]^{-1/2}$$

$$= c|A|^2 \frac{\kappa^2}{m} \frac{1}{(\kappa L)^{3/2}} e^{-2\kappa L/\sqrt{3}} + \dots$$



✓

MTH and Sharpe (2017)



Outline

Warm up and definitions

- Basic set-up
 - Finite-volume correlator
 - Three non-interacting particles
-

Two particles in a box

- Alternative derivation
- Truncation and application
- Relating matrix elements

Three particles in a box

- 3-to-3 scattering
 - (Sketch of) derivation
 - An unexpected infinite-volume quantity
 - Relating energies to scattering
-

Testing the result

- Know issues
- Large-volume expansion
- Effimov state in a box

Other methods

Numerical explorations

- Truncation at low energies
- Toy solutions for various systems
- Unphysical solutions

Looking forward

Other methods...

A. Rusetsky, H.W. Hammer, J.-Y. Pang:

- Non-relativistic
- Based in a specific EFT, focuses on extracting LECs,
- Simpler derivation and formulae, can handle K-matrix poles
- Argued to be “diagrammatically equivalent”
- No t-channel cut, integrals go infinitely far below threshold

M. Mai and M. Döring

- Relativistic
- Built on unitary constrains, replace imaginary cuts with volume cuts
- Cannot see the dropping of $\mathcal{O}(e^{-mL})$
- Connection to our approach is not yet well understood

See also Polejaeva, Rusetsky (2012) and Briceño, Davoudi (2013)

Usability?

“Despite this success, the quantization condition in these papers is not yet given in a form suitable for the analysis of the real lattice data”

Hammer, Pang and Rusetsky (2017)

We were motivated to challenge this claim...

We find that the “degree of usability” is comparable between the two approaches, provided one applies similar approximations.

How do we make the two-particle formalism usable?

Truncate partial waves

$$\mathcal{M}_2(E_2^*, \theta^*) \approx \sum_{\ell=0}^N P_\ell(\cos \theta^*) \mathcal{M}_{2,\ell}(E_2^*)$$

Single partial wave

$$\mathcal{M}_2(E_2^*, \theta^*) \approx \mathcal{M}_{2,s}(E_2^*) \propto \frac{1}{p^* \cot \delta_0(p^*) - ip^*}$$

Is there a three-particle analog?

$$\mathcal{K}_{\text{df},3}(E^*, \Omega'_3, \Omega_3) \approx \sum_{n=0}^N \mathcal{P}_n(\Omega'_3, \Omega_3) \mathcal{K}_{\text{df},3,n}(E^*)$$

$$\mathcal{K}_{\text{df},3}(E^*, \Omega'_3, \Omega_3) \approx \mathcal{K}_{\text{df},3}^{\text{iso}}(E^*) \in \mathbb{R}$$

At fixed energy $\frac{\mathcal{M}_2(E_2^*, \theta^*)}{\mathcal{K}_{\text{df},3}(E^*, \Omega'_3, \Omega_3)}$ **is a smooth function on a compact space.**

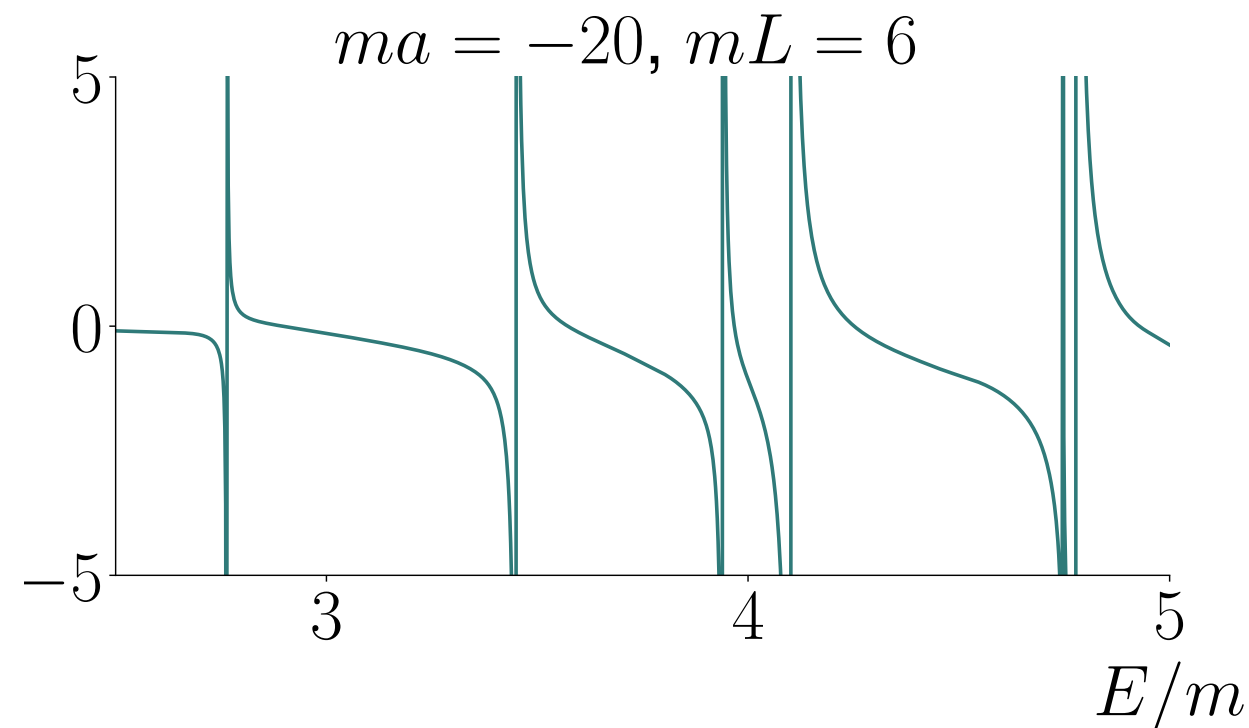
Further investigation is needed to understand suppression of higher $\mathcal{K}_{\text{df},3,n}(E^*)$.

Numerics (keeping only s-wave and $\mathcal{K}_{\text{df},3}(E^*, \Omega'_3, \Omega_3) \approx \mathcal{K}_{\text{df},3}^{\text{iso}}(E^*)$)

$$1/\mathcal{K}_{\text{df},3}^{\text{iso}}(E^*) = -F_3^{\text{iso}}[E, \vec{P}, L, \mathcal{M}_2^s] \quad \mathcal{M}_3(E^*, \Omega'_3, \Omega_3) = \mathcal{S} \left[\mathcal{D} + \mathcal{L} \frac{1}{1/\mathcal{K}_{\text{df},3}^{\text{iso}} + F_{3,\infty}^{\text{iso}}} \mathcal{R} \right]$$

For the numerical approach we restrict attention to... $p^* \cot \delta_0(p^*) = -\frac{1}{a}$, $\vec{P} = 0$

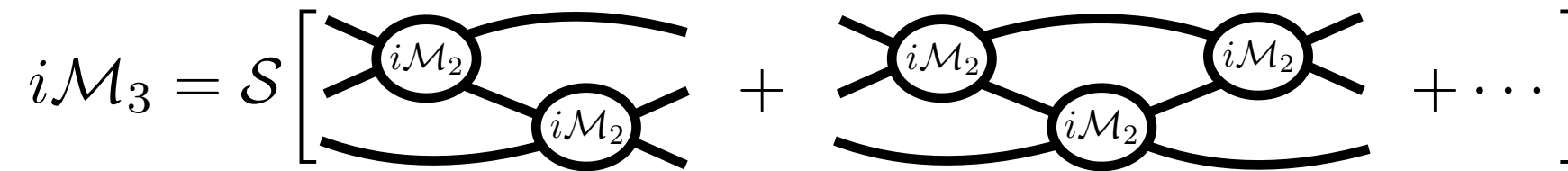
Then the quantization condition is based on $F_3^{\text{iso}}(E, L, a)$



Finite-volume energies wherever these curves intersect $-1/\mathcal{K}_{\text{df},3}^{\text{iso}}(E)$

$\mathcal{K}_{\text{df},3}^{\text{iso}}(E) = 0$ solutions

- Provides a useful benchmark: Deviations measure three-particle physics
- Meaning for three-to-three scattering is clear

$$i\mathcal{M}_3 = \mathcal{S} \left[\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} + \dots \right]$$


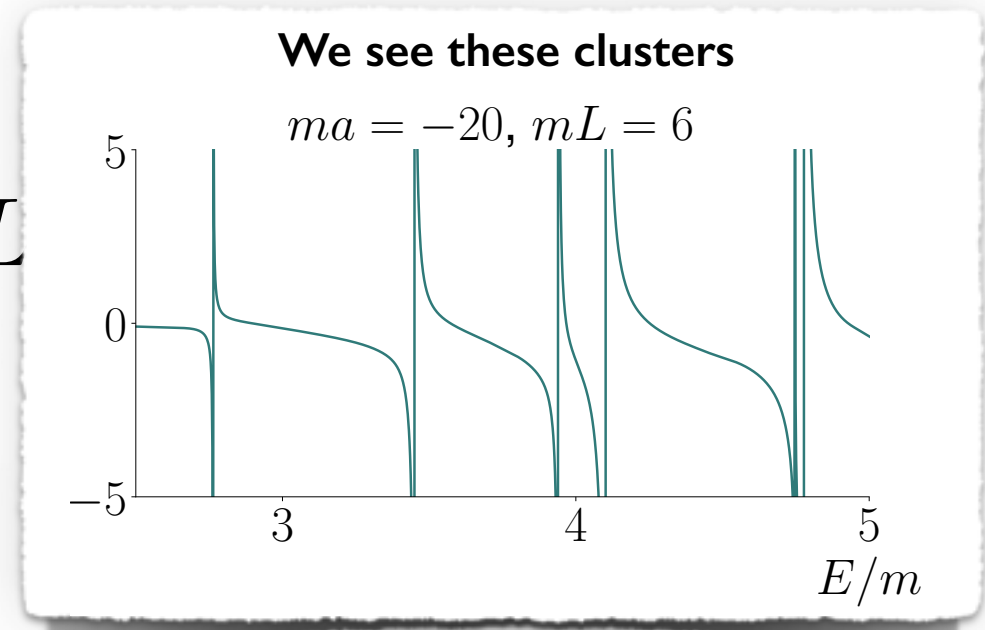
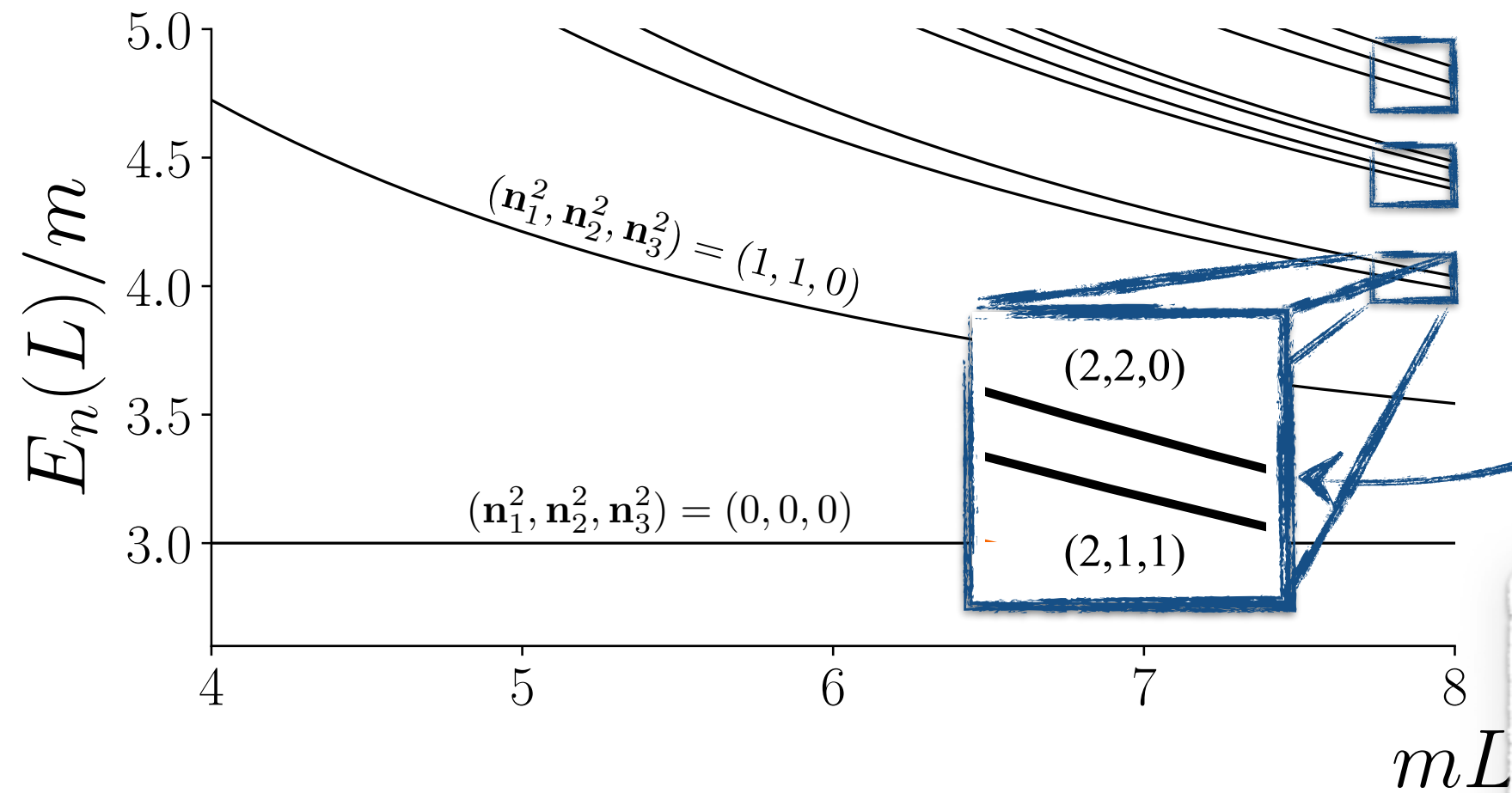
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$$i\mathcal{M}_3 = \mathcal{S} \left[\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \dots \end{array} \right]$$

The diagrams show Feynman-like diagrams for three-particle scattering. The first diagram has two interaction vertices labeled $i\mathcal{M}_2$. The second diagram has three interaction vertices labeled $i\mathcal{M}_2$. The diagrams are enclosed in large square brackets.

Again: **Non-interacting states** ($\mathcal{M}_2 = \mathcal{M}_3 = 0$)

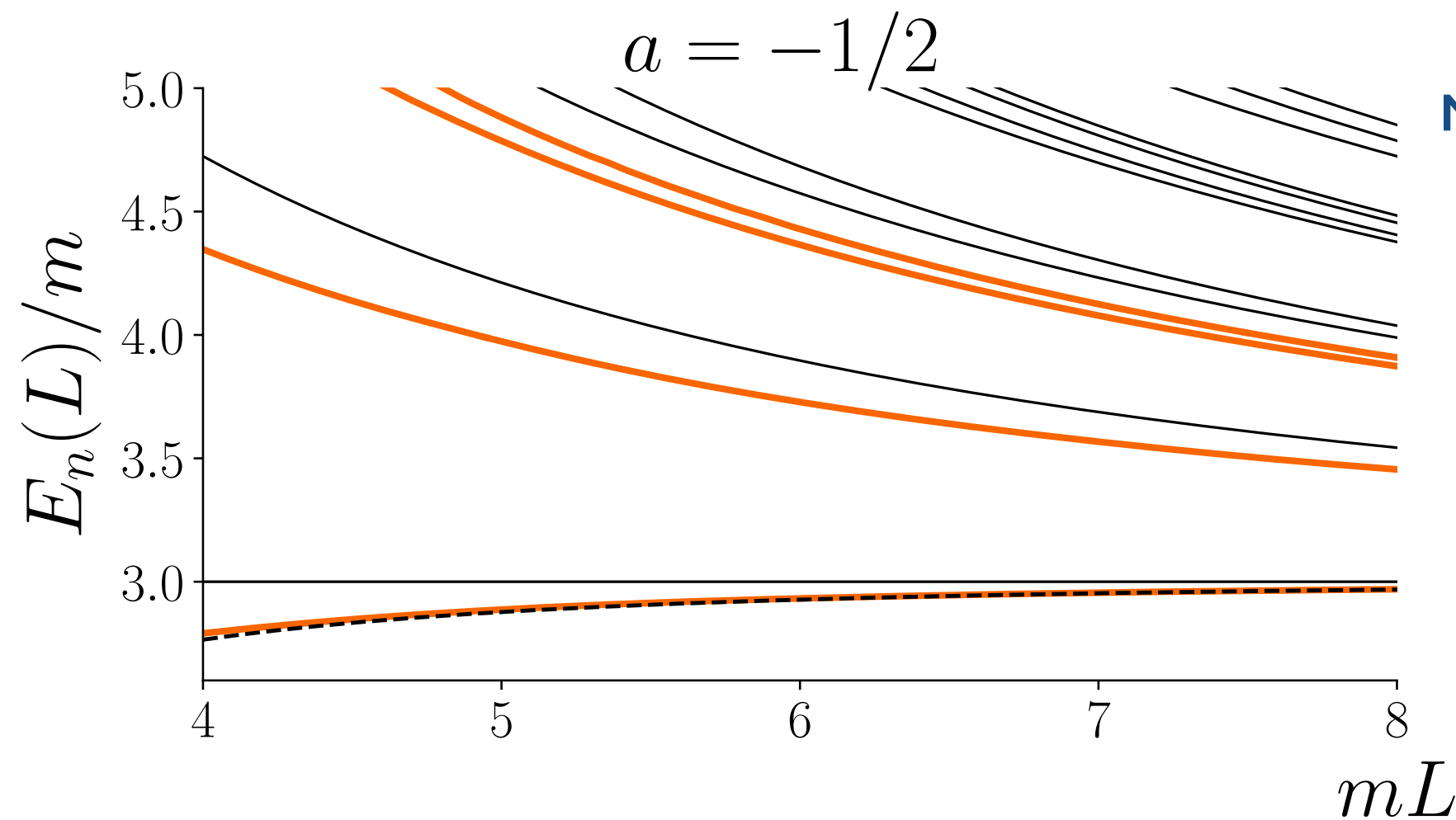


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$$i\mathcal{M}_3 = \mathcal{S} \left[\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} + \dots \right]$$

The diagrams show two Feynman-like diagrams for three-particle scattering. Each diagram consists of three external lines (two incoming from the left, one outgoing to the right) and two internal interaction vertices labeled $i\mathcal{M}_2$. The first diagram shows a loop structure where the two incoming lines meet at a vertex, then split into two lines that meet at another vertex, and finally recombine into the outgoing line. The second diagram shows a different loop structure where the two incoming lines meet at a vertex, then split into two lines that meet at another vertex, and finally recombine into the outgoing line in a different configuration.



Now we turn on the interactions

$$1/F_3^{\text{iso}}(E, L, a) = 0$$

$$F_3^{\text{iso}} = \frac{1}{L^3} \sum_{\vec{k}, \vec{p}} \left[\frac{\tilde{F}_s}{3} - \tilde{F}_s \frac{1}{\mathcal{H}} \tilde{F}_s \right]_{kp}$$

$$\mathcal{H} = 1/(2\omega\mathcal{K}_2) + \tilde{F}_s + \tilde{G}_s$$

known functions

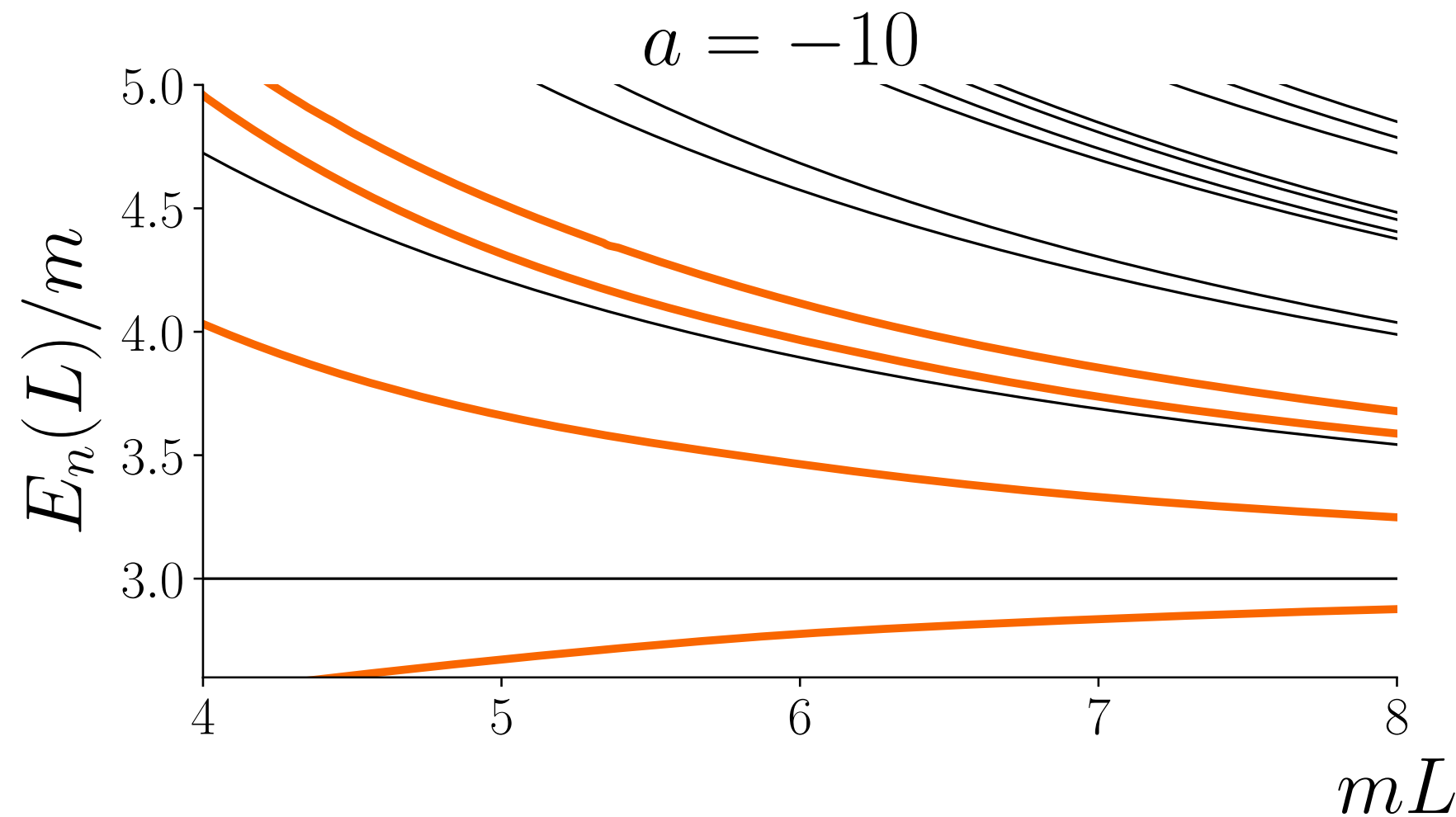
----- $1/L$ expansion

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The diagrams show two Feynman-like diagrams for three-particle scattering. The first diagram has two external lines on the left and two on the right, with two internal circles labeled $i\mathcal{M}_2$. The second diagram is similar but with an additional internal circle labeled $i\mathcal{M}_2$.



Can also accommodate large a

$$1/F_3^{\text{iso}}(E, L, a) = 0$$

$$F_3^{\text{iso}} = \frac{1}{L^3} \sum_{\vec{k}, \vec{p}} \left[\frac{\tilde{F}_s}{3} - \tilde{F}_s \frac{1}{\mathcal{H}} \tilde{F}_s \right]_{kp}$$

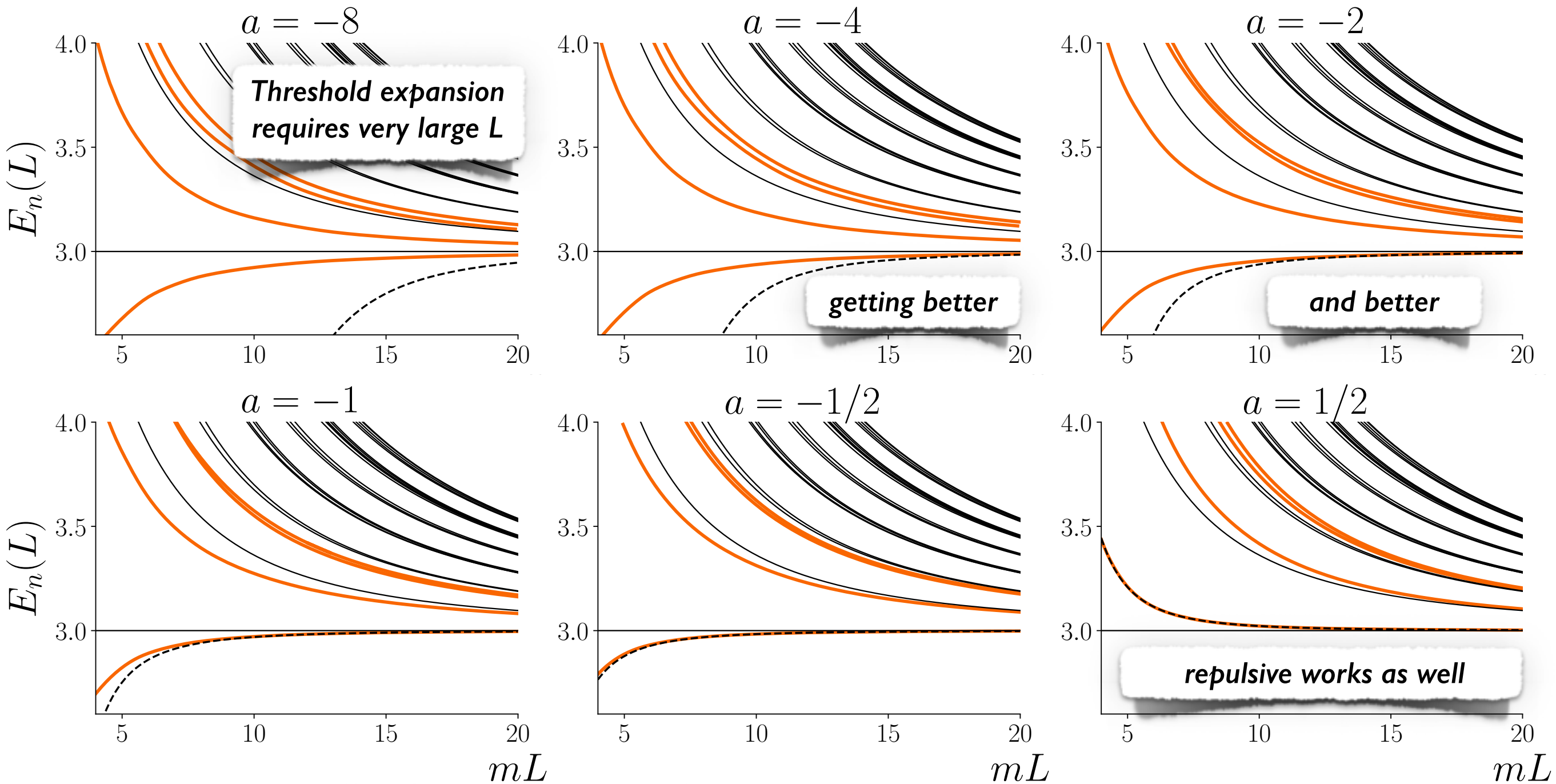
$$\mathcal{H} = 1/(2\omega\mathcal{K}_2) + \tilde{F}_s + \tilde{G}_s$$

known functions

----- $1/L$ expansion

$\mathcal{K}_{\text{df},3}^{\text{iso}}(E) = 0$ solutions

Straightforward to vary a and to study large volumes



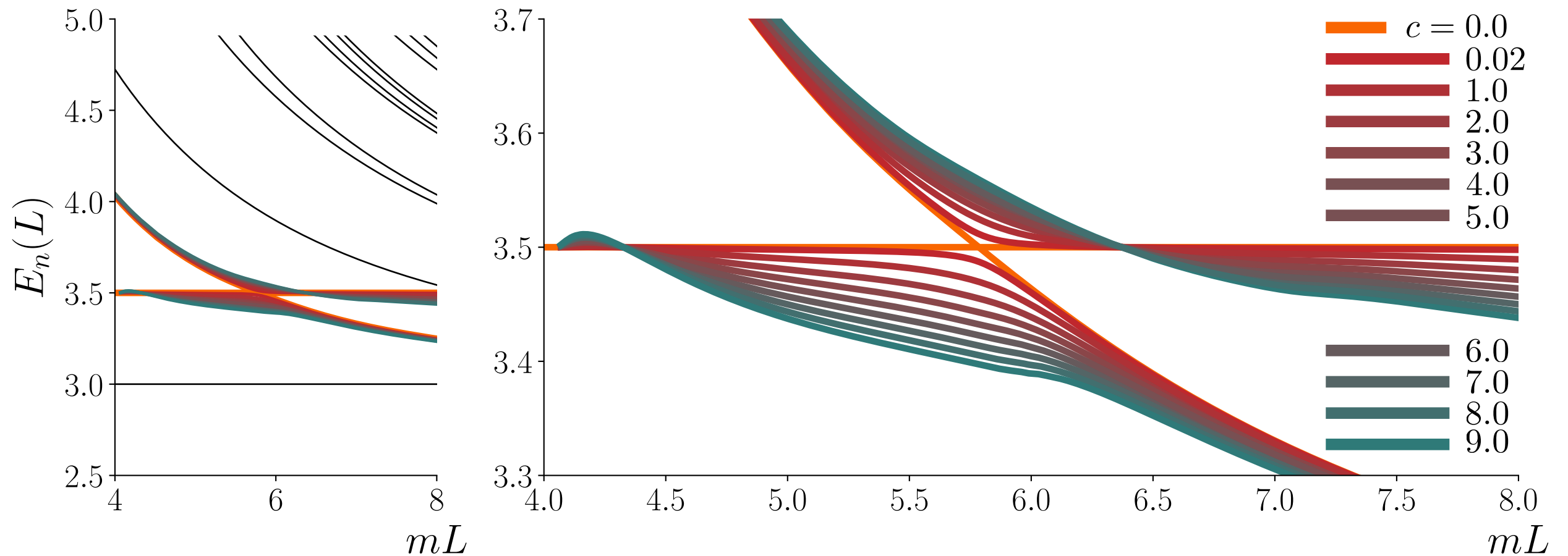
But, to avoid poles in \mathcal{K}_2 , we must require $a < 1/m$

Non-zero $\mathcal{K}_{\text{df},3}^{\text{iso}}(E)$: Toy resonance

Here we consider a fun example for non-zero $\mathcal{K}_{\text{df},3}^{\text{iso}}$

$$a = -10 \quad \mathcal{K}_{\text{df},3}^{\text{iso}}(E) = -\frac{c \times 10^3}{E^2 - M_R^2}$$

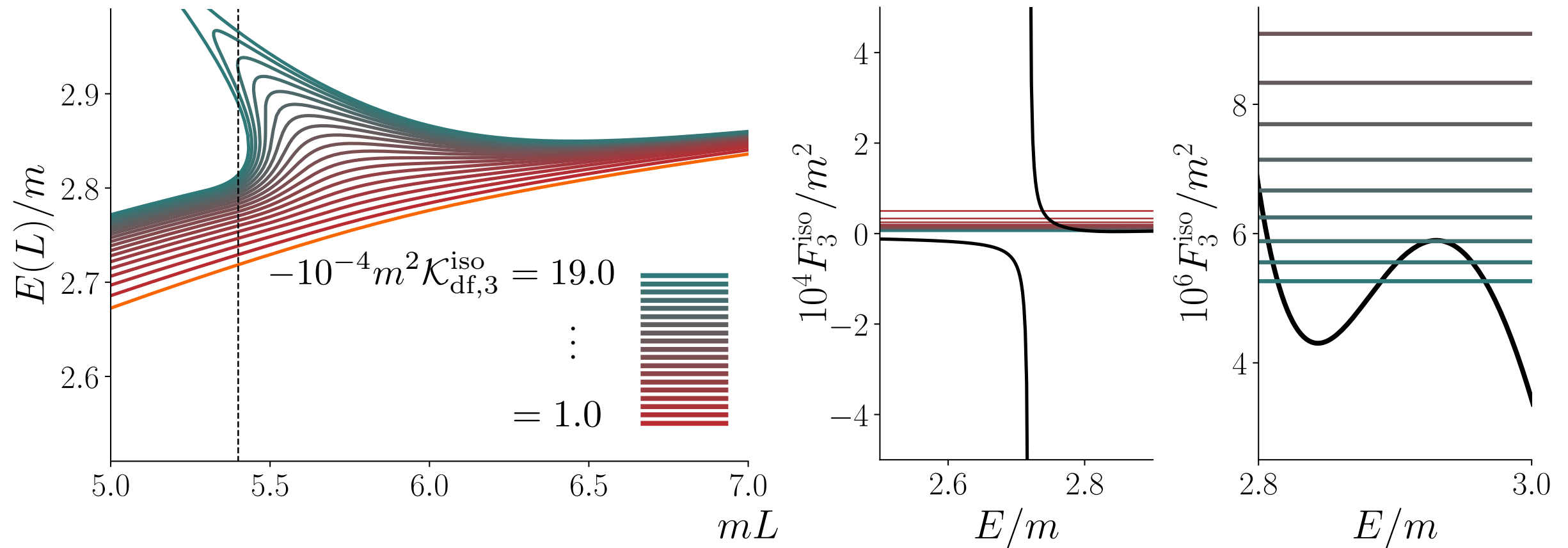
For small c we expect a narrow avoided level crossing, as c increases the gap grows



Further investigation is needed to see if this gives a physical resonance description

Unphysical solutions

Very large values of $\mathcal{K}_{\text{df},3}^{\text{iso}}$ can lead to unphysical solutions



Unphysical input? Enhanced $\mathcal{O}(e^{-mL})$ effects? Under investigation...

Non-zero $\mathcal{K}_{\text{df},3}(E)$: Unitary bound state

The parameters $a = -10^4$, $\mathcal{K}_{\text{df},3}^{\text{iso}}(E) = 2500$ lead to a shallow bound state

$$\kappa \approx 0.1m \text{ where } E_B = 3m - \kappa^2/m$$

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Finite-volume behavior of this state has a known asymptotic form

Meißner, Rios, Rusetsky (2015)

$$E_B(L) = 3m - \frac{\kappa^2}{m} - (98.35 \dots) |A|^2 \frac{\kappa^2}{m} \frac{e^{-2\kappa L/\sqrt{3}}}{(\kappa L)^{3/2}} \left[1 + \mathcal{O}\left(\frac{1}{\kappa L}, \frac{\kappa^2}{m^2}, e^{-\alpha\kappa L}\right) \right]$$

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$$\kappa L \approx 0.1mL \gg 1$$

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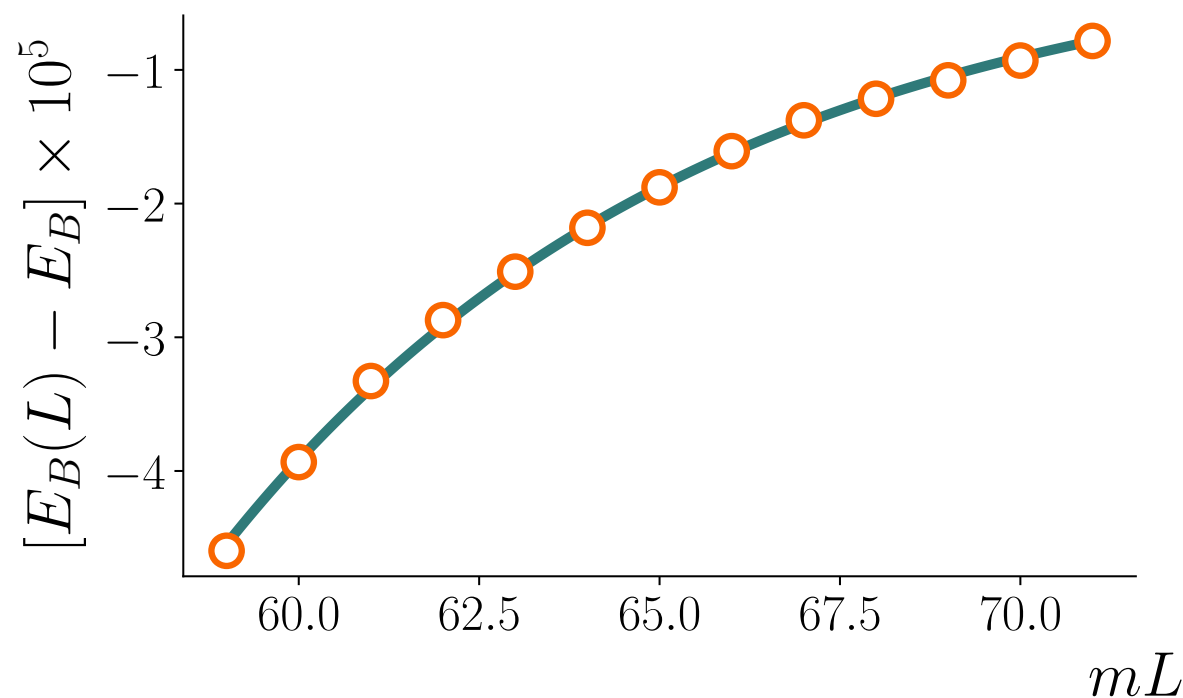
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This describes the bound state for

$$\kappa L \approx 0.1mL \gg 1$$

We fit our q.c. data over $60 < mL < 70$

→ $\kappa = 0.1068$, $|A|^2 = 0.948$

Close to one ✓

Non-zero $\mathcal{K}_{\text{df},3}(E)$: Unitary bound state

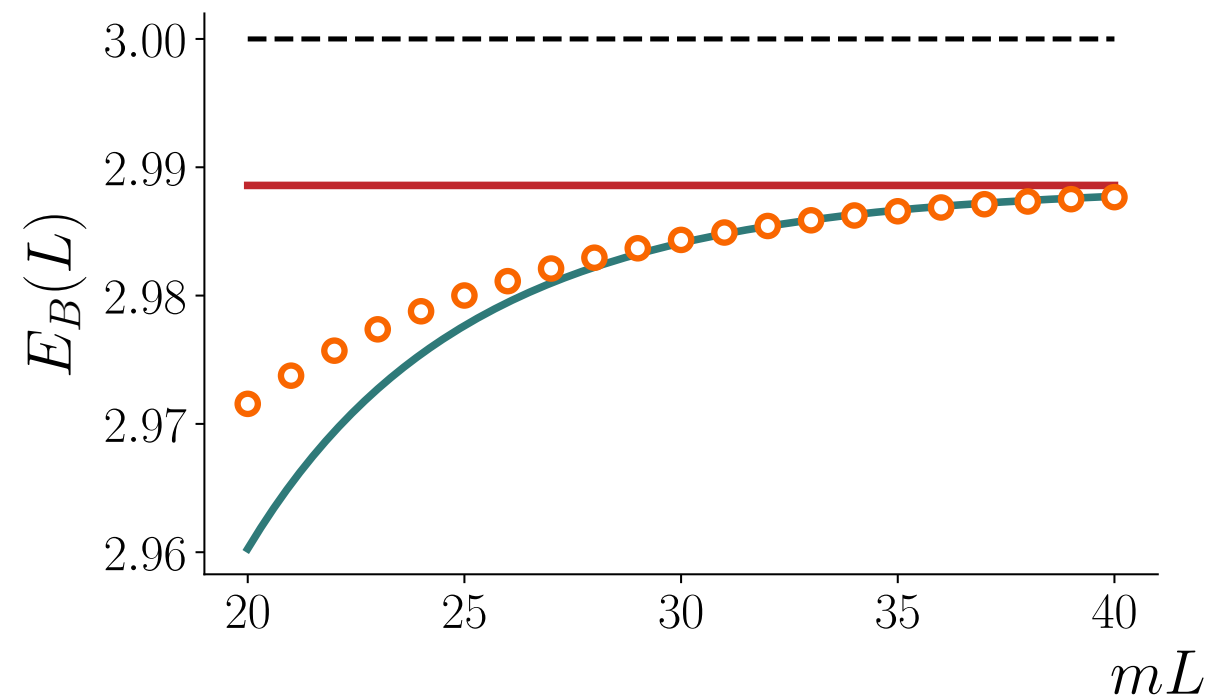
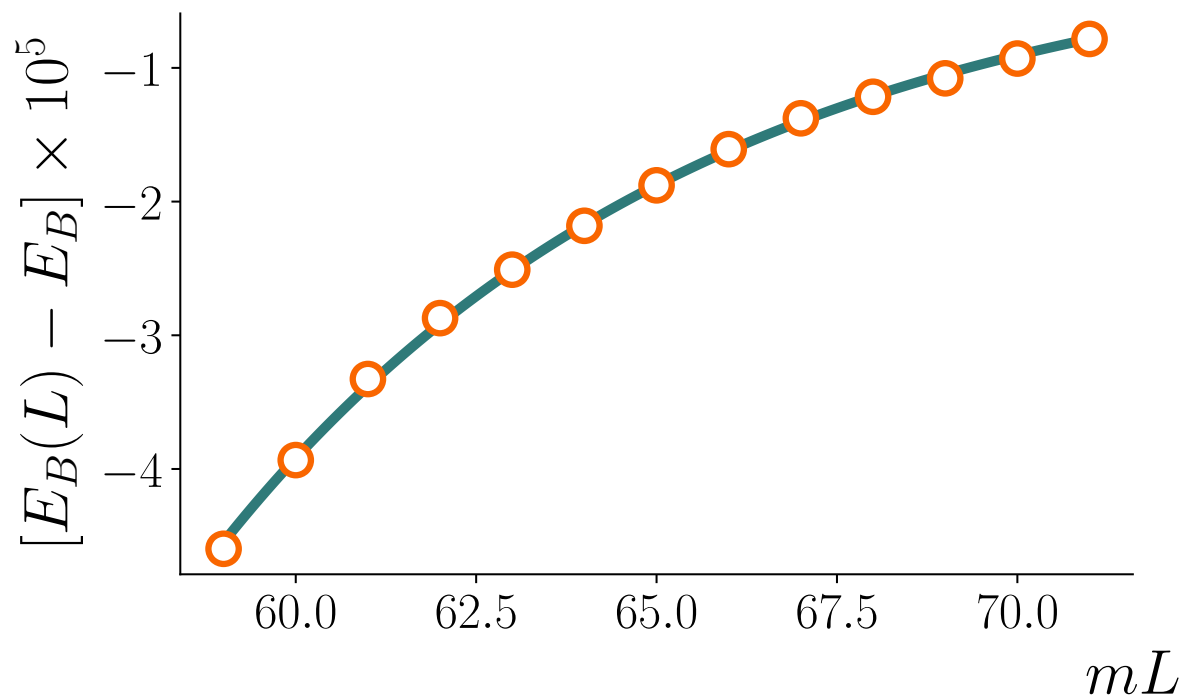
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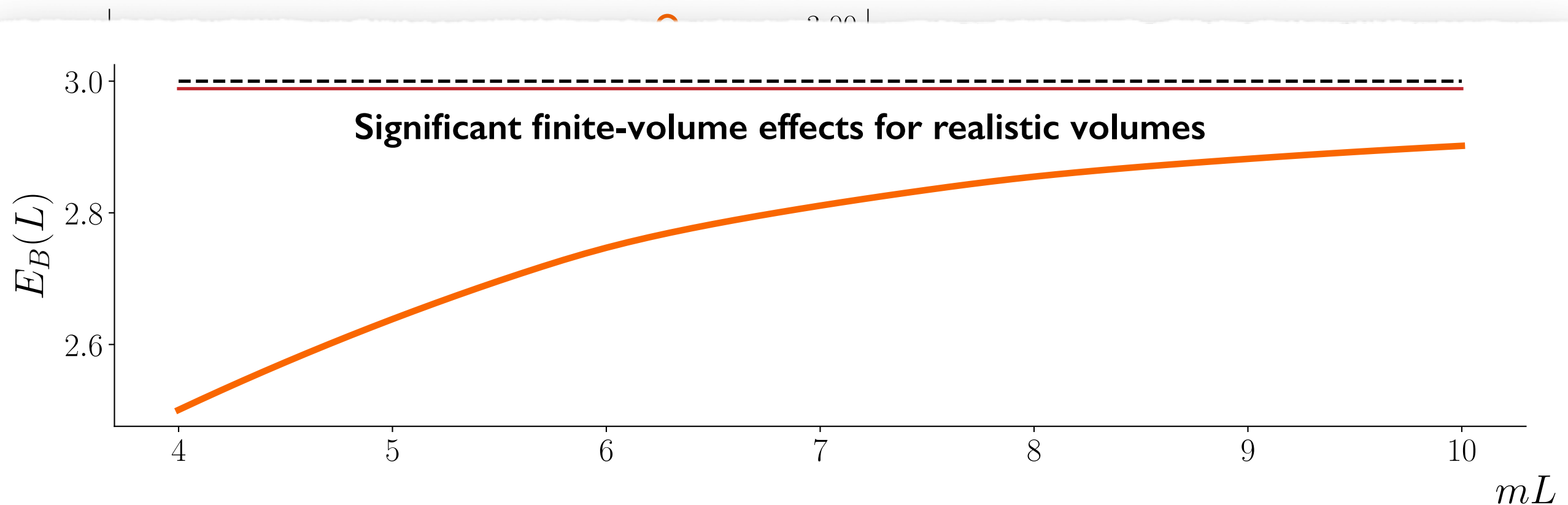
The parameters $a = -10^4$, $\mathcal{K}_{\text{df},3}^{\text{iso}}(E) = 2500$ lead to a shallow bound state

$$\kappa \approx 0.1m \text{ where } E_B = 3m - \kappa^2/m$$

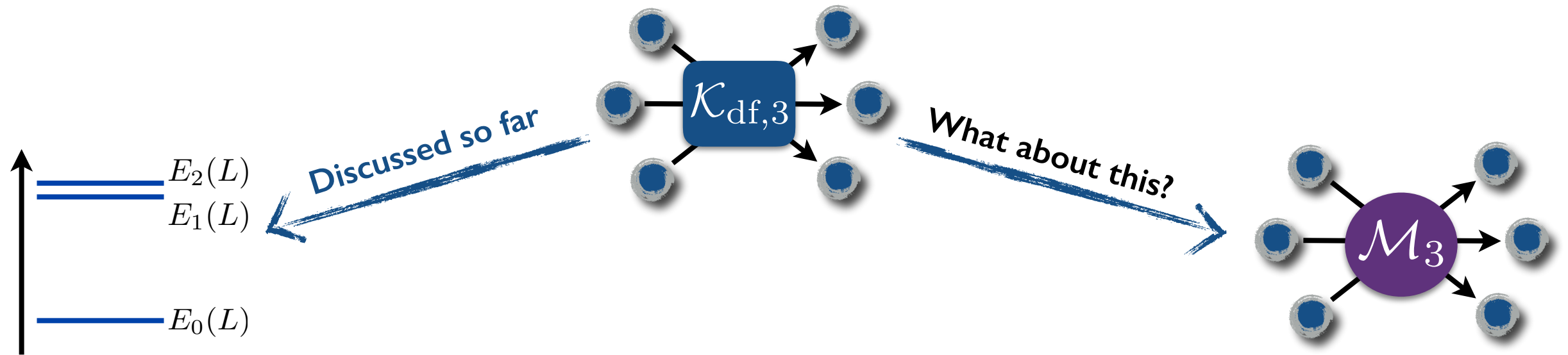
Finite-volume behavior of this state has a known asymptotic form

Meißner, Rios, Rusetsky (2015)

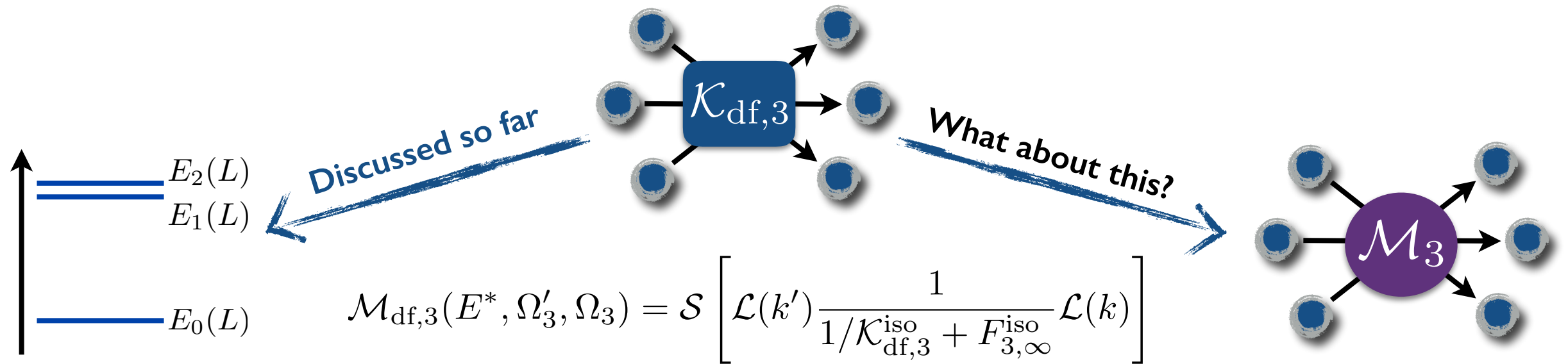
$$E_B(L) = 3m - \frac{\kappa^2}{m} - (98.35 \dots) |A|^2 \frac{\kappa^2}{m} \frac{e^{-2\kappa L/\sqrt{3}}}{(\kappa L)^{3/2}} \left[1 + \mathcal{O}\left(\frac{1}{\kappa L}, \frac{\kappa^2}{m^2}, e^{-\alpha\kappa L}\right) \right]$$



Converting to scattering amplitudes



Converting to scattering amplitudes

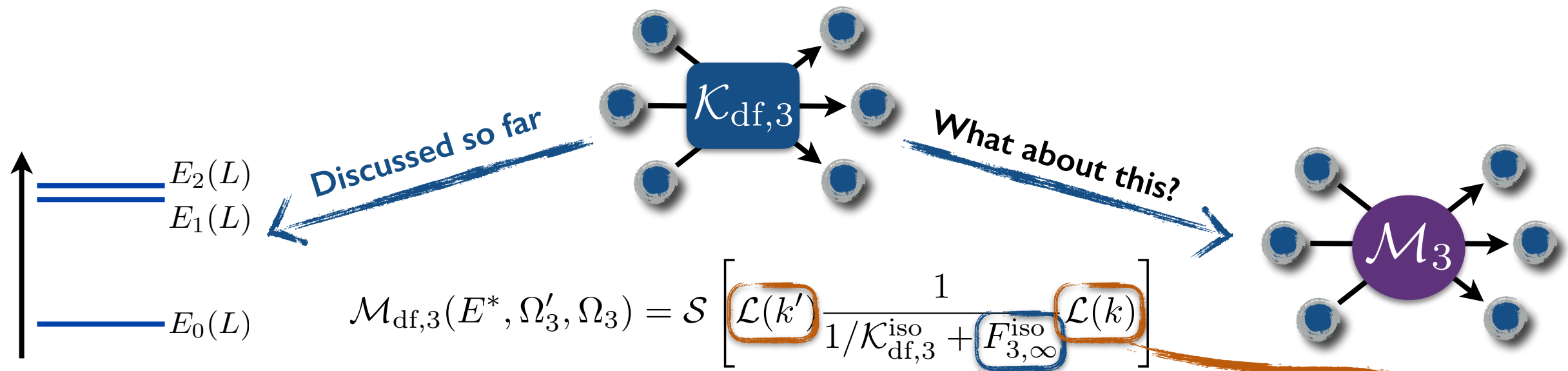


$$\mathcal{M}_{\text{df},3}(E^*, \Omega'_3, \Omega_3) = \mathcal{S} \left[\mathcal{L}(k') \frac{1}{1/\mathcal{K}_{\text{df},3}^{\text{iso}} + F_{3,\infty}^{\text{iso}}} \mathcal{L}(k) \right]$$

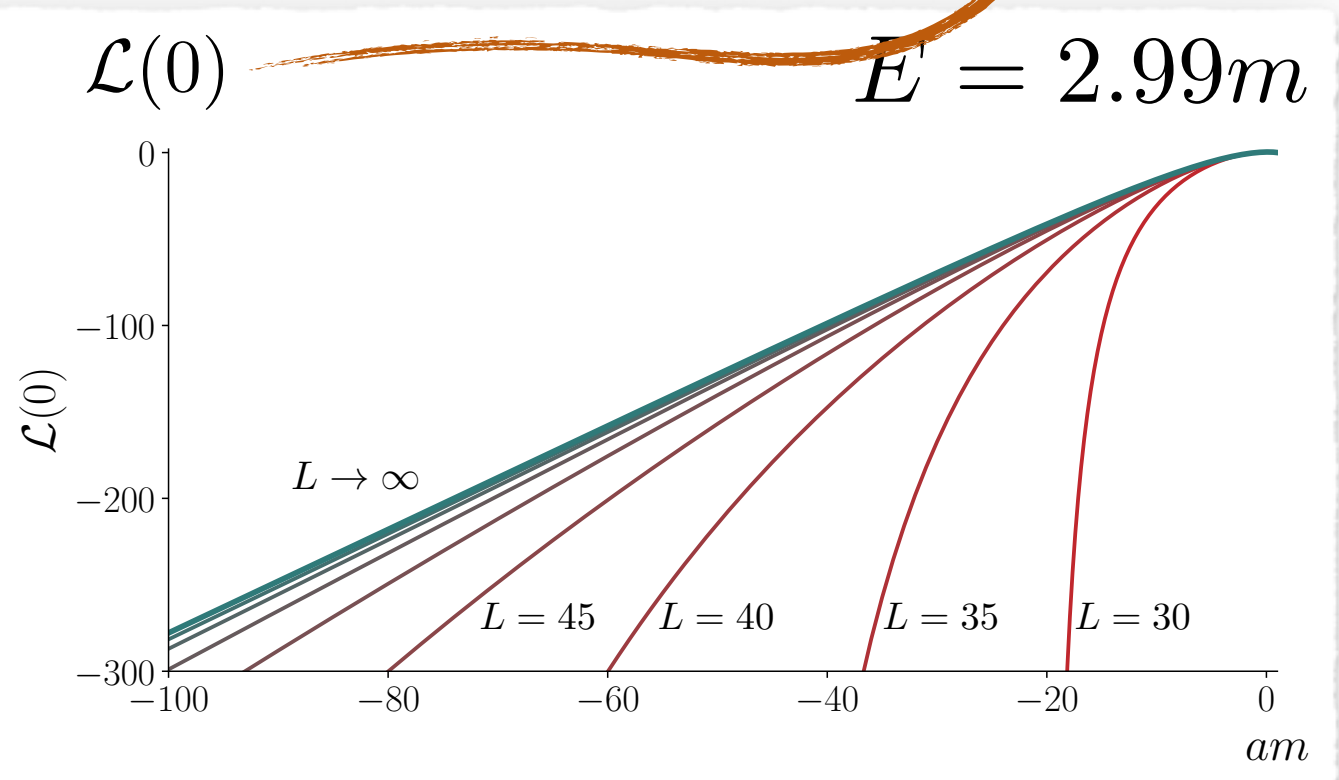
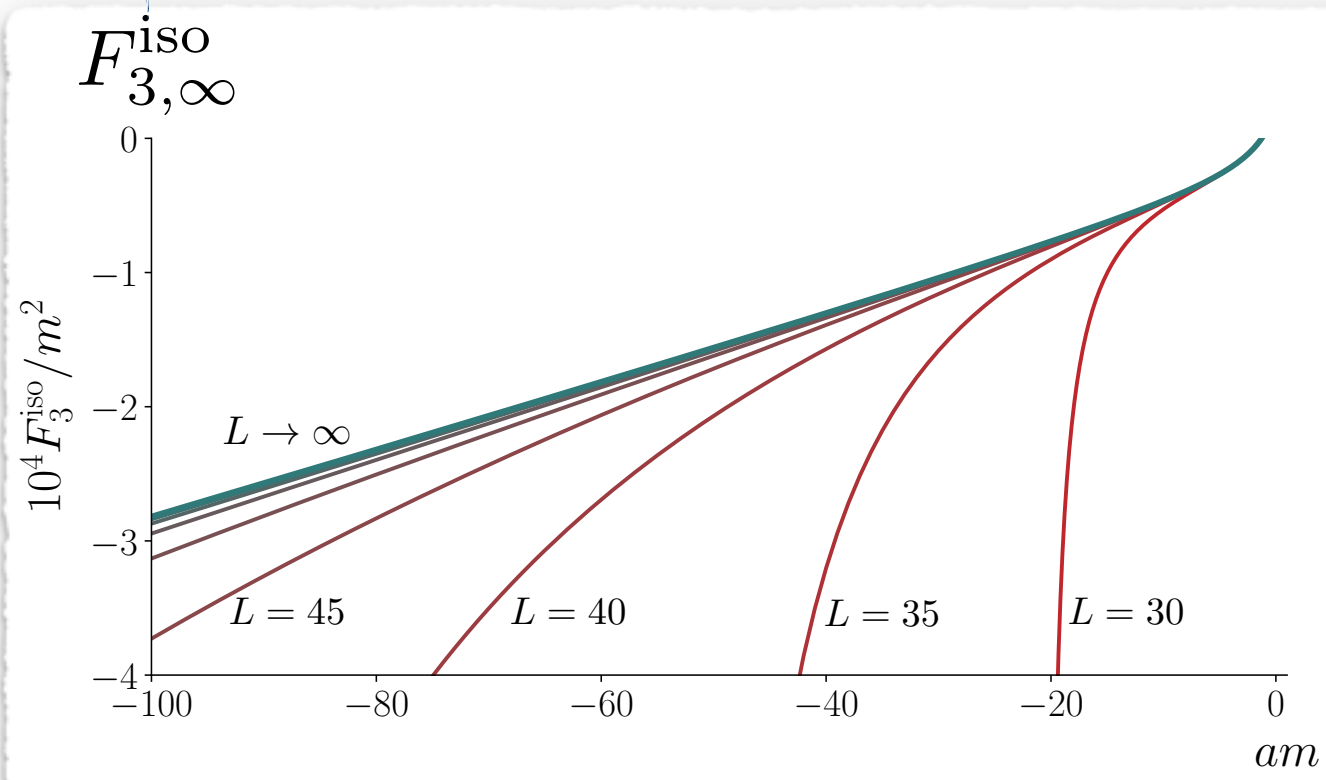
$$i\mathcal{M}_{\text{df},3} = i\mathcal{M}_3 - \mathcal{S} \left[\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \dots \end{array} \right]$$

The diagrams in the brackets represent a series of scattering processes involving two intermediate particles, each labeled $i\mathcal{M}_2$. The first diagram shows two particles interacting via two $i\mathcal{M}_2$ vertices. The second diagram shows two particles interacting via two $i\mathcal{M}_2$ vertices in a different configuration. The series continues with an ellipsis.

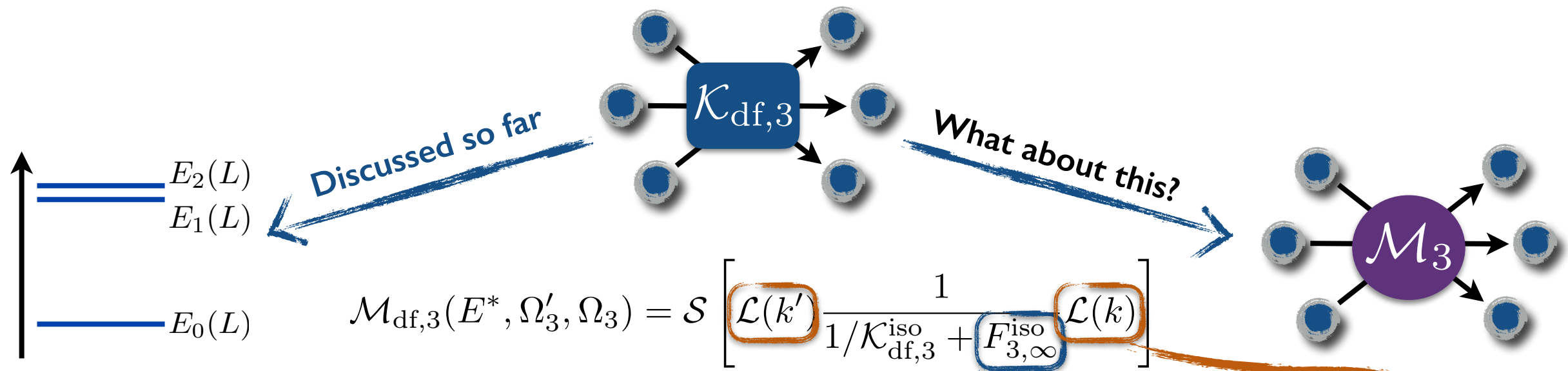
Converting to scattering amplitudes



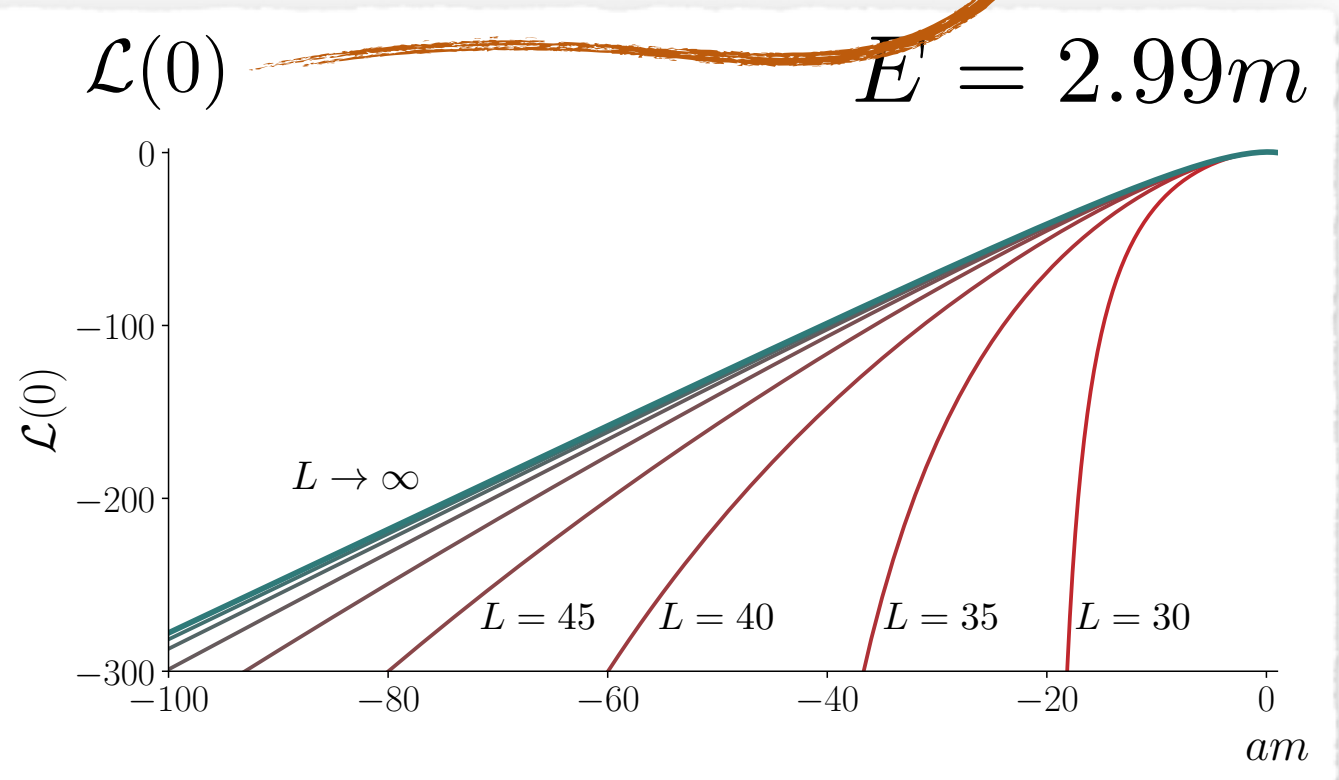
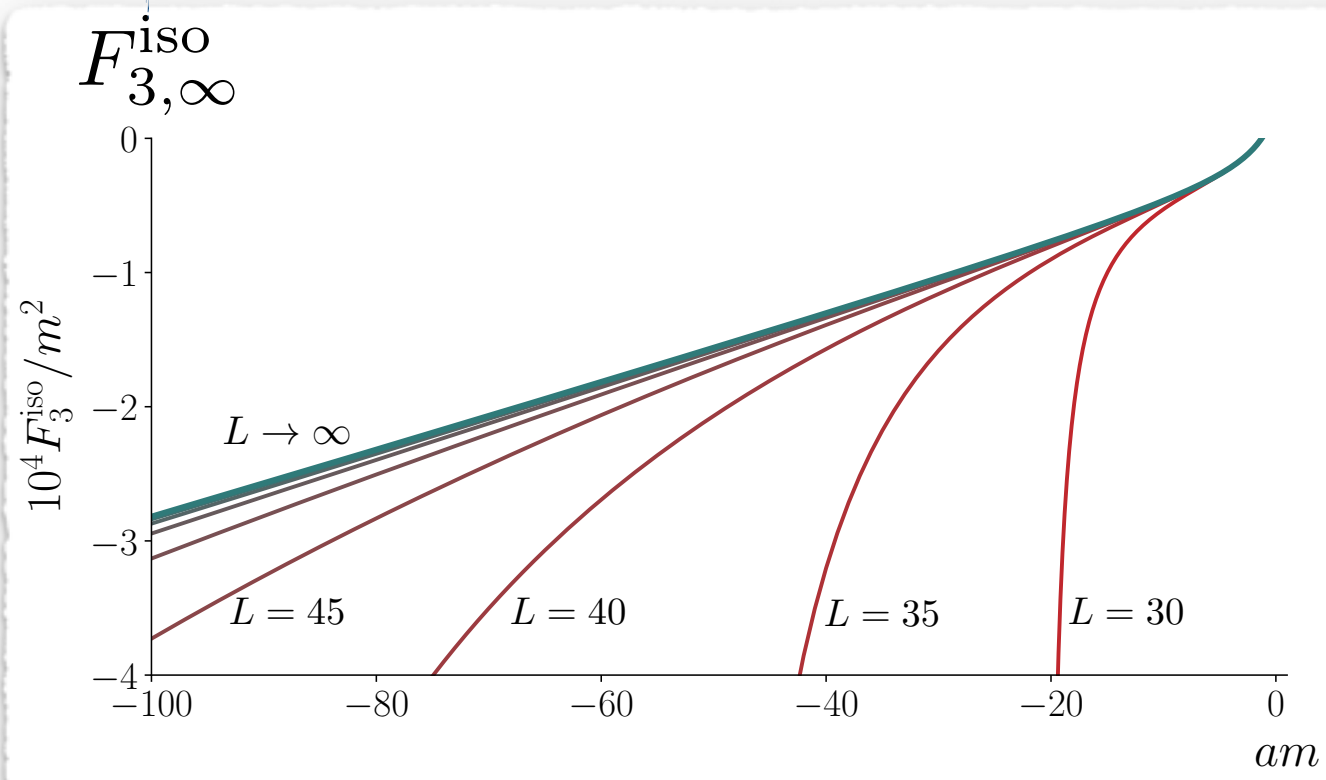
$$i\mathcal{M}_{\text{df},3} = i\mathcal{M}_3 - \mathcal{S} \left[\text{Diagram 1} + \text{Diagram 2} + \dots \right]$$



Converting to scattering amplitudes

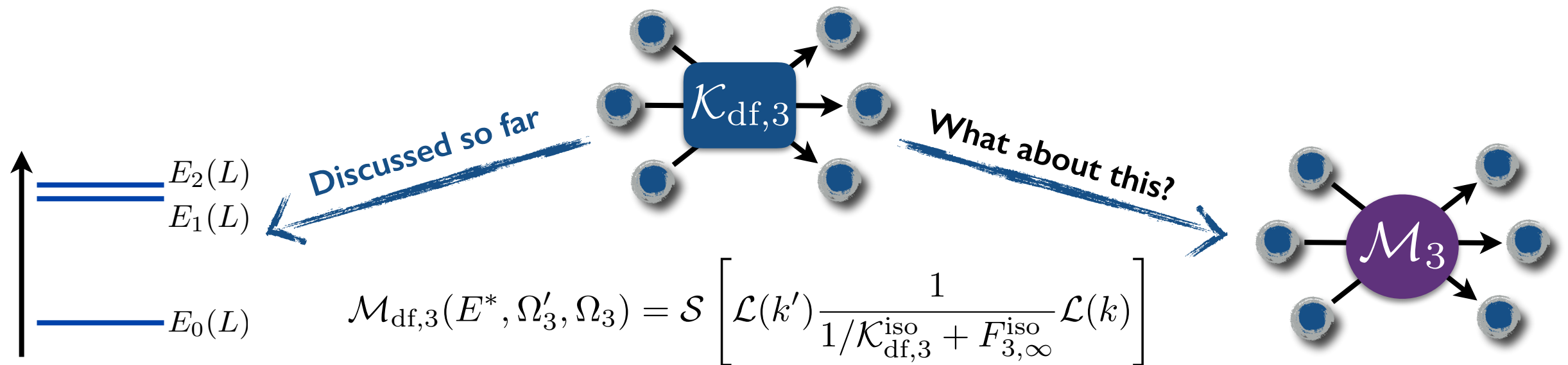


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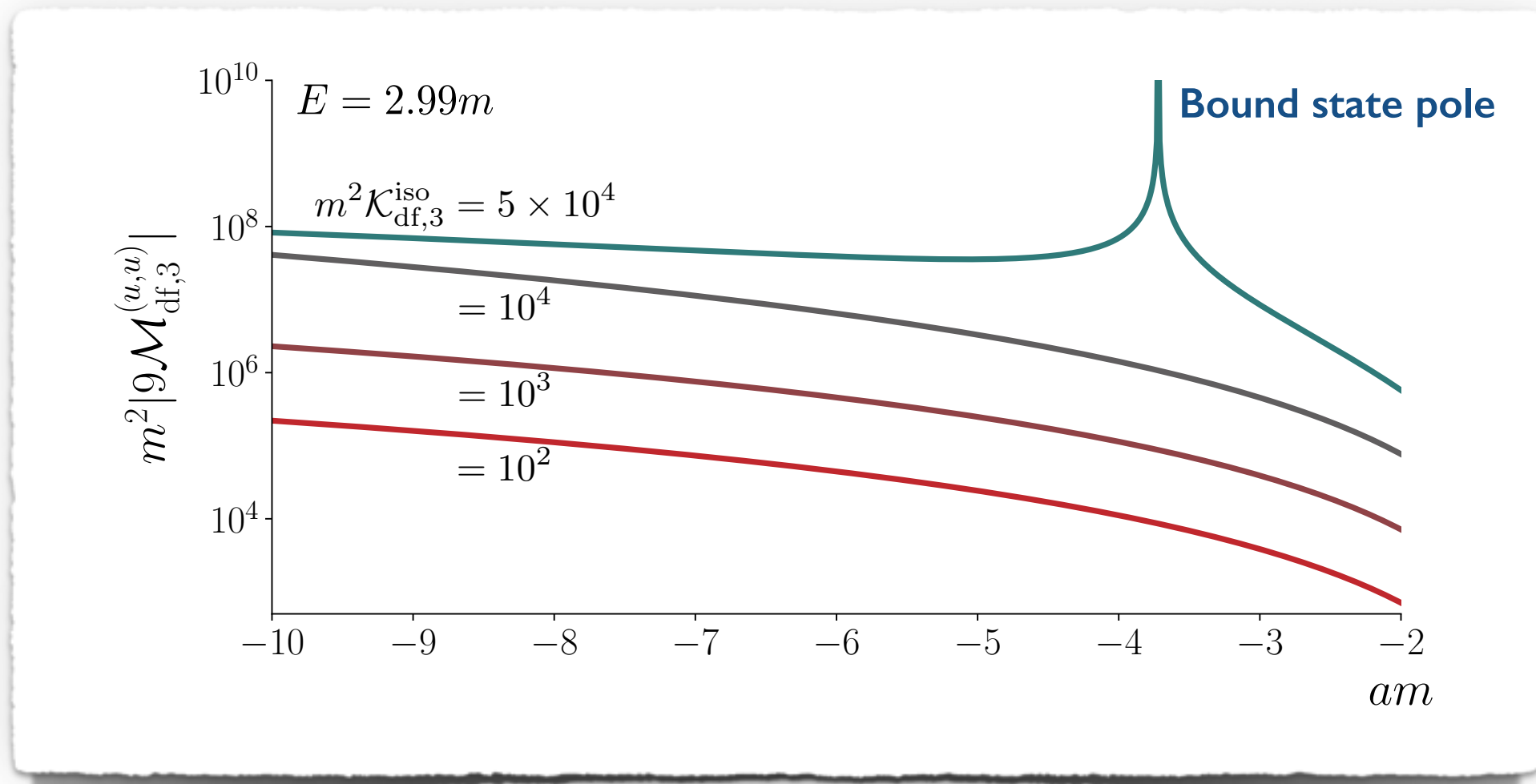
This only works below threshold... Relation above threshold crucially needed

Converting to scattering amplitudes

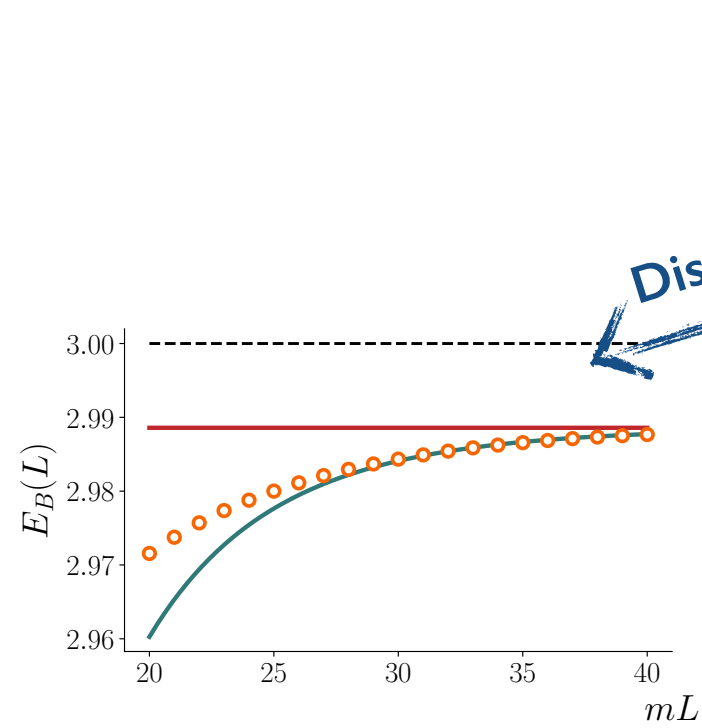


$$i\mathcal{M}_{\text{df},3} = i\mathcal{M}_3 - \mathcal{S} \left[\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \dots \end{array} \right]$$

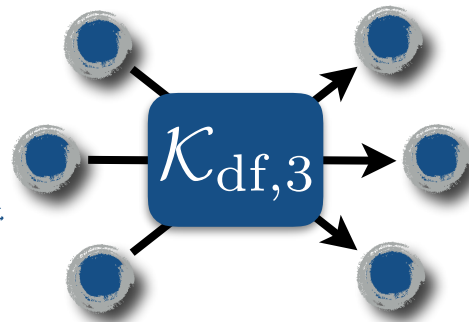
The diagrams in the brackets represent Feynman diagrams with internal vertices labeled $i\mathcal{M}_2$.



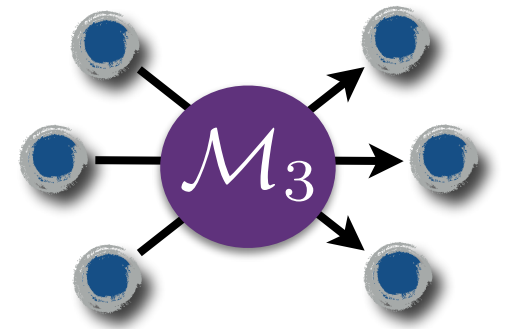
Back to the bound state



Discussed so far

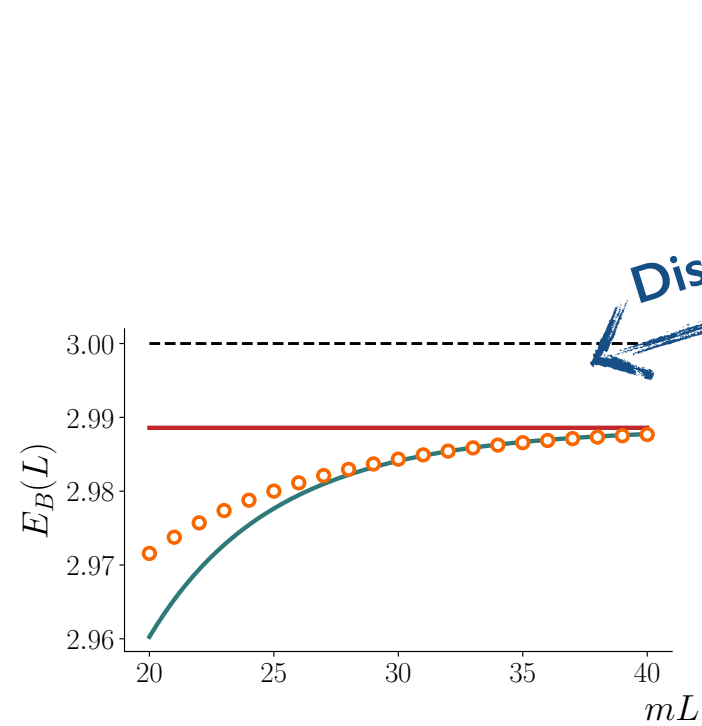


What about this?

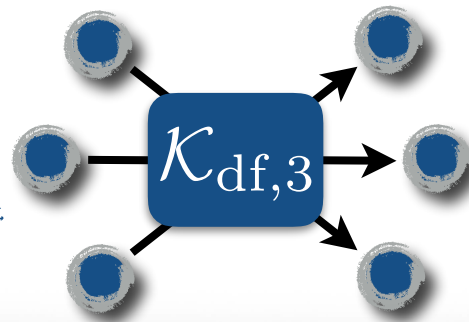


$$\mathcal{M}_3 \sim -\frac{\Gamma(\Omega'_3)\bar{\Gamma}(\Omega_3)}{E^2 - M_B^2}$$

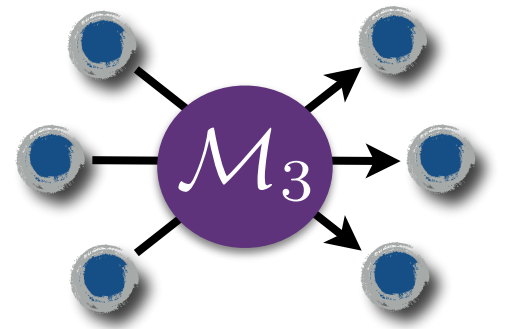
Back to the bound state



Discussed so far



What about this?



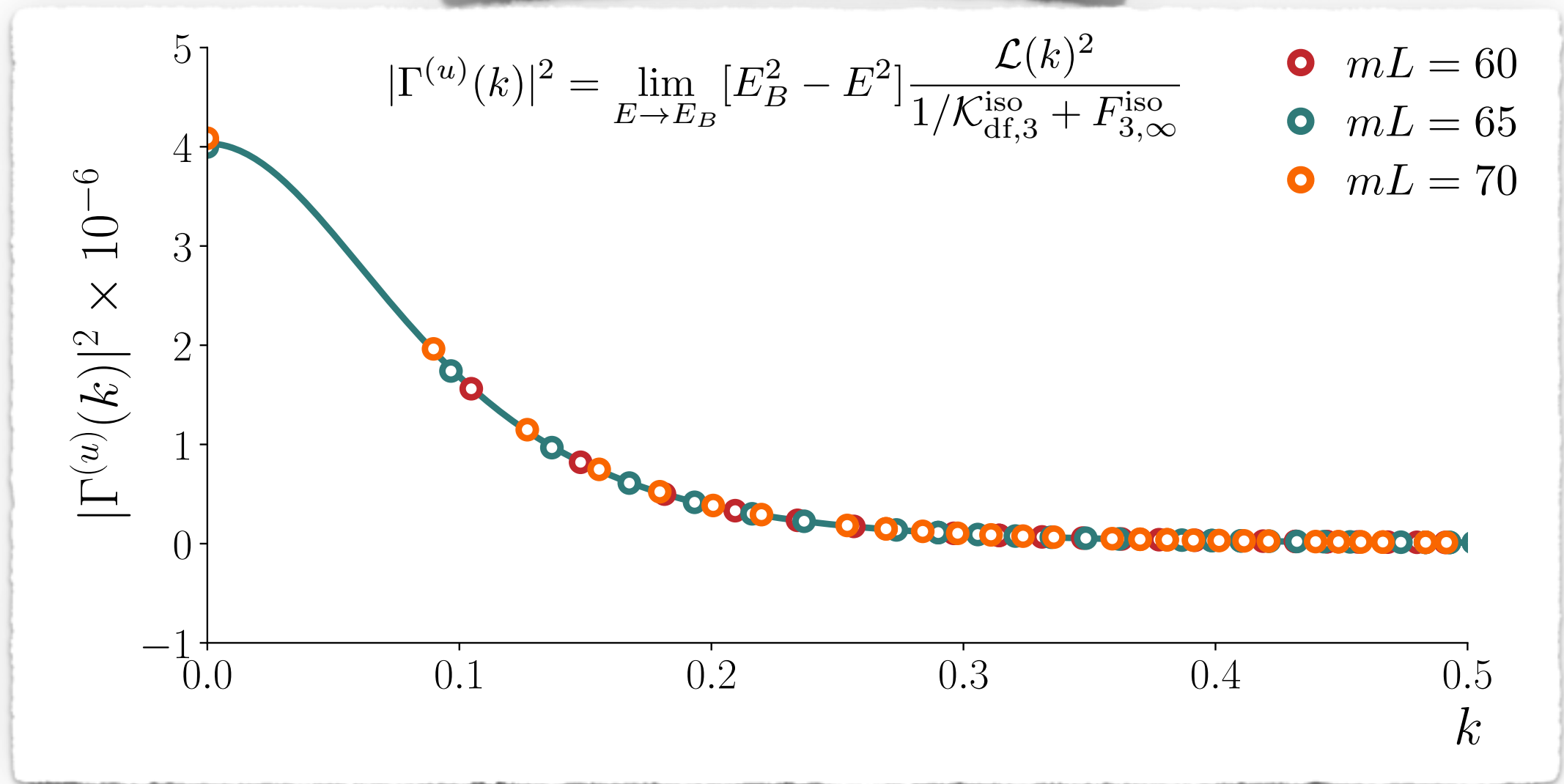
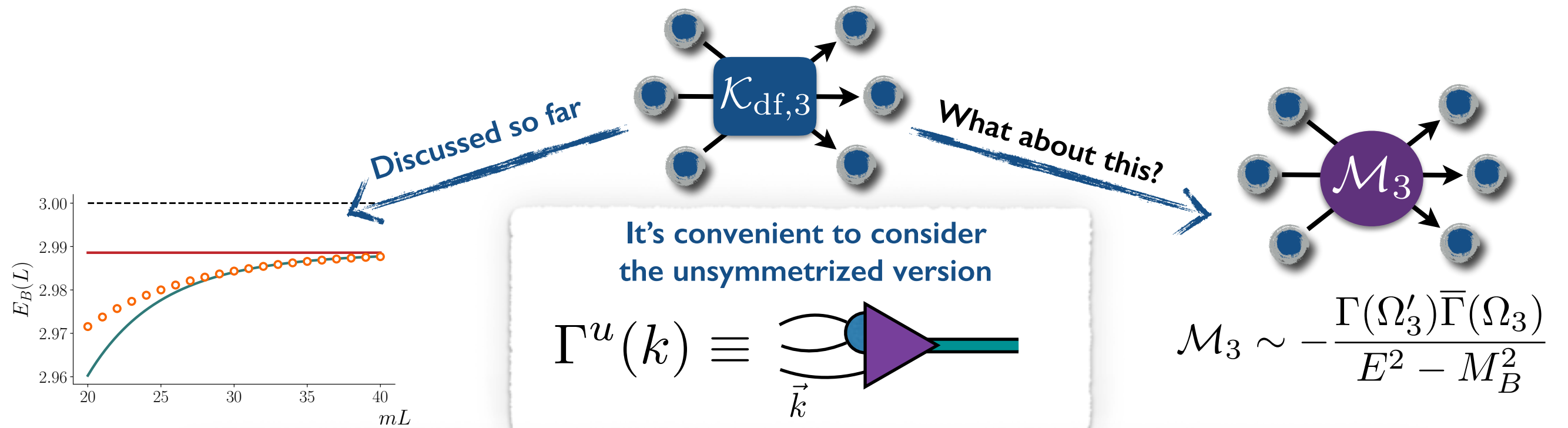
It's convenient to consider the unsymmetrized version

$$\Gamma^u(k) \equiv \text{Diagram of a purple triangle with two incoming lines and one outgoing line, labeled with } \vec{k}$$

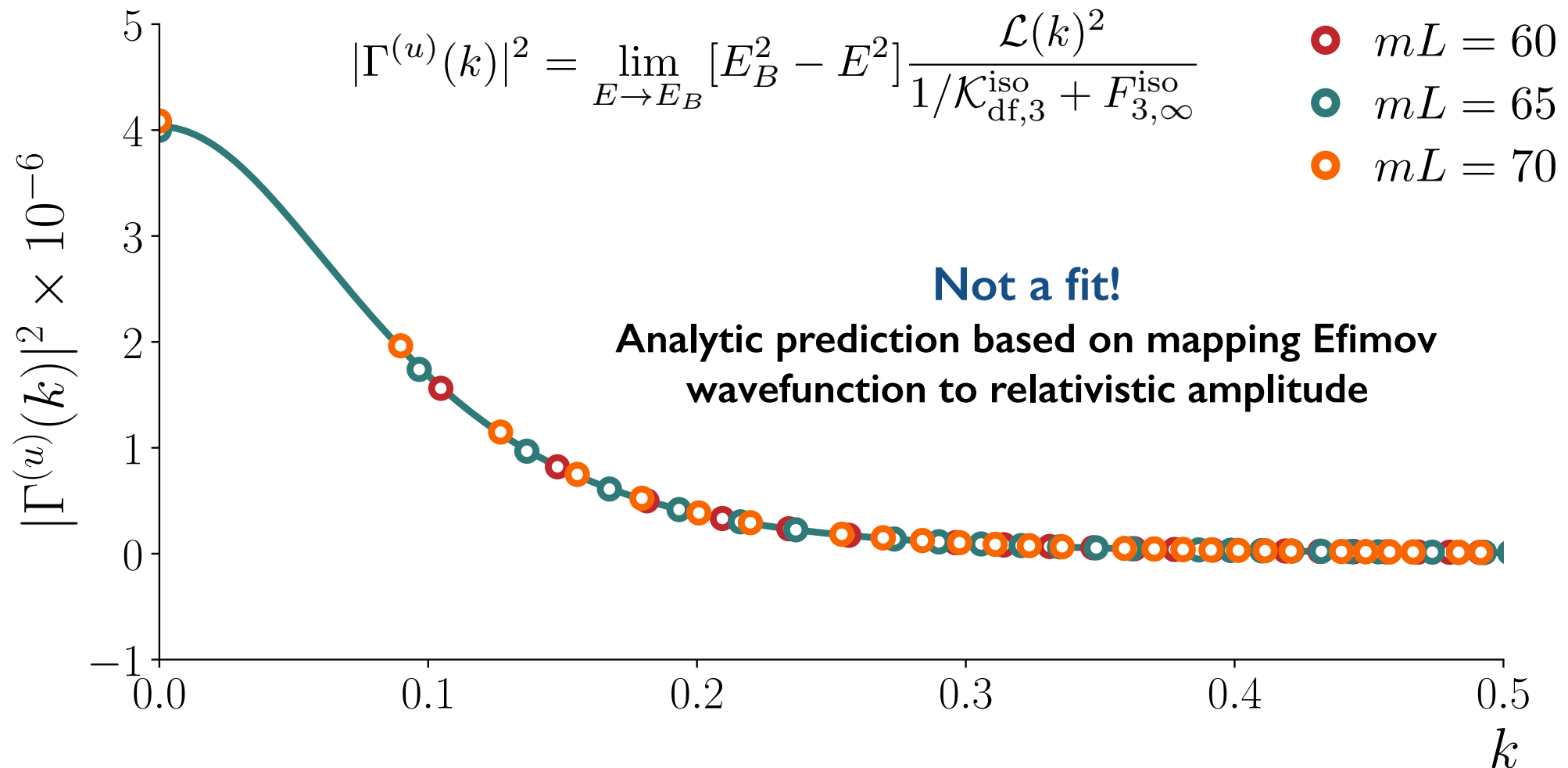
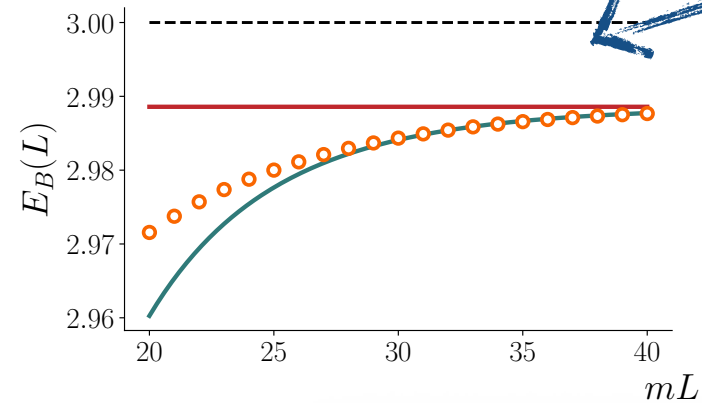
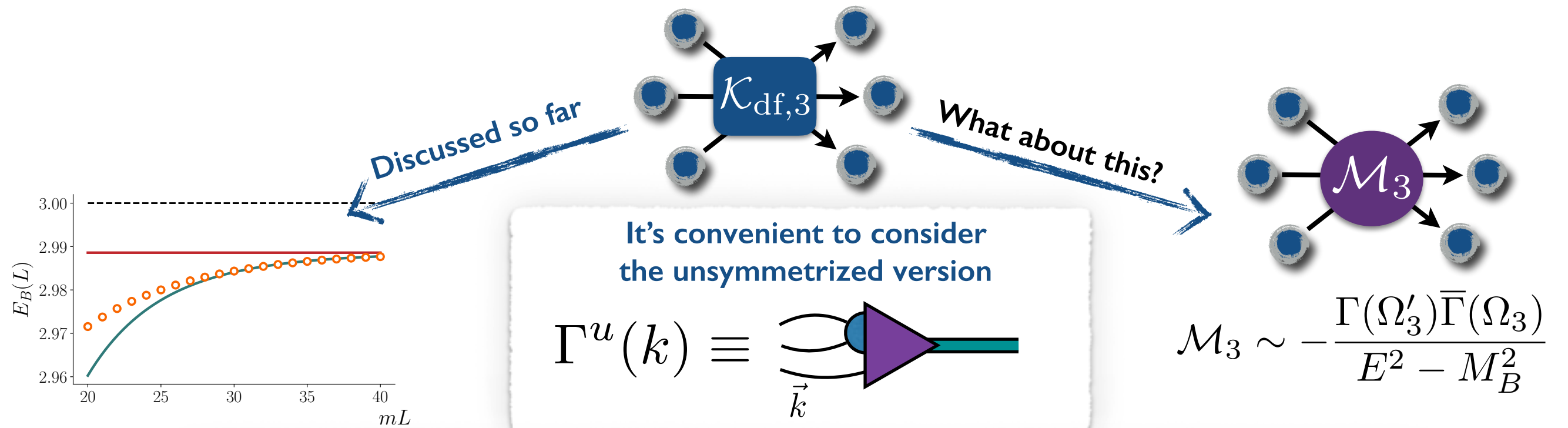
$$\mathcal{S}[\Gamma^u(k)] \equiv \langle \pi(k) \pi \pi | E_B \rangle$$

$$\mathcal{M}_3 \sim -\frac{\Gamma(\Omega'_3) \bar{\Gamma}(\Omega_3)}{E^2 - M_B^2}$$

Back to the bound state



Back to the bound state





Outline

Warm up and definitions

- Basic set-up
 - Finite-volume correlator
 - Three non-interacting particles
-

Two particles in a box

- Alternative derivation
- Truncation and application
- Relating matrix elements

Three particles in a box

- 3-to-3 scattering
 - (Sketch of) derivation
 - An unexpected infinite-volume quantity
 - Relating energies to scattering
-

Testing the result

- Know issues
- Large-volume expansion
- Effimov state in a box

Other methods

Numerical explorations

- Truncation at low energies
- Toy solutions for various systems
- Unphysical solutions

Looking forward

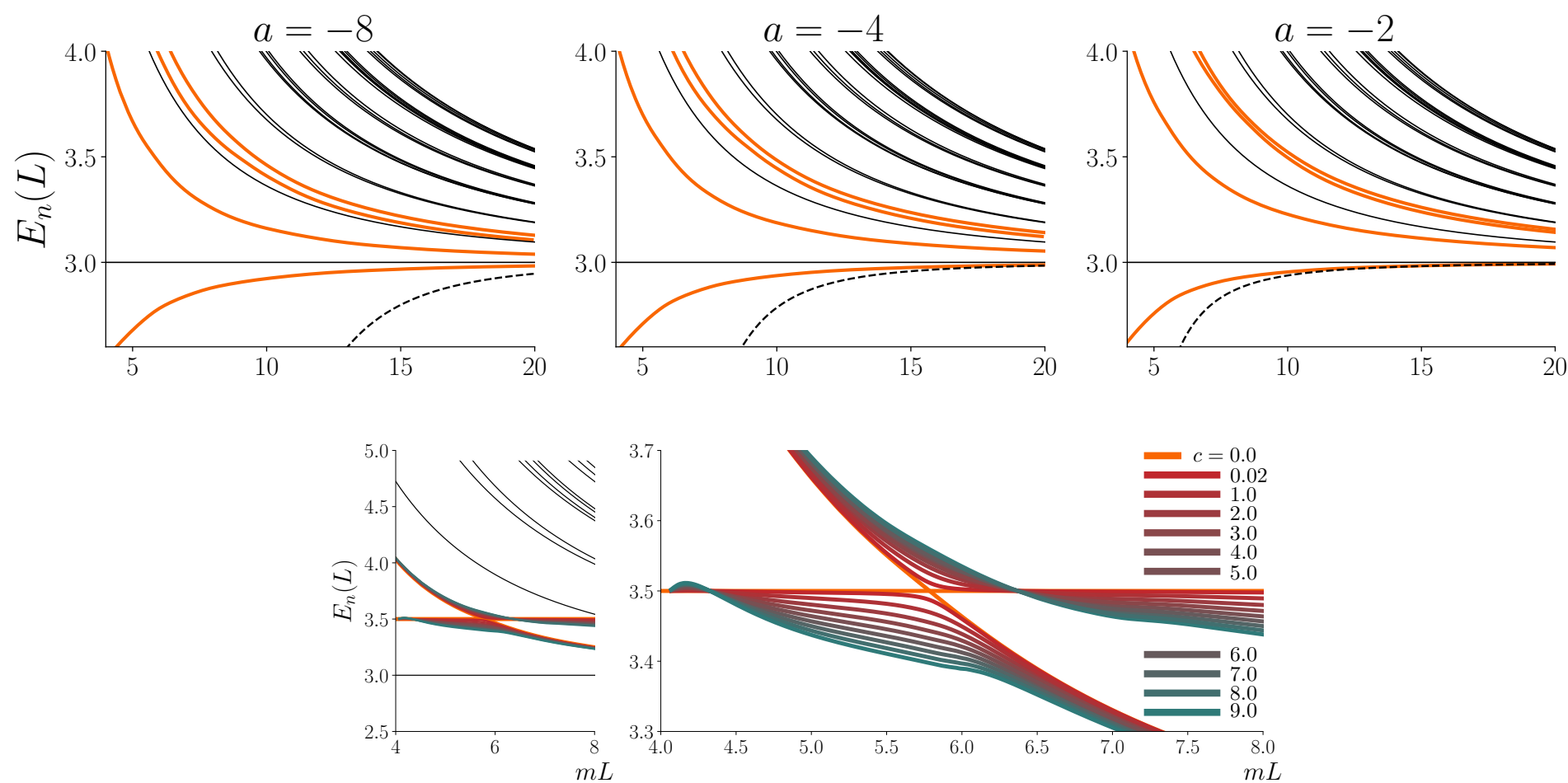
Still lots to do

- Finish result with intermediate two-particle resonances
- Understand unphysical solutions
- Extend to non-identical, non-degenerate, multiple channels, spin
- Study subduction to finite-volume irreps
- Understand rigorous parametrizations for the infinite-volume observables
- Convince practitioners that the formalism is mature
- Reliably measure finite-volume spectra
- Extract three-particle scattering from LQCD

Still lots to do

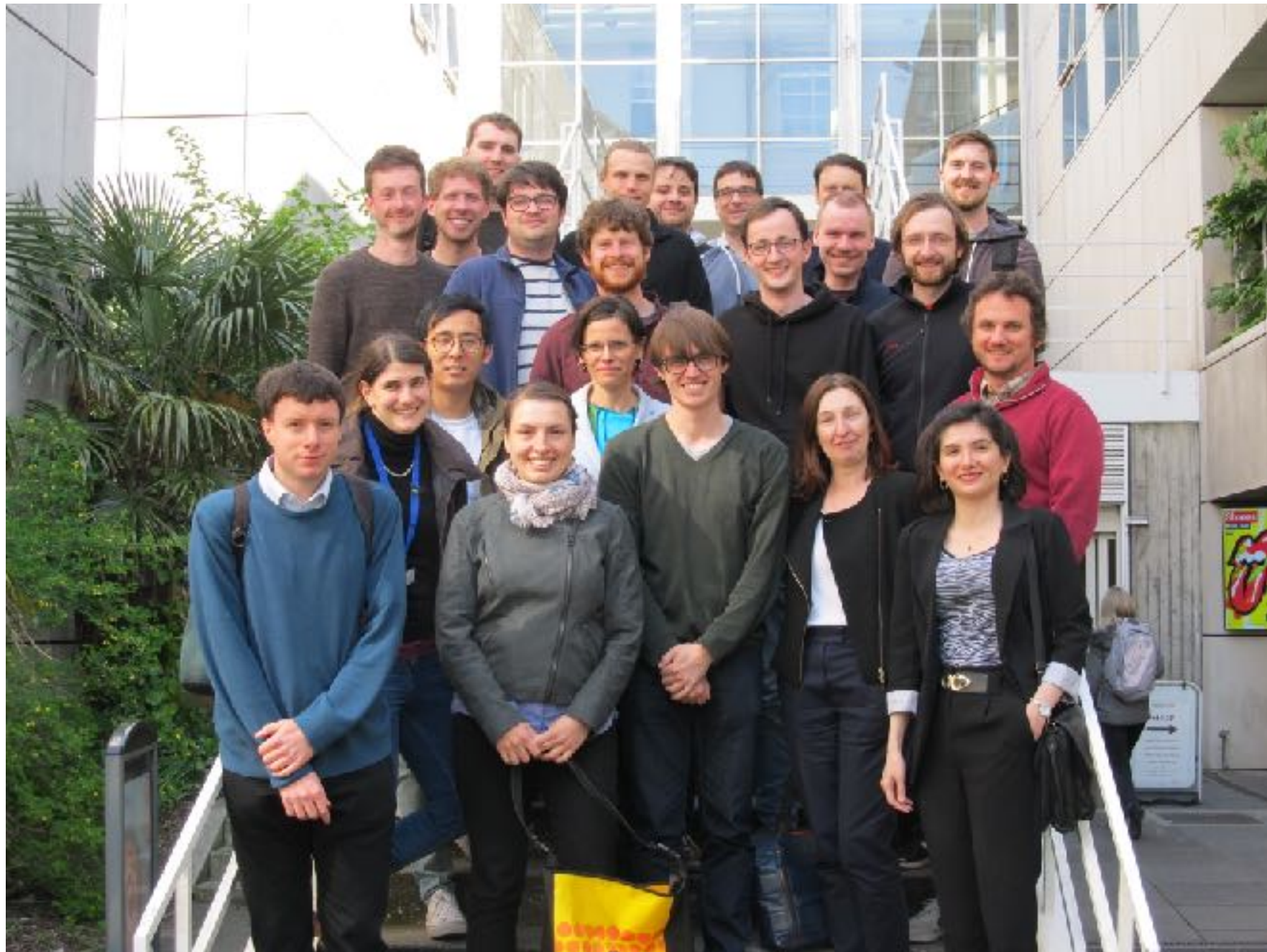
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- Reliably measure finite-volume spectra
- Extract three-particle scattering from LQCD

Big picture: making progress, but not quite there yet



Thanks!

That's all folks!



Thanks to all the participants!