Perspectives on bulk reconstruction
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Work with Gilad Lifschytz, emphasizing the papers 1703.06523, 1801.08101 and using results from 1505.03755.

1 intro

Bulk reconstruction, from CFT $\rightarrow$ AdS. Many approaches. Take a direct route and represent local interacting fields in AdS as operators in the dual CFT.

Lorentzian AdS$_{d+1}$, Poincaré coordinates

$$\phi_{\text{bulk}}(t, \vec{x}, z) \leftrightarrow \text{operator } \phi_{\text{CFT}}(t, \vec{x}, z)$$

We'll assume the CFT has a matrix-type $1/N$ expansion. It's known how to proceed in perturbation theory.

$\mathcal{O}(N^0)$: define a free bulk field

$$\phi^{(0)}(x, z) = \int K(x, z|x')\mathcal{O}(x')$$

$\mathcal{O}$ = single-trace local operator in the CFT (scalar, current, stress tensor, etc.)

this lets us reproduce bulk 2-point functions from the CFT

$\mathcal{O}(1/N)$: add corrections

$$\phi = \phi^{(0)} + \frac{1}{N}\sum_n a_n \int K_n \mathcal{O}_n$$

$\mathcal{O}_n$ = tower of double-trace operators, $\mathcal{O}_n \sim \mathcal{O}\partial^{2n}\mathcal{O}$

this lets us reproduce tree-level bulk 3-point functions

$\mathcal{O}(1/N^2)$: add a tower of triple-trace corrections

this lets us reproduce tree-level bulk 4-point functions

$\vdots$

The kernels K, K_n obey free wave equations in the bulk, appropriate to the dimension of the operator they're integrated against. It's a kind of spectral representation – we're representing a local interacting field in the bulk as a superposition of free fields with different masses.

The building blocks are the free fields $\phi^{(0)}$ ("kinematics"). Then we need to take appropriate superpositions, fixing the coefficients a_n ("dynamics").

Different approaches have been pursued by many authors.

1. Solving bulk e.o.m.
2. Symmetries / representation theory (as representations primary fields in the CFT correspond to on-shell field in AdS; identify the appropriate little group inside SO(d,2) or Virasoro, etc.)
3. Radon transform / bulk geodesics and OPE blocks

$$\vdots$$

It'll concentrate on two approaches Gilad and I have been pursuing: getting the free-field building blocks from reduced density matrices, and assembling free fields to get a well-defined bulk. The philosophy is to look for approaches that are intrinsic to the CFT. Suppose we didn't know about gravity but were really good at solving large-N CFT's. Could we discover gravity in AdS?

2 building blocks from modular Hamiltonians

Work in AdS_3, although the results should generalize.

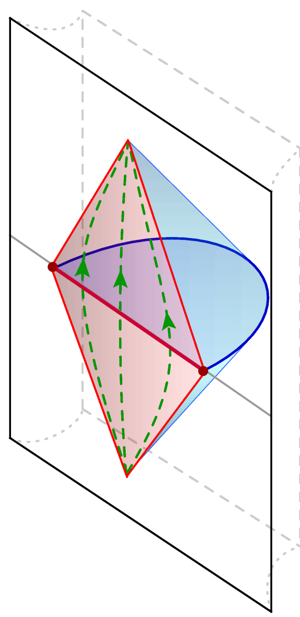
Natural object to consider in the CFT: the reduced density matrix associated with a spatial region (an interval R).

$$\mathcal{H} = \mathcal{H}_R \otimes \mathcal{H}_{\bar{R}}$$
$$\text{reduced density matrix } \rho = \mathrm{Tr}_{\mathcal{H}_{\bar{R}}} |\psi\rangle\langle\psi|$$
$$\text{modular Hamiltonian } H_{\text{mod}} = -\log\rho$$

The modular Hamiltonian generates a sort of time evolution in the boundary causal development of R. Better: use the "total modular Hamiltonian" $\tilde{H}_{\text{mod}} = H_{\text{mod,R}} - H_{\text{mod},\bar{\text{R}}}$. These objects have a nice bulk dual.

> Region R in the CFT defines an extremal RT surface γ in the bulk.
>
> Boundary modular flow is dual to bulk modular flow. (Jafferis, Lewkowycz, Maldacena, Suh)

For vacuum AdS_3 the flow looks like (figure from de Boer, Haehl, Heller & Myers, 1606.03307)



These reduced density matrices are basic for characterizing entanglement. Claim: they also provide a starting point for constructing bulk observables. Idea is to look for CFT operators which are invariant under modular flow.

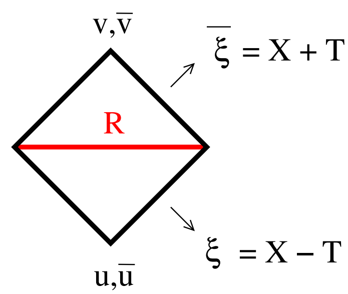
> If $[\tilde{H}_{\text{mod}}, \mathcal{O}] = 0 \Rightarrow \mathcal{O}$ lives on RT surface.
>
> If $[\tilde{H}^{(1)}_{\text{mod}}, \mathcal{O}] = [\tilde{H}^{(2)}_{\text{mod}}, \mathcal{O}] = 0 \Rightarrow \mathcal{O}$ lives on intersection of RT surfaces. Picks out a point in the bulk.

Operators invariant under modular flow have been considered before, for example

$$Q = \int_\gamma ds\, \phi^{(0)} \qquad \gamma = \text{RT surface}$$

(appears in geodesic Witten diagrams, OPE blocks). Here we'll use it to construct our building blocks $\phi^{(0)}$. Careful: we're building operators which are invariant under modular flows. They're *not* guaranteed to be local fields in the bulk. That will be part 3 of the talk.

Extended modular Hamiltonian for an interval $R = [y_1, y_2]$ in the CFT?



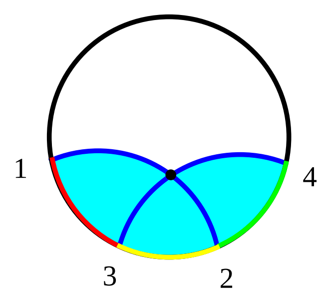
In the ground state (Casini - Huerta - Myers)

$$\tilde{H}_{\rm mod} = \int_{-\infty}^{\infty} d\xi \frac{(u-\xi)(\xi - v)}{u - v} T_{\xi\xi} + \text{left-movers}$$

This acts on CFT primaries as expected, via conformal Killing flow

$$[\tilde{H}_{\rm mod}, \mathcal{O}] = \frac{2\pi i}{y_2 - y_1}\Big((\bar\xi - \xi)\Delta - y_1 y_2(\partial_\xi - \partial_{\bar\xi}) + (y_1 + y_2)(\xi\partial_\xi - \bar\xi\partial_{\bar\xi}) + \bar\xi^2\partial_{\bar\xi} - \xi^2\partial_\xi\Big)\mathcal{O}$$

Suppose we have two overlapping regions, with RT surfaces that intersect in the bulk.



Let's solve $[\tilde{H}^{12}_{\rm mod}, \Phi] = [\tilde{H}^{34}_{\rm mod}, \Phi] = 0$. (Similar to Miyaji et al., Nakayama & Ooguri who imposed invariance under the little group that leaves a point fixed in the bulk.)

Start with the ansatz

$$\begin{aligned}\Phi &= \int dt' dy' g(q,p) \mathcal{O}(q,p) \\ q &= iy' - t,\ p = iy' + t\end{aligned}$$

and impose $[\tilde{H}^{12}_{\rm mod}, \Phi] = [\tilde{H}^{34}_{\rm mod}, \Phi] = 0$. The difference of the two conditions gives

$$\begin{aligned}g(q,p) &= f((p - X_0)(q - X_0)) \\ X_0 &= \frac{y_1 y_2 - y_3 y_4}{y_1 + y_2 - y_3 - y_4}\end{aligned}$$

then the sum fixes

$$\begin{aligned}f &= \text{const.}(Z^2 + (p - X_0)(q - X_0))^{\Delta - 2} \\ Z^2 &= (X_0 - y_1)(y_2 - X_0)\end{aligned}$$

In fact

- X_0 is the spatial coordinate where the two RT surfaces intersect
- Z is the radial coordinate of the intersection
- f is the smearing kernel in light-front coordinates
- One has to do some integration by parts. Requiring no surface terms restricts the region of integration as expected, to $t'^2 + y'^2 < Z^2$.

3 assembling bulk operators

We've built CFT operators invariant under the modular flows that leave a point in the bulk fixed. It's an output (a consequence of our ansatz in terms of a single-trace primary, but still, did this have to be the case?) that $\phi^{(0)}$

obeys a free wave equation in the bulk. These free fields are peculiar objects, since the CFT is dual to an interacting bulk theory. The loophole in the intuition is that nothing in the construction guarantees that $\phi^{(0)}$ respects bulk locality. In fact, nothing guarantees that $\phi^{(0)}$ is well-defined, i.e. has well-defined correlators. We have to check.

Upshot: we'll find that correlators are badly singular. One has to superimpose a specific tower of higher-dimension multi-trace operators in order to cancel the problematic singularity. This construction singles out the local field of bulk effective field theory and determines the local bulk equations of motion in terms of CFT data.

For example consider a massless field in AdS_3. The CFT 3-point function is

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle = \frac{\gamma}{N} \prod_{i<j} \frac{1}{-T_{ij}^2 + X_{ij}^2}$$

$\gamma =$ OPE coefficient

$T_{ij} = T_i - T_j$, $X_{ij} = X_i - X_j$

Wightman $i\epsilon$ prescription, $T_i \to T_i - i\epsilon_i$

operators ordered by decreasing ϵ

Defining the free bulk field $\phi^{(0)} = \int\limits_{t'^2+y'^2<Z^2} dt'dy'\, \mathcal{O}(t+t', x+iy')$ we find

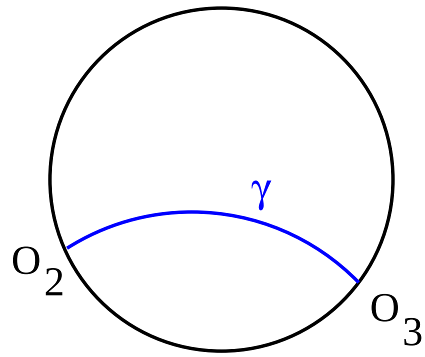
$$\langle \phi^{(0)} \mathcal{O}_2 \mathcal{O}_3 \rangle = \frac{\gamma}{2N} \frac{1}{(-T_{23}^2 + X_{23}^2)^2} \log \frac{\chi}{\chi - 1}$$

in terms of the AdS-invariant cross-ratio

$$\chi = \frac{(-T_{12}^2 + X_{12}^2 + Z^2)(-T_{13}^2 + X_{13}^2 + Z^2)}{(-T_{23}^2 + X_{23}^2)Z^2}$$

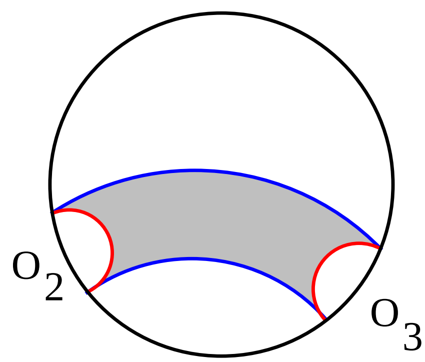
The correlator is singular at $\chi = 0$ and $\chi = 1$. Can study these singularities algebraically but it's easier to draw pictures. With the boundary operators inserted at spacelike separation on the $T = 0$ slice

$T = 0$:



γ = spacelike geodesic connecting $\mathcal{O}_2$ and $\mathcal{O}_3$
recall OPE block $P_{\mathcal{O}}\mathcal{O}_2\mathcal{O}_3 = \int_\gamma ds\, \phi^{(0)}$

$T > 0$:



Red curves correspond to $\chi = 0$, null to the boundary operators

Blue curves correspond to $\chi = 1$, null to the spacelike geodesic

From the bulk perspective the $\chi = 1$ singularities occur at spacelike separation so they violate locality. Worse, they make the correlator ambiguous. Starting from a local correlator in the CFT, to continue into the shaded region you have to cross a singularity.

Cross a $\chi = 0$ singularity $\Rightarrow$ inherit $i\epsilon$ prescription of the relevant light-cone.

Cross a $\chi = 1$ singularity $\Rightarrow$ result depends on the details of the continuation if ϵ_{12} and ϵ_{13} have different signs

So the result is ambiguous if ϵ_{12} and ϵ_{13} have different signs, i.e. if the bulk operator is in the middle of the correlator. Reflects a failure of operator associativity?

In $1/N$ perturbation theory the cure for this disease is to correct the definition of the bulk field.

$$\phi = \phi^{(0)} + \frac{1}{N}\sum_{n=0}^{\infty} a_n \int K_n \mathcal{O}_n$$

$$\text{double-trace operators } \mathcal{O}_n \sim \mathcal{O}\partial^{2n}\mathcal{O}$$

The individual terms on the right hand side have 3-point correlators with two boundary operators that have branch cuts at $0 < \chi < 1$. For $\phi^{(0)}$ itself the discontinuity across the cut is given by a Legendre function $P_0(2\chi - 1)$. For a double-trace operator the discontinuity is proportional to $P_{n+1}(2\chi-1)$. Fortunately from 1505.03755 we have the (formal) sum

$$P_0(x) = -\sum_{k=1}^{\infty}(-1)^k(2k+1)P_k(x)$$

Keeping track of numerical factors, this means we can cancel the cut by setting the coefficients a_n in the expansion according to (γ_n and γ are the OPE coefficients appearing in $\langle \mathcal{O}_n\mathcal{O}\mathcal{O}\rangle$ and $\langle \mathcal{O}\mathcal{O}\mathcal{O}\rangle$ respectively)

$$\frac{a_n\gamma_n}{\gamma} = (-1)^{n+1}\frac{\Gamma^2(n+2)}{\Gamma(2n+3)}$$

With a bit of work you can show that the resulting bulk field obeys a local equation of motion, $\nabla\phi = -4\pi^2\gamma\phi^2$. So we've recovered bulk ϕ^3 theory in terms of CFT data, just by requiring that bulk observables are well-defined.