HOT TOPICS IN COSMOLOGY

PRE-SUSY SCHOOL 2018, IFAE BARCELONA

The plan of the lectures is to deal with the physics of Primordial Black Holes (PBH's) and to use them as an excuse to touch other hot topics (e.g. 21 cm cosmology).

The basic reference is "PBH's - perspectives in GW astronomy" by M. Sasaki et. al, astro-ph/1801.05235.

- PBH's first discussed in the 60's by Zeldovich and Novikov who pointed out that BH's in the early universe may grow catastrophically by accreting the surrounding radiation.
- In 1971 Hawking proposed that highly overdense regions of inhomogeneities can go gravitational collapse to form PBH's. Contrary to stellar BH's whose mass is bounded from below ($\sim 3 M_\odot$), PBH's can be as light as the Planckian mass $M_p \sim 10^{19}$ GeV $\sim 10^{19} \cdot 10^{-24}$ gr $\sim 10^{-5}$ gr

$\left(1 M_\odot \sim 10^{33}\text{ gr} \sim \underset{\sim 10^{38} M_p}{10^{57}}\text{ GeV}\right) \Rightarrow$ PBH if seen at $z \gtrsim 30$ (after formation), or $M \leq M_\odot$. Chandrasekhar limit to overcome the e-degeneracy pressure)

However, PBH's are subject to evaporation. Let us see how this comes about.

$M \geq (10^5 \div 10^{10}) M_\odot$ probably from PBH. For smaller masses $\exists$ spin

Let us start with SR and an accelerated observer in Minkowski space-time. She/he has coordinates $\xi^\mu(\tau)$ such that $u^\mu = d\xi^\mu/d\tau$ and

$$u^\mu u_\mu = \eta_{\mu\nu} u^\mu u^\nu = -1 .$$

The corresponding acceleration is

$$a^\mu = \frac{du^\mu}{d\tau} = \frac{d^2\xi^\mu}{d\tau^2} \, .$$

The four-vectors u^μ and a^μ are orthogonal

$$\frac{d}{d\tau}(u^\mu u_\mu) = 2 a^\mu u_\mu = 0 \Rightarrow$$ if we take only ξ^1 and τ

we can write $(\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1))$

$$-(u^0)^2 + (u^1)^2 = -1 \quad \Rightarrow u^0(\tau) = \cosh F(\tau), \, u^1 = \sinh F(\tau)$$

$$\Rightarrow a^\mu(\tau) = \dot{F}(\tau)\left(\sinh F(\tau), \cosh F(\tau), 0, 0\right),$$

$$a^2 = a^\mu a_\mu = \dot{F}^2(\tau) \quad \Rightarrow F(\tau) = a\tau$$ if we take

a constant acceleration. Then we have

$$u^\mu = \left(\cosh a\tau, \sinh a\tau, 0, 0\right), \tag{1}$$

$$\xi^\mu(\tau) = \frac{1}{a}\left(\sinh a\tau, \cosh a\tau, 0, 0\right),$$

worldline: $$\eta_{\mu\nu}\xi^\mu\xi^\nu = -(\xi^0)^2 + (\xi^1)^2 = \frac{1}{a^2} \, .$$

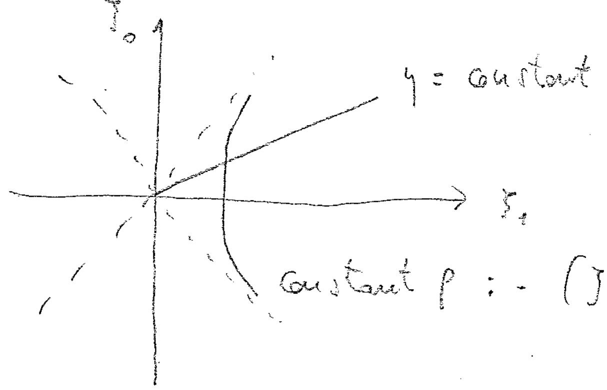
We can go now to rest-frame coordinate system (η, ρ) such that $\rho =$ constant (the new metric will not be $\eta_{\mu\nu}$). Comparing to Eq. (1), one finds:

$$\xi^0(\eta, \rho) = \rho \sinh\eta, \quad \xi^1(\eta, \rho) = \rho\cosh\eta,$$

$$ds^2 = -(d\xi^0)^2 + (d\xi^1)^2 = -\left(d\rho\sinh\eta + \rho\cosh\eta\, d\eta\right)^2$$
$$+ \left(d\rho\cosh\eta + \rho\sinh\eta\, d\eta\right)^2 = -d\rho^2\left[(\sinh\eta)^2 - (\cosh\eta)^2\right]$$
$$-\rho^2 d\eta^2\left[(\cosh\eta)^2 - (\sinh\eta)^2\right] = -\rho^2 d\eta^2 + d\rho^2$$

and η is the proper-time of the observer.

Since $\xi^0/\xi^1 = \tanh y$, we have



constant ρ : $-(\xi^0)^2 + (\xi^1)^2 = \rho^2 = \frac{1}{a^2}$

Now: $ds^2 = -\rho^2 dy^2 + \rho^2$. Translation $y \to y +$ constant is a symmetry and is a boost around the origin:

$$\begin{cases} \tilde{\xi}^0 = \xi^0 \cosh J + \xi^1 \sinh J , \\ \tilde{\xi}^1 = \xi^1 \cosh J + \xi^0 \sinh J , \end{cases} \qquad J = \text{rapidity} = \tanh^{-1}\beta$$

$$\tilde{\xi}^0 = \rho \sinh(y + \delta y) = \rho \sinh(y) \cosh(\delta y) + \rho \cosh(y) \sinh(\delta y)$$
$$\tilde{\xi}^1 = \rho \cosh(y + \delta y) = \rho \cosh(y) \cosh(\delta y) + \rho \sinh(y) \sinh \delta y$$
$$\tilde{\xi}^0 = \xi^0 \cosh(\delta y) + \xi^1 \sinh(\delta y)$$
$$\tilde{\xi}^1 = \xi^1 \cosh(\delta y) + \xi^0 \sinh(\delta y)$$

The boost operator is therefore the Hamiltonian and in Euclidean space it continues to rotations, such that $e^{2\pi i H} = 1$. To restore dimensions one remembers $[\omega \sim \frac{1}{dt}]$ that the frequency goes like $1/\sqrt{g_{yy}} = \frac{1}{\rho} = a$ and therefore one has $e^{2\pi i H \rho} = e^{2\pi i \frac{H}{a}}$. Going back to Minkowski one gets $e^{-H/T} \Rightarrow T = \frac{a}{2\pi}$.

This is the UNRUH effect: an accelerated observer sees the vacuum of an inertial observer as a state containing particles in thermal equilibrium.

Going back to the BH's:

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2 + r^2 d\Omega^2 .$$

Close to the event horizon $r = 2M$ an observer accelerates and therefore sees a thermal bath of particles popping out.

Choose $r = 2M + \frac{\rho^2}{8M} \Rightarrow dr = \frac{1}{4M}\rho\, d\rho$, $\frac{2M}{r} \simeq 1 - \frac{\rho^2}{8Mr} = 1 - \frac{\rho^2}{16M^2}$

$$\Rightarrow ds^2 = -\frac{\rho^2}{16M^2}dt^2 + \frac{1}{\frac{\rho^2}{16M^2}}\frac{1}{16M^2}\rho^2 d\rho^2 = -\frac{\rho^2}{16M^2}dt^2 + d\rho^2$$

$$= -\rho^2 d\eta^2 + d\rho^2$$

$$\eta = \frac{t}{4M}$$

$$\Rightarrow T = \frac{1}{2\pi\rho} = \frac{1}{4\pi\sqrt{2M(r-2M)}} .$$

The gravitational red shift is given by $\sqrt{g_{44}}$

$$\Rightarrow \quad T(r') = \frac{1}{4\pi\sqrt{2M}\sqrt{r-2M}}\sqrt{\frac{1-2M/r}{1-2M/r'}} \qquad T_O = \frac{\omega_E}{\omega_O} \sim \frac{dt_o}{dt_E} \sim \frac{\sqrt{g_{44E}}}{\sqrt{g_{44obs}}}$$

$$= \frac{1}{4\pi}\frac{1}{\sqrt{2Mr(1-2M/r')}} \xrightarrow{r'\to\infty} \frac{1}{4\pi\sqrt{2Mr}}$$

$$\xrightarrow{r=2M} \frac{1}{8\pi M} \quad \text{(Hawking's temperature)}$$

$$= T_H .$$

Restoring G: $\quad T_H = \frac{1}{8\pi G M}$

The power emitted by a BH is therefore

$$P = (\text{Area}) \cdot \sigma T_H^4 \quad (\sigma = \text{Stefan-Boltzmann constant})$$

$$\sigma = \frac{\pi^2 k_B^2}{60 \hbar^3}, \quad \text{Area} = 4\pi R_s^2 = 4\pi (2GM)^2 \qquad \hbar = 1,\ c = 1$$

$$P = -\frac{dE}{dt} = -\frac{d}{dt} M \Rightarrow -\dot{M} = \frac{K}{M^2}, \quad K = \frac{1}{15360 G^2}$$

$$\int_M^0 dM' M'^2 = -K \int_0^{t_{ev}(M)} dt \Rightarrow$$

$$t_{ev}(M) = \frac{M^3}{3K} = \frac{M^3}{3} \frac{15360 \pi G^2}{\hbar} \qquad 10^{15} g = 10^{-18} M_\odot$$

$$\Rightarrow M_c = 10^{15} g \left(\frac{t_0}{13,8 \text{ Gyr}}\right)^{1/3}$$ have already evaporated by today $t_0 \sim 10^{17}$ sec but they might have left some traces: for instance for $M \sim 10^9$ gr, $t_{ev} \sim 10^{-18}$ to $\sim 0,1$ seconds they might have changed the abundance of light elements during primordial nucleosynthesis.

The Ligo discovery of the merger event GW150914 of binary BH's of masses $\sim 30 M_\odot$ triggered interest in PHB's especially regarding the possibility that they are the DM. GW's provide a magnificent tool to understand PBH's.

PBH formation: Consider the standard FRW metric

$$ds^2 = -dt^2 + a^2(t)\, \delta_{ij}\, dx^i dx^j$$

and we can derive the standard FRW equation

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\bar{\rho}\,.$$

Consider now a region with an overdensity ($\rho > \bar{\rho}$) with size $> H^{-1}$. In this region we can write the metric as

$$ds^2 = -dt^2 + a^2(t)\, e^{2\psi(r)}\, \delta_{ij}\, dx^i dx^j,$$

where $\psi > 0$ and $\psi(r)$ decreases to zero at $r \to \infty$.
We have considered a spherical region. We have

$$ds^2 = -dt^2 + a^2(t)\, e^{2\psi(r)} \Big[dr^2 + r^2 \underbrace{(d\theta^2 + \sin^2\theta\, d\phi^2)}_{d\Omega^2} \Big].$$

Let us choose $R = r e^{\psi(r)} \Rightarrow dR = dr\, e^{\psi(r)} + r\, e^{\psi(r)} \psi'(r)\, dr$

$$\Rightarrow dr = \frac{dR}{e^{\psi}[1 + r\psi']}$$

$$\Rightarrow e^{2\psi}\left(dr^2 + r^2 d\Omega^2\right) = e^{2\psi}\left(\frac{dR^2}{e^{2\psi}(1+r\psi')^2} + \frac{R^2}{e^{2\psi}} d\Omega^2\right)$$

$$= \frac{dR^2}{(1+r\psi')^2} + R^2 d\Omega^2 = \frac{dR^2}{1 - kR^2} + R^2 d\Omega^2$$

$$\Rightarrow \quad 1 - \kappa R^2 = (1 + r\psi')^2 = 1 + r^2\psi'^2 + 2r\psi'$$

$$= 1 - \kappa r^2 e^{2\psi} \Rightarrow \kappa = \frac{-r^2\psi'^2 - 2r\psi'}{r^2 e^{2\psi}} = -\frac{\psi'}{r} \frac{(2 + r\psi')}{e^{2\psi}} .$$

The three-curvature at $t =$ constant is given by

$$^{(3)}R = -\frac{e^{-2\psi}}{3a^2} \delta^{ij} \left[2\partial_i \partial_j \psi + \partial_i \psi \partial_j \psi \right]$$

$$= \frac{\kappa}{a^2} \left(1 + \frac{d \ln \kappa(R)}{3 d \ln R} \right) \simeq \frac{\kappa}{a^2} + \text{gradients}$$

$$\Rightarrow \quad H^2 + \frac{\kappa(r)}{a^2} = \frac{8\pi G}{3} \rho = \left(\frac{\dot a}{a}\right)^2 + \frac{\kappa(r)}{a^2} = \frac{8\pi G_N}{3} \bar\rho + \frac{\kappa(r)}{a^2}$$

$$\Rightarrow \frac{\rho - \bar\rho}{\bar\rho} = \frac{3\kappa}{8\pi G_N \bar\rho a^2}$$

$\Rightarrow$ we can define a density contrast

$$\Delta = \frac{\rho - \bar\rho}{\bar\rho} = \frac{3\kappa}{8\pi G_N \bar\rho a^2} = \frac{\kappa}{H^2 a^2} .$$

In a radiation phase $\bar\rho \sim a^{-4} \sim H^2$ and Δ grows like a^2. If $\kappa > 0$ the region would eventually stop expanding and collapse when $\frac{3\kappa}{a^2} = 8\pi G\rho$, that is when

$$\frac{8\pi G}{3}\rho = H^2 = \frac{\kappa}{a^2} = \frac{1}{L_\kappa^2 a^2} \Rightarrow L_\kappa = \frac{H^{-1}}{a} \Rightarrow$$ the comoving scale of the region (recall r is comoving) becomes of the order of H^{-1}.

Also, since $\Delta = \dfrac{3K}{8\pi G\bar{\rho} a^2}$, $\Delta = 1$ is the time when the universe stops expanding if it were homogeneous and isotropic ($\equiv$ taking K = constant), let us assume this time to be t_c.

A perturbation smaller than the Jeans length cannot collapse, one has $c_s^2 k^2/a^2 = H^2$ at $t = t_c$ ($\equiv$ the perturbation $c_s k$ has most to be inside the horizon $\lambda > \lambda_J = \frac{1}{c_s k_J} \Rightarrow \frac{1}{c_s H} \leq H^{-1}$)

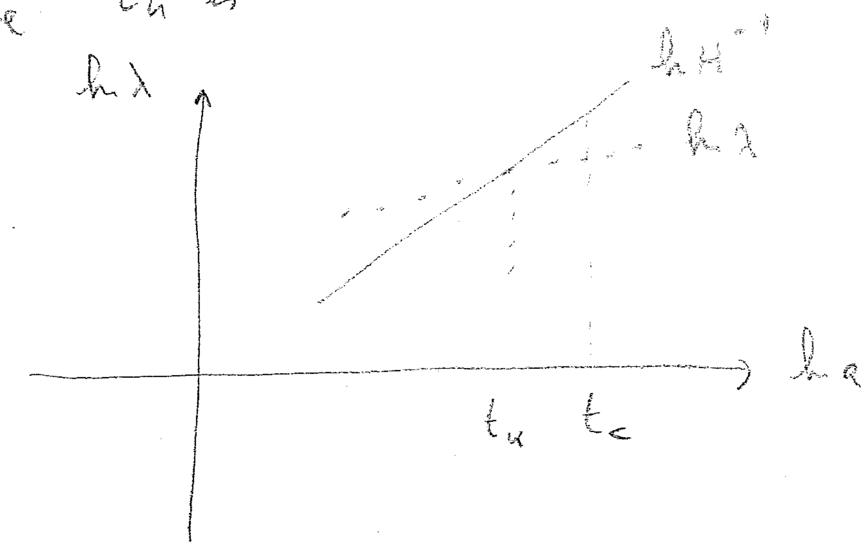
$$1 = \Delta(t_c) = \frac{K}{a^2 H^2} = \frac{K}{k^2} \cdot \frac{k^2}{a^2 H^2} = \frac{K}{c_s^2 k^2} \Rightarrow \text{we can}$$

$K \geq \frac{1}{c_s H}$

identify K with $c_s^2 k^2 \Rightarrow$ the condition is that the density contrast at the time of horizon re-entry is greater than $\Delta_c = c_s^2$: $(t_k < t_c)$

$$\Delta(t_k) = \frac{K}{H^2(t_k)\, a^2(t_k)} = \frac{c_s^2 k^2}{H^2(t_k)\, a^2(t_k)} \geq \Delta_c = c_s^2 = \frac{1}{3},$$

where t_k is the time at which $k = a(t_k) H(t_k)$.



$\frac{c_s^2 k^2}{a^2} \leq H^2$

$\Delta(t_c) = 1 \Rightarrow$
$a^2 H^2 = \underbrace{K \geq c_s^2 k^2}$

$\Delta(t_k) = \frac{K}{a_k^2 H_k^2} \geq \frac{c_s^2 k^2}{a_k^2 H_k^2} = c_s^2$

Crudely speaking $H(t_c) \simeq \frac{8}{3}\pi G\bar{\rho}\, H^{-3}(t_k)$.

Of course, the computation we have made is crude and details are important, nevertheless the actual value is in the correct ballpark.

Jeans' instability:

$$\begin{cases} \rho = \rho_0 + \delta\rho \\ u = u + \delta u \end{cases} \Rightarrow \quad \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = -\frac{\partial P}{\partial x} + \rho\frac{\partial \phi}{\partial x}$$

$$\underline{P} = c_s^2 \rho$$

$$\begin{cases} \frac{\partial \delta\rho}{\partial t} + \rho_0 \frac{\delta u}{\partial x} = 0 \qquad (u_0 = 0) \\ \rho_0 \frac{\partial \delta u}{\partial t} = -c_s^2 \frac{\partial \delta\rho}{\partial x} + \rho_0 \frac{\partial \phi}{\partial x} \\ \frac{\partial^2 \phi}{\partial x^2} = 4\pi G \delta\rho \end{cases}$$

$$\Rightarrow \quad \frac{\partial^2 \delta\rho}{\partial t^2} = c_s^2 \frac{\partial^2 \delta\rho}{\partial x^2} + 4\pi G \rho_0 \delta\rho$$

$$\Rightarrow \delta\ddot{\rho}_k = (-c_s^2 k^2 + 4\pi G_0 \rho_0)\delta\rho_k \Rightarrow c_s^2 k_J^2 = 4\pi G \rho_0$$

INSTABILITY $k < k_J$, $\lambda > \lambda_J$

Abundance of PBH's:

Let us define the mass (or energy density) fraction at formation:

$$\beta = \left.\frac{\rho_{PBH}}{\rho_{tot}}\right|_{formation} = \frac{\rho_{PBH}|_{formation}}{3H_{form}^2/8\pi G_N} =$$

$$= \frac{\rho_{PBH}\big|_{form}\, a_{form}^3}{3H_{form}^2/8\pi G_N} \frac{1}{a_{form}^3}$$

$$= \rho_{PBH}(t_0) \frac{a_0^3}{a_{form}^3} \frac{1}{3H_{form}^2/8\pi G_N}$$

$$= f_{PBH}\, \rho_{DM}(t_0) \frac{a_0^3}{a_{form}^3} \frac{1}{3H_{form}^2/8\pi G_N}$$

$$= f_{PBH}\, \Omega_{DM}\, \rho_c(t_0) \frac{a_0^3}{a_{form}^3} \frac{1}{3H_{form}^2/8\pi G_N}$$

$$= f_{PBH}\, \Omega_{DM} \left(\frac{H_0}{H_{form}}\right)^2 \frac{a_0^3}{a_{form}^3} .$$

The mass at formation is

$$M = \gamma M_H\big|_{formation} = \gamma \frac{4}{3}\pi \rho_{form} H_{form}^{-3}$$

$$= \gamma \frac{4\pi}{3} \frac{3H_{form}^2}{8\pi G_N} H_{form}^{-3} = \frac{1}{2G} \gamma H_{form}^{-1} ,$$

where γ accounts for the efficiency $\Rightarrow$

$$\beta = f_{PBH}\, \Omega_{DM} \frac{H_0^2}{\gamma^2} M^2 4 G^2 \frac{a_0^3}{a_{form}^3} \text{ (total entropy conservation)}$$

$$= f_{PBH}\, \Omega_{DM} \frac{H_0^2}{\gamma^2} 4 M^2 G^2 \frac{g_{*s}(T_f)}{g_{*s}(T_0)} \frac{T_{form}^3}{T_0^3}$$

$$= 3.7\, 10^{-9} \left(\frac{\gamma}{0,2}\right)^{-1/2} \left(\frac{g_*(T_f)}{10.75}\right)^{1/2} \left(\frac{M}{M_\odot}\right)^{1/2} f_{PBH} .$$

Once the probability distribution function is given, β can be regarded as the probability that the density contrast is larger than the threshold for PBH formation :

$$\beta = \gamma \int_{\Delta_c}^{1} d\Delta \, P(\Delta) \; ;$$

$$P(\Delta) = \frac{1}{\sqrt{2\pi}\, \sigma_\Delta(\Pi)} \, e^{-\frac{\Delta^2}{2\sigma_\Delta^2(\Pi)}} \qquad (\text{if Gaussian})$$

$$\sigma_\Delta^2(\Pi) = \int \frac{dk}{k} \, P_\Delta(k) \, W^2(k R_H),$$

$$W(x) = e^{-x^2/2}$$

$$R_H = \frac{1}{aH}$$

$$\Delta = \frac{4}{9} \frac{\nabla^2 \zeta}{a^2 H^2} \qquad (\text{see } 1A)$$

$$\beta \simeq \frac{\gamma}{\sqrt{2\pi}} \frac{\sigma_\Delta(\Pi)}{\Delta_c} \, e^{-\frac{\Delta_c^2}{2\sigma_\Delta^2(\Pi)}}$$

Where is the $\frac{4}{9}$ coming from :

1) Poisson equation $\quad \nabla^2 \phi = 4\pi G \Delta = \frac{3}{2} H^2 \Delta$

2) $\zeta = \frac{5+3w}{3(1+w)} \phi \Rightarrow \zeta = \frac{6}{4} \phi \; (RD) = \frac{3}{2} \phi_{RD} \Rightarrow$

$$\Phi_{RAD} = \frac{2}{3} J \Rightarrow \Delta = \frac{2}{3}\frac{\nabla^2 \Phi}{H^2} = \frac{2}{3}\cdot\frac{2}{3}\frac{\nabla^2 J}{H^2} = \frac{4}{9}\frac{\nabla^2 J}{H^2}$$

Complement with the notes from 1A

Generating seeds of PBH's:

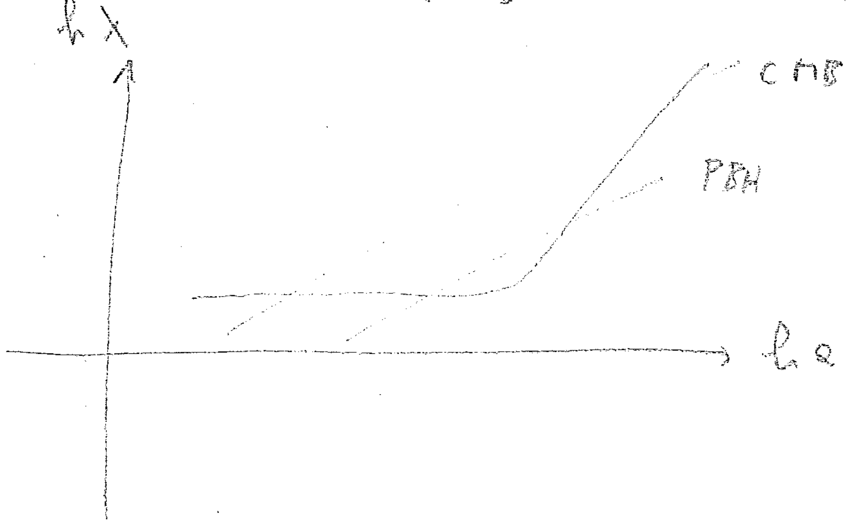
If PBH's are generated by an overdensity at horizon re-entry, then the wave-number at formation is

$$K = aH\big|_f .$$

During radiation $H^2 \sim a^{-4} \Rightarrow a \sim H^{-1/2} \Rightarrow H_{form} \sim K^2$

$$\Rightarrow M = \frac{\gamma}{2G} H_{form}^{-1} \Rightarrow M(k) \simeq 30\, M_\odot \left(\frac{\gamma}{0.2}\right)\left(\frac{g_*(T_f)}{10.75}\right)^{-1/6} \times \left(\frac{k}{2.9\ 10^5\,\mathrm{Mpc}^{-1}}\right)^{-2}$$

$$\Rightarrow N_{CMB-30M_\odot PBH} = \ln \frac{K_{30M_\odot PBH}}{K_{CMB}} = \ln \frac{2.9\ 10^5}{0.002} \simeq 26$$



Non-attractor models:

$$\mathcal{R} = \psi + H\frac{\delta\phi}{\dot{\phi}_0} \quad ; \quad \text{in the flat gauge } \psi = 0$$

one has $$\mathcal{R} = H\frac{\delta\phi}{\dot{\phi}_0}$$

$\mathcal{R}$ satisfies the equation of motion

$$\mathcal{R}'' + 2\frac{z'}{z}\mathcal{R}' + k^2\mathcal{R} = 0$$

$$z = \frac{a\dot{\phi}_0}{H}$$

$$\frac{z''}{z} = 2a^2H^2\left(1 + \frac{5}{2}\epsilon + \epsilon^2 - 2\epsilon\eta - \frac{1}{2}\frac{V''}{H^2}\right)$$

$$\epsilon = -\frac{\dot{H}}{H^2}, \quad \eta = -\frac{\ddot{\phi}_0}{H\dot{\phi}_0}, \quad \xi^2 = 3(\epsilon+\eta) - \eta^2 - \frac{V''}{H^2}$$

$$\mathcal{R} = \frac{u}{z} \Rightarrow u'' - \frac{z''}{z}u + k^2 z = 0.$$

During slow-roll $\frac{z'}{z} = aH(1+\epsilon-\eta) \simeq aH$

and $\frac{z''}{z} \simeq 2a^2H^2.$

In the flat gauge $\delta\tilde{\phi}_k'' + \left(k^2 - \frac{a''}{a}\right)\delta\tilde{\phi}_k = 0, \quad \delta\tilde{\phi} = \delta\phi\, a$

where $a = -\frac{1}{H\tau}, \quad \frac{a''}{a} = \frac{2}{\tau^2} \Rightarrow \delta\phi_k = \frac{e^{-ik\tau}H(1+ik\tau)}{\sqrt{2k^3}}$

and therefore on long wavelengths $(-k\tau) \ll 1$

$$|\delta\phi_k| \simeq \frac{H}{\sqrt{2k^3}} \quad \text{and} \quad P_{\delta\phi} = \frac{k^3}{2\pi^2} |\delta\phi_k|^2 = \left(\frac{H}{2\pi}\right)^2 .$$

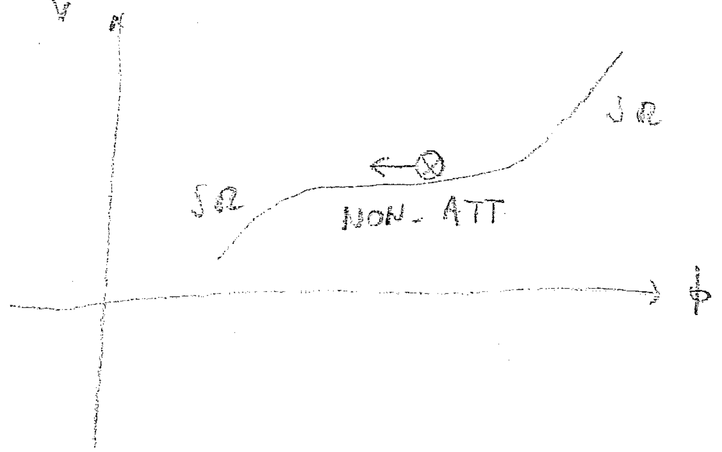
Suppose now that there is a period during which

$$\frac{z'}{z} < 0 \Rightarrow 1 + \epsilon - \eta < 0 \Rightarrow \eta > 1 + \epsilon .$$

If the potential is very flat:

$$\ddot{\phi}_0 + 3H\dot{\phi}_0 + V' \simeq \ddot{\phi}_0 + 3H\dot{\phi}_0 = 0$$

$$\Rightarrow \dot{\phi}_0 \sim e^{-3Ht}, \quad \eta = -\frac{H\ddot{\phi}_0}{\dot{\phi}_0} = 3$$



Dual transformation:

$$\tilde{R} = \frac{\tilde{z}}{z} R \Rightarrow \tilde{R}'' + 2\frac{\tilde{z}'}{\tilde{z}}\tilde{R}' + k^2\tilde{R} = 0 \Leftrightarrow \frac{\tilde{z}''}{\tilde{z}} = \frac{z''}{z}$$

$$\tilde{z}(\tau) = c_1 z(\tau) + c_2 z(\tau) \int^{\tau} \frac{d\tau'}{z^2(\tau')}$$

$$\Rightarrow P_R \Big|_{\text{end of inflation}} = \frac{\tilde{z}}{z} P_{\tilde{R}} \Big|_{\text{end of inflation}}$$

During slow-roll

$$\frac{z''}{z} \simeq 2H^2a^2 = \frac{2}{\tau^2}, \quad z \sim \frac{1}{\tau},$$

during the non-attractor phase: $\phi_0'' + 2\mathcal{H}\phi_0' = 0 \Rightarrow \phi_0' \sim \tau^2$

$$z = \frac{a\phi_0'}{\mathcal{H}} \sim \tau^2$$

$$\Rightarrow \quad z(\tau)\int \frac{d\tau'}{z^2(\tau')} \sim \tau^2 \int^{\tau} \frac{d\tau'}{\tau'^4} \sim \frac{1}{\tau}$$ like in SR:

one can make a dual transformation and reduce to the slow-roll one $\Rightarrow$ the power spectrum must be flat:

$$P_R = \left(\frac{H}{2\pi}\right)^2 \frac{1}{\dot\phi_0} \sim e^{6N}, \quad N = Ht$$

grows like e^{3N} and can get large values:

1) the decaying mode counts, adiabaticity is broken $\frac{\delta\phi}{\dot\phi_0} \neq \frac{\delta\dot\phi}{\ddot\phi_0}$,

$$\ddot\phi_0 + 3H\dot\phi_0 + V' = 0$$
$$\dddot\phi_0 + 3H\ddot\phi_0 + V''\dot\phi_0 = 0 \Rightarrow \delta\phi = C(\vec{x})\dot\phi_0$$
$$\delta\ddot\phi + 3H\delta\dot\phi + V''\delta\phi = 0$$

2) one needs to compute P_R at the end of inflation,

3) ζ is not equal to R -

The diffusion problem:

the equation of motion of the long-mode contains a noise term:

$$\phi = \int \frac{d^3k}{(2\pi)^{3/2}} e^{-i\vec{k}\vec{x}} \theta(aH-k)\, \phi_k + \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\vec{k}\vec{x}} \theta(k-aH)\, \phi_k$$

long   short

$$\dot{\phi} = \dot{\phi}_{long} - \dot{a}H \int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}\vec{x}} \delta_D(k-aH)\, \phi_k$$

$$\ddot{\phi}_l + 3H\dot{\phi}_{long} = 3H^2\dot{a} \int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}\vec{x}} \delta_D(k-aH)\phi_k - V'$$

$t \to N = Ht$

$$\phi_l'' + 3\phi' = \underbrace{3\dot{a} \int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}\vec{x}} \delta_D(k-aH)\phi_k}_{\xi} - \frac{V'}{H^2}$$

$$\langle \xi(N_1)\xi(N_2)\rangle = 9\dot{a}_1\dot{a}_2 \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3} \delta_D(k_1 - a_1H)\, \delta_D(k_2 - a_2H)\, |\phi_k|^2 (2\pi)^3 \delta^{(3)}(\vec{k}_1 - \vec{k}_2)$$

$$\delta_D(e^{N_1}H - e^{N_2}H) = \frac{\delta_D(N_1-N_2)}{aH}$$

$$= 9\dot{a}_1\dot{a}_2 \frac{(4\pi)}{(2\pi)^3} \frac{H^2a_1^2}{2a_1^3H^3(=k_1^3)} \delta_D(a_1H - a_2H)$$

$$= \frac{9H^2a_1^2}{4\pi^2} \frac{H^2a_1^2}{a_1^3H^3} \frac{1}{a_1H_1} \delta_D(N-N')$$

$\dot{a} = aH$

$$= \frac{9H^2}{4\pi^2} \delta_D(N-N')$$

$$\Pi = \frac{d\phi_0}{dN}$$

$$\begin{cases} \dfrac{d\phi_0}{dN} = \Pi \\ \dfrac{d\Pi}{dN} + 3\Pi + \dfrac{V'}{H^2} = \xi \end{cases}, \qquad \langle \xi(N)\xi(N') \rangle = D\,\delta(N-N')$$

$$D = \frac{9H^2}{4\pi^2}$$

$\Rightarrow$ KΠ equation

$$\frac{\partial P}{\partial N} = -\frac{\partial}{\partial \phi}(\Pi P) + \frac{\partial}{\partial \Pi}\left(3\Pi P + \frac{V_\phi}{H^2} P\right) + \frac{D}{2}\frac{\partial^2}{\partial \Pi^2} P$$

$\Rightarrow$ if $P(\phi, \Pi, 0) = \delta_D(\phi - \phi_0)\,\delta_D(\Pi - \Pi_0)$

for a linear potential $V(\phi) = V_0\left(1 + \sqrt{2\epsilon_V}\,(\phi - \phi_0)\right) + \ldots$

$$\sqrt{2\epsilon_V} = V_\phi / V_0$$

$$P = \frac{1}{\pi}\left(\frac{27}{2D^2N}\right)^{1/2} e^{-\frac{9}{2DN}(\phi - \langle\phi\rangle)^2}$$

$$e^{\frac{3}{DN}(\phi - \langle\phi\rangle)(\Pi - \langle\Pi\rangle)}\, e^{-\frac{3}{D}(\Pi - \langle\Pi\rangle)^2}$$

$$\langle\phi\rangle = \phi_0 + \frac{1}{3}(\Pi_0 - \Pi) - \sqrt{2\epsilon_V}\,N$$

$$\langle\Pi\rangle = \sqrt{2\epsilon_V}\left(e^{-3N} - 1\right) + \Pi_0 e^{-3N}$$

$$\langle (\Pi - \langle\Pi\rangle)^2 \rangle \overset{N \gg 1}{=} \frac{D}{6}$$

$\beta(M)$ is now a stochastic quantity and it turns out that, by defining

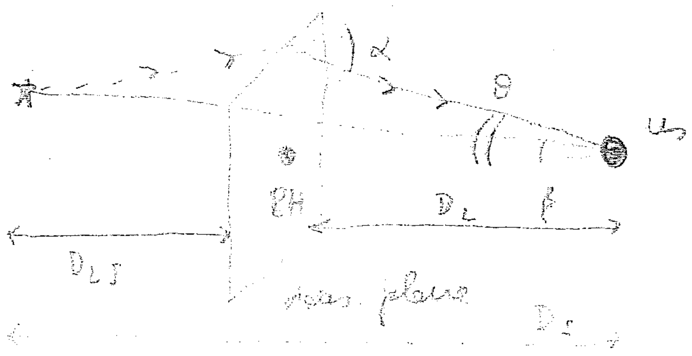
$$\Delta_{qd} = \ln \frac{\beta^{qd}}{\beta^{cl}}$$

Δ_{qd} is distributed like a Gaussian, but with a $\sqrt{\text{variance}} \sim \mathcal{O}(10) \Rightarrow$ the classical calculation is not reliable.

Limits: broadly speaking the constraints on PBH's are direct and indirect; direct (lensing, dynamical effect, accretion, growth of LSS) are those directly triggered by the gravitational potential of PBH's; indirect are those not by PBH's, but something else connected to it.

- $M \lesssim 10^{15}$ gr : they have evaporated, but bounds come from, e.g., change of the abundance of light elements during nucleosynthesis, extragalactic photon background and damping of CMB anisotropies at small scales by modifying the ionization history

- $M \gtrsim 10^{15}$ gr : lensing



$$\theta D_S = D_S \beta + D_{LS} \alpha \quad , \alpha = \frac{4GM}{D_L \theta} .$$ (10)

Indicating by $r = D_L \theta$ and $r_0 = D_L \beta$ one obtains

$$r^2 - r_0 r = R_E^2 , \qquad R_E = \sqrt{\frac{4GM\, D_L D_{LS}}{D_S}} =$$ (2)

= Einstein's radius

$$\Rightarrow r_{1,2} = \frac{1}{2} \left(r_0 \pm \sqrt{r_0^2 + 4R_E^2} \right)$$

If the two images cannot be resolved we speak about microlensing and one sees a superposition of images which is more brighter than the original source; the magnification is maximal if the BH's have a non-vanishing velocities and at $\beta = 0$; the time scale for the rise up and down is

$$T \approx 2yr \left(\frac{M}{10 M_\odot} \right)^{1/2} \left(\frac{D_S}{100kpc} \right)^{1/2} \left(\frac{v}{200km/s} \right)^{-1} .$$

An important quantity is the optical depth τ: how likely a background source is micro-lensed by the BH with magnification larger than a critical number depending on the experiment (typically ~1.3):

$$\tau = 10^{-6} f_{PBH} .$$

If $f_{PBH} = 1$ $1/10^6$ stars will be microlensed with $T \sim 2yr$. EROS has reported $\tau \lesssim 10^{-7}$. Now GAIA, an optical space-based observatory by ESA, should be able to detect compact objects even in small fraction.

Femtolensing appears when $\lambda_{light} \gtrsim R_S$, diffraction changes the amplification causing a difference in time of the two images $\Delta T \sim 4GM$, and an oscillatory pattern in the magnification. One looks for oscillatory behaviours of γ-ray bursts occurring at cosmological distances for $10\,keV \lesssim E_\gamma \lesssim MeV$ for which the angular separation of the source lensed by PBH's is around the femto-scale.

Dynamical constraints: disruption of white dwarfs ($M_{WD} \sim M_\odot$, $r_{WD} \sim r_{Earth}$) and the passage of a PBH inside a WD increases the energy of the particles thus igniting the thermonuclear reactions and making the WD explode.

The same for neutron stars which captures PBH with

$$t \sim 10^4\,yr \left(\frac{M}{10^{u}\,gr}\right)^{-3/2}$$

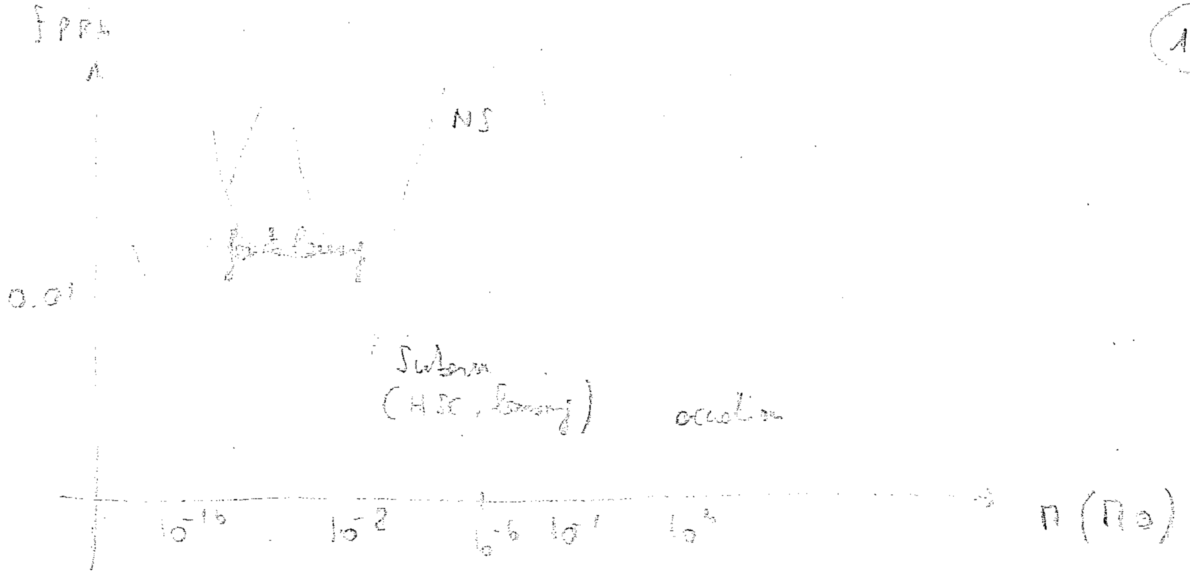
and are destroyed by their accretion.

Dynamical friction: PBH's accumulate at the Galactic center whose mass is however bounded $\lesssim 3\, 10^6 M_\odot$

LSS: PBH's provide an isocurvature component to the perturbation and they are Poisson-distributed for which $P_{PBH} = \frac{u^3}{2\pi^2} \frac{f_{PBH} M}{\rho_0}$, $n = 3$.

This is constrained by dynamical forest giving

$$f_{PBH} \lesssim \left(\frac{10^4 M_\odot}{M}\right).$$



Indirect limits: PBH's if generated by spikes can give rise to GW's at second-order in perturbation theory:

$$ds^2 = a^2(\tau)\left[-(1+2\Phi)\,d\tau^2 + (1-2\Psi)\left(\delta_{ij} + \frac{1}{2}h_{ij}\right)dx^i dx^j\right].$$

During radiation

$$\Psi'' + \frac{4}{\tau}\Psi' - \frac{\nabla^2}{3}\Psi = 0$$

$$\Rightarrow \Psi_k(\tau) = D_k(\tau)\,\Psi_k(0), \quad D_k(\tau) = \frac{3}{(k\tau)^3}\left[\frac{\sqrt{3}}{k\tau}\sin\left(\frac{k\tau}{\sqrt{3}}\right) - \cos\left(\frac{k\tau}{\sqrt{3}}\right)\right]$$

and

$$h''_{ij} + 2\mathcal{H}h'_{ij} - \nabla^2 h_{ij} = -4\,\underbrace{\hat{T}^{\ell m}_{ij}}_{\text{projector}}\,S_{\ell m}$$

$$S_{ij} = 2\Psi\,\partial_i\partial_j\Psi - \left(\partial_i\Psi + \frac{1}{\mathcal{H}}\partial_i\Psi'\right)\left(\partial_j\Psi + \frac{\partial_j\Psi'}{\mathcal{H}}\right).$$

$$\rho_{GW} = \frac{\langle \dot h_{ij}\dot h_{ij}\rangle}{128\pi G} \Rightarrow \Omega_{GW} \sim \frac{k^2}{24 H^2}\,P_h(k)$$

$$\Omega_{GW} \sim A_\zeta^2, \quad f = 10^{-9}\,\mathrm{Hz}\left(\frac{M}{30 M_\odot}\right)^{-1/2}.$$

Pulsars are rapidly rotating neutron stars emitting radio waves. Arrival times of pulses generated by millisecond pulsars have been measured with accuracy and frequency changes as

$$\frac{\delta\nu}{\nu} = -\underbrace{H^{ij}}_{\text{geometrical factor}}\left[h_{ij}(t,\underbrace{\vec{x}_E}_{\text{Earth}}) - h_{ij}(t-D,\underbrace{\vec{x}_p}_{\text{pulsar}})\right]$$

It seems that current bounds already rule out $M \sim 3\cdot M_\odot$.
However one should take into account NG for which the variance of ζ and therefore A_ζ may be smaller.

CMB spectral distortions: the CMB oscillations are damped by photon diffusion (photons are not totally coupled to baryons) at $k \simeq 10^{-6}(1+z)^{3/2}\ \mathrm{Mpc}^{-1}$. The oscillation energy is transferred to the background plasma causing a μ-distortion.
If $P_\zeta \sim A_\zeta\, \delta(\ln k - \ln k_*)$ and $10^{4} \leq M \leq 10^{13} M_\odot$ are ruled out for $A \sim 0{,}01$. For $M \leq 10^{4} M_\odot$ Silk damping dissipates energy after nucleosynthesis, but before recombination ($\eta = n_b/n_\gamma$, $\eta_{BBN} > \eta_{CMB}$) giving $A_\zeta \leq 10^{-2}$.

$\Downarrow$

$A_\zeta \lesssim 10^{-2}$

NB: let us go back to the non-attractor models; (12)

$$ds^2 = -dt^2 + a^2 e^{-2\psi} dx^2$$

$\Rightarrow$ on super-Hubble scales $N = \ln \tilde{a} = \ln a e^{-\psi}$

$$\Rightarrow \mathcal{R} = \psi = -\delta N = -N + N_0 = -\ln a e^{-\psi} + \ln a = -\ln e^{-\psi}.$$

During the non-attractor phase

$$\begin{cases} \dot\phi_0 = \dot\phi_* e^{-3N}, & * = \text{end of the non attractor phase} \\ \phi_0(N) = \phi_* + \dfrac{\dot\phi_*}{3}\left(1 - e^{-3N}\right) \end{cases}$$

$$N = -\frac{1}{3} \ln \frac{\dot\phi_0(N)}{\dot\phi_*} \Rightarrow \mathcal{R} = -\frac{1}{3} \ln \left(\frac{\dot\phi_0 + \delta\dot\phi_0}{\dot\phi_*}\right) + \frac{1}{3} \ln\left(\frac{\dot\phi_0}{\dot\phi_*}\right).$$

$$\Rightarrow \mathcal{R}(t_*) = -\frac{1}{3} \ln \left[1 + \frac{\delta\dot\phi_{0*}}{\dot\phi_*}\right].$$

Using $\dot\phi_* = 3\left[\phi_0(N) - \phi_*\right] + \dot\phi_0(N) \left(= \dot\phi_* e^{-3N}\right)$

$$\Rightarrow \mathcal{R} = -\frac{1}{3} \ln \left[1 + 3\frac{\delta\phi}{\dot\phi_*}\right] + \ldots$$

Now, $\delta\phi$ is a random scalar field in de Sitter and is therefore a Gaussian field

$$\Rightarrow P(\mathcal{R}) = \left|\frac{\partial \delta\phi}{\partial \mathcal{R}}\right| P(\delta\phi).$$

One gets $$P(R) = \frac{|\dot{\phi}_*|}{\sqrt{2\pi}} \frac{1}{\sigma_{\delta\phi}} e^{-\left[3R + \frac{\dot{\phi}_*^2}{18\sigma_{\delta\phi}^2}\left(1-e^{-3R}\right)^2\right]}$$

and $$\beta(R) = \int_{R_c} dR\, P(R) \simeq -\frac{1}{3} \frac{\dot{\phi}_*\, e^{-3R_c}}{\sqrt{2\pi}\,\sigma_{\delta\phi}} e^{-\dot{\phi}_*^2/18\sigma_{\delta\phi}^2}$$

which matches the β_{GAUSS} for smaller values of R _

HAWKING EVAPORATION
(HEURISTIC)

Consider an of the vacuum within a constant electric field. Pairs of masses m can be produced within a time Δt:

$$\Delta t \sim \frac{\hbar}{\Delta E} \sim \frac{\hbar}{m} . \qquad (c=1)$$

The limiting velocity of light allows this to occur at a distance

$$\Delta x \sim \Delta t \sim \frac{\hbar}{2m} .$$

If the electric field does a work (on each particle)

$$(eE)\cdot(\Delta x) \geq m$$

then particles become real $\Rightarrow$

$$E_c \sim \frac{2m^2}{e\hbar}$$

Unruh Radiation

Instead of E consider the inertial force on each virtual particle in the vacuum in a frame with proper acceleration a:

$$(ma)(\Delta x) \sim m$$

$$\Rightarrow \quad a \sim \frac{2m}{\hbar} .$$

If these particles are thermally distributed with temperature T $\Rightarrow$ $(ma)(\Delta x) \sim 3kT \Rightarrow T \sim \frac{\hbar a}{6k}$.

BH radiation:

hold a particle of mass m_0 at distance r from a BH:

$$x^\mu = (t, r, \theta, \phi), \quad u^\mu = (\dot{t}, 0, 0, 0)$$

$$ds^2 = -f(r)\,dt^2 = -d\tau^2 \Rightarrow \dot{t} = dt/d\tau = \frac{1}{\sqrt{f}}, \quad f = 1 - \frac{2Gm}{r}$$

$$\Rightarrow u^\mu = \left(\frac{1}{\sqrt{f}}, 0, 0, 0\right)$$

$$\Rightarrow a^\mu = \nabla_u u^\mu = u^\nu \nabla_\nu u^\mu$$

$$\Rightarrow a^r = \frac{f'}{2} \Rightarrow a = \sqrt{g_{\mu\nu} a^\mu a^\nu} = \frac{1}{2} \frac{f'}{\sqrt{f}}$$

$$a(r) = \frac{Gm}{r^2} \frac{1}{\sqrt{1 - \frac{2Gm}{r}}}$$

Invoking the EP:

$$T \sim \frac{\hbar a}{6k} \sim \frac{\hbar}{\sqrt{g_{00}}} \frac{1}{6k} \frac{Gm}{4G^2m^2} = \frac{\hbar}{\sqrt{g_{00}}} \frac{1}{24kGm}$$

at $r = r_s$;

for an observer at infinity $T_\infty = \frac{\omega_0}{\omega_E} T_E$

$$\sim \frac{d\tau_E}{d\tau_0} T_E = \sqrt{\frac{g_{00}(E)}{g_{00}(0)}}\, T_E = \frac{\hbar}{24kGm} \sim \frac{\hbar}{8\pi kGm}$$

Some complement : $dx^0 = dt/a$

$$ds^2 = -(1+2\phi)(dx^0)^2 + a^2(t)(1-2\psi)\,d\vec{x}^2$$

$$x^0 \to \tilde{x}^0 = x^0 + \alpha \quad , \quad d\tilde{x}^0 = dx^0 + \alpha' dx^0$$

$$a^2(\tilde{x}^0) = a^2(x^0) + 2aa'\alpha \quad , \quad \cancel{\phi \to \tilde{\phi} = \phi + \alpha}$$

$$\widetilde{ds^2} = ds^2 \Rightarrow \quad \tilde{\phi} = \phi - \alpha' - \frac{a'}{a}\alpha$$

Similary :

$$a^2(x^0)\left(1 + 2\,{}^{a'}\!/_{a}\,\alpha\right)(1-2\tilde{\psi}) = a^2(x^0)(1-2\psi)$$

$$\tilde{\psi} = \psi + {}^{a'}\!/_{a}\,\alpha \; -$$

For scalars: $\tilde{f}(\tilde{x}) = f(x)$

$$\widetilde{\delta f} = \tilde{f}(\tilde{x}) - \tilde{f}_0(\tilde{x}^0) = \tilde{f}(\tilde{x}) - f_0(\tilde{x}^0)$$

$$= f(x^0) - f_0(\tilde{x}^0) = f(x^0) - f_0(x^0) - f_0'\alpha$$

$$= \delta f - f_0'\alpha \quad , \quad \mathcal{H} = a'/a$$

$$\Rightarrow \zeta = \psi + \mathcal{H}\frac{\delta\rho}{\rho_0'} \to \tilde{\zeta} = \tilde{\psi} + \mathcal{H}\frac{\widetilde{\delta\rho}}{\rho_0'}$$

$$= \psi + \mathcal{H}\alpha - \frac{\mathcal{H}\delta\rho}{\rho_0'} - \mathcal{H}\alpha = \zeta$$

COMOVING CURVATURE PERTURBATION

$$\zeta = \psi + \mathcal{H}\frac{\delta\rho}{\rho_0'} = \psi - \frac{1}{3(1+w)}\frac{\delta\rho}{\rho_0}$$

$$\rho_0' = -3\mathcal{H}(\rho_0 + P_0) = -3(1+w)\rho_0$$

$$\psi = \phi \Rightarrow \quad \zeta = \phi - \frac{1}{3(1+w)}\frac{\delta\rho}{\rho_0}$$

$$ds^2 = -(1+2\phi)dt^2 + (1-2\psi)a^2 d\vec{x}^2$$

On long wavelengths $\gg H^{-1}$

$$ds^2 = -e^{2\phi}dt^2 + e^{-2\psi}a^2 d\vec{x}^2$$

$$= -d\bar{t}^2 + \bar{a}^2 d\vec{x}^2 \qquad d\bar{t} = e^{\phi}dt, \quad \bar{a} = e^{-\psi}a$$

$$H^2 = \frac{8\pi G}{3}\rho \Rightarrow \bar{H}^2 = \frac{8\pi G}{3}\bar{\rho}_0$$

$$\bar{H} = \frac{d\bar{a}}{d\bar{t}} = \frac{(\dot{a} - \dot{\psi})e^{-\phi}}{a} \simeq \frac{\dot{a}}{a}(1-\phi) = H(1-\phi)$$

$$\bar{H}^2 \simeq H^2(1-2\phi) = \frac{8\pi G}{3}\rho_0\left(1+\frac{\delta\rho}{\rho_0}\right)$$

$$= H^2\left(1+\frac{\delta\rho}{\rho_0}\right) \Rightarrow \frac{\delta\rho}{\rho_0} = -2\phi$$

$$\Rightarrow \zeta = \phi + \frac{1}{3(1+w)}2\phi = \frac{5+3w}{3(1+w)}\phi$$

$$\zeta(RD) = \frac{6\phi}{3\cdot\frac{4}{3}} = \frac{3}{2}\phi \qquad \frac{\nabla^2\phi}{a^2} = 4\pi G\,\delta\rho = \frac{3}{2}H^2\Delta = \frac{\nabla^2\zeta}{a^2}\frac{2}{3}$$

$$\Rightarrow \Delta = \frac{4}{9}\frac{\nabla^2\zeta}{a^2H^2}$$

ζ is conserved on super-Hubble scales?

$$\bar{\rho}' + 3\bar{H}(\bar{\rho} + \bar{P}) = 0$$

$$\bar{\rho} = \rho_0 + \delta\rho$$

$$\bar{H} = H - \dot{\psi}$$

$$\delta\dot{\rho} + 3H(\delta\rho + \delta P) - 3\dot{\psi}(\rho_0 + P_0) = 0 .$$

In the uniform energy density gauge $\delta\rho = 0$,

$$\delta P = \delta P_{NAD} + \frac{\partial P}{\partial \rho}\delta\rho = \delta P_{NAD}$$

$$\Rightarrow \quad \dot{\zeta} = \dot{\psi} = \frac{H \, \delta P_{NAD}}{\rho_0 + P_0}$$

Furthermore one can use $\mathcal{R} = \psi + H\dfrac{\delta\phi}{\dot{\phi}}$;

during inflation $\rho_\phi \simeq V(\phi)$

$$\Rightarrow \quad \delta\rho_\phi = V'\delta\phi, \qquad 3H\dot{\phi}_0 = -V'$$

$$\Rightarrow \zeta = \psi + H\frac{\delta\rho_\phi}{\dot{\rho}_\phi} = \psi + H\frac{V'\delta\phi}{-3H\dot{\phi}_0^2}$$

$$= \psi + H\frac{\delta\phi}{\dot{\phi}_0} = \mathcal{R} .$$

Valid in slow-roll

$$S = \int d^4x \sqrt{-g}\ \frac{1}{2}\left[g^{\mu\nu}\partial_\mu \chi \partial_\nu \chi\right] \qquad , \ d\tau\, a = dt$$

$$g_{\mu\nu} = \left(-a^2, a^2, a^2, a^2\right)$$

$$S = \int d^4x\ \frac{a^4}{2a^2}\left[\chi'^2 - (\nabla\chi)^2\right]$$

$$\chi = \frac{\sigma}{a}\ ,\quad \chi' = \frac{\sigma'}{a} - \left(\frac{a'}{a^2}\right)\sigma$$

$$S = \int d^4x\ \frac{1}{2}a^2\left[\frac{\sigma'^2}{a^2} + \frac{a'^2}{a^4}\sigma^2 - 2\,\overbrace{\sigma\sigma'}^{(\sigma^2)'}\frac{a'}{a^3} - \frac{(\nabla\sigma)^2}{a^2}\right]$$

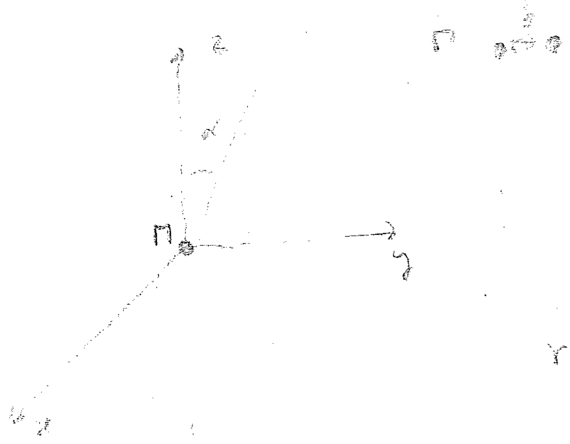
$$= \int d^4x\ \frac{1}{2}\cancel{a^2}\left[\frac{\sigma'^2}{\cancel{a^2}} + \frac{a'^2}{a^2}\sigma^2 + \sigma^2\left(\frac{a'}{a}\right)' - (\nabla\sigma)^2\right]$$

$$= \frac{1}{2}\int d^4x\ \left[\sigma'^2 + \cancel{\left(\frac{a'}{a}\right)^2\sigma^2} + \sigma^2\frac{a''}{a} - \cancel{\sigma^2\frac{a'^2}{a^2}} - (\nabla\sigma)^2\right]$$

$$= \frac{1}{2}\int d^4x\ \left[\sigma'^2 - (\nabla\sigma)^2 + \sigma^2\frac{a''}{a}\right]$$

$$\sigma'' - \nabla^2\sigma - \frac{a''}{a}\sigma = 0$$

$$\sigma'' + \left(k^2 - \frac{a''}{a}\right)\sigma = 0$$



$$\phi = -\frac{Gm}{r}$$

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{b^2+z^2}, \quad b^2 = x^2+y^2$$

$$\vec{\alpha} = 2\int_{\lambda_A}^{\lambda_B} \vec{\nabla}_\perp \phi \, d\lambda \qquad d\lambda \simeq dz$$

$\perp$ to the light path

$$\vec{\nabla}_\perp \phi = \begin{pmatrix} \partial_x \phi \\ \partial_y \phi \end{pmatrix} = \frac{Gm}{r^3}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\vec{\alpha} = 2Gm \begin{pmatrix} x \\ y \end{pmatrix} \int_{-\infty}^{\infty} \frac{dz}{(b^2+z^2)^{3/2}} = 4Gm \begin{pmatrix} x \\ y \end{pmatrix} \int_0^{\infty} \frac{dz}{(b^2+z^2)^{3/2}}$$

$$= 4Gm \begin{pmatrix} x \\ y \end{pmatrix} \left[\frac{z}{b^2(b^2+z^2)^{1/2}} \right]_0^{\infty} = \frac{4Gm}{b^2}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{4Gm}{b}\begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}$$

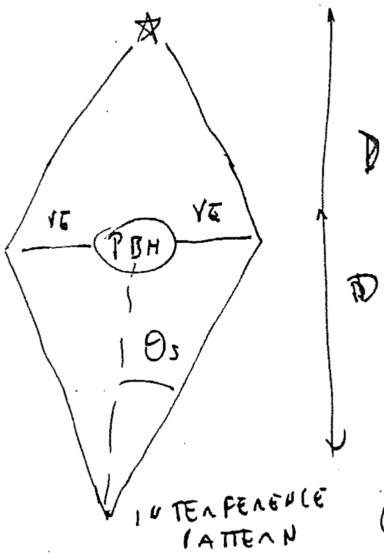
(2): $\theta D_S = D_S \beta + D_{LS}\,\alpha$

$$\frac{r}{D_L} D_S = D_S \frac{r_0}{D_L} + D_{LS}\frac{4Gm}{r}$$

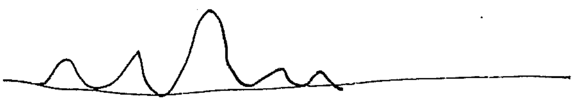
$$r^2 - r_0 r = 4Gm\frac{D_L D_{LS}}{D_S}$$

If $\lambda \gtrsim r_S$ the geometrical optic stops being a good approximation.

Since $r_E \sim \sqrt{r_S D}$ $\quad (D_{LS} \sim D \sim D_S)$



(double slit interference experiment)



$$\theta_1 = \frac{\lambda}{r_E}$$

$$\theta_S \sim \frac{r_E}{D}$$

If $\theta_S \lesssim \theta_1$ the pattern is not resolvable, usual procedure

If $\theta_S \gtrsim \theta_1$ the pattern is there and geometrical optic breaks down $\Rightarrow \theta_S - \frac{\lambda}{r_E} \gtrsim \frac{r_E}{D} \Rightarrow \lambda \gtrsim \frac{r_E^2}{D} \sim r_S$.

The hoop conjecture (Thorne 1972, Misner et al 1973): (1D)

BH's with horizons form when and only when a mass M gets compactified into a region whose circumference in every direction is smaller than $2\pi \cdot 2GM$ – The hoop C of a region is defined as the maximum of its circumferences in all directions. We have then

$$C \lesssim 2\pi \cdot 2GM \qquad p_i = \partial_i \psi$$

Zel'dovich approx: $r_i = a(t) q_i + b(t) p_i(q_j)$ (p_i = deviation vector).

Deformation tensor: $D_{ik} = \frac{\partial r_i}{\partial q_k} = a(t)\delta_{ik} + b(t) \frac{\partial p_i}{\partial q_k}$,

$\frac{\partial p_i}{\partial q_k} = \mathrm{diag}(-\alpha, -\beta, -\gamma)$ defines fundamental axes.

$D_{ik} = \mathrm{diag}(a - \alpha b, a - \beta b, a - \gamma b)$ – The mass is conserved

$dm = \rho d^3 r = \bar\rho a^3 d^3 q \Rightarrow \rho = \frac{a^3}{(a-\alpha b)(a-\beta b)(a-\gamma b)} \bar\rho$.

In the linear regime: $\delta_L = \left(\frac{\rho - \bar\rho}{\bar\rho}\right)_L = \frac{1}{(1-\alpha b/a)(1-\beta b/a)(1-\gamma b/a)} - 1$

$\simeq (\alpha + \beta + \gamma) \frac{b}{a} \sim a \Rightarrow b \sim a^2$.

Define local coordinates $r_1 = (a - \alpha b) q_1$, $r_2 = (a - \beta b) q_2$, $r_3 = (a - \gamma b) q_3$

and $\infty > \alpha \geq \beta \geq \gamma > -\infty$ and $\alpha > 0$ (at least one axis collapses).

t_i = horizon entry, t_f maximum expansion, t_c collapse. At t_i:

$a(t_i) q = H^{-1}(t_i) \Rightarrow m = \frac{4}{3}\pi \bar\rho(t_i) a_i^3 q^3 = \frac{1}{2G} H_i^2 \frac{1}{H_i^3} = \frac{1}{2GH_i} \Rightarrow a_i q = \frac{1}{H_i}$

$= 2Gm = r_g$ and $\delta_i = (\alpha + \beta + \gamma) \frac{b_i}{a_i}$. At t_f $\dot r_1(t_f) = 0 \Rightarrow$

$\dot a_f = \alpha \dot b_f \Rightarrow \frac{b_f}{a_f} = \frac{1}{2\alpha}$ (since $\dot b \overset{=\kappa}{=} 2a\dot a \Rightarrow b_f = 2a_f \dot a_f = \alpha^{-1} \dot a_f$, $\kappa = \frac{\alpha^{-1}}{2a_f}$

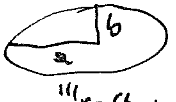
$b_f = \kappa a_f^2 = \frac{\alpha^{-1}}{2a_f} a_f^2 \Rightarrow b_f/a_f = \frac{1}{2\alpha}$) $\Rightarrow r_f = r_1(t_f) = (a_f - \alpha b_f) q = \frac{1}{2} a_f q$

$\Rightarrow \frac{r_g}{r_f} = 2\frac{a_i}{a_f} = \frac{2a_i^2}{a_f a_i} = \frac{2b_i}{\kappa a_f a_i} = 2 \cdot 2\alpha \frac{b_i \dot a_f}{a_i} = 4\alpha \frac{b_i}{a_i} = \frac{4\alpha}{(\alpha+\beta+\gamma)} \delta_i$.

At t_c, $r_1(t_c) = 0 \Rightarrow \frac{b_c}{a_c} = \frac{1}{\alpha} \Rightarrow$ pancake $\Rightarrow r_2(t_c) = (a_c - \beta b_c) q = a_c (1 - \beta/\alpha) q$

$\frac{b_c}{a_c} = \frac{1}{\alpha} = 2 \frac{b_f}{a_f} \Rightarrow \kappa a_c = 2\kappa a_f \Rightarrow a_c = 2a_f \Rightarrow a_c q = 2a_f q = 4r_f$, $r_3(t_c) = 4\left(1 - \frac{\gamma}{\alpha}\right) r_f$

Hoop conjecture:



$= r_1(t_c)$ e = eccentricity $= \sqrt{1-b^2/a^2}$

$r_3(t_c)$

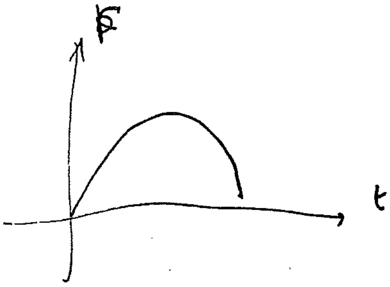
C = perimeter of the ellipse $= 4\, r_3(t_c)\, E(e)$ (elliptic integral of the II kind)

$$C = 4\cdot 4\, (1-\delta/\alpha)^{r_f}\, E\left[\sqrt{1-\frac{r_2^2(t_c)}{r_3^2(t_c)}}\right] \le 2\pi \cdot 2M$$

If not satisfied $\Rightarrow$ relaxation by bouncing and virialization.

$$\frac{r_2(t_c)}{r_3(t_c)} = \frac{1-\beta/\alpha}{1-\delta/\alpha}$$

$r_f = r_1(t_f)$ $\quad t_f : \dot r_1(t_f) = 0$



$r(\theta) = A(1-\cos\theta)$

$t(\theta) = B(\theta - \sin\theta)$

Turn-around $\theta = \pi$,

$t = B\pi$

$$\Delta = \frac{\rho-\bar\rho}{\bar\rho} = \frac{3}{20}\left(\frac{6t}{B}\right)^{2/3} \simeq 1.06$$