

# Outline

PART 1 : Effective description of BSM physics

1.1 "Integrating out" fields

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PART 2 : SM as an effective field theory

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# EFFECTIVE FIELD THEORIES

①

We have seen how to obtain an effective field theory by integrating out fields with wavelength smaller than some cutoff length  $\frac{1}{\Lambda}$  in the Wilsonian approach. This is however not the only way to obtain an effective theory valid at energies much below the cutoff  $\Lambda$ . A more standard procedure consists in "integrating out" all fields whose mass is above the cutoff. It is intuitive, indeed, that at low energies these states cannot be produced and thus never appear as external lines in scattering processes. However, they can contribute to observables through virtual effects.

EFT a la Wilson : integrating out fields with wavelength  $< 1/\Lambda$

Standard EFT : integrating out fields with mass  $> \Lambda$   
(hence onshell wavelength  $< 1/\Lambda$ )

The action of the effective theory can be defined through the functional integral. Let  $\Phi$  be a collection of fields with mass  $m_\Phi \sim \Lambda$ , and  $\varphi$  be fields with mass much lighter than  $\Lambda$ . Then (on the Euclidean)

$$Z[J] = \int D\varphi D\Phi e^{-S[\Phi, \varphi] - \int J \cdot \varphi} \equiv \int D\varphi e^{-S_{\text{eff}}[\varphi] - \int J \cdot \varphi}$$

where

$$e^{-S_{\text{eff}}[\varphi]} \equiv \int \mathcal{D}\Phi e^{-S[\Phi, \varphi]}$$

Notice that  $S_{\text{eff}}[\varphi]$  is in general a non-local function of  $\varphi$ . However, at energies  $E \ll \Lambda$ , it can be expanded as a series of local terms suppressed by powers of the cutoff scale  $\Lambda$ . Higher-dimensional terms will thus be suppressed at low energy by powers  $(E/\Lambda)^{4-[0]}$ , and can thus be neglected. Such truncated effective Lagrangian is thus local.

The reason why higher-terms are suppressed is essentially connected to the uncertainty principle:

high-energy modes propagate for very short distances in virtual exchanges.

Notice that although our effective Lagrangian is valid up to energies of order of the cutoff scale  $\Lambda$ , loop contributions to low-energy observables are computed by allowing the internal momenta of the light fields to assume arbitrarily large values (including values above  $\Lambda$ ).

This contribution from high internal momenta, on the other hand, will be equivalent to that of local operators, again due to the uncertainty principle. It will thus renormalize the coefficients of local operators but will not affect the long-distance behavior of observables, which is what is truly calculable in an effective theory.

The effective Lagrangian thus include a series of local operators of increasing dimensionality, built by adding powers of fields and (covariant) derivatives. (3)

One can in fact derive the most general form of an effective theory by writing down all possible operators allowed by the symmetries. The coefficients appearing in front of the operators will encode all the details of the specific UV theory above the cutoff.

In order to compute the operators' coefficients, an alternative to the calculation of the functional integral is the so-called matching.

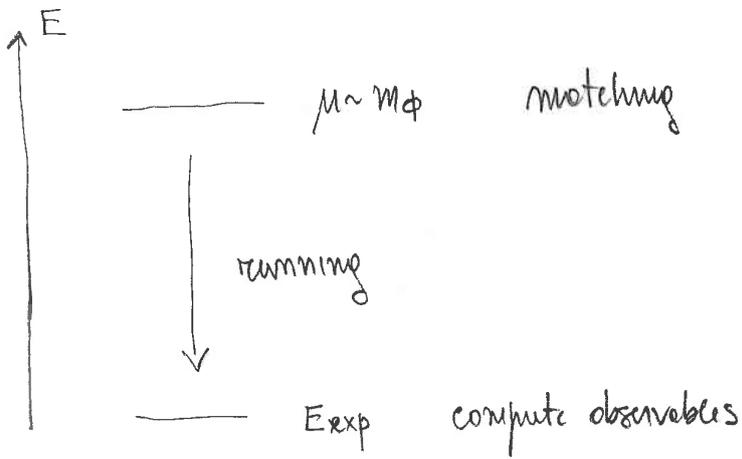
The idea is to compute (a set of) Green Functions in both the full and the effective theory and adjust the coefficients so as to obtain identical results (up to higher orders in the  $1/m_\phi$  expansion).

- Notice that:
- i) the GF's do not have to be on-shell nor to correspond to physical observables
  - ii) once the matching is done, the effective theory so obtained can be used to compute any GF

It is important to notice the following:

- i) in performing the matching, the subtraction scale  $\mu$  should be set to be of order of the cutoff scale,  $\mu \sim m_\phi$ , to avoid large logs.
- ii) when computing observables at low energy, one should set  $\mu \sim E$  to avoid large logs.
- iii) the evolution of the operators' coefficients from  $\mu \sim m_\phi$  down to  $\mu \sim E$  should be done through the RG

Thus, the common procedure is that of matching and running:



The renormalization group is thus crucial to resum large logs and deal with large separation of scales.

Before working out an explicit example of running and matching, it is useful to notice that equations of motions can be used to simplify the list of higher-dimensional operators.

Consider for example an effective theory whose Lagrangian can be organized into a series

$$\mathcal{L} = \mathcal{L}^{(4)} + \mathcal{L}^{(6)} + \mathcal{L}^{(8)} + \dots$$

where  $\mathcal{L}^{(m)}$  denotes terms with (classical) dimension  $m$ . Then, the Equations of Motion (EOM) implied by  $\mathcal{L}^{(m)}$  can be used to simplify  $\mathcal{L}^{(n+2)}$ .

To prove it, let us notice that a non-linear field redefinition

$$\phi = \chi F(x) \quad \text{where} \quad \begin{cases} \phi = \chi F(x) \\ F(0) = 1 \end{cases}$$

does not affect S-matrix elements and observables.

For a discussion see: Donoghue et al. "Dynamics of the Standard Model" section IV-1, pag 110

and references therein.

Let us then consider the field redefinition

$$\phi = \phi' + \delta\phi' \quad \text{where} \quad \delta\phi' = O(\phi'^3)$$

This is a change of variables in the functional integral that does not affect the physics. The Lagrangian is modified as follows:

$$\begin{aligned} \mathcal{L}_4(\phi, \partial_\mu\phi) &= \mathcal{L}_4(\phi', \partial_\mu\phi') + \frac{\delta\mathcal{L}_4}{\delta\phi} \delta\phi' + \frac{\delta\mathcal{L}_4}{\delta(\partial_\mu\phi)} \partial_\mu(\delta\phi') \\ &= \mathcal{L}_4(\phi', \partial_\mu\phi') + \underbrace{\left( \frac{\delta\mathcal{L}_4}{\delta\phi} - \partial_\mu \frac{\delta\mathcal{L}_4}{\delta(\partial_\mu\phi)} \right)}_{\text{dim}-6} \delta\phi' + \text{total derivative} \end{aligned}$$

$$\mathcal{L}_6(\phi, \partial_\mu\phi) = \mathcal{L}_6(\phi', \partial_\mu\phi') + \text{dim}-8$$

⋮

Hence, since  $\delta\phi'$  is  $O(\phi'^3)$ , the variation of  $\mathcal{L}^{(4)}$  has dimension 6, while  $\mathcal{L}^{(6)}$  varies by higher-order terms. One can thus choose  $\delta\phi'$  so as to simplify  $\mathcal{L}^{(6)}$  by adding to them a term proportional to the equations of motion derived from  $\mathcal{L}^{(4)}$ .

For example: 
$$\mathcal{L}^{(4)} = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{m_\phi^2}{2} \phi^2 - \frac{\lambda}{4!} \phi^4$$

The EOM are:

$$\left. \begin{aligned} \frac{\delta \mathcal{L}^{(4)}}{\delta \phi} &= -m_\phi^2 \phi - \frac{\lambda}{3!} \phi^3 \\ \frac{\delta \mathcal{L}^{(4)}}{\delta (\partial_\mu \phi)} &= \partial_\mu \phi \end{aligned} \right\} \Rightarrow \boxed{\square \phi = -m_\phi^2 \phi - \frac{\lambda}{3!} \phi^3}$$

One can thus rewrite

$$\phi^3 \square \phi = -m_\phi^2 \phi^4 - \frac{\lambda}{3!} \phi^6$$

We said that matching can be used to extract the coefficients of the local operators and thus fully determine the form of the low-energy effective Lagrangian. On the other hand, effective Lagrangians are useful even when the UV theory is unknown (as in the case of gravity) or cannot be easily matched because it is strongly coupled (as in the case of QED).

Indeed, the coefficients of the local operators can be measured by using some observables as inputs, so as to be able to predict other observables. This is the case, for example, of the effective theory of pions (chiral Lagrangian).

Even if the coefficients of local operators are unknown, effective theories make distinctive predictions: they allow one to compute the non-analytic dependence of observables on momenta (e.g.  $\log(-p^2)$ ) and on Lagrangian's parameters (e.g.  $\sqrt{m_\pi^2}$ ).

These are in fact the contributions which cannot be affected by local operators and are thus genuine "long-distance" predictions of the effective theory in terms of light modes.

At energies much smaller than the cutoff scale  $\Lambda$ , the effects of the non-renormalizable operators are suppressed by powers of  $(E/\Lambda)$ . This goes under the name of "decoupling theorem": heavy particles contribute to the renormalization of the parameters of the Lagrangian and to corrections which are power suppressed at low energy ( $E \ll \Lambda$ ).

Since non-renormalizable operators are suppressed at low energies, renormalizability appears as an emergent phenomenon.

The fact that the predictions of the renormalizable Lagrangian are in good agreement with experimental data, then indicate that the cutoff scale is much higher than the energies probed so far.

A lower bound on the scale of new physics (cutoff scale  $\Lambda$ ) can be set by making a naive estimate of the coefficient of higher-dimensional operators.

Consider for example the electrodynamics of electrons and muons: the  $(g-2)$  gets a contribution from the dipole operator (8)

$$\frac{e}{\Lambda} \bar{\psi} \sigma^{\mu\nu} F_{\mu\nu} \psi$$

$$\sigma^{\mu\nu} \equiv \frac{i}{2} [\gamma^\mu, \gamma^\nu]$$

This has dimension 5 hence it is naively suppressed by one power of the cutoff scale. The power of the coupling constant  $e$  in the numerator has been included to account for the interaction strength of any new field of mass  $\sim \Lambda$  with the electromagnetic field.

The above dipole operator gives a contribution  $(4e/\Lambda)$  to the  $(g-2)$ , while the experimental value of the latter agrees with the predictions of the renormalizable theory of QED to a precision of order a few  $\times 10^{-10} e/2mc$ . This sets the lower bound

$$\Lambda \gtrsim 8 \times 10^{10} mc = 4 \times 10^7 \text{ GeV}$$

which is much higher than the energy scales probed so far.

It has to be noticed that symmetry considerations play an important role in estimating the bound on  $\Lambda$ . For example, one could assume that chirality

$$\psi \rightarrow \gamma^5 \psi$$

is an invariance of the full (QED + non-renormalizable operators) theory except for an explicit violation coming from the fermion mass. We thus assume that the only breaking of such symmetry comes from the mass. A useful way to incorporate this assumption in our formalism in a systematic way is to treat

the mass  $m_\mu$  as a field (or "spurion") and assign to it a transformation rule

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$$M_\mu \rightarrow -M_\mu$$

under chiral transformations.

In this way the mass term is formally invariant. We can thus construct all higher-order operators by including the spurion and requiring invariance under chiral transformations. This implies that the dipole operator now reads

$$e \frac{M_\mu}{\Lambda^2} \bar{\psi} \sigma^{\mu\nu} F_{\mu\nu} \psi$$

and is thus suppressed by two powers of  $\Lambda$ . Because of the proportionality to  $m_\mu$ , the strongest constraint now comes from the  $(g-2)$  of the muon. One gets

$$\Lambda > \sqrt{8 \times 10^{10}} m_\mu = 3 \times 10^3 \text{ GeV}$$

We have seen how imposing symmetries on the effective Lagrangian modifies our estimate of the cutoff scale. There is another important situation, on the other hand, in which symmetries emerge accidentally: it can indeed happen that the renormalizable Lagrangian is invariant under a larger global invariance which is violated by the non-renormalizable operators. Since these latter are suppressed at low energy, such accidental symmetry emerges as an approximate one up to small violations.

A notable example of accidental invariance is the custodial symmetry (10) of the scalar sector in the Standard Model.

Given the  $SU(2)_L$  Higgs doublet  $H$

$$H = \begin{bmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{bmatrix} \quad H \rightarrow U \cdot H \quad U = e^{i\alpha^a \sigma^a} \in SU(2)_L$$

invariance under  $SU(2)_L$  implies that the potential can only depend on  $(H^\dagger H)$ :

$$V(H) = f(H^\dagger H) = f\left(\sum_i \phi_i^2\right)$$

It has thus a larger  $SO(4)$  invariance under which the four real components  $\phi_i$  rotate into each other

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} \rightarrow R \cdot \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} \quad R \in SO(4)$$

Similarly, the kinetic term is also invariant under  $SO(4)$ :

$$(\partial_\mu H^\dagger)(\partial^\mu H) = \sum_i (\partial_\mu \phi_i)^2$$

After  $H$  gets a vev,  $\langle H \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$ , that is to say  $\langle \phi_3 \rangle = v/\sqrt{2}$ , the  $SO(4)$  is spontaneously broken to  $SO(3) \sim SU(2)$ .

This latter is called custodial symmetry, and it is an accidental invariance of the vacuum.

## Example of Running and Matching

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As an example of running and matching, let us consider the following theory of one fermion  $\psi$  and one real scalar  $\phi$ :

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m\psi)\psi + i\gamma_5 \bar{\psi}\gamma_5\phi\psi + \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m_\phi^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$$

The Lagrangian is invariant under parity transformations

$$\psi(\vec{x}, t) \rightarrow \gamma^0 \psi(-\vec{x}, t)$$

$$\phi(\vec{x}, t) \rightarrow -\phi(-\vec{x}, t)$$

The field  $\phi$  is thus a pseudo-scalar.

We consider the limit in which  $m_\psi$  is large (compared to energies experimentally probed), so that the fermion can be integrated out.

In doing so, one generates a list of higher-dimensional operators built with the field  $\phi$  and its derivatives. Given the parity invariance, there are no operators with dimension 5 (and in general no operators with an odd number of fields  $\phi$ ). To find the complete and minimal list of dimension-6 operators we can analyze all possible ones compatible with the symmetries of the Lagrangian (parity and Lorentz).

We have:

$$\mathcal{L}_{eff} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m_\phi^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 + \mathcal{L}^{(6)} + \mathcal{L}^{(8)} + \dots$$

with 0 derivatives :  $\phi^6$

with 2 derivatives :  $(\phi \partial_\mu \phi)^2$

$$\phi (\partial_\mu \phi) \partial_\mu \phi^2 = 2 \phi^2 (\partial_\mu \phi)^2 \rightarrow \text{not new}$$

$$\partial_\mu \phi \partial_\mu \phi^3 = (\partial_\mu \phi)^2 \phi^2 + (\partial_\mu \phi) \phi (\partial_\mu \phi^2) \rightarrow \text{not new}$$

$$\phi^3 \square \phi \underset{\text{EOM}}{=} -m_\phi^2 \phi^4 - \frac{\lambda}{3!} \phi^6 \rightarrow \text{not new}$$

$$\phi \square \phi^3 = \partial_\mu \phi \partial_\mu \phi^3 + \text{tot. derivative} \rightarrow \text{not new}$$

$$\partial_\mu \phi^2 \partial_\mu \phi^2 = (2 \phi \partial_\mu \phi)^2 \rightarrow \text{not new}$$

$$\phi^2 \square \phi^2 = \partial_\mu \phi^2 \partial_\mu \phi^2 + \text{tot. derivative} \rightarrow \text{not new}$$

with 4 derivatives :

$$\partial_\mu \phi \square \partial_\mu \phi \underset{\text{EOM}}{=} \partial_\mu \phi \partial_\mu \left( -m_\phi^2 \phi - \frac{\lambda}{3!} \phi^3 \right) \rightarrow \text{not new}$$

$$\square \phi \square \phi = \partial_\mu \phi \partial_\mu \square \phi + \text{tot. derivative} \rightarrow \text{not new}$$

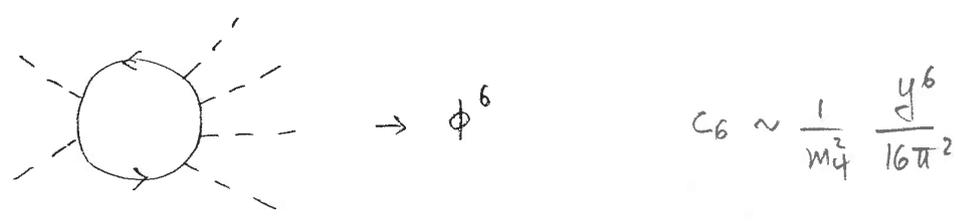
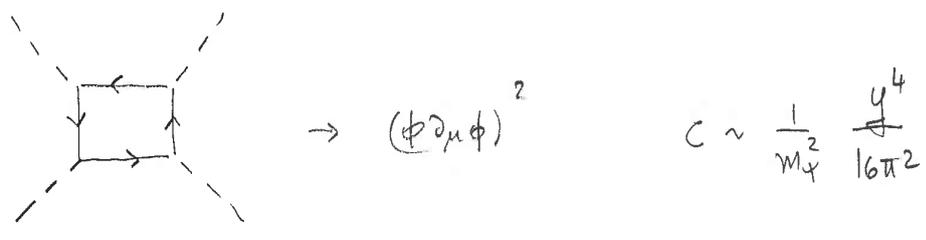
$$\phi \square \square \phi = \partial_\mu \phi \square \partial_\mu \phi + \text{tot. derivative} \rightarrow \text{not new}$$

$$\partial_\mu \partial_\nu \phi \partial_\mu \partial_\nu \phi = \partial_\mu \phi \square \partial_\nu \phi + \text{tot. derivative} \rightarrow \text{not new}$$

Hence

$$\mathcal{L}^{(6)} = C (\phi \partial_\mu \phi)^2 + C_6 \phi^6$$

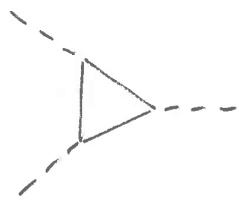
where  $C, C_6$  are two coefficients to be determined through the matching. For example, we can use a 4-point and a 6-point function to extract  $C$  and  $C_6$  respectively :



Besides generating  $c$  and  $c_6$ , integrating out  $\psi$  also gives a correction to the mass and kinetic term of the light field  $\phi$ :



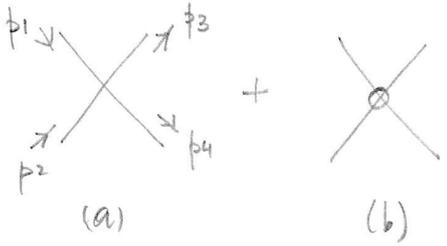
Notice that all GFs with an odd number of external  $\phi$  legs vanish due to parity.

Exercise: prove that  vanishes.

Let us for example extract  $c$  by matching the full and effective theories at order  $\lambda$  (1-loop matching).

We need to compute the 4-point function both in the full and in the effective theory:

Effective theory :

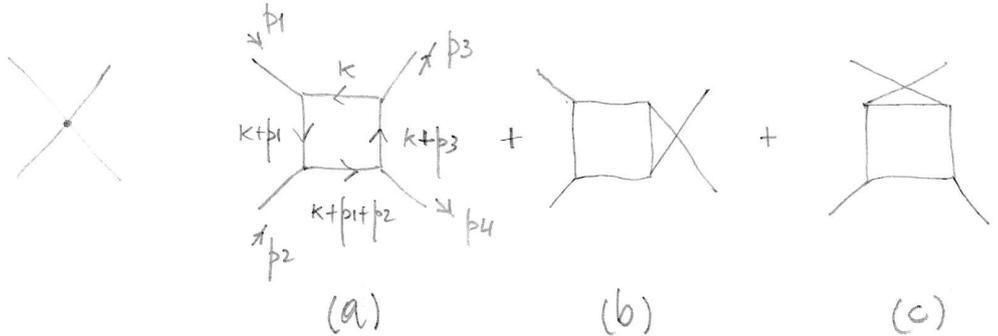


$$M = \underbrace{-i\lambda}_{\text{diagram a}} + 2iC \underbrace{(p_1^2 + p_2^2 + p_3^2 + p_4^2)}_{\text{diagram b}}$$

combinatorial factor for diagram b :

$$\begin{aligned} & i 2 \times 2 (-ip_1) \cdot (-ip_2) + i 2 \times 2 (-ip_1) \cdot (ip_3) + i 2 \times 2 (-ip_1) \cdot (ip_4) \\ & + i 2 \times 2 (ip_3) \cdot (ip_4) + i 2 \times 2 (-ip_2) \cdot (ip_3) + i 2 \times 2 (-ip_2) \cdot (ip_4) \\ & = (-i) \cdot 4 [ p_1 \cdot p_2 - p_1 \cdot p_3 - p_1 \cdot p_4 + p_3 \cdot p_4 - p_2 \cdot p_3 - p_2 \cdot p_4 ] \\ & = -4i [ (s+t+u) - \frac{3}{2} (p_1^2 + p_2^2 + p_3^2 + p_4^2) ] = +2i (p_1^2 + p_2^2 + p_3^2 + p_4^2) \end{aligned}$$

Full theory :



$$M = -i\lambda - 2y^4 (A^{(a)} + A^{(b)} + A^{(c)})$$

fermion loop  
 there are in total six Wick contractions equal in pairs, hence the factor 2

where

$$A^{(a)} = \int \frac{d^4 k}{(2\pi)^4} \text{Tr} \left[ \gamma^5 \frac{i}{k + \not{p}_1 - m_4} \gamma^5 \frac{i}{k - m_4} \gamma^5 \frac{i}{k + \not{p}_3 - m_4} \gamma^5 \frac{i}{k + \not{p}_1 + \not{p}_2 - m_4} \right]$$

$$= A(s, t, u)$$

$A^{(b)} = A(s, u, t)$  obtained from (a) by exchanging  $p_3 \leftrightarrow p_4$

$A^{(c)} = A(u, t, s)$  obtained from (a) by exchanging  $p_1 \leftrightarrow p_3$

Combinatorial factor for diagram (a) :  $(iy)^4 \cdot \frac{i^4}{4!} \underbrace{4}_{\text{first leg}} \times \underbrace{3}_{\text{second leg}} \times \underbrace{2}_{\text{third leg}}$

In order to compute the diagrams we choose a simple kinematic configuration :

$$\left\{ \begin{array}{l} p_3 = p_4 = 0 \\ p_1 = -p_2 = p \end{array} \right\}$$

Let us first compute  $A^{(a)}$  :

$$A^{(a)} = \int \frac{d^4 k}{(2\pi)^4} \text{Tr} \left[ \gamma^5 \frac{1}{k + \not{p} - m_4} \gamma^5 \frac{1}{k - m_4} \gamma^5 \frac{1}{k - m_4} \gamma^5 \frac{1}{k - m_4} \right]$$

$$= \int \frac{d^4 k}{(2\pi)^4} \text{Tr} \left[ \frac{1}{k + \not{p} - m_4} \frac{1}{k + m_4} \frac{1}{k - m_4} \frac{1}{k + m_4} \right]$$

$$= \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2 - m_4^2)} \text{Tr} \left[ \frac{1}{k + \not{p} - m_4} \frac{1}{k + m_4} \right]$$

Since we are interested in the  $O(p^2)$  term, we can expand for small external momentum :

$$\frac{1}{k+\not{p}-m} = \frac{1}{k-m} + \not{p}^\mu \frac{\partial}{\partial p^\mu} \frac{k+\not{p}+m}{(k+\not{p})^2-m^2} \Big|_{p=0}$$

$$+ \frac{1}{2} \not{p}^\mu \not{p}^\nu \frac{\partial^2}{\partial p^\mu \partial p^\nu} \frac{k+\not{p}+m}{(k+\not{p})^2-m^2} \Big|_{p=0} + O(p^3)$$

$$\frac{\partial}{\partial p^\mu} \frac{k+\not{p}+m}{(k+\not{p})^2-m^2} = - \frac{\gamma^\mu}{(k+\not{p})^2-m^2} + \frac{(k+\not{p}+m) 2(k^\mu+\not{p}^\mu)}{[(k+\not{p})^2-m^2]^2}$$

$$\frac{\partial^2}{\partial p^\mu \partial p^\nu} \frac{k+\not{p}+m}{(k+\not{p})^2-m^2} = - \frac{2\gamma^\mu (k^\nu+\not{p}^\nu)}{[(k+\not{p})^2-m^2]^2} - \frac{2\gamma^\nu (k^\mu+\not{p}^\mu) + 2\eta^{\mu\nu}(k+\not{p}+m)}{[(k+\not{p})^2-m^2]^2}$$

$$+ 2 \frac{(k+\not{p}+m) 4(k^\mu+\not{p}^\mu)(k^\nu+\not{p}^\nu)}{[(k+\not{p})^2-m^2]^3}$$

$$\frac{1}{k+\not{p}-m} = \frac{1}{k-m} - \not{p}^\mu \left[ - \frac{\gamma^\mu}{k^2-m^2} + \frac{2k^\mu(k+m)}{(k^2-m^2)^2} \right]$$

$$+ \frac{1}{2} \not{p}^\mu \not{p}^\nu \left[ - \frac{2\gamma^\mu k^\nu + 2\gamma^\nu k^\mu + 2\eta^{\mu\nu}(k+m)}{(k^2-m^2)^2} \right.$$

$$\left. + 8 \frac{k^\mu k^\nu (k+m)}{(k^2-m^2)^3} \right] + O(p^3)$$

This gives (dropping  $O(p)$  terms which vanish upon integration):

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$$\begin{aligned} \text{Tr} \left[ \frac{1}{k+p-m} \frac{1}{k+m} \right] &= \text{Tr} \left[ \left\{ \frac{1}{k-m} + \frac{1}{2} \not{p} \not{p} \left( -2 \frac{\gamma^\mu k^\nu + \gamma^\nu k^\mu + \eta^{\mu\nu} (k+m)}{(k^2-m^2)^2} \right. \right. \right. \\ &\quad \left. \left. \left. + 8 k^\mu k^\nu \frac{(k+m)}{(k^2-m^2)^3} \right) \right\} \frac{1}{k+m} \right] + O(p^4) \\ &= \text{Tr} \left[ \frac{1}{k^2-m^2} + \frac{1}{2} \not{p} \not{p} \left( -2 \frac{(\gamma^\mu k^\nu + \gamma^\nu k^\mu) (k-m)}{(k^2-m^2)^3} \right. \right. \\ &\quad \left. \left. + \frac{8 k^\mu k^\nu}{(k^2-m^2)^3} - 2 \frac{\eta^{\mu\nu}}{(k^2-m^2)^2} \right) \right] + O(p^4) \end{aligned}$$

Let us use dimensional regularization to perform the integral.  
We can thus drop terms which are odd in  $k^\mu$  and use

$$\int d^4k f(k^2) k^\mu k^\nu = \frac{1}{4} \eta^{\mu\nu} \int d^4k f(k^2) \cdot k^2$$

It follows

$$\begin{aligned} \text{Tr} [\dots] &= \text{Tr} \left[ \frac{1}{k^2-m^2} + \not{p} \not{p} \left( - \frac{(\gamma^\mu k^\nu + \gamma^\nu k^\mu) k}{(k^2-m^2)^3} + \eta^{\mu\nu} \frac{m^2}{(k^2-m^2)^3} \right) \right] + O(p^4) \\ &= \frac{4}{k^2-m^2} - 8 \not{p} \not{p} \frac{k^\mu k^\nu}{(k^2-m^2)^3} + \frac{4 \not{p}^2 m^2}{(k^2-m^2)^3} + O(p^4) \\ &= \frac{4}{k^2-m^2} + 2 \not{p}^2 \left( - \frac{1}{(k^2-m^2)^2} + \frac{m^2}{(k^2-m^2)^3} \right) + O(p^4) \end{aligned}$$

Hence it follows

$$A^{(a)} = O(p^0) + \int \frac{d^4 k}{(2\pi)^4} 2p^2 \left( -\frac{1}{(k^2 - m^2)^3} + \frac{m^2}{(k^2 - m^2)^4} \right) + O(p^4)$$

For  $m > 2$  and in dimensional regularization one has

$$\begin{aligned} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2 - m^2)^m} &= i (-1)^m \frac{\pi^2}{\Gamma(2)} \frac{1}{(2\pi)^4} \int_0^{+\infty} dk^2 k^2 \frac{1}{\Gamma(n)} \times \\ &\quad \times \int_0^{+\infty} dt t^{m-1} e^{-t(k^2 + m^2)} \\ &= i (-1)^m \frac{1}{16\pi^2} \frac{\Gamma(2)}{\Gamma(n)} \int_0^{+\infty} dt t^{m-3} e^{-tm^2} \end{aligned}$$

Wick rotation

$$\boxed{\int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2 - m^2)^m} = \frac{i (-1)^m}{16\pi^2} \frac{\Gamma(m-2)}{\Gamma(n)} (m^2)^{2-n}}$$

Hence it follows (using  $\Gamma(m-1) = m!$  for  $m$  integer)

$$\begin{aligned} A^{(a)} &= O(p^0) + 2i \frac{p^2}{16\pi^2} \left[ \frac{\Gamma(1)}{\Gamma(3)} (m^2)^{-1} + \frac{m^2}{\Gamma(4)} m^{-2} \right] + O(p^4) \\ &= O(p^0) + \frac{p^2}{m^2} \frac{1}{16\pi^2} \left( 1 + \frac{1}{3} \right) + O(p^4) \end{aligned}$$

$$\boxed{A^{(a)} = O(p^0) + \frac{4}{3} \frac{p^2}{m^2} \frac{1}{16\pi^2} + O(p^4)}$$

It is easy to see that with our choice of external momenta one has

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$$A^{(b)} = A^{(a)}$$

Hence we are left with the calculation of diagram (c) :

$$\begin{aligned} A^{(c)} &= \int \frac{d^4 k}{(2\pi)^4} \text{Tr} \left[ \gamma^5 \frac{i}{k-m} \gamma^5 \frac{i}{k-m} \gamma^5 \frac{i}{k-p-m} \gamma^5 \frac{i}{k-p-m} \right] \\ &= \int \frac{d^4 k}{(2\pi)^4} \text{Tr} \left[ \frac{1}{k-m} \frac{1}{k+m} \frac{1}{k-p-m} \frac{1}{k-p+m} \right] \\ &= 4 \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2-m^2} \frac{1}{(k-p)^2-m^2} \end{aligned}$$

By expanding at second order in  $p^\mu$  :

$$\boxed{\frac{1}{(k-p)^2-m^2} = \frac{1}{k^2-m^2} + p^\mu \frac{2k_\mu}{(k^2-m^2)^2} + \frac{1}{2} p^\mu p^\nu \left[ \frac{8k^\mu k^\nu}{(k^2-m^2)^3} - \frac{2\eta^{\mu\nu}}{(k^2-m^2)^2} \right] + O(p^3)}$$

Hence (dropping  $O(p)$  terms which vanish upon integration)

$$\begin{aligned} A^{(c)} &= 4 \int \frac{d^4 k}{(2\pi)^4} \left[ \frac{1}{(k^2-m^2)^2} + p^\mu p^\nu \left( \frac{4k^\mu k^\nu}{(k^2-m^2)^4} - \frac{\eta^{\mu\nu}}{(k^2-m^2)^3} \right) \right] + O(p^4) \\ &= O(p^0) + 4 p^2 m^2 \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2-m^2)^4} + O(p^4) \end{aligned}$$

$$A^{(c)} = O(p^0) + 4 p^2 m^2 \frac{i}{16\pi^2} \frac{\Gamma(2)}{\Gamma(4)} (m^2)^{-2} + O(p^4)$$

$$A^{(c)} = O(p^0) + \frac{2}{3} \frac{1}{16\pi^2} \frac{p^2}{m^2} + O(p^4)$$

Hence :

$$M = O(p^0) - 2 \frac{y^4}{J} \frac{1}{16\pi^2} \frac{p^2}{m^2} \left( \frac{4}{3} + \frac{4}{3} + \frac{2}{3} \right) + O(p^4)$$

$$M = O(p^0) - \frac{20}{3} i \frac{y^4}{16\pi^2} \frac{p^2}{m^2} + O(p^4)$$

Comparing with the result in the effective theory

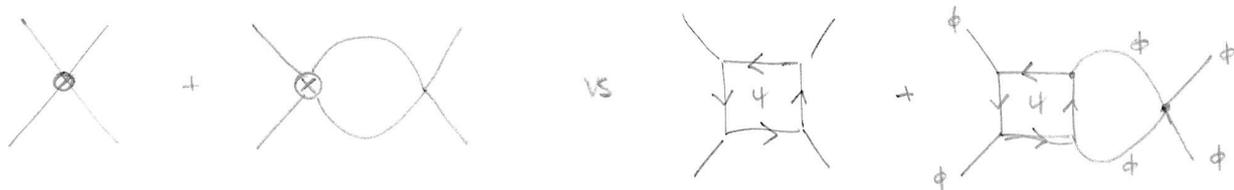
$$M_{\text{eff}} = O(p^0) + 4i c p^2$$

gives

$$c = -\frac{5}{3} \frac{y^4}{16\pi^2} \frac{1}{m^2}$$

We have seen that the  $\mathcal{O}(p^2)$  contribution to the 4-point function is finite at 1-loop level. Had we included  $\mathcal{O}(\lambda)$  corrections (hence considering matching at two loops), we would have found a divergence on the side of the effective theory:

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Removing the divergence requires introducing a counterterm. This can be done by regarding  $c$  in the original Lagrangian as a "bone" coefficient and define

$$c = c(\mu) \mu^{-2+\epsilon} Z_c \quad (d = 4 - \epsilon)$$

where  $c(\mu)$  is dimensionless and  $\mu$  is the subtraction point in dimensional regularization. We can define

$$Z_c \equiv 1 + \delta_c$$

where  $\delta_c$  is the counterterm.

Explicitly performing the matching, we would find an expression for  $c(\mu)$  which contains logarithms of the form  $\log \mu/m_4$ .

For these logs to be small, so that perturbation theory is not invalidated, we need to set  $\mu \sim m_4$  in doing the matching.

In other words, the coefficient  $c(\mu)$  extracted by the matching is evaluated at a scale  $\mu \sim m_4$  near the cutoff.

Schematically, one expects

$$A_{\text{full}} \sim O(p^0) + p^2 \left\{ \frac{\lambda(\mu)^2}{16\pi^2} \frac{1}{m_q^2} + \frac{y_4^4(\mu)}{16\pi^2} \frac{1}{m_4^2} \left( 1 + \frac{\lambda(\mu)}{16\pi^2} \log \frac{p^2}{m_4^2} \right) \right\} + O(p^4)$$

$$A_{\text{eff}} \sim O(p^0) + p^2 \left\{ \frac{\tilde{\lambda}(\mu)}{16\pi^2} \frac{1}{m_q^2} + c(\mu) \left( 1 + \frac{\tilde{\lambda}(\mu)}{16\pi^2} \log \frac{p^2}{\mu^2} \right) \right\} + O(p^4)$$

where  $\tilde{\lambda}(\mu)$  is the scalar quartic coupling in the effective theory.

Upon matching, one finds

$$c(\mu) \sim \frac{y_4^4}{16\pi^2} \frac{1}{m_4^2} \left( 1 + \frac{\lambda(\mu)}{16\pi^2} \log \frac{\mu^2}{m_4^2} \right)$$

where the  $\log \mu/m_4$  accounts for the 1-loop dependence of the effective coefficient  $c(\mu)$  on  $\mu$ . Hence, choosing  $\mu \approx m_4$  avoids large logs which could invalidate the perturbative expansion used in performing the matching.

On the other hand, when computing observables at energies  $E \ll M_4$ , the result will contain logarithms of the form  $\log E/\mu$ , so that  $\mu \sim E$  is a necessary choice if one wants to use perturbation theory. Hence, in order to compute low-energy observables one needs  $c(\mu)$  evaluated at  $\mu \sim E \ll M_4$ .

The evolution of  $c(\mu)$  from  $\mu \sim M_4$  down to  $\mu \sim E$  is obviously done through the renormalization group.

It is easy to find the RG equation obeyed by  $c(\mu)$  by using the fact that the bare coupling is  $\mu$ -independent:

$$0 = \mu \frac{d}{d\mu} c = \mu \frac{d}{d\mu} c(\mu) \mu^{-2+\epsilon} Z_c + (-2+\epsilon) c(\mu) \mu^{-2+\epsilon} Z_c + c(\mu) \mu^{-2+\epsilon} Z_c \left( \frac{1}{Z_c} \mu \frac{d}{d\mu} Z_c \right)$$

Hence:

$$\mu \frac{d}{d\mu} c(\mu) = c(\mu) \left( 2 - \epsilon - \frac{1}{Z_c} \mu \frac{d}{d\mu} Z_c \right)$$

In the limit  $\epsilon \rightarrow 0$  one obtains

$$\mu \frac{d}{d\mu} c(\mu) = (2 + \gamma_c(\lambda)) c(\mu)$$

where  $\gamma_c(\lambda) \equiv -\frac{1}{Z_c} \mu \frac{d}{d\mu} Z_c$

The solution of the RG equation is

$$C(\mu) = C(M) \exp \left( \int_{\log M}^{\log \mu} d \log y (2 + \gamma_c(\lambda(y))) \right)$$

$$= C(M) \exp \left( 2 \log(\mu/M) + \int_{\lambda(M)}^{\lambda(\mu)} d\lambda' \frac{\gamma_c(\lambda')}{\beta(\lambda')} \right)$$

$$C(\mu) = C(M) \left( \frac{\mu}{M} \right)^2 \exp \left( \int_{\lambda(M)}^{\lambda(\mu)} d\lambda' \frac{\gamma_c(\lambda')}{\beta(\lambda')} \right)$$

It is useful to derive a slightly more explicit expression by expanding  $\gamma_c$  and  $\beta$  at leading orders in  $\lambda$ :

$$\left\{ \begin{aligned} \gamma_c(\lambda) &= \frac{\gamma_c^{(0)}}{16\pi^2} \lambda + o(\lambda^2) \\ \beta(\lambda) &= \frac{\beta_0}{16\pi^2} \lambda^2 + o(\lambda^3) \end{aligned} \right. \quad \text{where } \beta_0 = 3$$

Thus

$$\int_{\lambda(M)}^{\lambda(\mu)} d\lambda' \frac{\gamma_c(\lambda')}{\beta(\lambda')} \approx \frac{\gamma_c^{(0)}}{\beta_0} \int_{\lambda(M)}^{\lambda(\mu)} d\lambda' \frac{1}{\lambda'} = \log \left( \frac{\lambda(\mu)}{\lambda(M)} \right)^{\gamma_c^{(0)}/\beta_0}$$

hence

$$C(\mu) \approx C(M) \left( \frac{\mu}{M} \right)^2 \left( \frac{\lambda(\mu)}{\lambda(M)} \right)^{\gamma_c^{(0)}/\beta_0}$$

↑  
classical running

As long as  $\lambda(H) \log \mu/M \ll 1$  one can expand the expression for  $c(\mu)$  at leading order in the logarithms:

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$$\mu \frac{d}{d\mu} \lambda(\mu) = \beta(\lambda) = \frac{\beta_0}{16\pi^2} \lambda^2 + o(\lambda^3)$$

$$\lambda(\mu) = \frac{\lambda(H)}{1 - \frac{\beta_0}{16\pi^2} \lambda(H) \log \mu/M}$$

$$\left( \frac{\lambda(\mu)}{\lambda(H)} \right)^{\frac{\gamma_c^{(0)}}{\beta_0}} \approx 1 + \frac{\gamma_c^{(0)}}{16\pi^2} \lambda(H) \log \mu/M$$

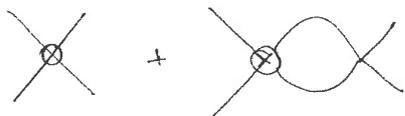
$$c(\mu) \approx c(H) \left( \frac{\mu^2}{M^2} \right) \left( 1 + \frac{\gamma_c^{(0)}}{16\pi^2} \lambda(H) \log \frac{\mu}{M} \right)$$

This formula is valid up to  $o(\lambda(H) \log \mu/M)$  and  $o(\lambda^2)$ .

This formula resums all  $(\lambda(\mu) \log \mu)^m$  terms and therefore removes the problem of large logs. (21)

For example, consider as an observable the scattering amplitude for the process  $\phi\phi \rightarrow \phi\phi$ .

At  $\mathcal{O}(\lambda)$  the contribution of the operator  $(\phi \partial_\mu \phi)^2$  will have the form, after performing a wave-function renormalization and neglecting  $m_\phi$  ( $m_\phi \ll E$ )



$$\begin{aligned} \delta A &\propto c(\mu) \mu^{-2+\epsilon} Z_c \left\{ 1 + \frac{\lambda(\mu) \mu^{-\epsilon}}{16\pi^2} \gamma_c^{(0)} \left( \frac{1}{\epsilon} - \log p + \dots \right) \right\} \\ &\propto c(\mu) \mu^{-2+\epsilon} \left\{ 1 + \delta_c + \frac{\lambda(\mu)}{16\pi^2} \gamma_c^{(0)} \left( \frac{1}{\epsilon} - \log p/\mu + \dots \right) \right\} \end{aligned}$$

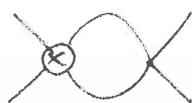
hence

$$\delta_c = - \frac{\gamma_c^{(0)}}{16\pi^2} \lambda(\mu) \frac{1}{\epsilon}$$

and

$$\delta A \propto c(\mu) \mu^{-2} \left( 1 + \frac{\gamma_c^{(0)}}{16\pi^2} \lambda(\mu) \log \mu/p + \dots \right)$$

Exercise : derive the value of  $\gamma_c^{(0)}$  by computing the 1-loop diagram

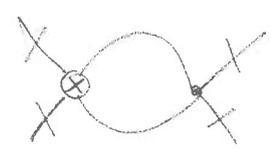


Notice that in principle extracting  $\delta_c$  (hence  $\gamma_c^{(0)}$ ) would require computing also the wavefunction renormalization of the field  $\phi$  :

$$C(\phi_0 \partial_\mu \phi_0)^2 = C(\mu) \mu^{-2+\epsilon} Z_c Z_\phi^2 (\phi \partial_\mu \phi)^2$$

where  $\phi_0 \equiv Z_\phi^{1/2} \phi$ .

The counterterm which absorbs the  $1/\epsilon$  pole of the amputated GF



is indeed  $\delta_c + 2\delta_\phi$ , since  $Z_c Z_\phi^2 = (1 + \delta_c)(1 + \delta_\phi)^2 \approx 1 + \delta_c + 2\delta_\phi$

To extract  $\delta_c$  one should then determine  $\delta_\phi$  from the two-point GF. In the special case of our theory (basically a  $\phi^4$  theory plus higher-dimensional operators), wave function normalization arises only at two-loop level. The 1-loop correction to the 2-point function only gives a correction to the mass term



→ no wave function, only mass correction

Hence  $\delta_\phi = O(\epsilon^2)$  and it does not enter our calculation.

We end by noticing that a complete matching requires computing the correction induced by loops of  $\psi$  to the mass, wave function and self-coupling of the field  $\phi$ .

In particular, the mass  $m_\phi$  will receive a contribution of order

$$\delta m_\phi^2 \approx m_\psi^2 \frac{y^2}{16\pi^2}$$



We thus see that in order to keep  $\phi$  light, as assumed so far in order to derive an effective theory, we need to start with a bare mass of order of  $\delta m_\phi^2$  and fine tune  $m_{\phi|bare}^2 + \delta m_\phi^2 \ll m_\psi^2$ .

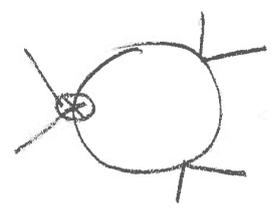
This is an explicit example of the fine-tuning problem which arises when trying to keep elementary scalars lighter than the cutoff scale of an effective theory.

# Operator mixing

At the level of 1 insertion, a given operator mixes with those with the same quantum numbers and equal or lower dimension

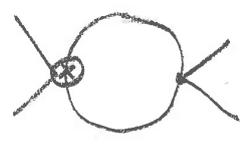
Ex:  $(\phi \partial_\mu \phi)^2$  mixes with  $\phi^6, \phi^4, \phi^2, (\partial_\mu \phi)^2$

$(\phi \partial \phi)^2 \rightarrow \phi^6$



$\sim c(\mu) \log \frac{\mu}{M} \frac{\lambda^2}{16\pi^2}$

$(\phi \partial \phi)^2 \rightarrow \phi^4$



$\sim c(\mu) \log \frac{\mu}{M} \frac{\lambda}{16\pi^2} m_\phi^2$

$(\phi \partial \phi)^2 \rightarrow (\partial \phi)^2$



$\sim c(\mu) \log \frac{\mu}{M} \frac{1}{16\pi^2} m_\phi^2$

$(\phi \partial \phi)^2 \rightarrow \phi^2$



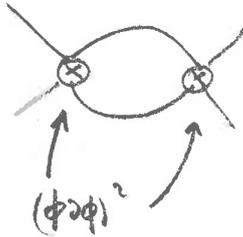
$\sim c(\mu) \log \frac{\mu}{M} \frac{1}{16\pi^2} m_\phi^4$

The mixing with lower-dimensional operators takes place only in presence of relevant operators (ex: mass terms) in the theory

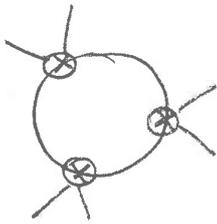
The dependence on  $\lambda$  of the anomalous dimension matrix can be easily understood by taking into account the  $k$  dimensionality of coupling constants.

Multiple insertions of the same operator can generate higher-dimension operators (counter terms). For example, neglecting masses and focusing on logarithmic divergent contributions:

Ex :



$$\sim c^2 \rightarrow (\partial\phi)^4 \times \log\mu \quad (\text{dim } 8)$$



$$\sim c^3 \rightarrow \phi^2 (\partial\phi)^4 \log\mu \quad (\text{dim } 10)$$

etc...

# $\hbar$ Dimensionality

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Example with scalar theory :

$$\frac{1}{\hbar} S = \frac{1}{\hbar} \int d^4x \mathcal{L}(\phi, \partial\phi)$$

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{m^2}{2} \phi^2 + \sum_{n,k} \phi^n \partial^k C_{n,k}$$

Define couplings through

$$C_{n,k}(\Lambda) \equiv \frac{g \bar{C}_{n,k}(\Lambda)}{\Lambda^{m+k-4}} \quad \bar{C}_{n,k} \text{ dimensionless}$$

Rescale  $\phi \rightarrow \phi \hbar^{1/2}$  so that

$$\frac{C_{n,k}}{\hbar} \rightarrow \frac{C_{n,k}}{\hbar} \hbar^{m/2} \equiv C'_{n,k} \Rightarrow \boxed{[g']_k = \frac{1}{2}}$$

A given quantity / observable is computed in a perturbative expansion in the number of loops

$$O = O_{\text{tree}} + O_{\text{1-loop}} + O_{\text{2-loop}} + \dots$$

Each term in the expansion must be homogeneous in the couplings (i.e. have a given dimensionality in  $\hbar$ ).

The relative dimensionality in  $\hbar$  of various terms is such that

$$[O_{(n+1)\text{-loop}}]_{\hbar} = [O_{n\text{-loop}}]_{\hbar} + 1$$