Three-particle scattering from numerical lattice QCD

Scattering from the lattice: applications to phenomenology and beyond

Maxwell T. Hansen

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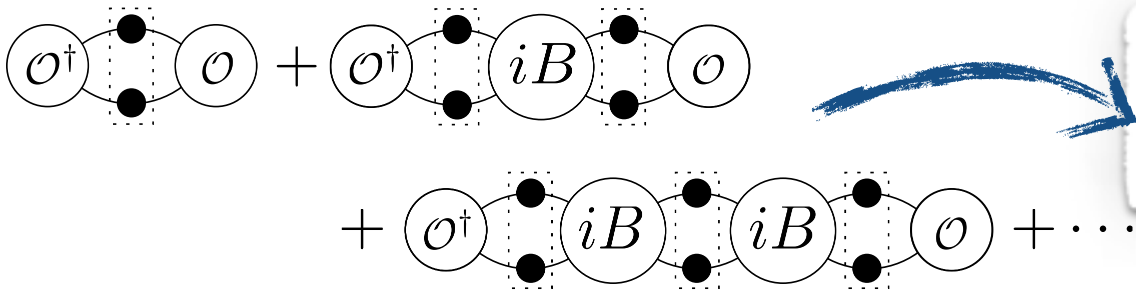
CERN

Review:

☑ Alternative two-particle derivation (Kim, Sachrajda, Sharpe)

$C_L(P) =$



$$C_L(P) = C_\infty(P) - A' F \frac{1}{1 + \mathcal{M}_{2\to 2} F} A$$

☑ Lellouch-Lüscher via pole matching

$$-\frac{L^3 \langle 0|\mathcal{O}(0)|n, \vec{P}, L\rangle \langle n, \vec{P}, L|\mathcal{O}^\dagger(0)|0\rangle}{E - E_n} \qquad -\frac{\langle 0|\mathcal{O}(0)|\pi\pi, \text{in}\rangle \mathcal{R}(E_n, \vec{P}, L) \langle \pi\pi, \text{out}|\mathcal{O}^\dagger(0)|0\rangle}{E - E_n}$$

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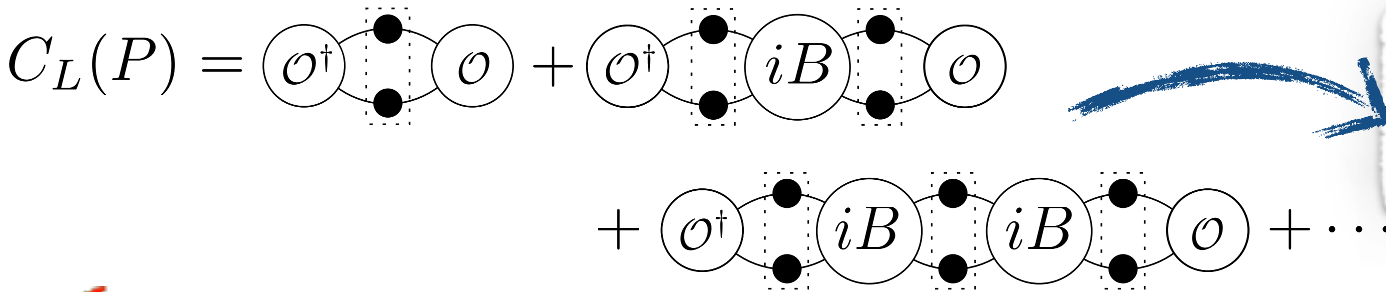

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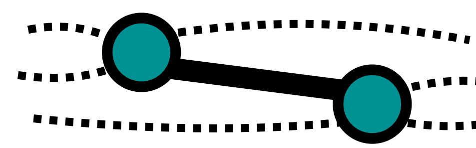
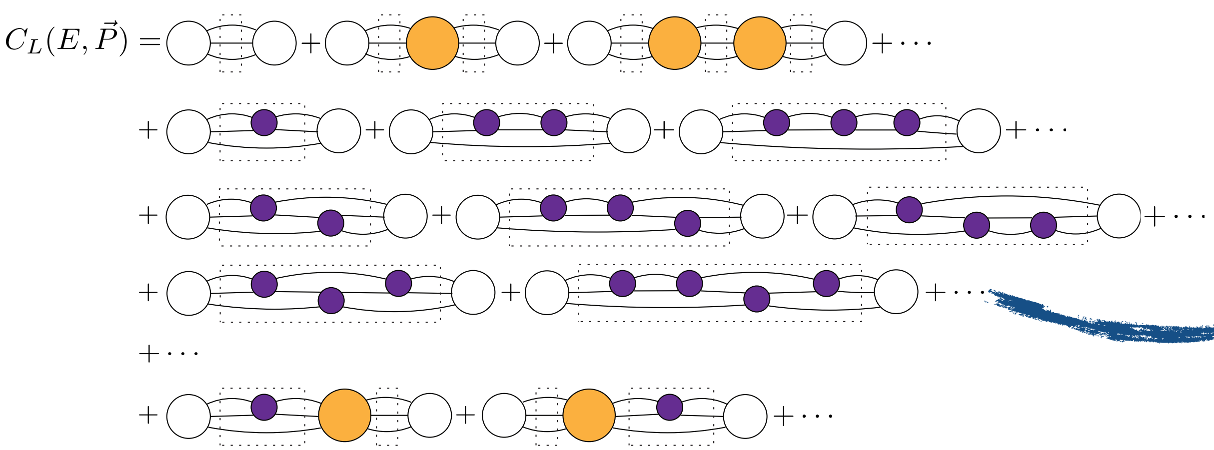
☑ A real, smooth, three-particle quantity… $\mathcal{K}_{\text{df},3} \not\supset$ 

☑ Three-particle quantization condition



$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) - A' F_3 \frac{1}{1 + \mathcal{K}_{\text{df},3} F_3} A$$

☑ Road to physics:

I. Use q.c. + energy levels to determine $\mathcal{K}_{\text{df},3}$

II. Use known integral equation to relate $\mathcal{K}_{\text{df},3}$ to $\mathcal{M}_3$

Outline

☑ Warm up and definitions
- ☑ Basic set-up
- ☑ Finite-volume correlator
- ☑ Three non-interacting particles

☑ Two particles in a box
- ☑ Alternative derivation
- ☑ Truncation and application
- ☑ Relating matrix elements

☑ Three particles in a box
- ☑ 3-to-3 scattering
- ☑ (Sketch of) derivation
- ☑ An unexpected infinite-volume quantity
- ☑ Relating energies to scattering

☐ Testing the result
- ☐ Know issues
- ☐ Large-volume expansion
- ☐ Effimov state in a box

☐ Other methods

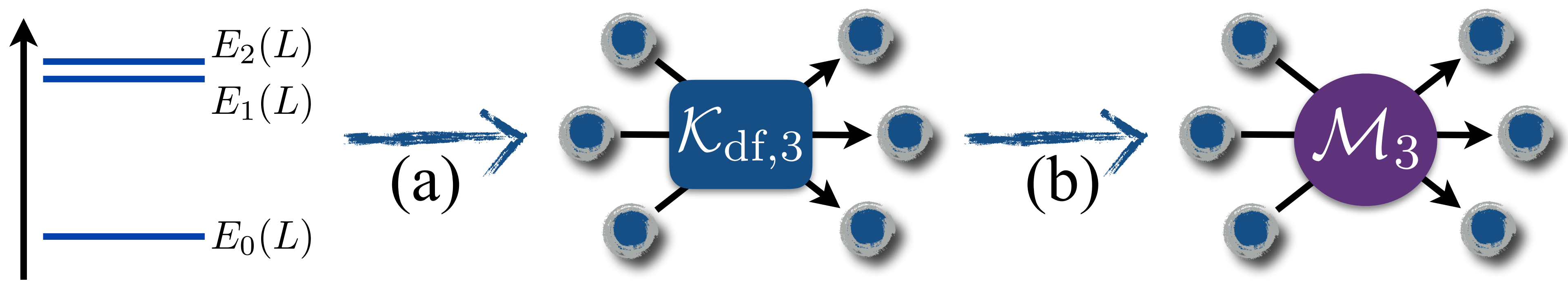
☐ Numerical explorations
- ☐ Truncation at low energies
- ☐ Toy solutions for various systems
- ☐ Unphysical solutions

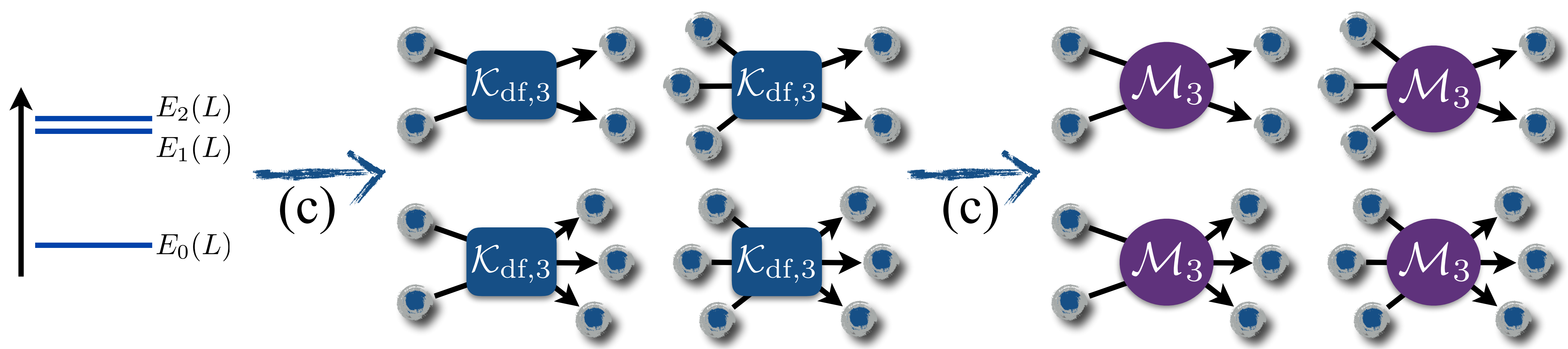
☐ Looking forward

Current status

Model- & EFT-independent relation between

finite-volume energies and relativistic two-and-three particle scattering





(a),(b) ***MTH and Sharpe (2015),(2016)***
(c) ***Briceño, MTH, Sharpe (2017)***

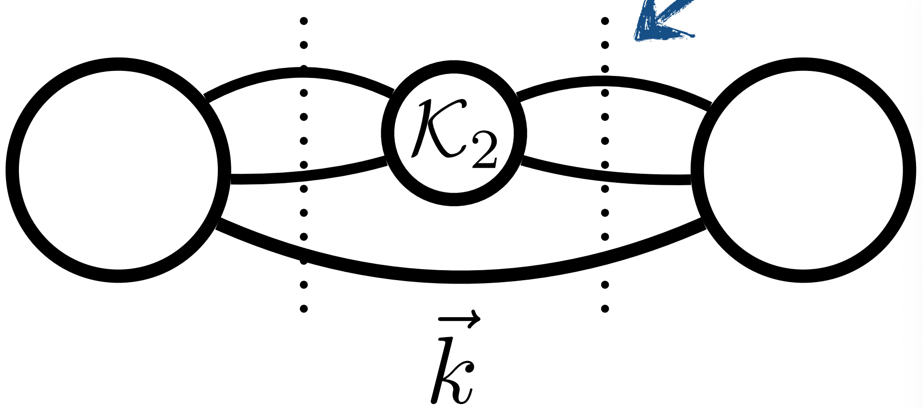
Smooth cutoff function

$\mathcal{K}_{\mathrm{df},3}$ **and** F_3 **depend on a smooth cutoff function**

To see why, consider one of the contributions to C_L…

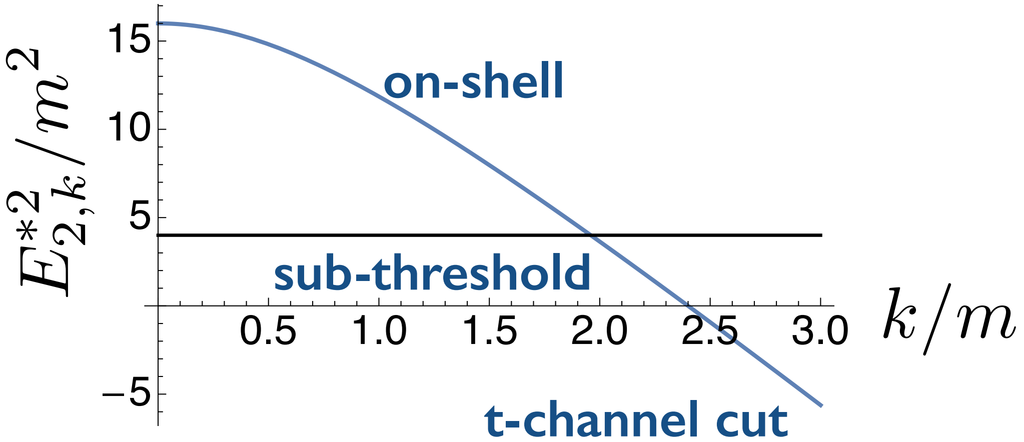
How do we define this on-shell cut?

$C_L(E, \vec{P}) \supset$



Energy of top two particles is:

$$E_{2,k}^{*2} = (E - \omega_k)^2 - (\vec{P} - \vec{k})^2$$



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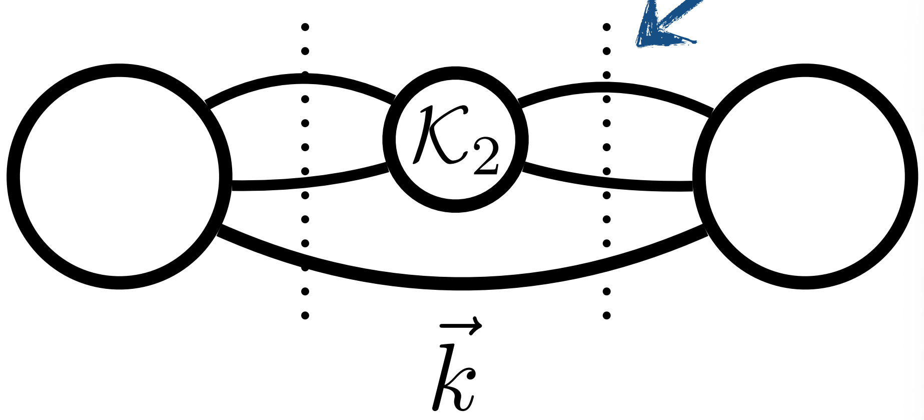

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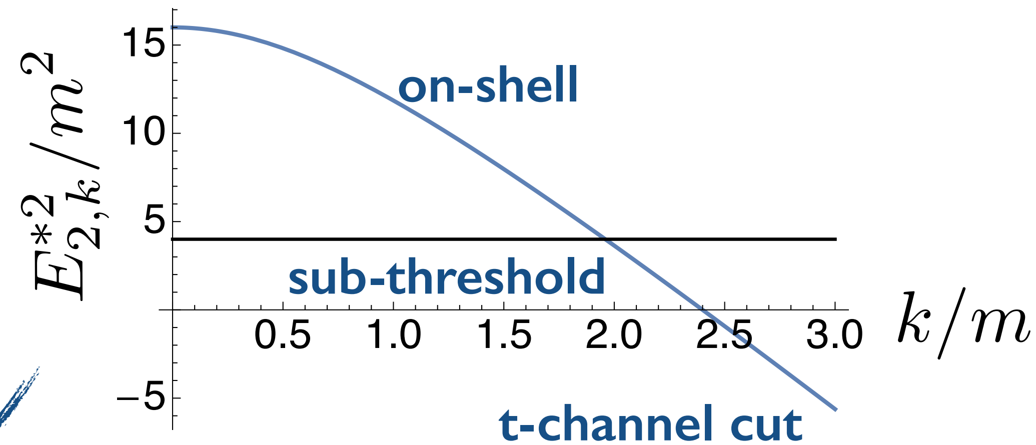
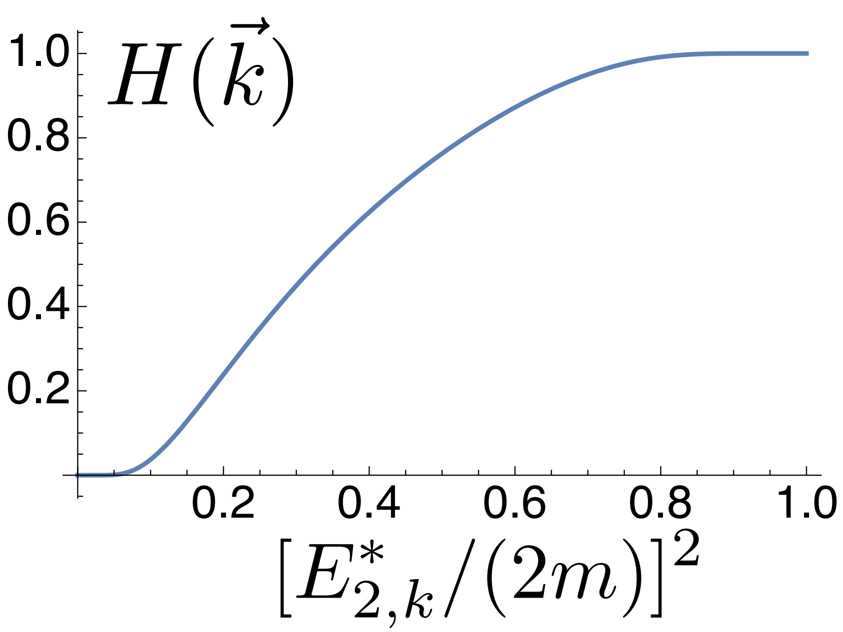


To keep on-shell states and avoid spurious off-shell contributions…

Cuts are defined with



🤔 Is this really necessary? 🤔

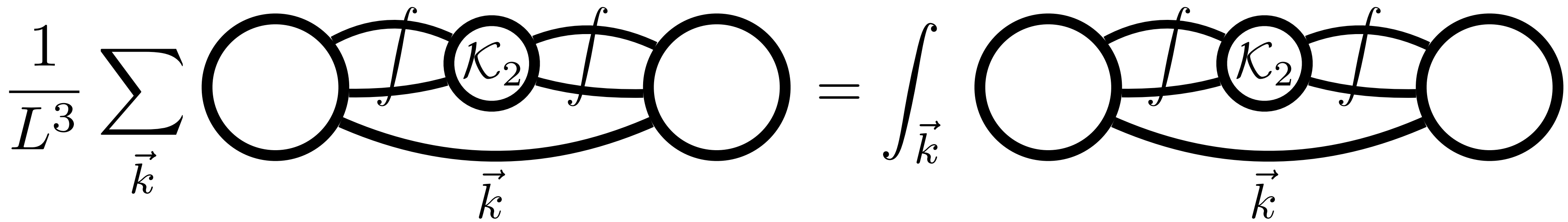
We choose to sum subdiagrams into $\mathcal{K}_2$

On-shell $\mathcal{K}_2$ gives important volume effects
No important effects far below threshold

Must connect the two regions

Important limitation

Current formalism requires no poles in $\mathcal{K}_2$ … Derivation assumes



Given that we are seeking an EFT-independent mapping…

Is it intuitive that $\mathcal{K}_2$ poles need special treatment?

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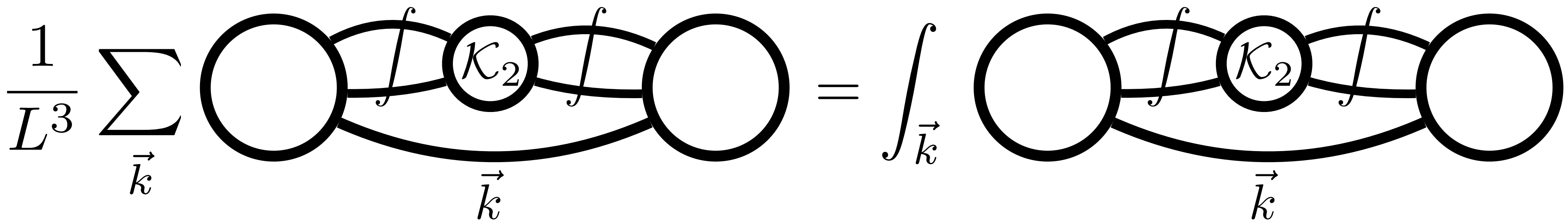

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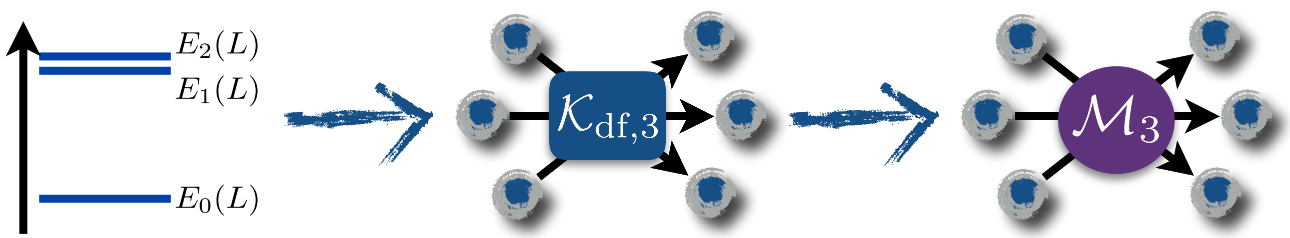


$M_\rho > E_2^* > 2M_\pi \longrightarrow \pi\pi\pi \to \pi\pi\pi$

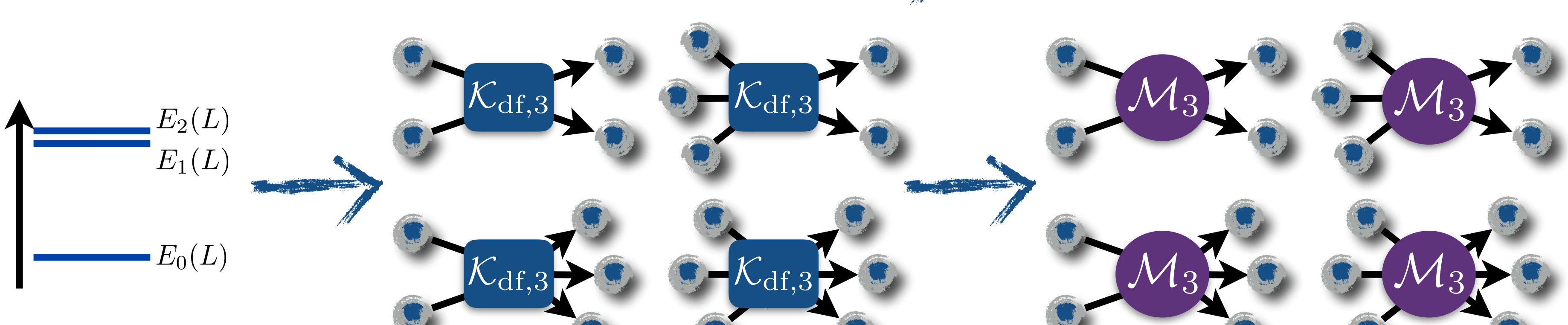
<u>Need to bridge the gap</u>

Update

<u>We now of a complete derivation of formalism that includes K_2 poles.</u>

Briceño, MTH, Sharpe (underway)

$E_2^* > 2M_\pi \gg M_\rho \longrightarrow \rho\pi \to \pi\pi\pi$



The most technical detail of all…

Far below threshold there is no ambiguity about which two-to-two scattering quantity appears in C_L

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Reason: $\frac{1}{L^3}\sum_{\vec{k}} \frac{1}{(2\omega_k)^2(E_{\rm sub}-2\omega_k)} = \int_{\vec{k}} \frac{1}{(2\omega_k)^2(E_{\rm sub}-2\omega_k)} = \text{Analytic Continuation}\left[\int_{\vec{k}} \frac{1}{(2\omega_k)^2(E-2\omega_k+i\epsilon)}\right]$

This means that our subthreshold $\mathcal{K}_2$ is non-standard

$$\mathcal{K}_2^{-1} \propto p^* \cot\delta(p^*) + [1-H(\vec{k})]\kappa(p^*)$$

K matrix above threshold, smooth at threshold, interpolates to the amplitude below threshold

🙈 ***Why are you telling me this?*** 🙈

It is important because our formalism breaks down when there are poles in this definition of $\mathcal{K}_2$.

Testing the formalism (Weak interactions)

☑ **Expansion is well known for small** a/L

$$E = 3m + \frac{12\pi a}{mL^3}\left(1 + c_4\frac{a}{L} + c_5\frac{a^2}{L^2}\right) + \mathcal{O}(1/L^6)$$

Huang and Yang (1957); Beane, Detmold, Savage, (2007); Tan(2007)

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$$\det[F_3(E,L)^{-1} + \mathcal{K}_{\mathrm{df},3}(E)] = 0$$

$$F_{3;00}(E,L) = \frac{1}{L^3}\left[\frac{\tilde{F}}{3} - \tilde{F}\mathcal{H}^{-1}\tilde{F}\right]_{00} \qquad \mathcal{H} = \tilde{\mathcal{K}}_2^{-1} + \tilde{F} + \tilde{G}$$

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  - The leading order follows from

$$\mathcal{H}_{00} = -\frac{1}{64\pi m^2 a} + \frac{1}{16L^3m^3\Delta E} + \frac{1}{8m^3L^3\Delta E} + \cdots$$

Testing the formalism (Weak interactions)

We reproduce known results through $1/L^5$ and derive a relation at $1/L^6$

Note: Relativistic effects enter at $1/L^6$, same order as three-to-three

$$E = 3m + \frac{12\pi a}{mL^3}\left(1 + c_4 \frac{a}{L} + \cdots\right) - \frac{\mathcal{M}_{\text{thr}}}{48 m^3 L^6} + \cdots$$

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- ☑ We checked this in $\lambda\phi^4$ through $\mathcal{O}(\lambda^4)$

MTH and Sharpe (2016), Sharpe (2017)

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Meißner, Rìos and Rusetsky, *Phys. Rev. Lett*. 114, 091602 (2015) + erratum

The infinite-volume boundstate energy, $E_B \equiv 3m - \frac{\kappa^2}{m}$
is shifted in finite volume by an amount

$$\Delta E(L) = c|A|^2 \frac{\kappa^2}{m} \frac{1}{(\kappa L)^{3/2}} e^{-2\kappa L/\sqrt{3}} + \cdots$$

$c = -96.351\cdots$

geometric constant from Effimov wavefunction

$|A|^2$: **normalization correction factor (close to one)**

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Our formalism gives a general relation between scattering amplitudes and energy levels. So we substitute…

$$\mathcal{M}_3 \sim -\frac{\Gamma\,\overline{\Gamma}}{E^2 - E_B^2} \qquad \mathcal{M}_2 = -\frac{16\pi E_2^*}{ip^*}$$

and study the lowest three-particle finite-volume level

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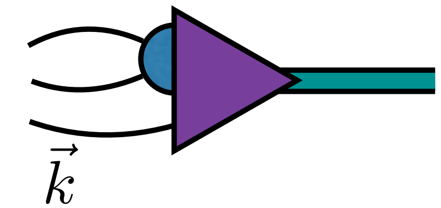
We aim to reproduce the exponent, leading power and overall constant using our relativistic formalism

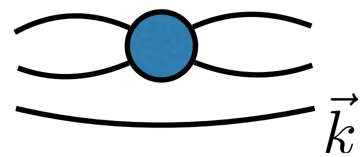
Reproducing the result…

1. Show that the relativistic quantization predicts (at leading order in 1/L)

$$\Delta E(L) = -\frac{1}{2E_B}\left[\frac{1}{L^3}\sum_{\vec{k}} - \int_{\vec{k}}\right]\frac{\overline{\Gamma}^{(u)}(k)\Gamma^{(u)}(k)}{2\omega_k \mathcal{M}_2(k)}$$

usymmetrized residue factor





s-wave scattering amplitude

2. Derive the functional forms of the infinite-volume quantities

$$\Gamma^{(u)}(k) = \frac{3^{3/8}\pi^{1/4}}{4}A\sqrt{-c}\,\mathcal{M}_2(k)$$

follows from matching to Effimov wavefunction

$$\mathcal{M}_2(k) = \frac{32\pi m}{\kappa}\left[1 + \frac{3k^2}{4\kappa^2}\right]^{-1/2}$$

unitary amplitude with spectator “stealing” some momentum

3. Evaluate the sum-integral difference with Poisson summation

$$\Delta E(L) = c|A|^2\frac{3^{3/4}\pi^{3/2}}{3\kappa}6\int_{\vec{k}} e^{iL\hat{x}\cdot\vec{k}}\frac{1}{2\omega_k}\left[1 + \frac{3k^2}{4\kappa^2}\right]^{-1/2}$$

$$= c|A|^2\frac{\kappa^2}{m}\frac{1}{(\kappa L)^{3/2}}e^{-2\kappa L/\sqrt{3}} + \cdots$$

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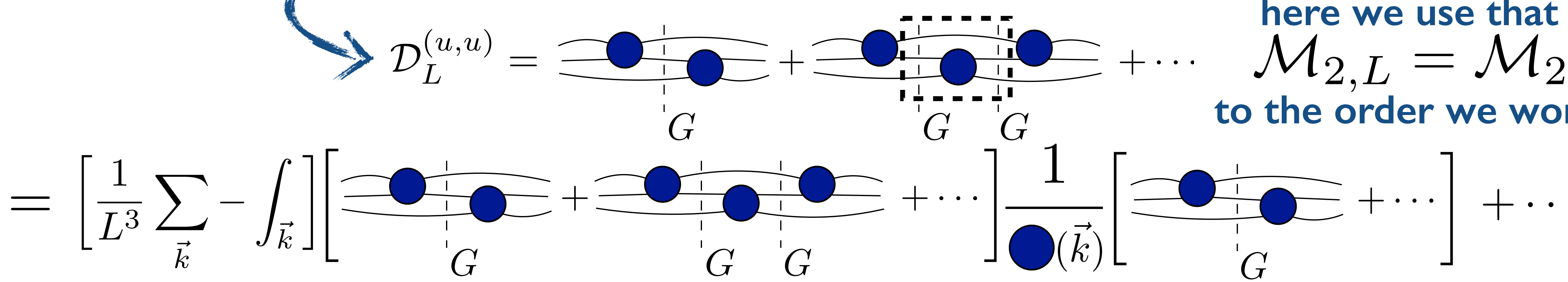
We use the second form of the finite-volume correlator

$$\mathcal{M}_{3,L} = \mathcal{S}\left[\mathcal{D}_L^{(u,u)} + \mathcal{L}_L^{(u)}\mathcal{K}_{\mathrm{df},3}\frac{1}{1+F_3\mathcal{K}_{\mathrm{df},3}}\mathcal{R}_L^{(u)}\right]$$



here we use that $\mathcal{M}_{2,L} = \mathcal{M}_2$ to the order we work

$$\mathcal{D}_L^{(u,u)} = \mathcal{D}_\infty^{(u,u)} + \left[\frac{1}{L^3}\sum_{\vec{k}} - \int_{\vec{k}}\right]\mathcal{D}_\infty^{(u,u)}\frac{1}{2\omega_k\mathcal{M}_2(\vec{k})}\mathcal{D}_L^{(u,u)}$$

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Substituting pole ansatz and solving gives the claimed result

$$-\frac{1}{E^2-[E_B+\Delta E(L)]^2} = -\frac{1}{E^2-E_B^2} + \left[\frac{1}{L^3}\sum_{\vec{k}} - \int_{\vec{k}}\right]\frac{1}{E^2-E_B^2}\frac{\overline{\Gamma}^{(u)}(k)\Gamma^{(u)}(k)}{2\omega_k\mathcal{M}_2(\vec{k})}\frac{1}{E^2-[E_B+\Delta E(L)]^2}$$

finite-volume pole   infinite-volume pole

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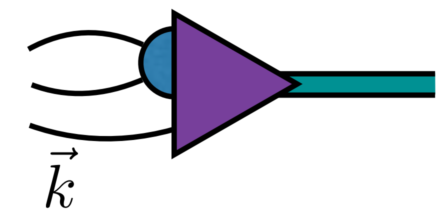
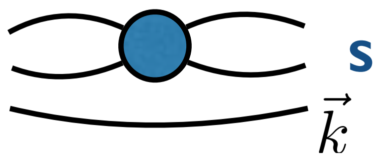




s-wave scattering amplitude

2. Derive the functional forms of the infinite-volume quantities

$$\Gamma^{(u)}(k) = \frac{3^{3/8}\pi^{1/4}}{4} A\sqrt{-c}\,\mathcal{M}_2(k) \qquad \mathcal{M}_2(k) = \frac{32\pi m}{\kappa}\left[1 + \frac{3k^2}{4\kappa^2}\right]^{-1/2}$$

follows from matching to Effimov wavefunction

unitary amplitude with spectator "stealing" some momentum

3. Evaluate the sum-integral difference with Poisson summation

$$\Delta E(L) = c|A|^2 \frac{3^{3/4}\pi^{3/2}}{3\kappa} 6 \int_{\vec{k}} e^{iL\hat{x}\cdot\vec{k}} \frac{1}{2\omega_k}\left[1 + \frac{3k^2}{4\kappa^2}\right]^{-1/2}$$

$$= c|A|^2 \frac{\kappa^2}{m} \frac{1}{(\kappa L)^{3/2}} e^{-2\kappa L/\sqrt{3}} + \cdots$$

MTH and Sharpe (2017)

Reproducing the result (more details)…

2. Derive the functional forms of the infinite-volume quantities

$$\mathcal{M}_2(k) = -\frac{16\pi E_2^*}{ip^*} \xrightarrow[\text{leading order in binding momentum}]{\text{subthreshold,}} \mathcal{M}_2(k) = \frac{32\pi m}{\kappa}\left[1+\frac{3k^2}{4\kappa^2}\right]^{-1/2}$$

$$p^* = \sqrt{[(E-\omega_k)^2 - \vec{k}^2]/4 - m^2}$$

To derive the residue factor we match to the non-relativistic wavefunction

$$\left[-\frac{1}{2m}\sum_i \frac{\partial^2}{\partial \mathbf{r}_i^2} + \sum_{ij} V(\mathbf{r}_i - \mathbf{r}_j)\right]\psi(\mathbf{r}_1,\mathbf{r}_2) = -\frac{\kappa^2}{m}\psi(\mathbf{r}_1,\mathbf{r}_2)$$

this can be re-expressed using the Faddeev equation

$$\psi = \phi_1 + \phi_2 + \phi_3 \qquad \left[-\frac{1}{2m}\sum_i \frac{\partial^2}{\partial \mathbf{r}_i^2} + \frac{\kappa^2}{m}\right]\phi_3(\mathbf{r}_1,\mathbf{r}_2) = V(\mathbf{r}_1 - \mathbf{r}_2)\,\psi(\mathbf{r}_1,\mathbf{r}_2)$$

We have found that the unsymmetrized residue factor is given by

$$\Gamma^{(u)}(k) = \lim_{\text{on shell}} 4\sqrt{3}m^2\left(-\frac{\kappa^2}{m} - H_0\right)\tilde{\phi}_3$$

Substituting the known wave function and expanding about the leading singularity, we find

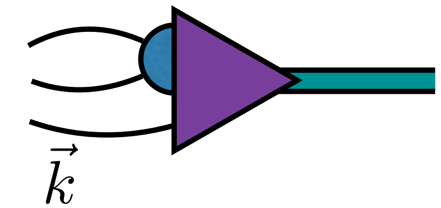
$$\Gamma^{(u)}(k) = \frac{3^{3/8}\pi^{1/4}}{4}A\sqrt{-c}\,\mathcal{M}_2(k)$$

Reproducing the result (more details)…

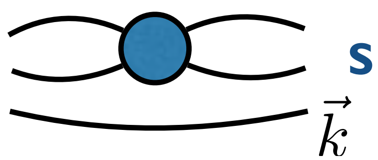
1. **Show that the relativistic quantization predicts (at leading order in 1/L)**

$$\Delta E(L) = -\frac{1}{2E_B}\left[\frac{1}{L^3}\sum_{\vec{k}} - \int_{\vec{k}}\right] \frac{\overline{\Gamma}^{(u)}(k)\Gamma^{(u)}(k)}{2\omega_k \mathcal{M}_2(k)}$$

usymmetrized residue factor



s-wave scattering amplitude



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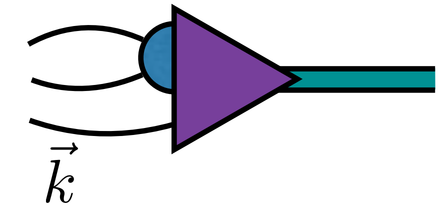

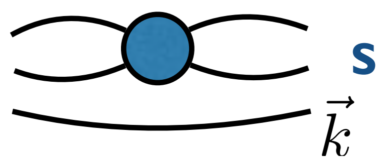

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unitary amplitude with spectator "stealing" some momentum

3. Evaluate the sum-integral difference with Poisson summation

singularity from two-to-two amplitude

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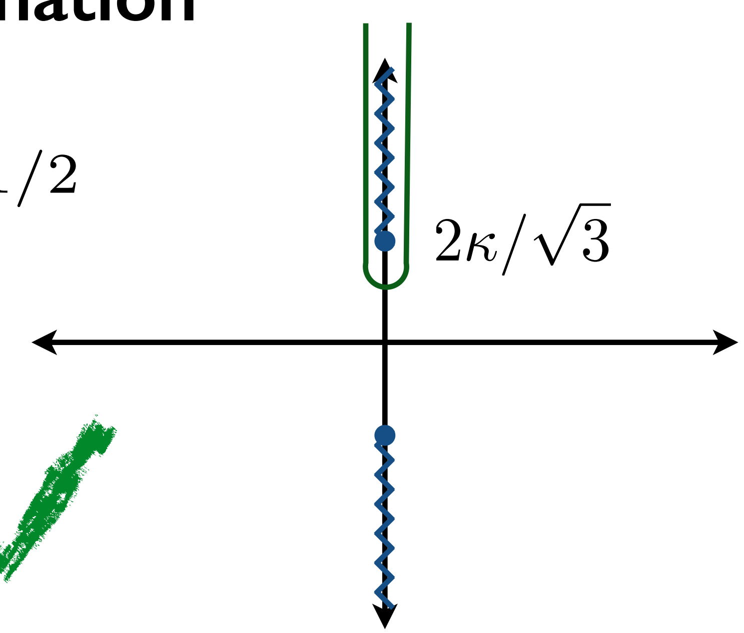


MTH and Sharpe (2017)

Outline

- [x] **Warm up and definitions**
  - [x] Basic set-up
  - [x] Finite-volume correlator
  - [x] Three non-interacting particles

- [x] **Two particles in a box**
  - [x] Alternative derivation
  - [x] Truncation and application
  - [x] Relating matrix elements

- [x] **Three particles in a box**
  - [x] 3-to-3 scattering
  - [x] (Sketch of) derivation
  - [x] An unexpected infinite-volume quantity
  - [x] Relating energies to scattering

- [x] **Testing the result**
  - [x] Know issues
  - [x] Large-volume expansion
  - [x] Effimov state in a box

- [ ] **Other methods**

- [ ] **Numerical explorations**
  - [ ] Truncation at low energies
  - [ ] Toy solutions for various systems
  - [ ] Unphysical solutions

- [ ] **Looking forward**

Other methods…

☑ A. Rusetsky, H.W. Hammer, J.-Y. Pang:

- ☑ Non-relativistic
- ☑ Based in a specific EFT, focuses on extracting LECs,
- ☑ Simpler derivation and formulae, can handle K-matrix poles
- ☑ Argued to be "diagramatically equivalent"
- ☑ No t-channel cut, integrals go infinitely far below threshold

☑ M. Mai and M. Döring

- ☑ Relativistic
- ☑ Built on unitary constrains, replace imaginary cuts with volume cuts
- ☑ Cannot see the dropping of $\mathcal{O}(e^{-mL})$
- ☑ Connection to our approach is not yet well understood

See also ***Polejaeva, Rusetksy (2012)*** *and* ***Briceño, Davoudi (2013)***

Usability?

"Despite this success, the quantization condition in these papers is not yet given in a form suitable for the analysis of the real lattice data"

Hammer, Pang and Rusetsky (2017)

We were motivated to challenge this claim…

We find that the "degree of usability" is comparable between the two approaches, provided one applies similar approximations.

How do we make the two-particle formalism usable?

Truncate partial waves → **Single partial wave**

$$\mathcal{M}_2(E_2^*,\theta^*) \approx \sum_{\ell=0}^{N} P_\ell(\cos\theta^*)\mathcal{M}_{2,\ell}(E_2^*) \longrightarrow \mathcal{M}_2(E_2^*,\theta^*) \approx \mathcal{M}_{2,s}(E_2^*) \propto \frac{1}{p^*\cot\delta_0(p^*) - ip^*}$$

Is there a three-particle analog?

$$\mathcal{K}_{\text{df},3}(E^*,\Omega_3',\Omega_3) \approx \sum_{n=0}^{N} \mathcal{P}_n(\Omega_3',\Omega_3)\mathcal{K}_{\text{df},3,n}(E^*) \longrightarrow \mathcal{K}_{\text{df},3}(E^*,\Omega_3',\Omega_3) \approx \mathcal{K}_{\text{df},3}^{\text{iso}}(E^*) \in \mathbb{R}$$

At fixed energy $\mathcal{M}_2(E_2^*,\theta^*)$, $\mathcal{K}_{\text{df},3}(E^*,\Omega_3',\Omega_3)$ **is a smooth function on a compact space.**

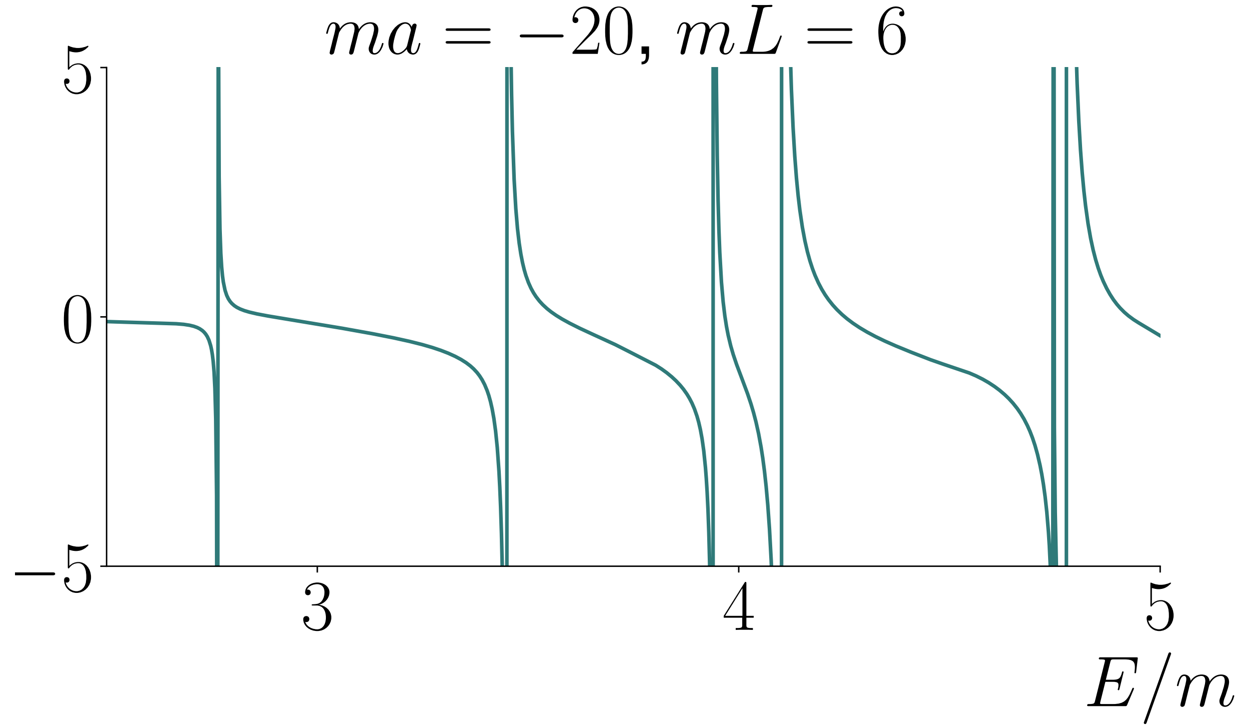
Further investigation is needed to understand suppression of higher $\mathcal{K}_{\text{df},3,n}(E^*)$.

Numerics (keeping only s-wave and $\mathcal{K}_{\mathrm{df},3}(E^*,\Omega_3',\Omega_3) \approx \mathcal{K}^{\mathrm{iso}}_{\mathrm{df},3}(E^*)$)

$$1/\mathcal{K}^{\mathrm{iso}}_{\mathrm{df},3}(E^*) = -F_3^{\mathrm{iso}}[E,\vec{P},L,\mathcal{M}_2^s] \qquad \mathcal{M}_3(E^*,\Omega_3',\Omega_3) = \mathcal{S}\left[\mathcal{D} + \mathcal{L}\frac{1}{1/\mathcal{K}^{\mathrm{iso}}_{\mathrm{df},3} + F^{\mathrm{iso}}_{3,\infty}}\mathcal{R}\right]$$

For the numerical approach we restrict attention to… $p^*\cot\delta_0(p^*) = -\frac{1}{a}$, $\vec{P}=0$

Then the quantization condition is based on $F_3^{\mathrm{iso}}(E,L,a)$

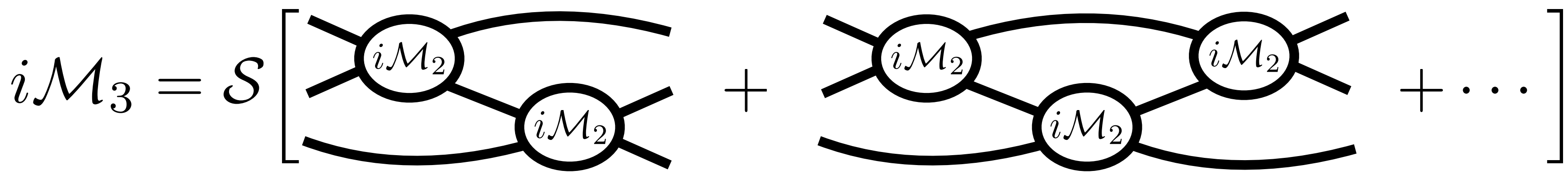


Finite-volume energies wherever these curves intersect $-1/\mathcal{K}^{\mathrm{iso}}_{\mathrm{df},3}(E)$

Briceño, Hansen and Sharpe (2018)

$\mathcal{K}_{\mathrm{df},3}^{\mathrm{iso}}(E) = 0$ solutions

- **Provides a useful benchmark: Deviations measure three-particle physics**
- **Meaning for three-to-three scattering is clear**



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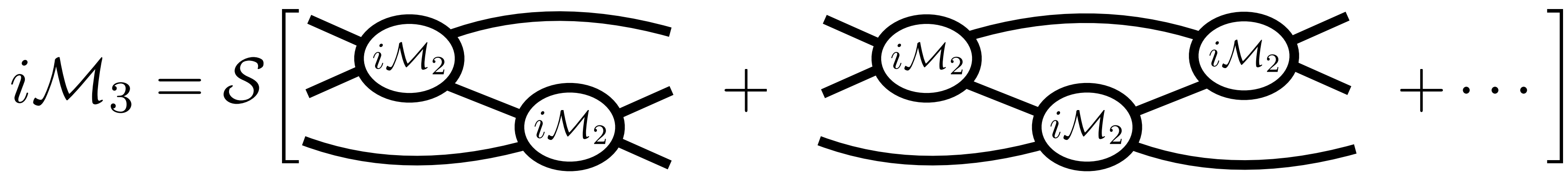
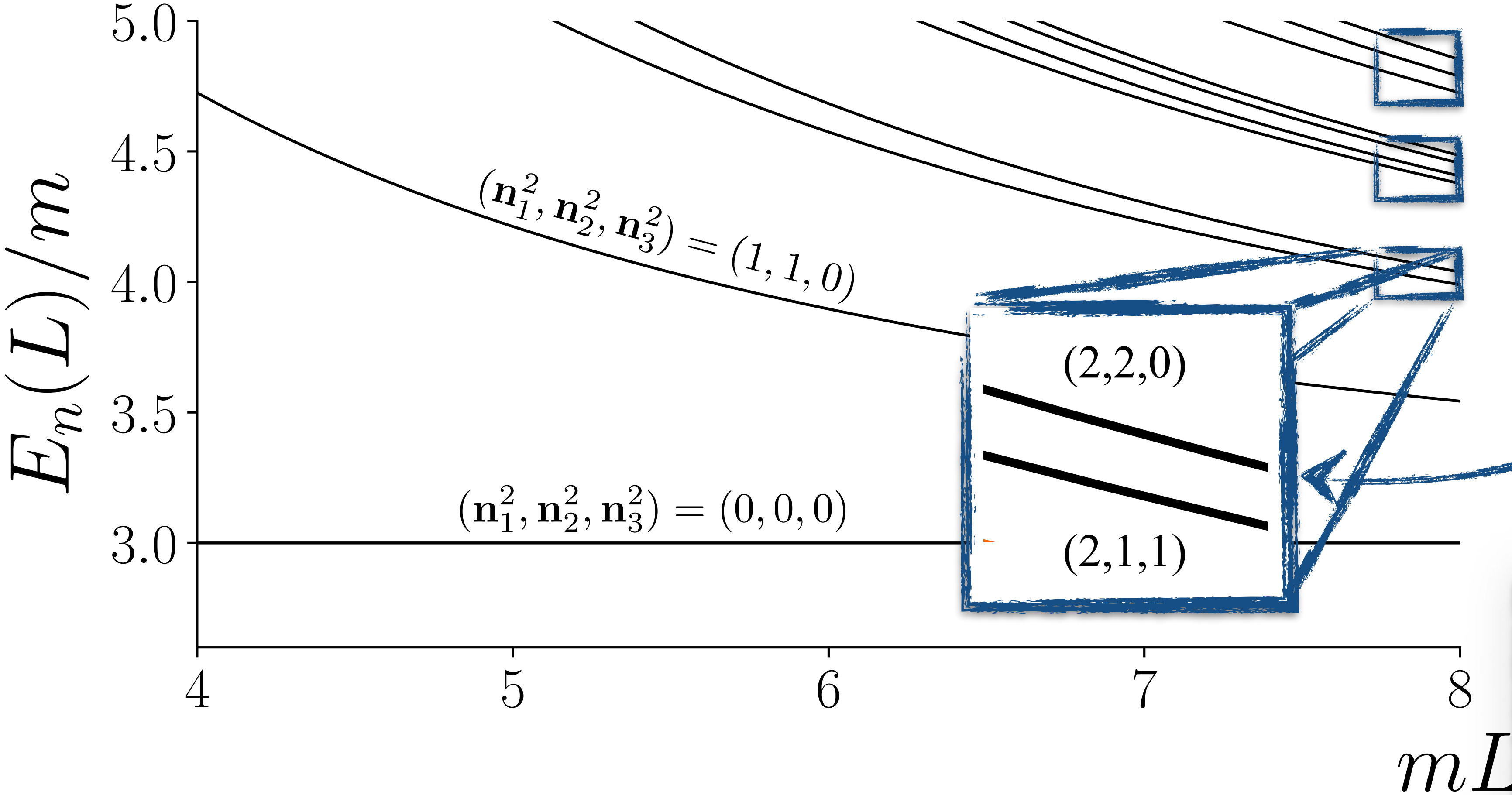
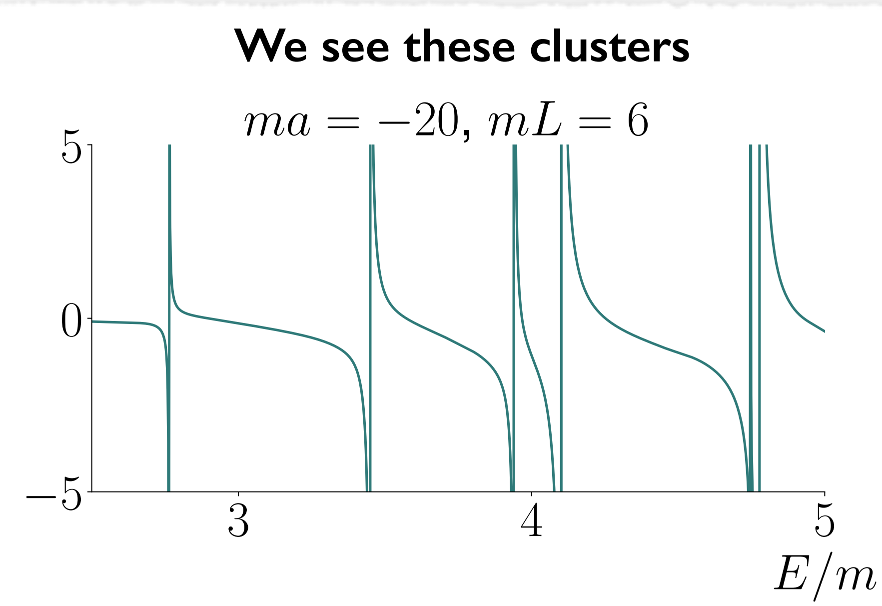


Again: Non-interacting states ($\mathcal{M}_2 = \mathcal{M}_3 = 0$)



These two states are degenerate in the NR theory



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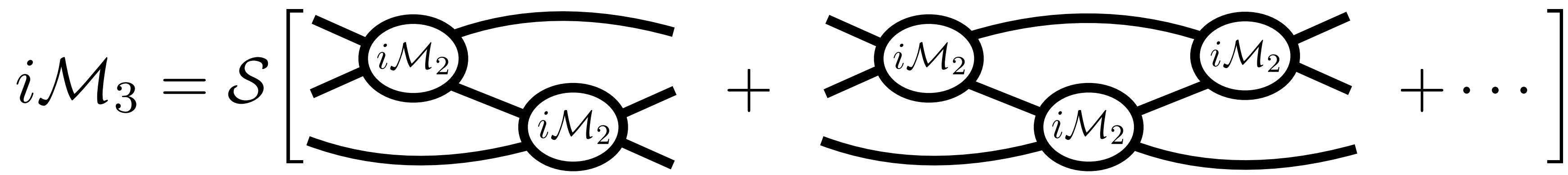

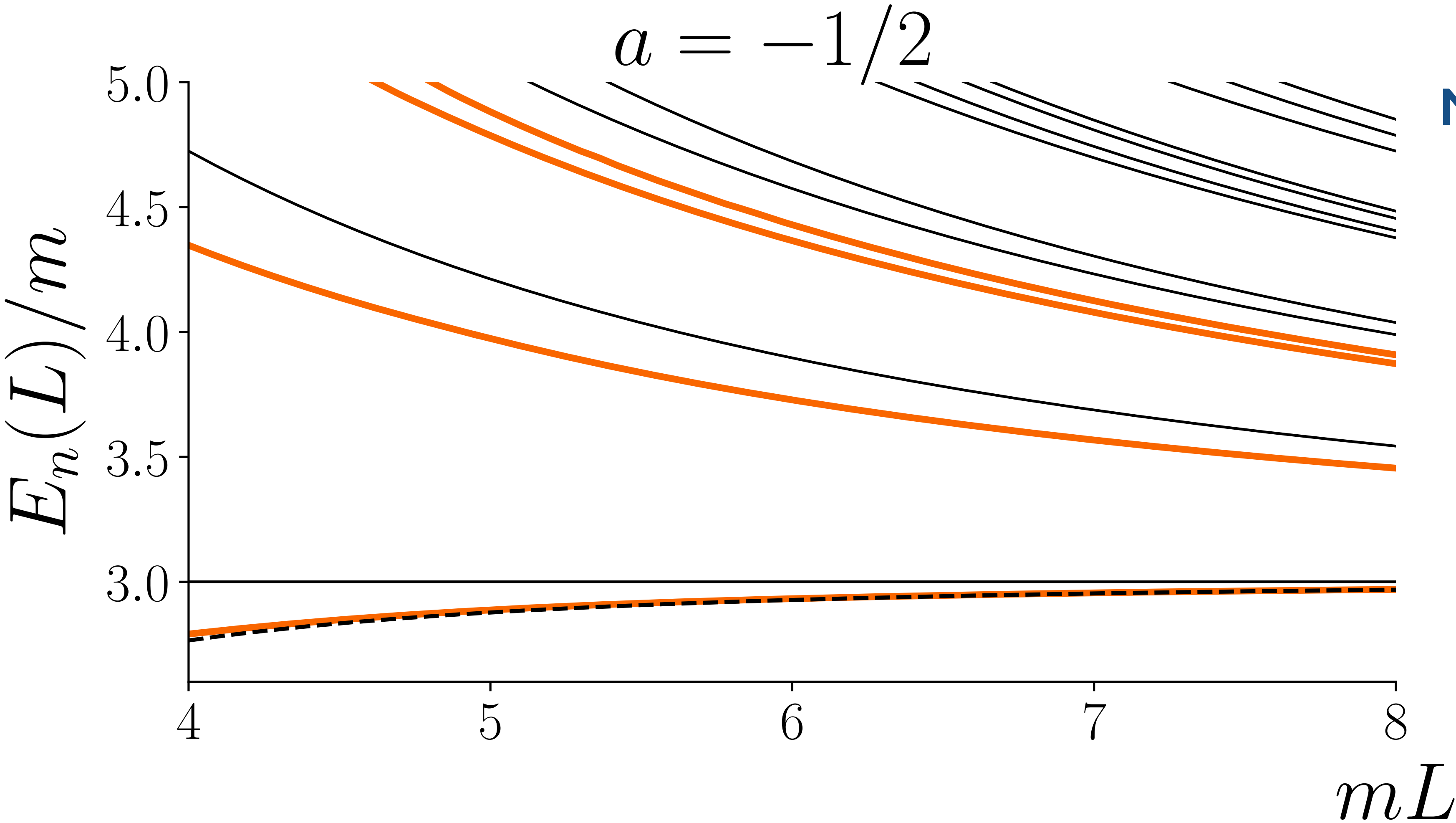

Now we turn on the interactions

$$1/F_3^{\mathrm{iso}}(E, L, a) = 0$$

$$F_3^{\mathrm{iso}} = \frac{1}{L^3} \sum_{\vec{k},\vec{p}} \left[\frac{\widetilde{F}_s}{3} - \widetilde{F}_s \frac{1}{\mathcal{H}} \widetilde{F}_s \right]_{kp}$$

$$\mathcal{H} = 1/(2\omega\mathcal{K}_2) + \widetilde{F}_s + \widetilde{G}_s$$

known functions

------ $1/L$ **expansion**

$\mathcal{K}^{\rm iso}_{\rm df,3}(E) = 0$ solutions

- **Provides a useful benchmark: Deviations measure three-particle physics**
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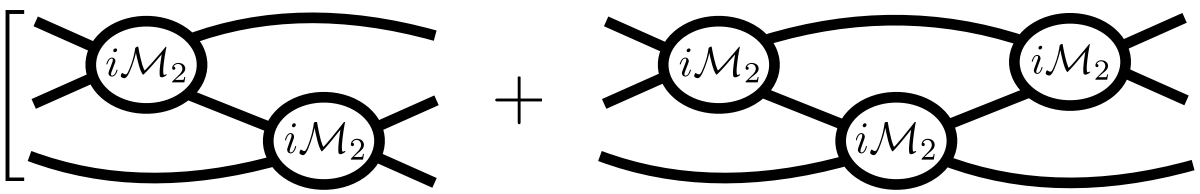
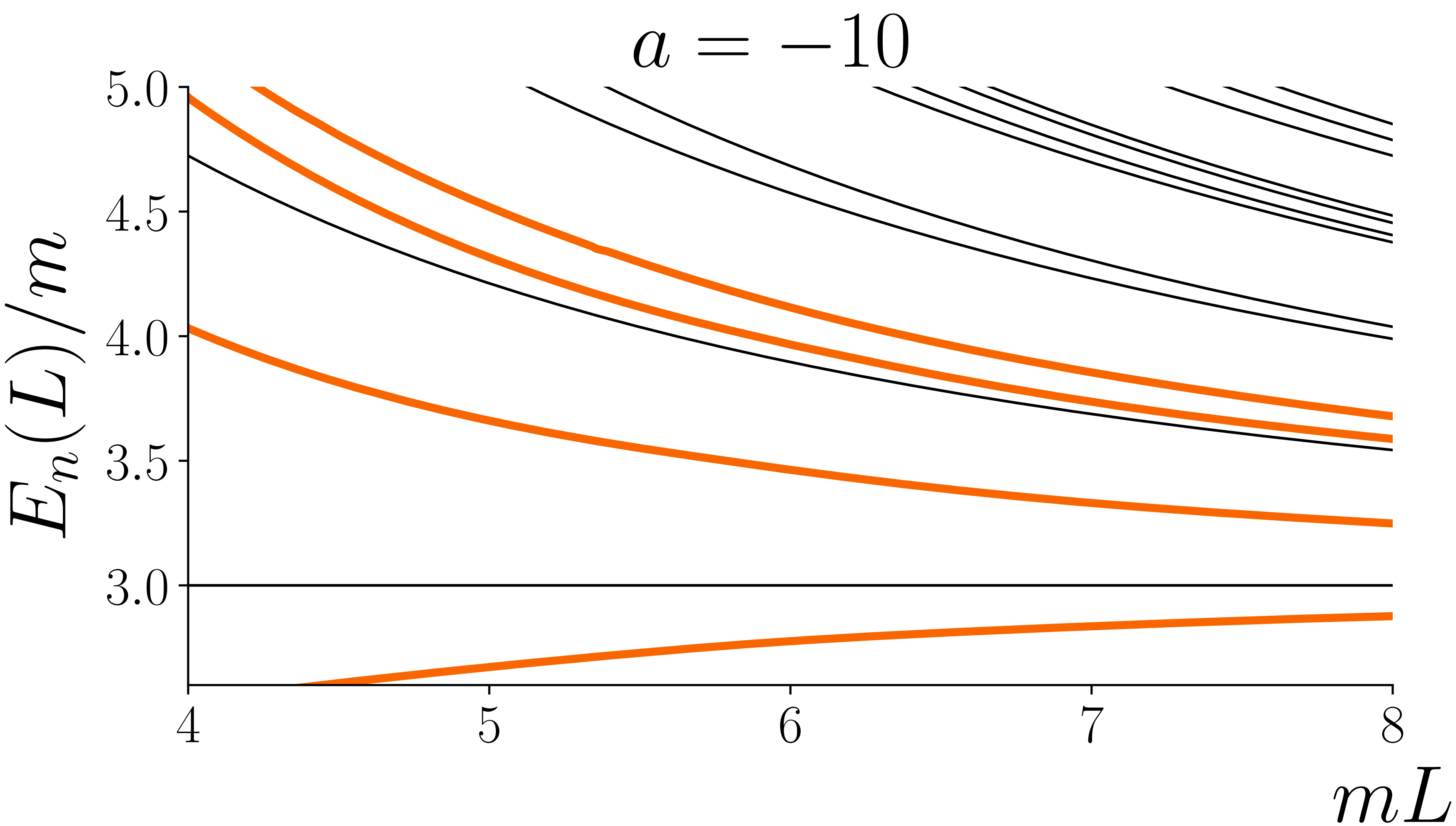
$i\mathcal{M}_3 = \mathcal{S}\Big[$  $+\cdots\Big]$



Can also accommodate large *a*

$$1/F_3^{\rm iso}(E,L,a) = 0$$

$$F_3^{\rm iso} = \frac{1}{L^3}\sum_{\vec k,\vec p}\left[\frac{\widetilde F_s}{3} - \widetilde F_s \frac{1}{\mathcal{H}} \widetilde F_s\right]_{kp}$$

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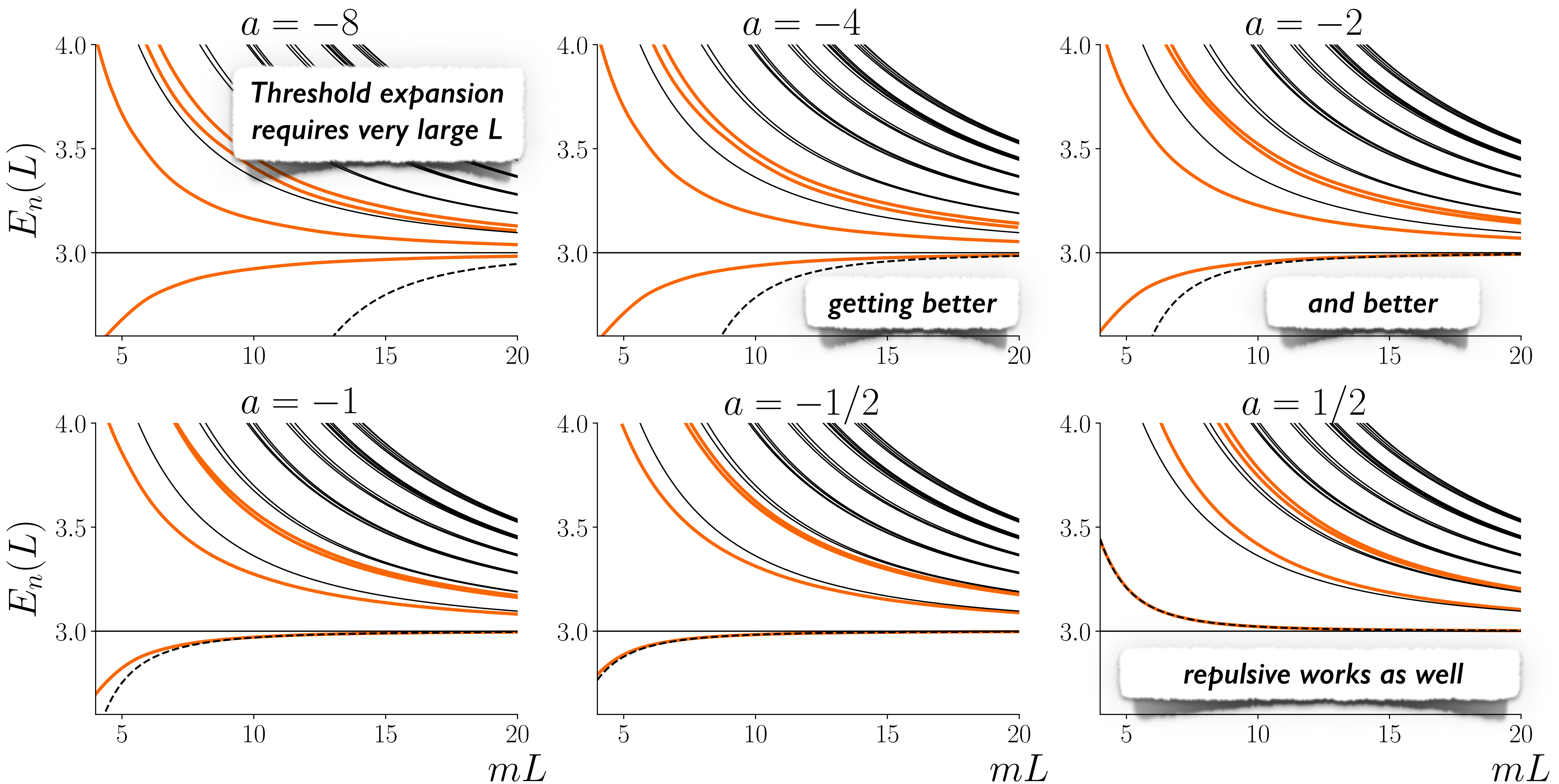
known functions

- - - - - - - $1/L$ **expansion**

$\mathcal{K}^{\rm iso}_{\rm df,3}(E) = 0$ solutions

Straightforward to vary *a* and to study large volumes



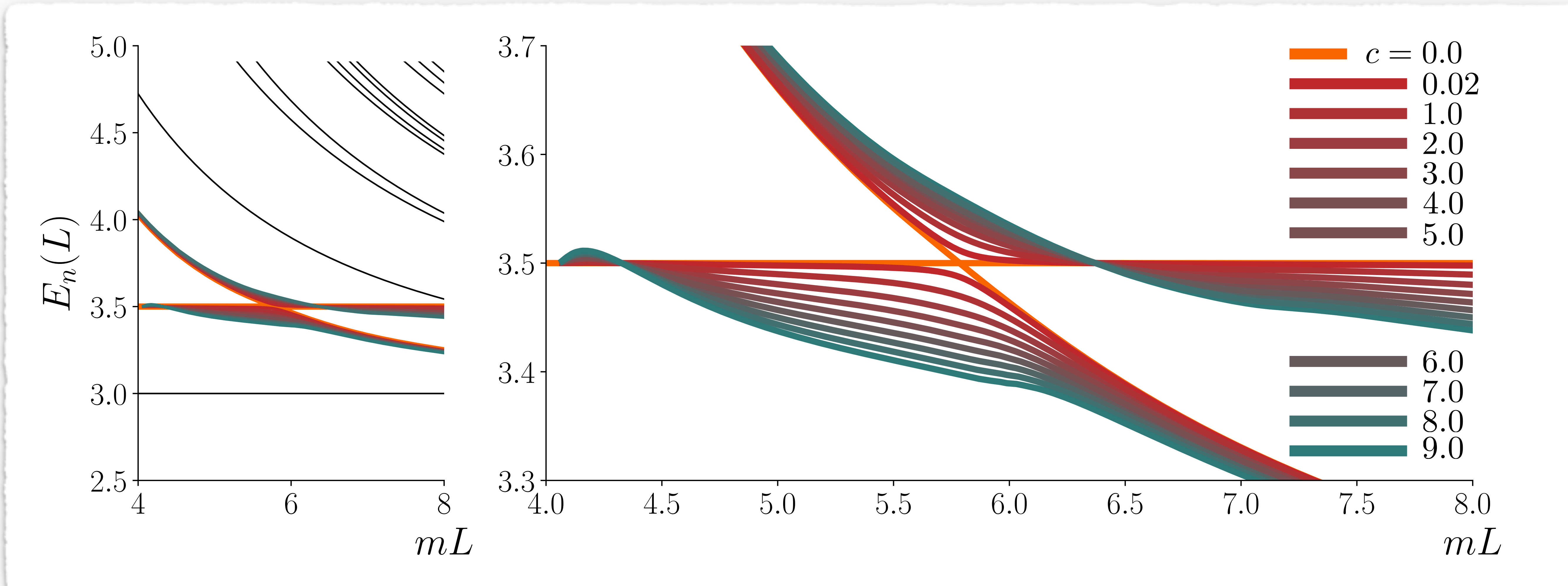
But, to avoid poles in $\mathcal{K}_2$, we must require $a < 1/m$

Non-zero $\mathcal{K}^{\text{iso}}_{\text{df},3}(E)$: Toy resonance

Here we consider a fun example for non-zero $\mathcal{K}^{\text{iso}}_{\text{df},3}$

$$a = -10 \qquad \mathcal{K}^{\text{iso}}_{\text{df},3}(E) = -\frac{c \times 10^3}{E^2 - M_R^2}$$

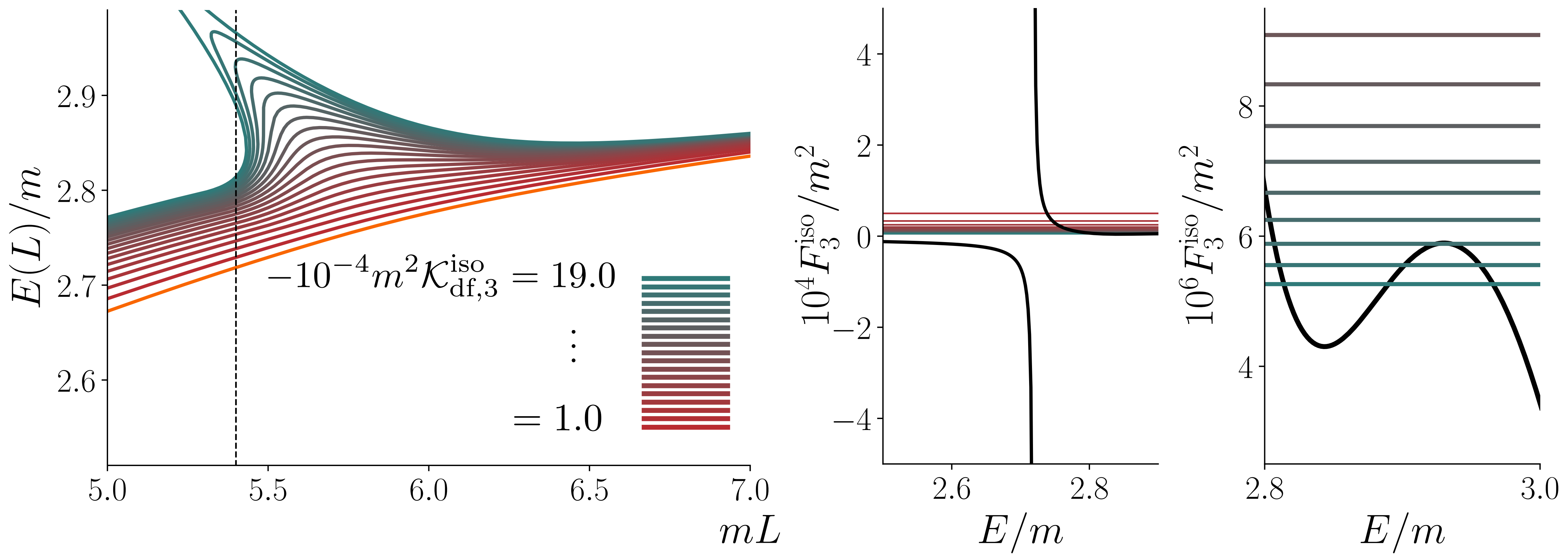
For small *c* we expect a narrow avoided level crossing, as *c* increases the gap grows



Further investigation is needed to see if this gives a physical resonance description

Unphysical solutions

Very large values of $\mathcal{K}_{\rm df,3}^{\rm iso}$ can lead to unphysical solutions



Unphysical input? Enhanced $\mathcal{O}(e^{-mL})$ effects? Under investigation…

Non-zero $\mathcal{K}_{\mathrm{df},3}(E)$: Unitary bound state

The parameters $a = -10^4$, $\mathcal{K}_{\mathrm{df},3}^{\mathrm{iso}}(E) = 2500$ lead to a shallow bound state

$\kappa \approx 0.1m$ **where** $E_B = 3m - \kappa^2/m$

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Finite-volume behavior of this state has a known asymptotic form

Meißner, Rios, Rusetsky (2015)

$$E_B(L) = 3m - \frac{\kappa^2}{m} - (98.35\cdots)|A|^2 \frac{\kappa^2}{m} \frac{e^{-2\kappa L/\sqrt{3}}}{(\kappa L)^{3/2}} \left[1 + \mathcal{O}\left(\frac{1}{\kappa L}, \frac{\kappa^2}{m^2}, e^{-\alpha\kappa L}\right)\right]$$

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This describes the bound state for

$\kappa L \approx 0.1 mL \gg 1$

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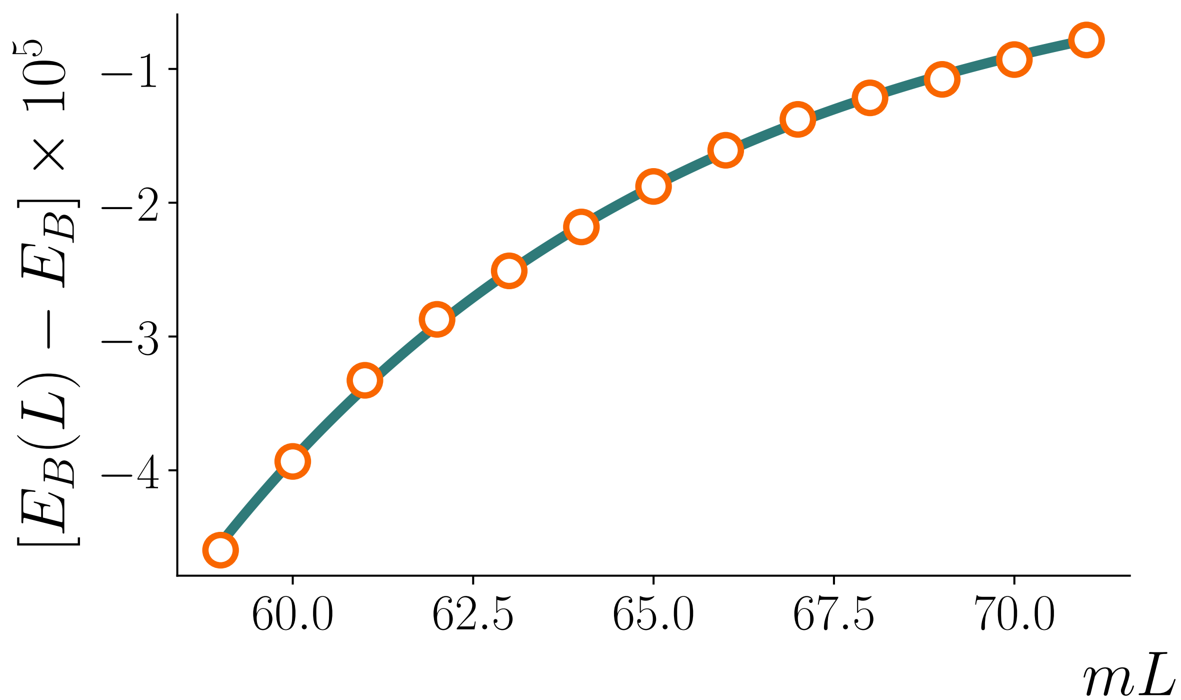

This describes the bound state for

$\kappa L \approx 0.1 mL \gg 1$

We fit our q.c. data over $60 < mL < 70$

$\longrightarrow \ \kappa = 0.1068, \ \ |A|^2 = 0.948$

Close to one ✓

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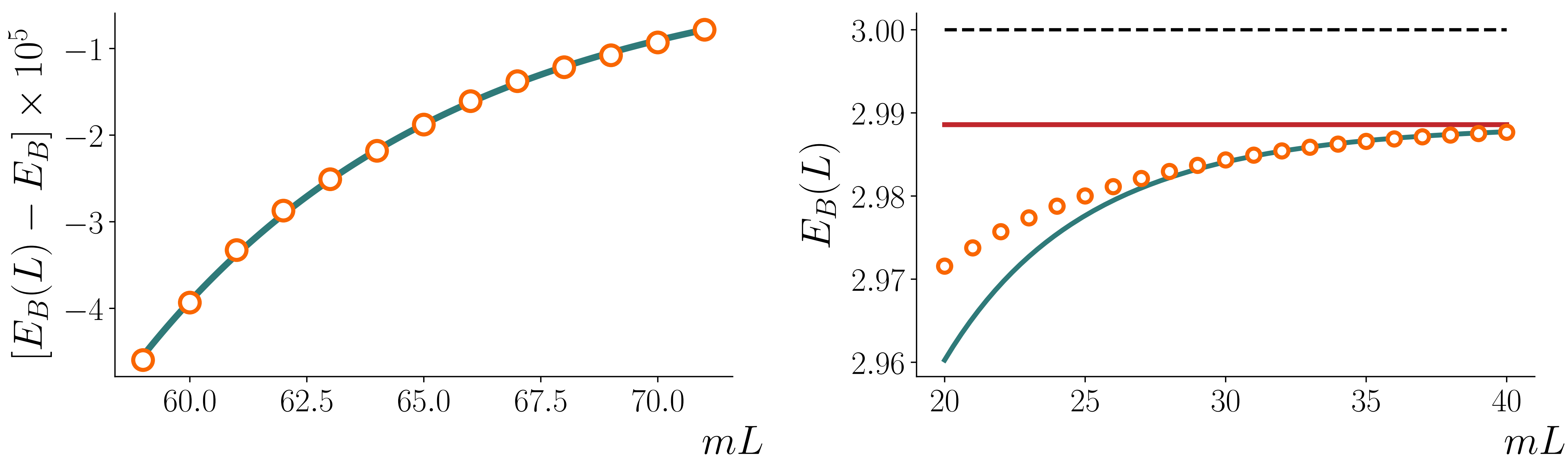

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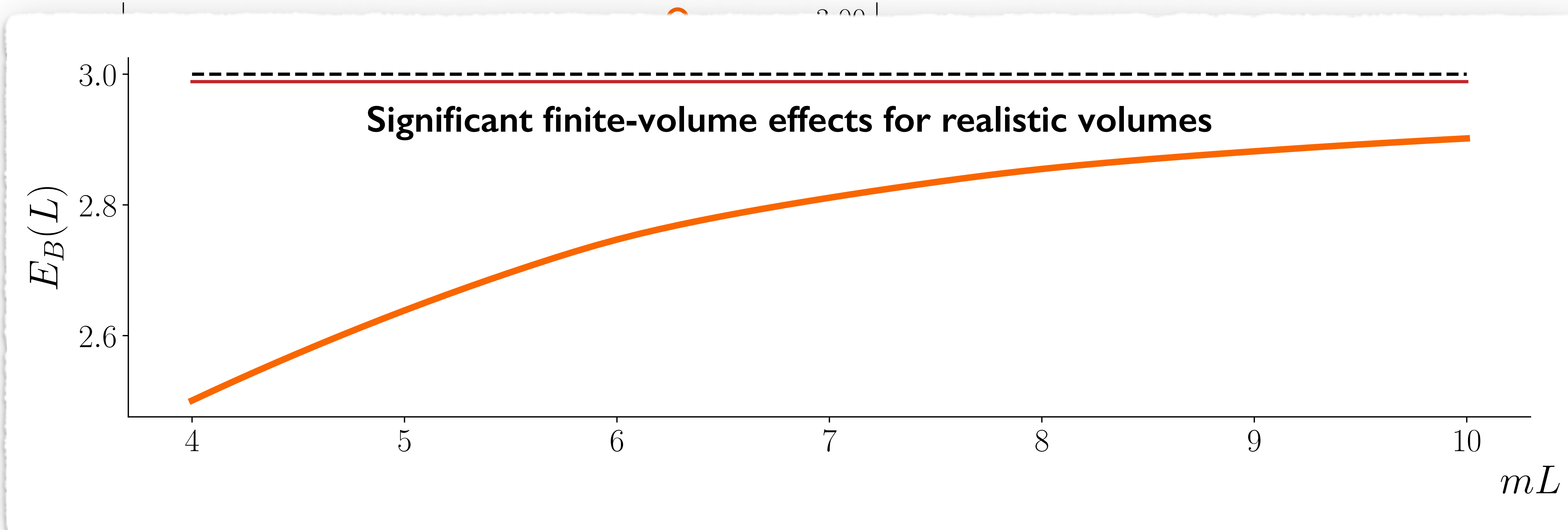
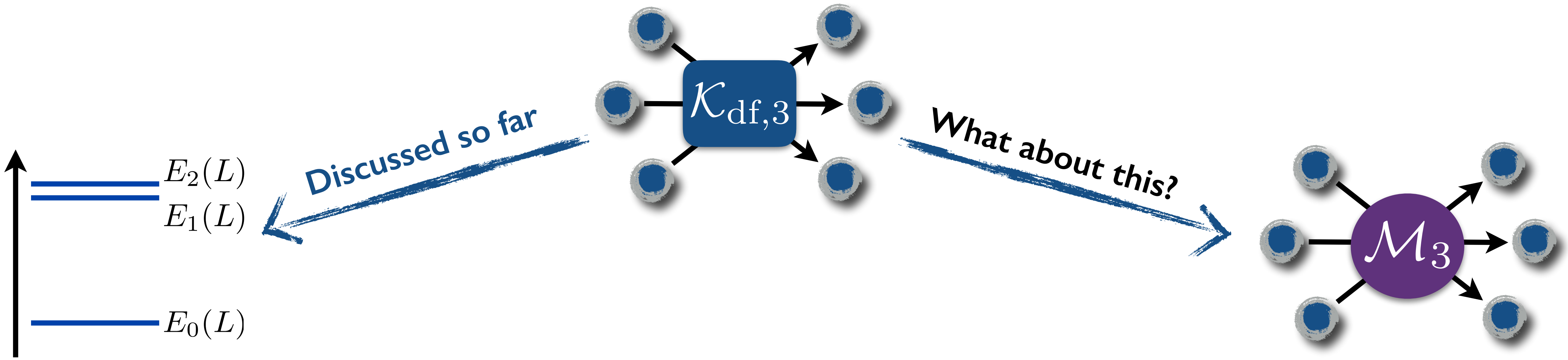


Converting to scattering amplitudes



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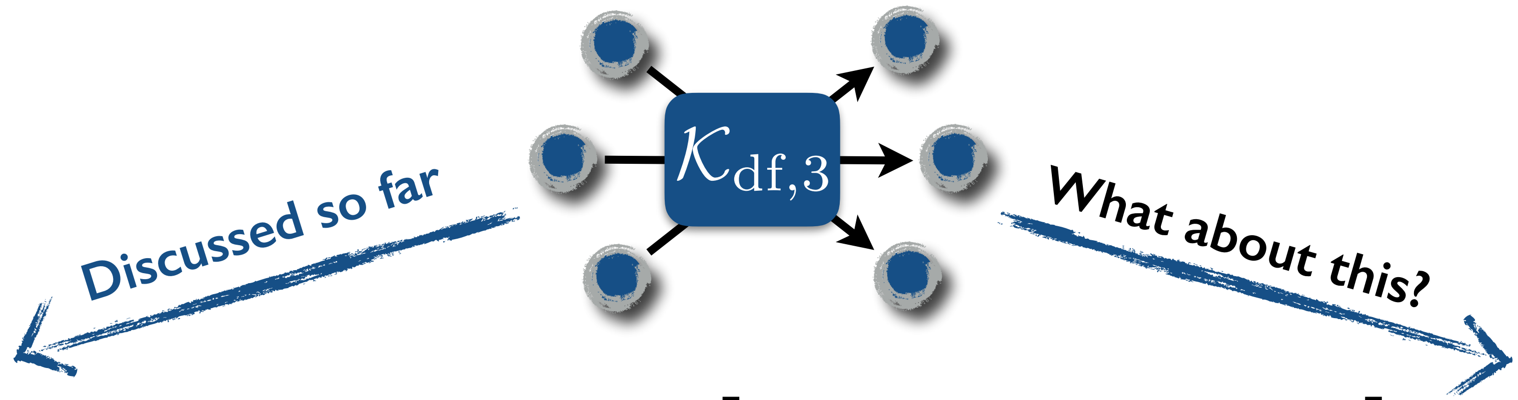
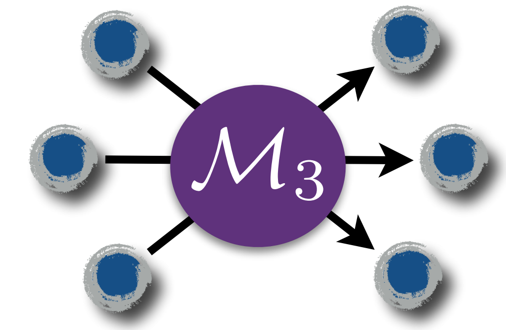
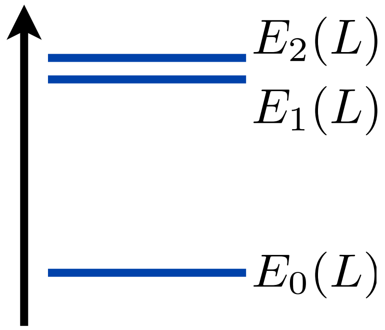
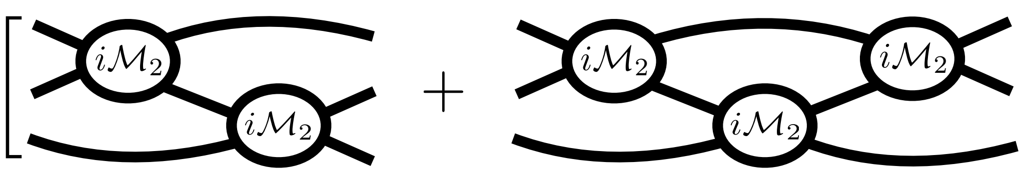






$$\mathcal{M}_{\mathrm{df},3}(E^*, \Omega_3', \Omega_3) = \mathcal{S}\left[\mathcal{L}(k')\frac{1}{1/\mathcal{K}_{\mathrm{df},3}^{\mathrm{iso}} + F_{3,\infty}^{\mathrm{iso}}}\mathcal{L}(k)\right]$$

$i\mathcal{M}_{\mathrm{df},3} = i\mathcal{M}_3 - \mathcal{S}\Big[$  $+\cdots\Big]$

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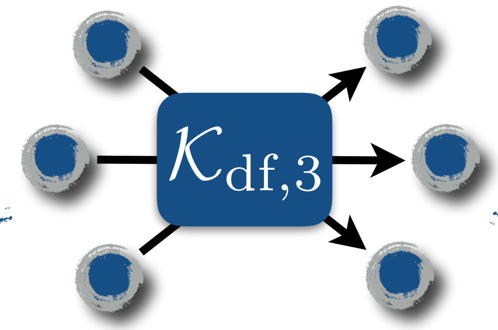
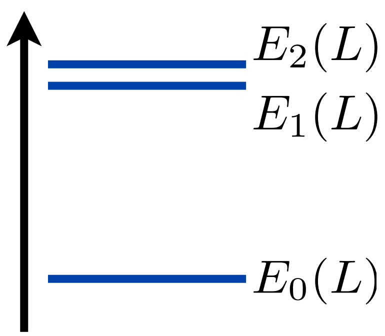
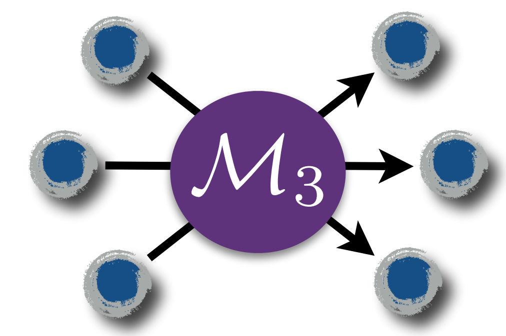


Discussed so far

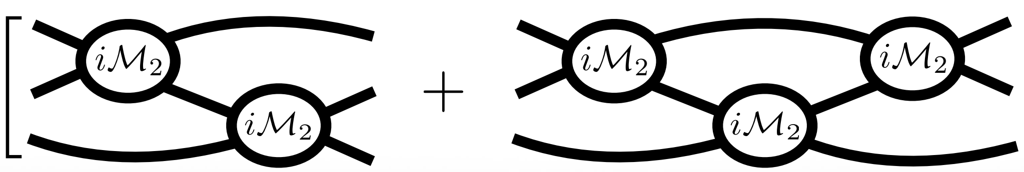


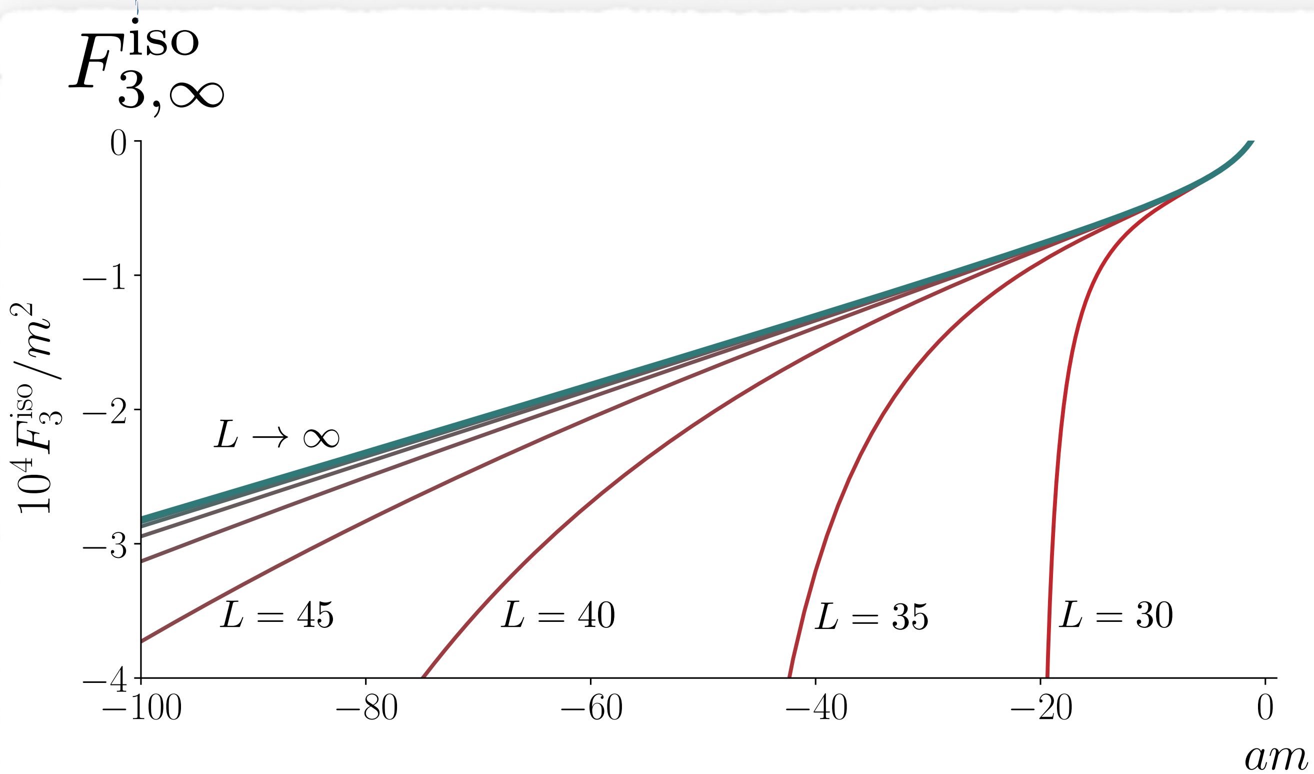
What about this?

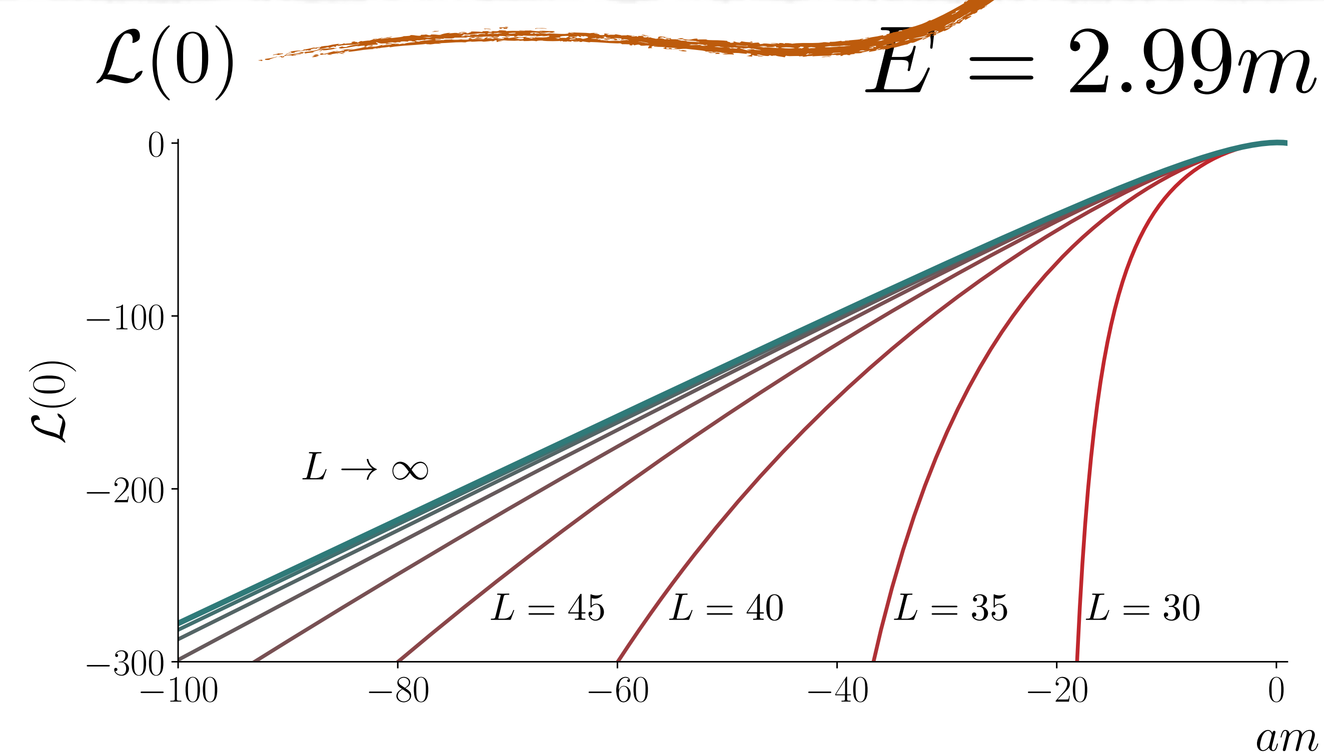


$$\mathcal{M}_{\mathrm{df},3}(E^*, \Omega_3', \Omega_3) = \mathcal{S}\left[\mathcal{L}(k') \frac{1}{1/\mathcal{K}^{\mathrm{iso}}_{\mathrm{df},3} + F^{\mathrm{iso}}_{3,\infty}} \mathcal{L}(k)\right]$$

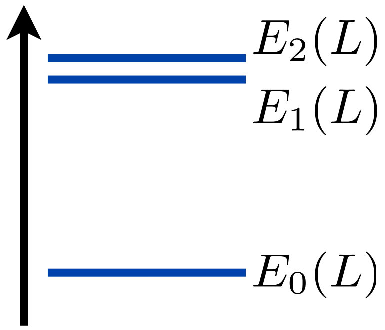
$$i\mathcal{M}_{\mathrm{df},3} = i\mathcal{M}_3 - \mathcal{S}\left[\quad + \quad + \cdots \right]$$



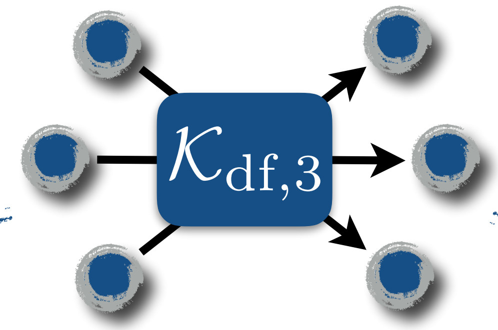




Converting to scattering amplitudes



Discussed so far



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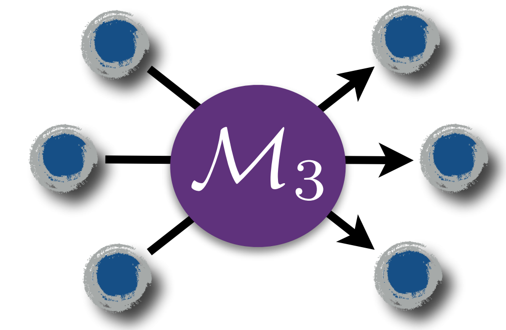

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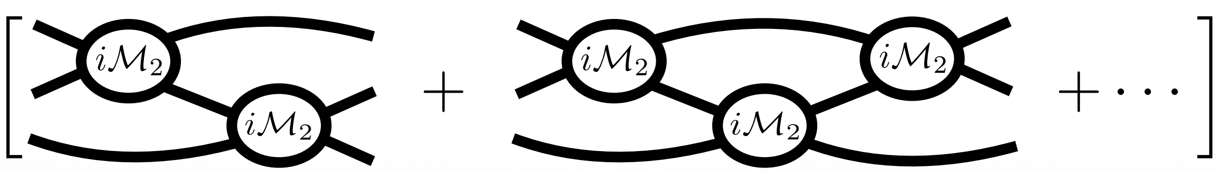
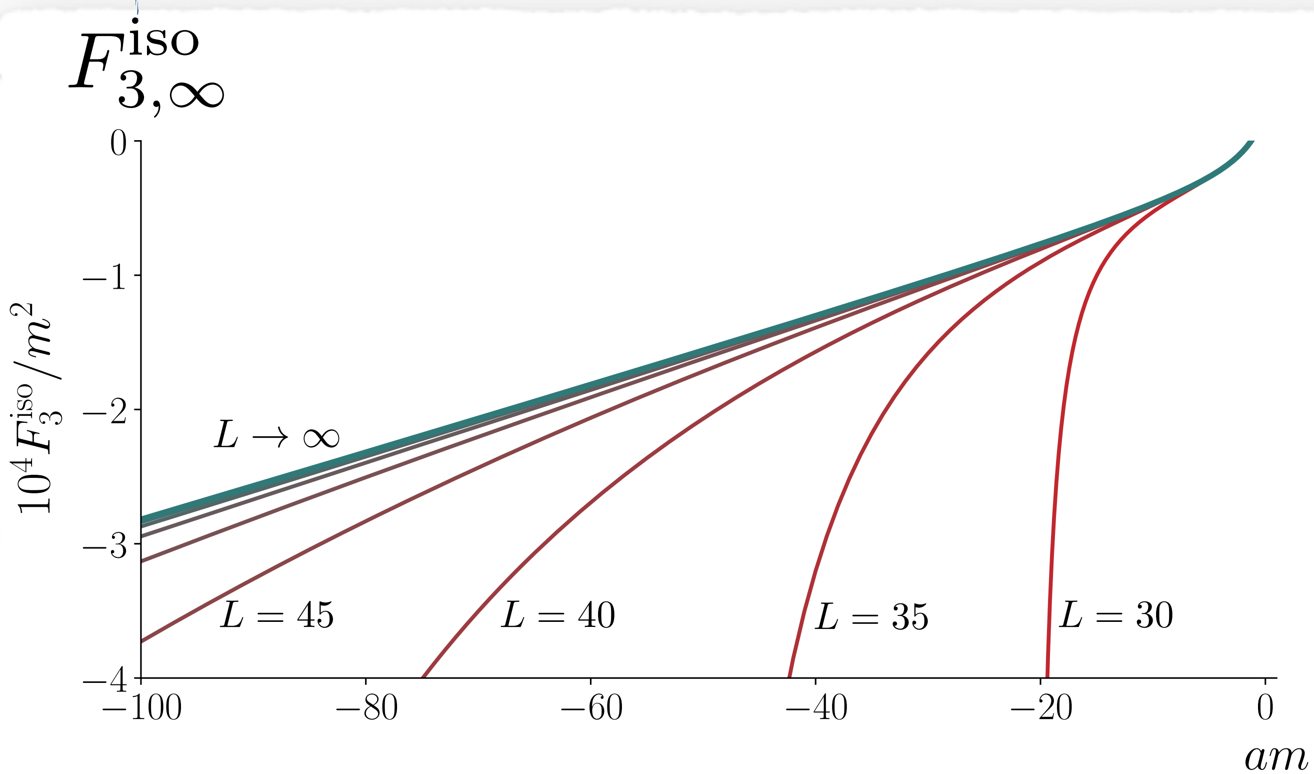
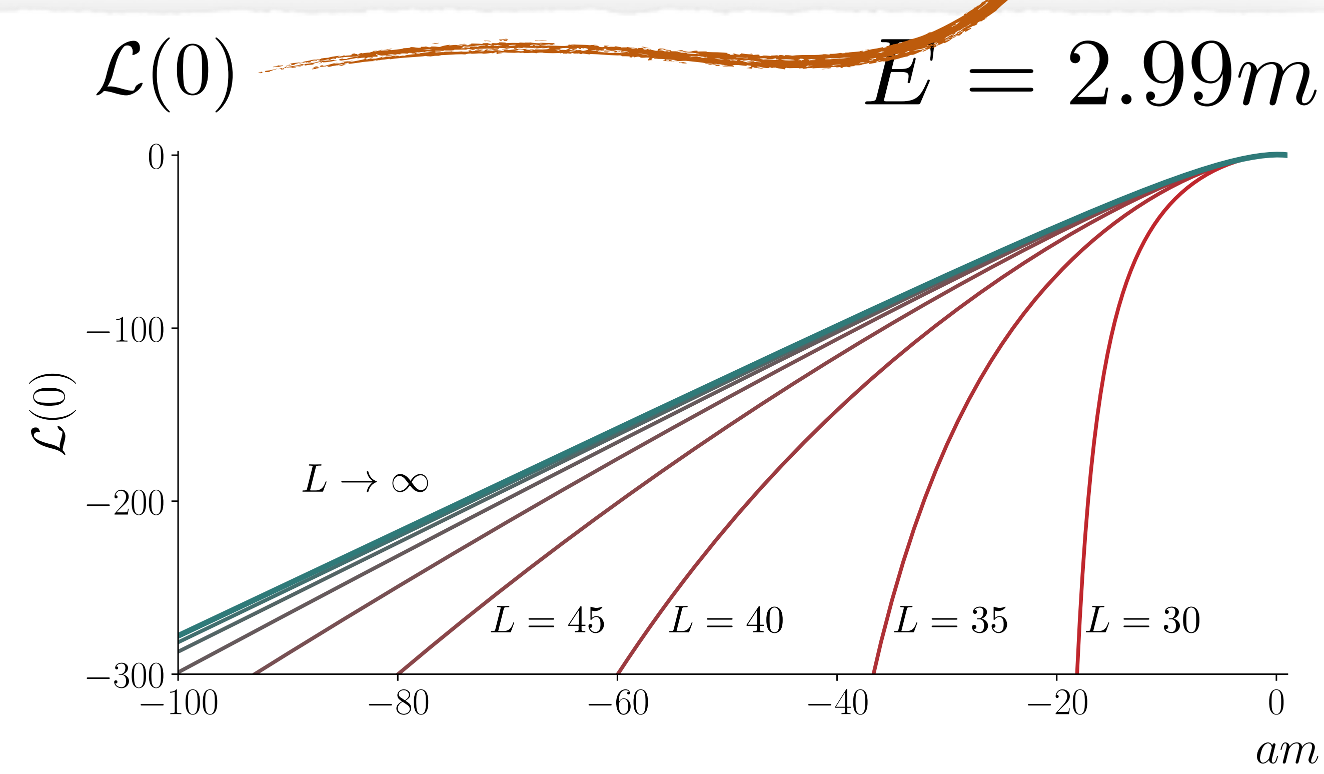
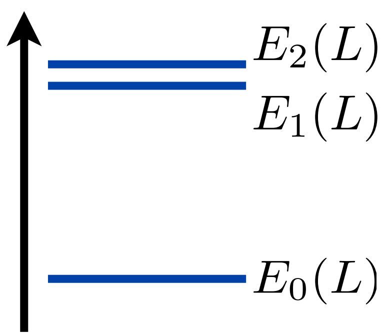


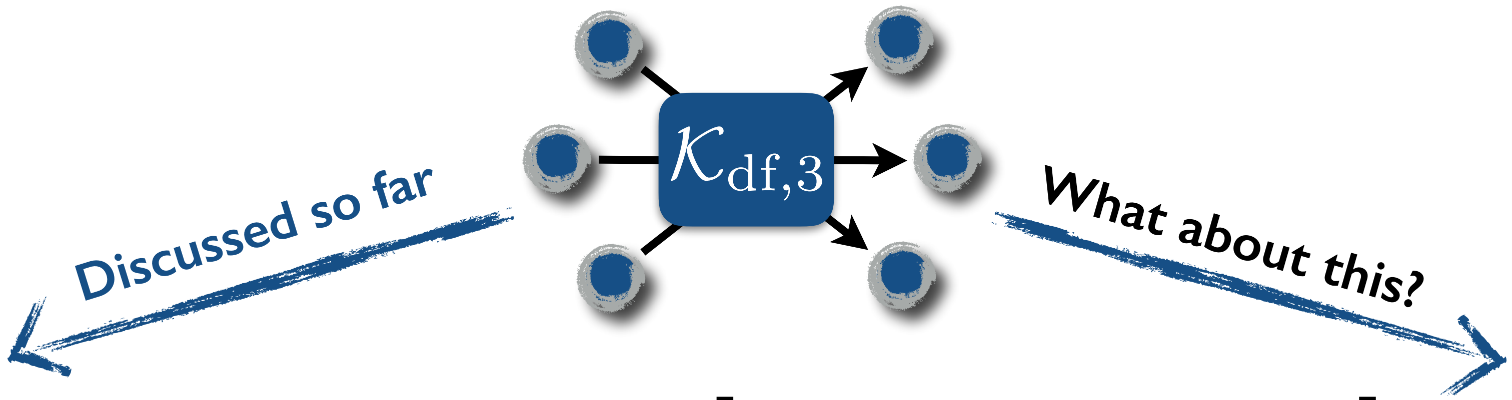


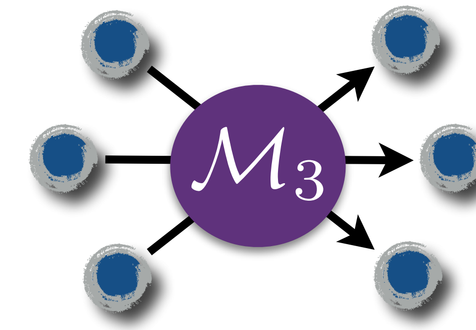


This only works below threshold… Relation above threshold crucially needed

Converting to scattering amplitudes







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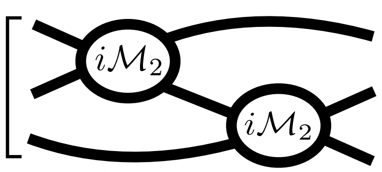
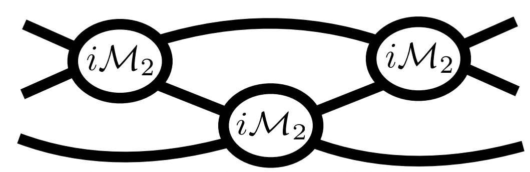
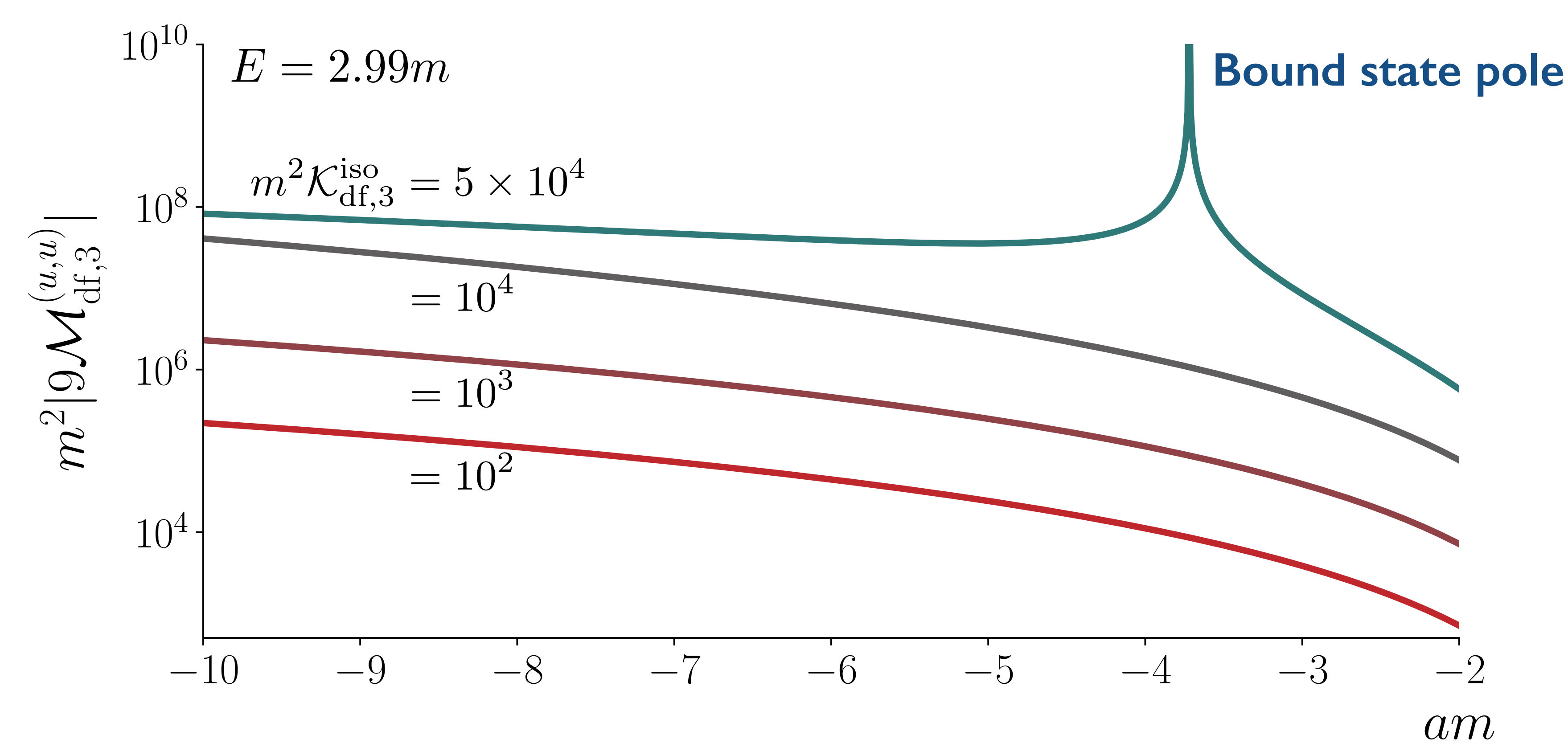
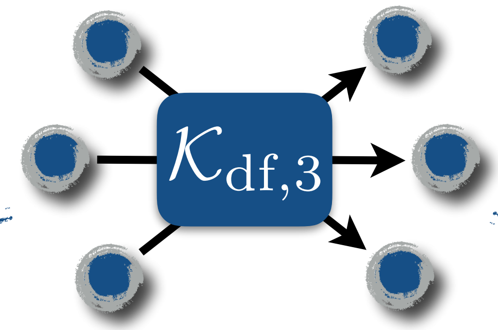
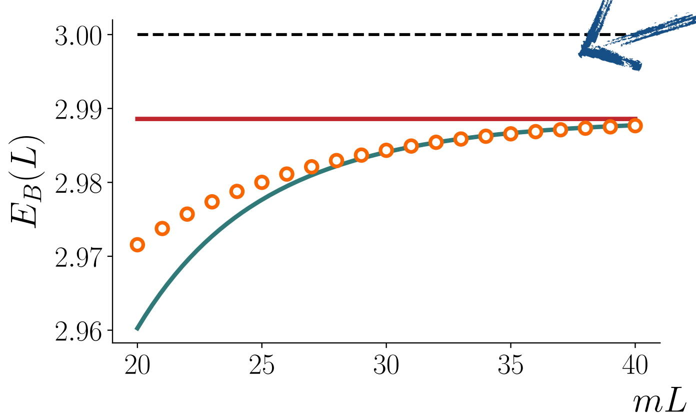
$i\mathcal{M}_{\mathrm{df},3} = i\mathcal{M}_3 - \mathcal{S}\Big[$  $+$  $+\cdots\Big]$



Back to the bound state



Discussed so far



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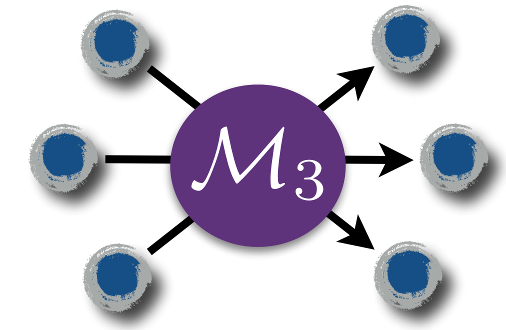

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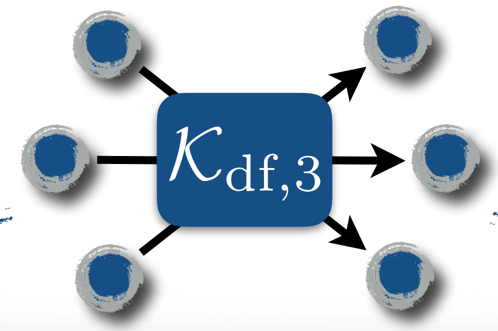

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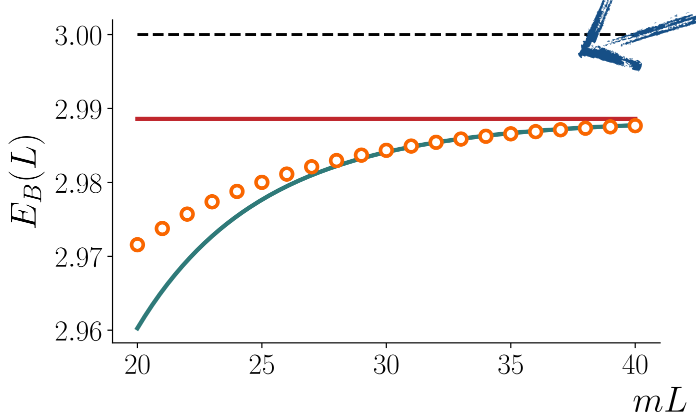

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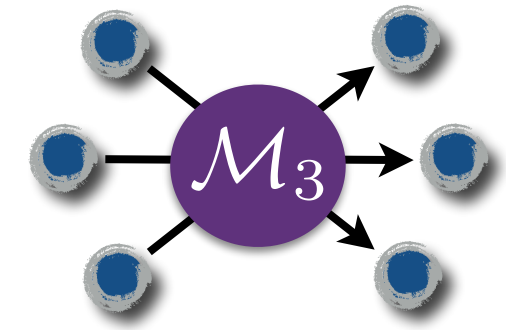

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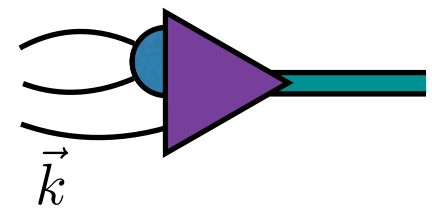
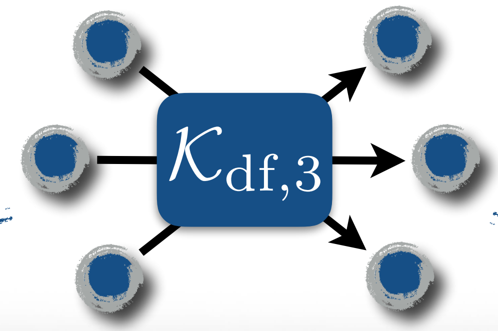
$$\Gamma^u(k) \equiv$$



$$\mathcal{S}[\Gamma^u(k)] \equiv \langle \pi(k)\pi\pi | E_B \rangle$$

Back to the bound state



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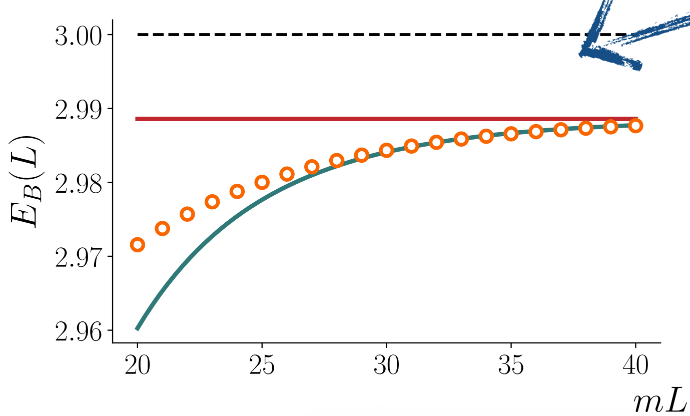

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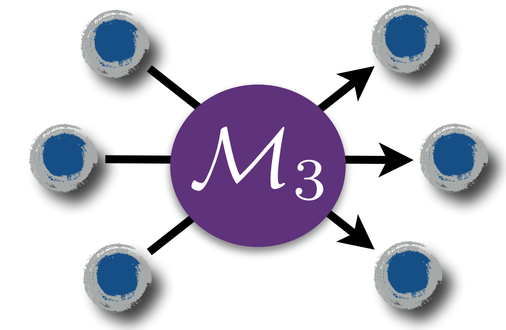

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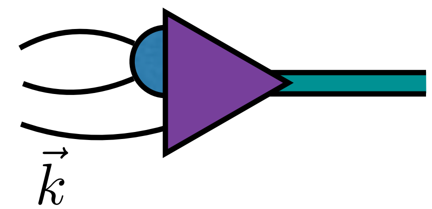

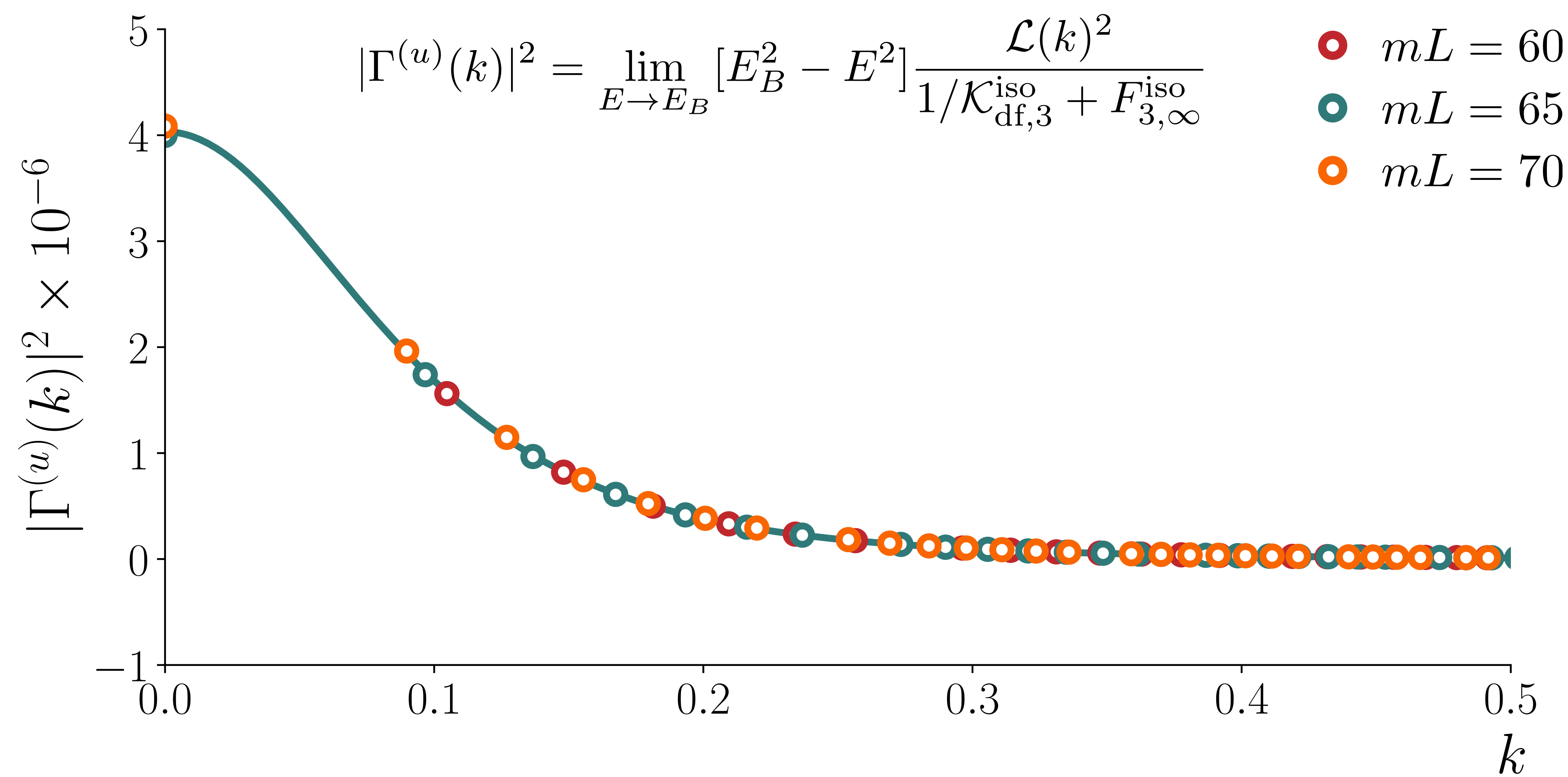

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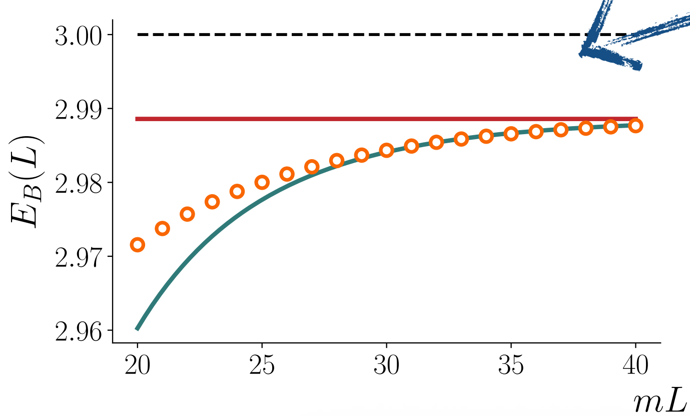

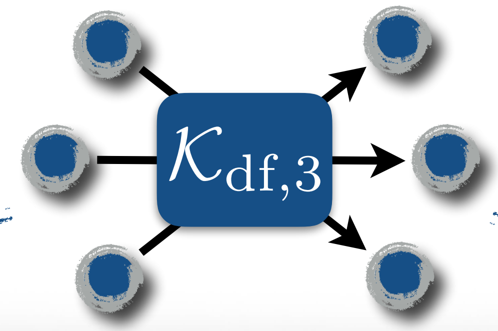

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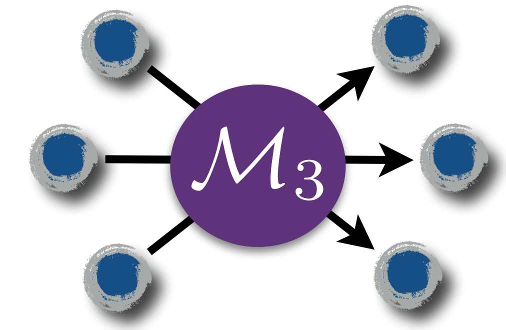

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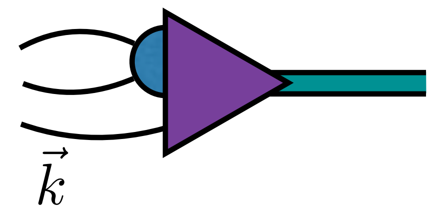

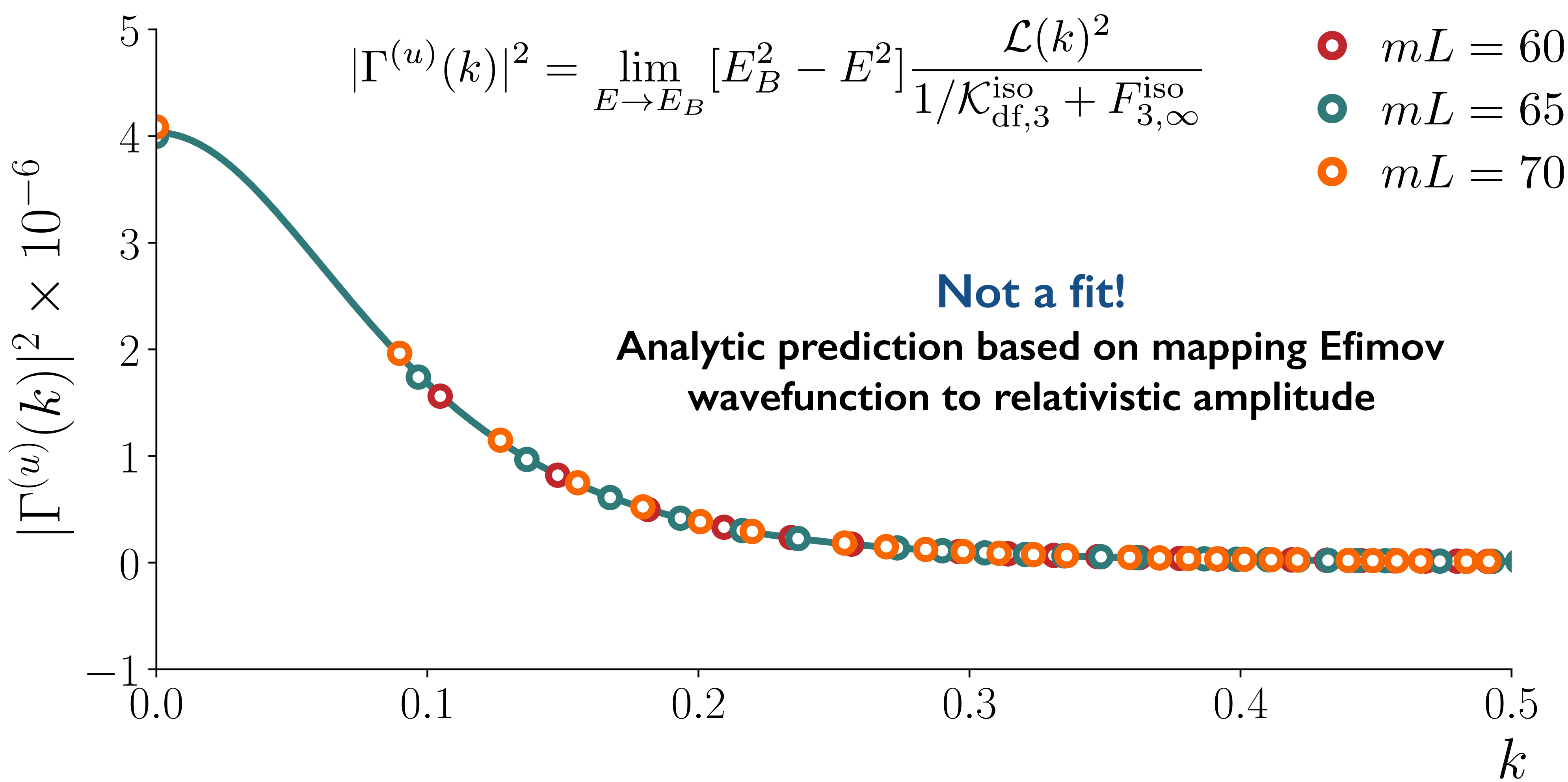

Outline

☑ Warm up and definitions
- ☑ Basic set-up
- ☑ Finite-volume correlator
- ☑ Three non-interacting particles

☑ Two particles in a box
- ☑ Alternative derivation
- ☑ Truncation and application
- ☑ Relating matrix elements

☑ Three particles in a box
- ☑ 3-to-3 scattering
- ☑ (Sketch of) derivation
- ☑ An unexpected infinite-volume quantity
- ☑ Relating energies to scattering

☑ Testing the result
- ☑ Know issues
- ☑ Large-volume expansion
- ☑ Effimov state in a box

☑ Other methods

☑ Numerical explorations
- ☑ Truncation at low energies
- ☑ Toy solutions for various systems
- ☑ Unphysical solutions

☐ Looking forward

Still lots to do

- ☐ Finish result with intermediate two-particle resonances
- ☐ Understand unphysical solutions
- ☐ Extend to non-identical, non-degenerate, multiple channels, spin
- ☐ Study subduction to finite-volume irreps
- ☐ Understand rigorous parametrizations for the infinite-volume observables
- ☐ Convince practitioners that the formalism is mature
- ☐ Reliably measure finite-volume spectra
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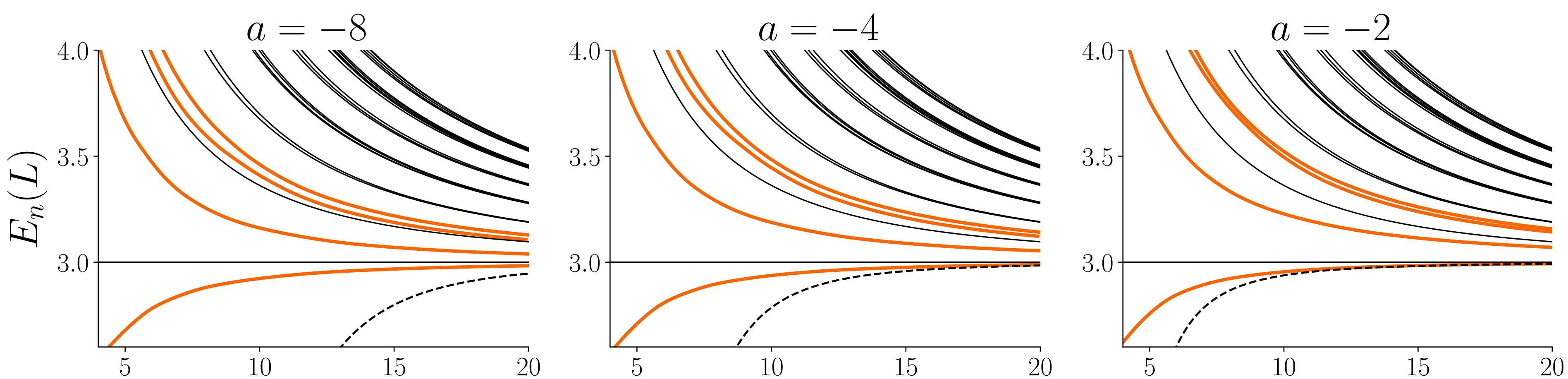
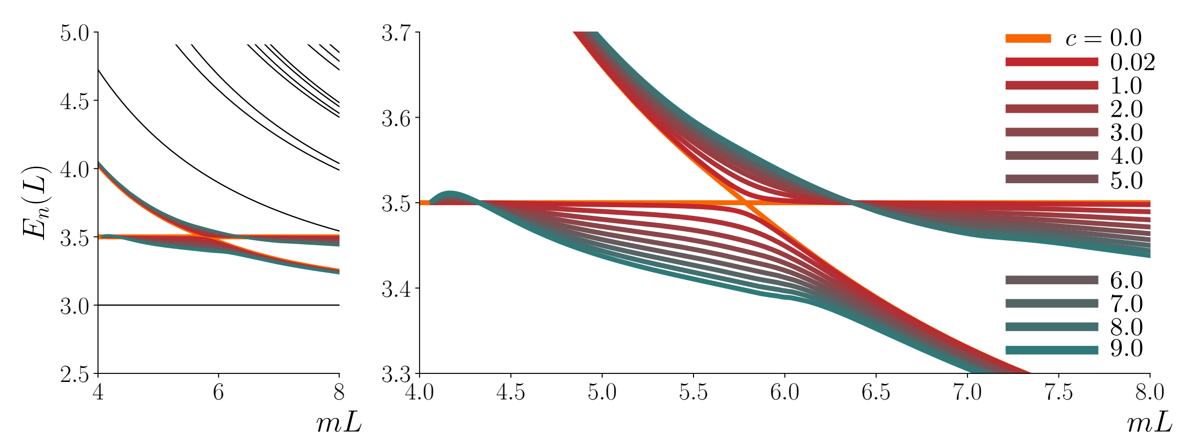
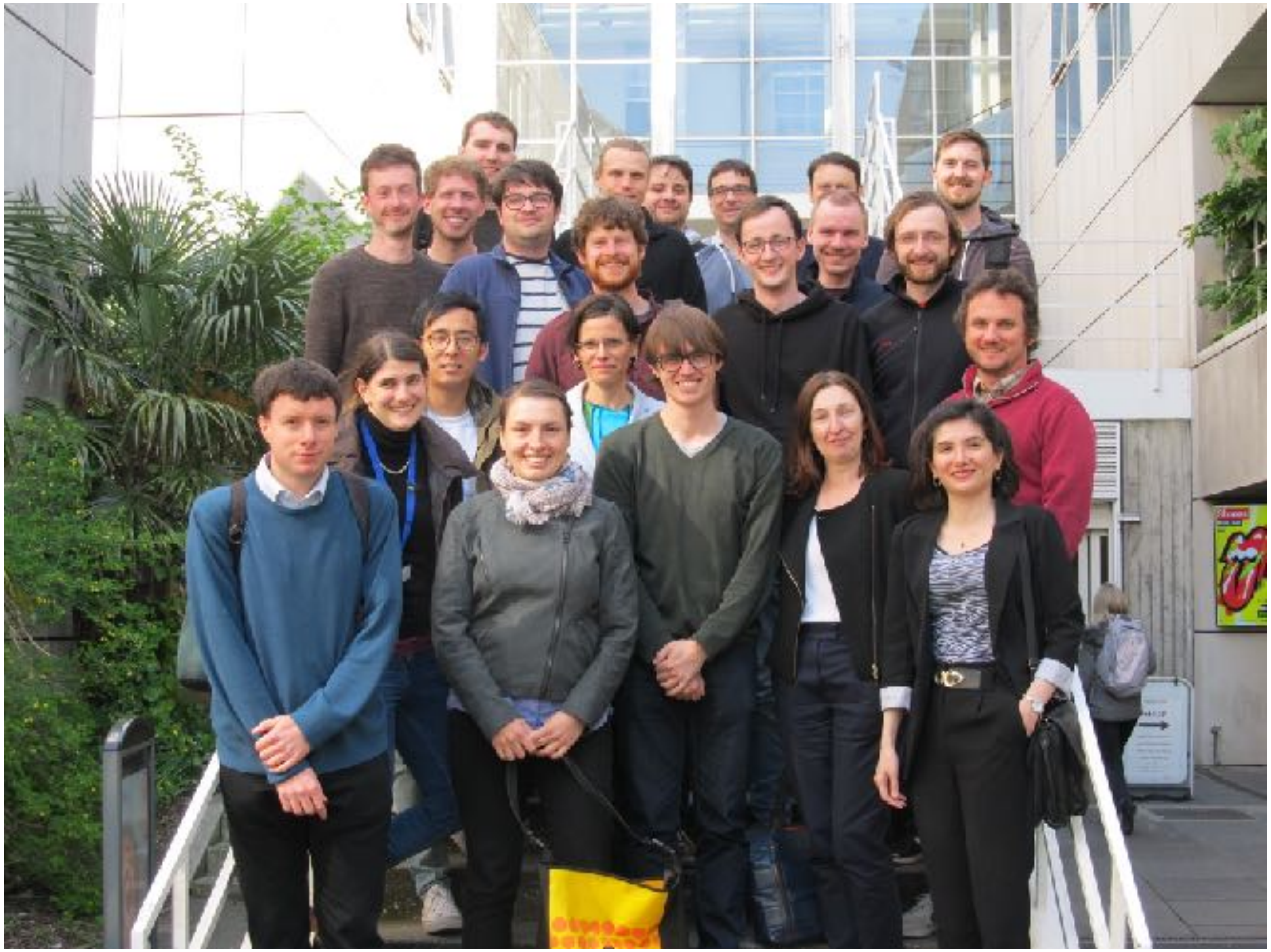
Big picture: making progress, but not quite there yet





Thanks!

That's all folks!



Thanks to all the participants!