THE CERN ACCELERATOR SCHOOL

Theory of Electromagnetic Fields

Part III: Travelling Waves

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In the previous lectures, we discussed:

- Maxwell's equations and their physical interpretation.
- Plane wave solutions for electromagnetic fields in free space.
- Electromagnetic potentials.
- Generation of electromagnetic waves.
- Energy in electromagnetic fields.
- Boundary conditions at interfaces between different media.
- Standing waves in cavities.

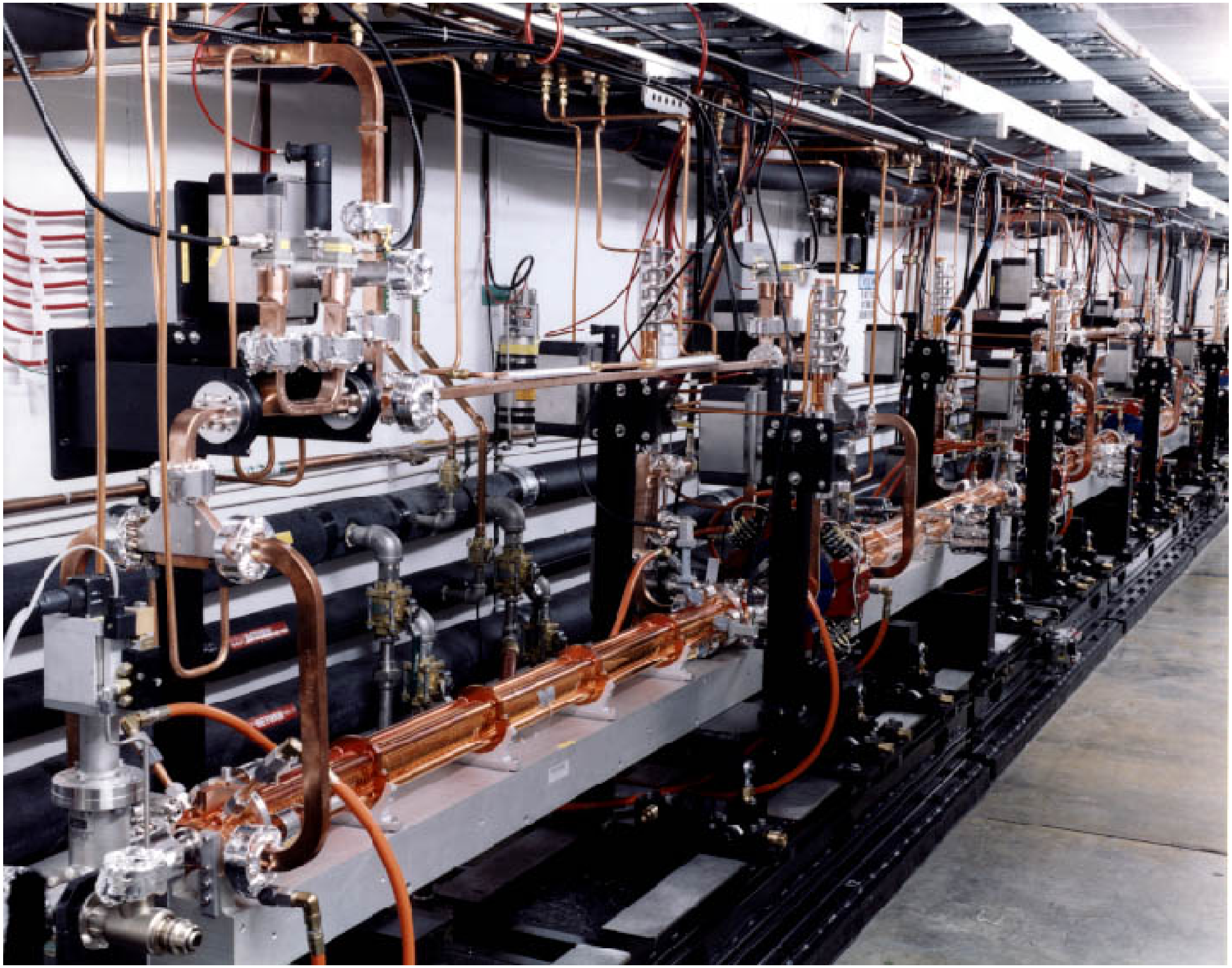
In this lecture we shall discuss how electromagnetic waves can be guided from one place to another.

In particular, we shall talk about:

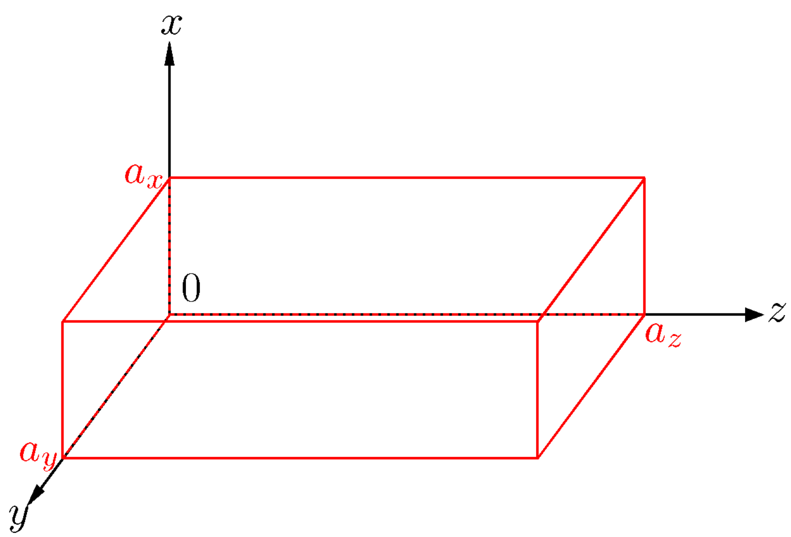
- Electromagnetic waves in rectangular waveguides.
- Electromagnetic waves in transmission lines.

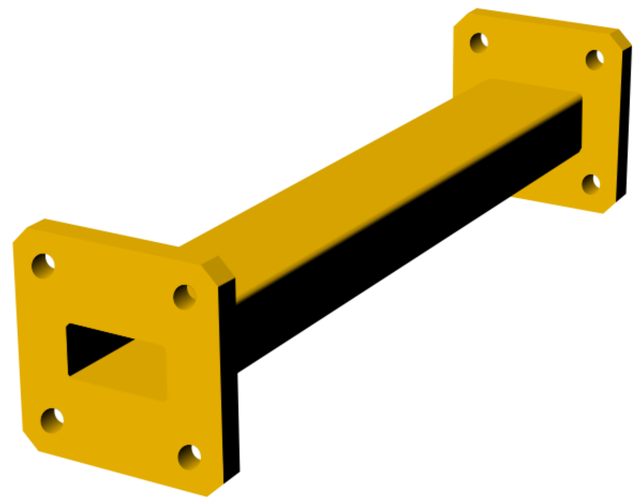
Waveguides are often used for carrying large amounts of energy (high power rf).

Transmission lines are generally used for carrying low power rf signals.



Consider a perfectly conducting tube with rectangular cross-section, of width a_x and height a_y. This is essentially a cavity resonator with length $a_z \to \infty$.





The electric field must solve the wave equation:

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0 \tag{1}$$

together (as usual) with Maxwell's equations.

By comparison with the rectangular cavity case, we expect to find standing waves in x and y, with plane wave solution in z. Therefore, we try a solution of the form:

$$E_x = E_{x0} \cos k_x x \, \sin k_y y \, e^{i(k_z z - \omega t)} \tag{2}$$

$$E_y = E_{y0} \sin k_x x \, \cos k_y y \, e^{i(k_z z - \omega t)} \tag{3}$$

$$E_z = iE_{z0} \sin k_x x \, \sin k_y y \, e^{i(k_z z - \omega t)} \tag{4}$$

To satisfy the wave equation, we must have:

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}. \tag{5}$$

We must of course also satisfy Maxwell's equation:

$$\nabla \cdot \vec{E} = 0 \tag{6}$$

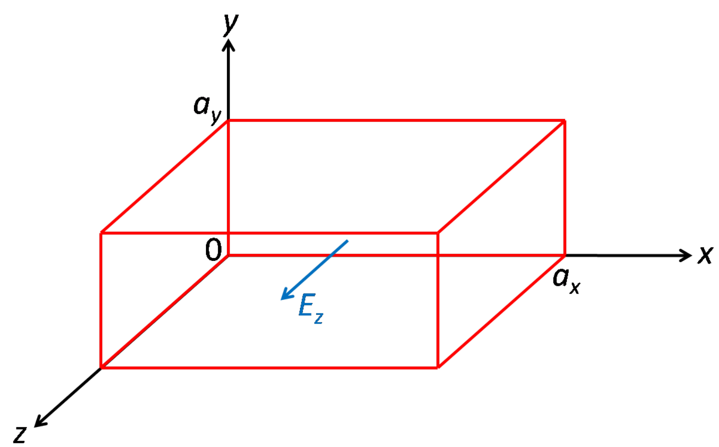
at all x, y and z. This leads to the condition:

$$k_x E_{x0} + k_y E_{y0} + k_z E_{z0} = 0, \tag{7}$$

which is the same as we found for a rectangular cavity.

Now we apply the boundary conditions. In particular, the tangential component of the electric field must vanish at all boundaries.

For example, we must have $E_z = 0$ for all t, z at $x = 0$ and $x = a_x$, and at $y = 0$ and $t = a_y$.



Therefore, $\sin k_x a_x = 0$, and $\sin k_y a_y = 0$, and hence:

$$k_x = \frac{m_x \pi}{a_x}, \qquad m_x = 0, 1, 2 \ldots \tag{8}$$

$$k_y = \frac{m_y \pi}{a_y}, \qquad m_y = 0, 1, 2 \ldots \tag{9}$$

These conditions also ensure that $E_x = 0$ at $y = 0$ and $y = a_y$, and that $E_y = 0$ at $x = 0$ and $x = a_x$.

We can find the magnetic field in the waveguide from:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}. \tag{10}$$

The result is:

$$B_x = \frac{1}{\omega}(k_y E_{z0} - k_z E_{y0}) \sin k_x x \, \cos k_y y \, e^{i(k_z z - \omega t)} \tag{11}$$

$$B_y = \frac{1}{\omega}(k_z E_{x0} - k_x E_{z0}) \cos k_x x \, \sin k_y y \, e^{i(k_z z - \omega t)} \tag{12}$$

$$B_z = \frac{1}{i\omega}(k_x E_{y0} - k_y E_{x0}) \cos k_x x \, \cos k_y y \, e^{i(k_z z - \omega t)} \tag{13}$$

The conditions (8) and (9):

$$k_x = \frac{m_x \pi}{a_x}, \qquad m_x = 0, 1, 2 \ldots$$

$$k_y = \frac{m_y \pi}{a_y}, \qquad m_y = 0, 1, 2 \ldots$$

ensure that the boundary conditions on the magnetic field are satisfied.

Also, the conditions (5):

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2},$$

and (7):

$$k_x E_{x0} + k_y E_{y0} + k_z E_{z0} = 0,$$

ensure that Maxwell's equation:

$$\nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \tag{14}$$

is satisfied.

In a rectangular cavity, the mode could be specified by a set of three numbers, m_x, m_y, and m_z, that determined the wave numbers and the frequency of oscillation of the field.

Because m_x, m_y and m_z had to be integers, there was a discrete spectrum of allowed frequencies of oscillation in a cavity.

In a waveguide, we only need two integer mode numbers, m_x and m_y, corresponding to the wave numbers in the transverse dimensions.

The third wave number, in the longitudinal direction, is allowed to take values over a continuous range: this is because of the absence of boundaries in the longitudinal direction. Therefore, there is a continuous range of frequencies also allowed.

However, if we want a wave to propagate along the waveguide, there is a minimum frequency of oscillation that is needed to achieve this, as we shall now show.

Let us write equation (5) in the form:

$$k_z^2 = \frac{\omega^2}{c^2} - k_x^2 - k_y^2. \tag{15}$$

If the wave is to propagate with no attentuation (i.e. propagate without any loss in amplitude), then k_z must be real. We therefore require that:

$$\frac{\omega^2}{c^2} > k_x^2 + k_y^2. \tag{16}$$

This can be written:

$$\frac{\omega}{c} > \sqrt{\left(\frac{m_x \pi}{a_x}\right)^2 + \left(\frac{m_y \pi}{a_y}\right)^2} \tag{17}$$

Note that at least one of the mode numbers m_x and m_y must be non-zero, for a field to be present at all.

Therefore, there is a minimum frequency of oscillation for a wave to propagate down the waveguide, given by:

$$\omega_{\text{cut-off}} = \frac{\pi}{a} c \tag{18}$$

where c is the speed of light, and a is the side length of the *longest* side of the waveguide cross-section.

Consider an electromagnetic wave that propagates down a waveguide, with longitudinal wave number k_z and frequency ω.

The phase velocity v_p is given by:

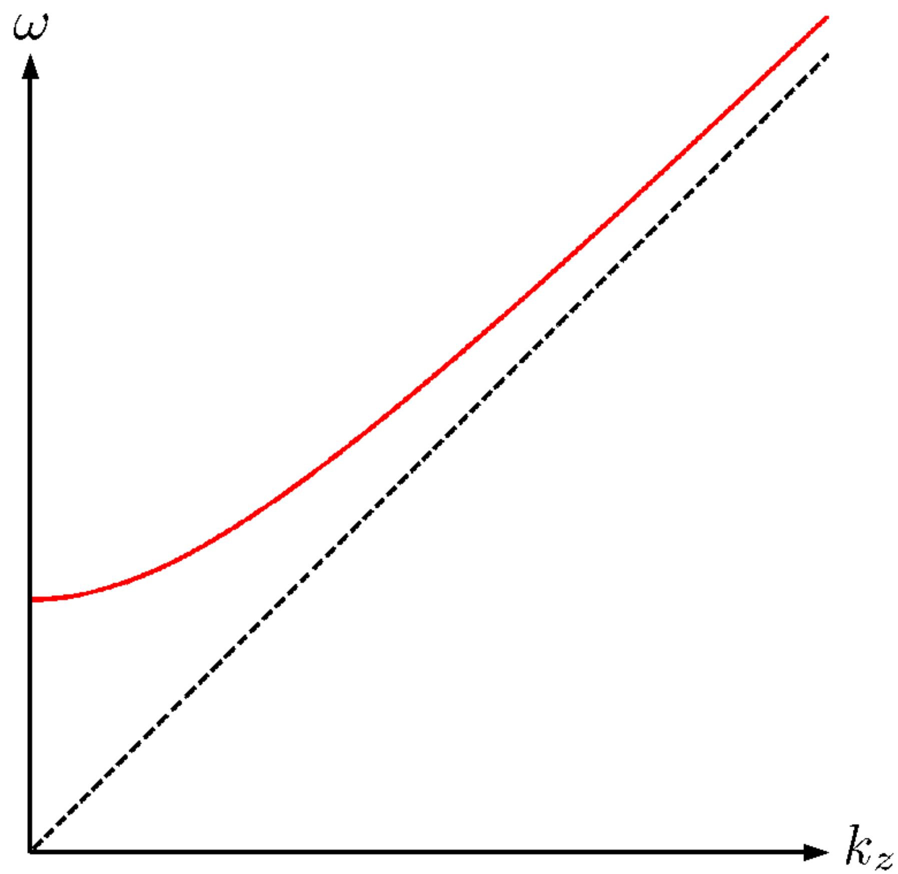
$$v_p = \frac{\omega}{k_z} = \frac{\sqrt{k_x^2 + k_y^2 + k_z^2}}{k_z} c \tag{19}$$

This is the speed at which one would have to travel along the waveguide to stay at a constant phase of the electromagnetic wave (i.e. $k_z z - \omega t =$ constant).

However, since k_x, k_y and k_z are all real, it follows that the phase velocity is greater than the speed of light:

$$v_p > c \tag{20}$$

The dispersion curve shows how the frequency ω varies with the longitudinal wave number k_z:



The phase velocity of a wave in a waveguide is greater than the speed of light.

However, this does not violate special relativity, since energy (and information signals) travel down the waveguide at the group velocity, rather than the phase velocity.

The group velocity v_g is given by:

$$v_g = \frac{d\omega}{dk_z} = \frac{k_z}{\sqrt{k_x^2 + k_y^2 + k_z^2}} c = \frac{k_z}{\omega} c^2, \tag{21}$$

so we have:

$$v_g < c. \tag{22}$$

Note that for a rectangular waveguide, the phase and group velocities are related by:

$$v_p v_g = c^2. \tag{23}$$

Using equation (5):

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2},$$

and the boundary conditions (8) and (9):

$$k_x = \frac{m_x \pi}{a_x} \qquad \text{and} \qquad k_y = \frac{m_y \pi}{a_y},$$

we can write the group velocity (21) for a given mode (m_x, m_y) in terms of the frequency ω:

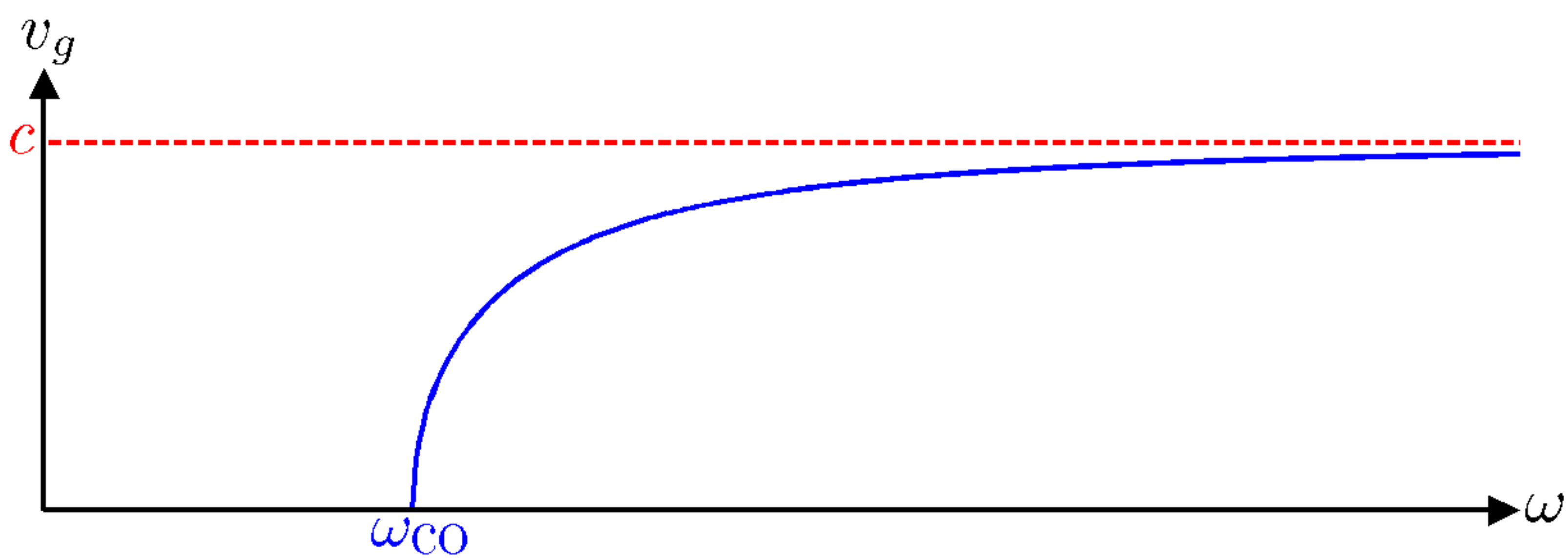
$$v_g = c\sqrt{1 - \frac{\pi^2 c^2}{\omega^2}\left(\frac{m_x^2}{a_x^2} - \frac{m_y^2}{a_y^2}\right)} \qquad (24)$$

Energy in the wave propagates along the waveguide only for $\omega > \omega_{\text{co}}$ (where ω_{co} is the cut-off frequency).

Note the limiting behaviour of the group velocity, for $\omega > \omega_{\text{co}}$:

$$\lim_{\omega \to \infty} v_g = c \quad (25)$$

$$\lim_{\omega \to \omega_{\text{co}}} v_g = 0 \quad (26)$$



The energy density and the energy flow in a waveguide are of interest.

The energy density in the waveguide can be calculated from:

$$U = \frac{1}{2}\varepsilon_0 \vec{E}^2 + \frac{1}{2}\frac{\vec{B}^2}{\mu_0}. \tag{27}$$

Using the expressions for the fields (2) − (4), and assuming that both k_x and k_y are non-zero, we see that the *time-average energy per unit length* in the electric field is:

$$\iint \bar{U}_E \, dx \, dy = \frac{1}{16}\varepsilon_0 A \left(E_{x0}^2 + E_{y0}^2 + E_{z0}^2 \right), \tag{28}$$

where A is the cross-sectional area of the waveguide.

The fields in a waveguide are just a superposition of plane waves in free space. We know that in such waves, the energy is shared equally between the electric and magnetic fields.

Using the expressions for the magnetic field (11) – (13), we can indeed show that the magnetic energy is equal (on average) to the electric energy - though the algebra is rather complicated.

The total time-average energy per unit length in the waveguide is then given by:

$$\iint \bar{U}\, dx\, dy = \frac{1}{8}\varepsilon_0 A \left(E_{x0}^2 + E_{y0}^2 + E_{z0}^2 \right). \tag{29}$$

The energy flow along the waveguide is given by the Poynting vector, $\vec{S} = \vec{E} \times \vec{H}$.

Taking the real parts of the field components, we find for the time-average transverse components:

$$\bar{S}_x = \bar{S}_y = 0. \qquad (30)$$

The longitudinal component is non-zero; taking the time average and integrating over the cross section gives the time average energy flow along the waveguide

$$\iint \bar{S}_z \, dx \, dy = \frac{1}{8\mu_0} A(E_{x0}^2 + E_{y0}^2 + E_{z0}^2)\frac{k_z}{\omega}. \qquad (31)$$

Using equation (21) for the group velocity v_g, the time-average energy flow along the waveguide can be written in the form:

$$\iint \bar{S}_z \, dx \, dy = \frac{1}{8}\varepsilon_0 A (E_{x0}^2 + E_{y0}^2 + E_{z0}^2) v_g. \tag{32}$$

Thus, the energy density per unit length and the rate of energy flow are related by:

$$\iint \bar{S}_z \, dx \, dy = v_g \iint \bar{U} \, dx \, dy. \tag{33}$$

This is consistent with our interpretation of the group velocity as the speed at which energy is carried by a wave.

When we discussed cylindrical cavities, we wrote down two sets of modes for the fields: TE modes ($E_z = 0$), and TM modes ($B_z = 0$).

We can also define TE and TM modes in rectangular cavities, and rectangular waveguides.

For a TE mode in a waveguide, the longitudinal component of the electric field is zero at all positions and times, which implies that:

$$E_{z0} = 0. \tag{34}$$

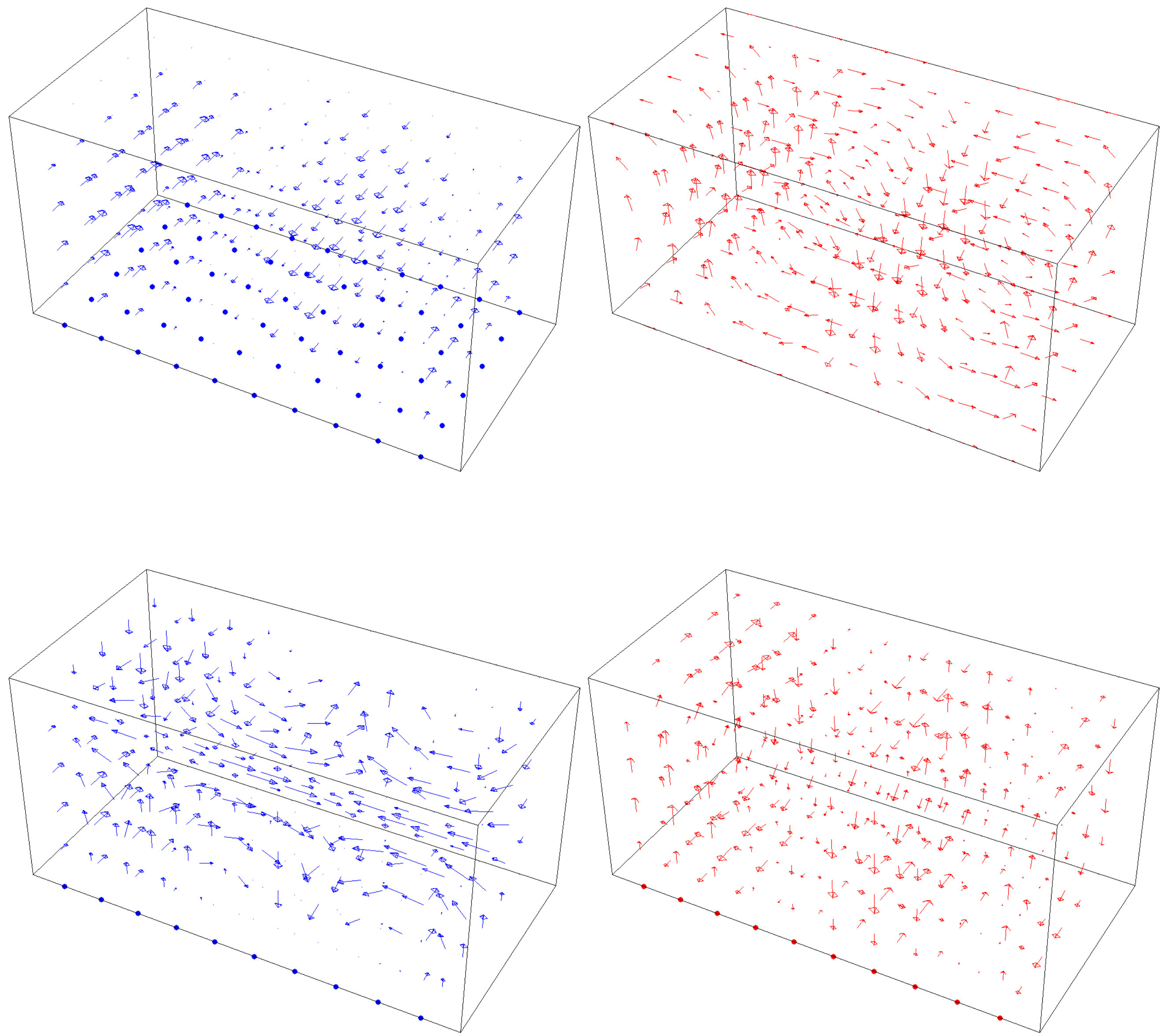
From (7), it follows that in a TE mode, the wave numbers and amplitudes must be related by:

$$k_x E_{x0} + k_y E_{y0} = 0. \tag{35}$$

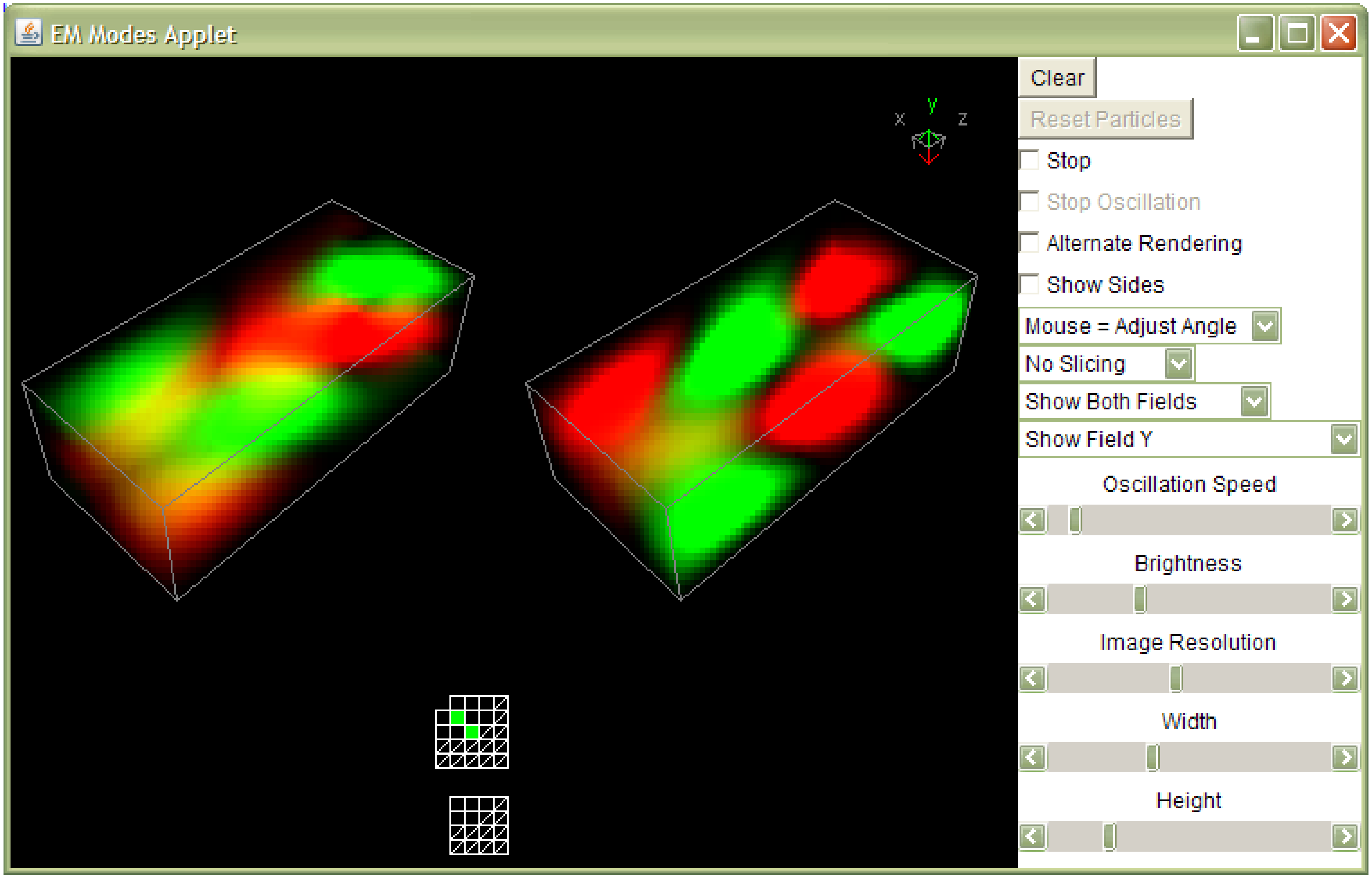
Similarly, for a TM mode, we have (using 13):

$$B_{z0} = 0, \quad k_x E_{y0} - k_y E_{x0} = 0. \tag{36}$$

Field plots for TE_{01} and TM_{11} modes

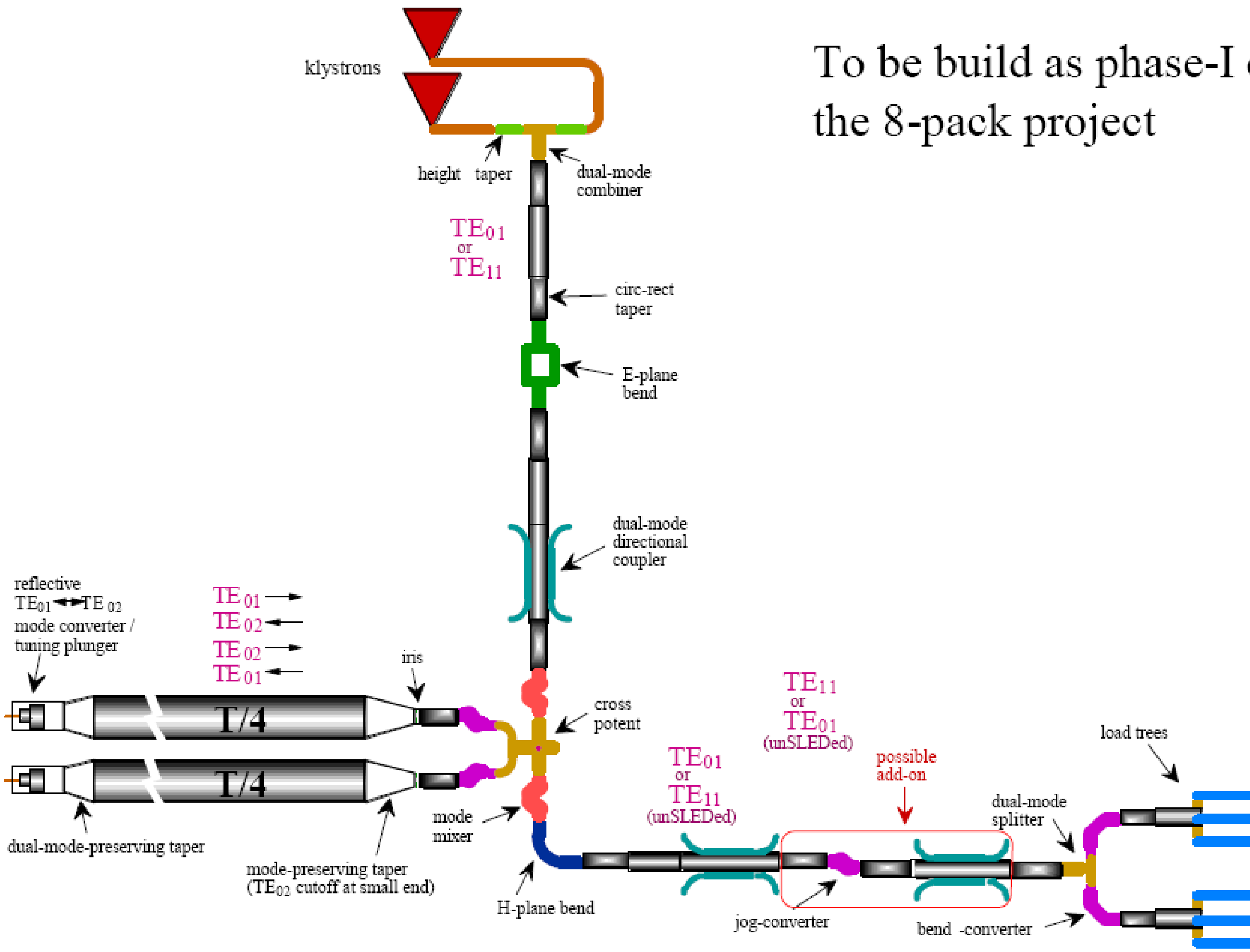


Field plots



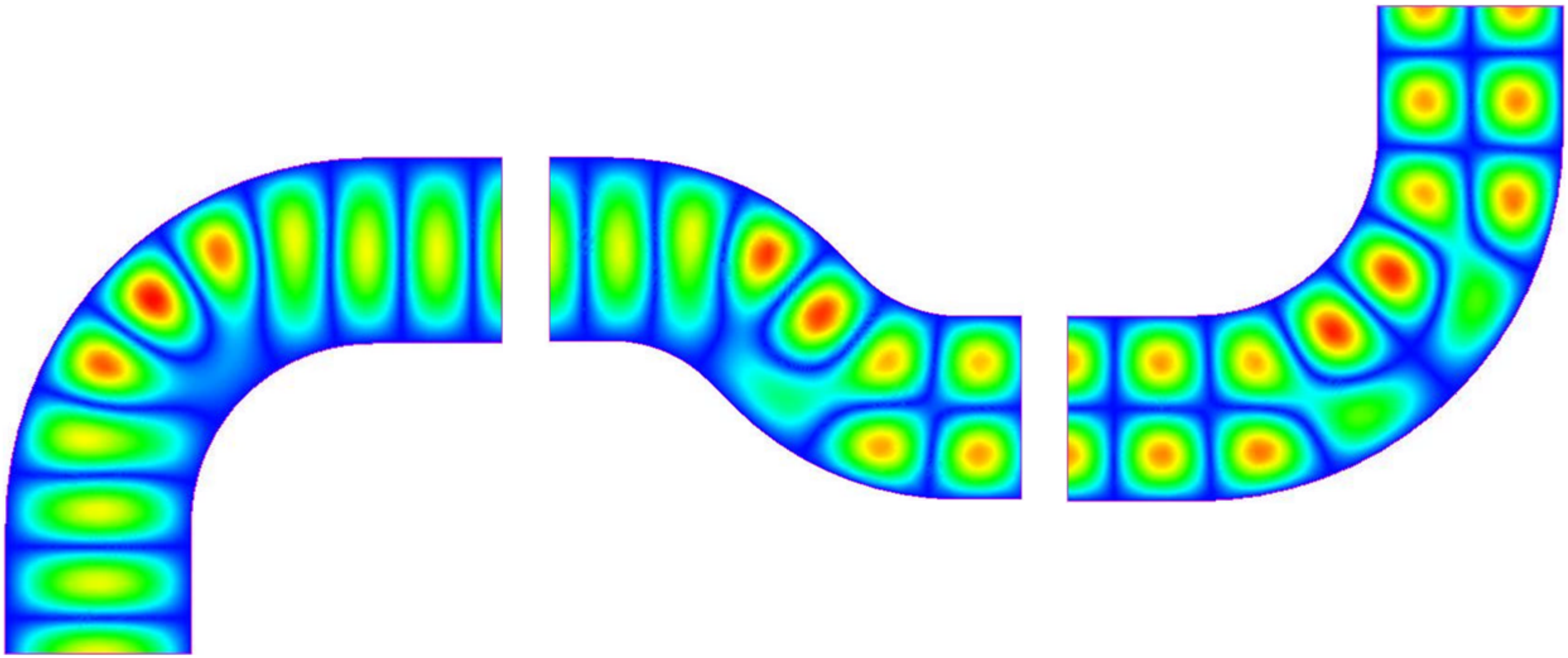
www.falstad.com

Dual-Moded Pulse Compression System



To be build as phase-I of the 8-pack project

Sami Tantawi, "SLED-II and DLDS Pulse Compression", presentation at ILC Machine Advisory Committee (2002).



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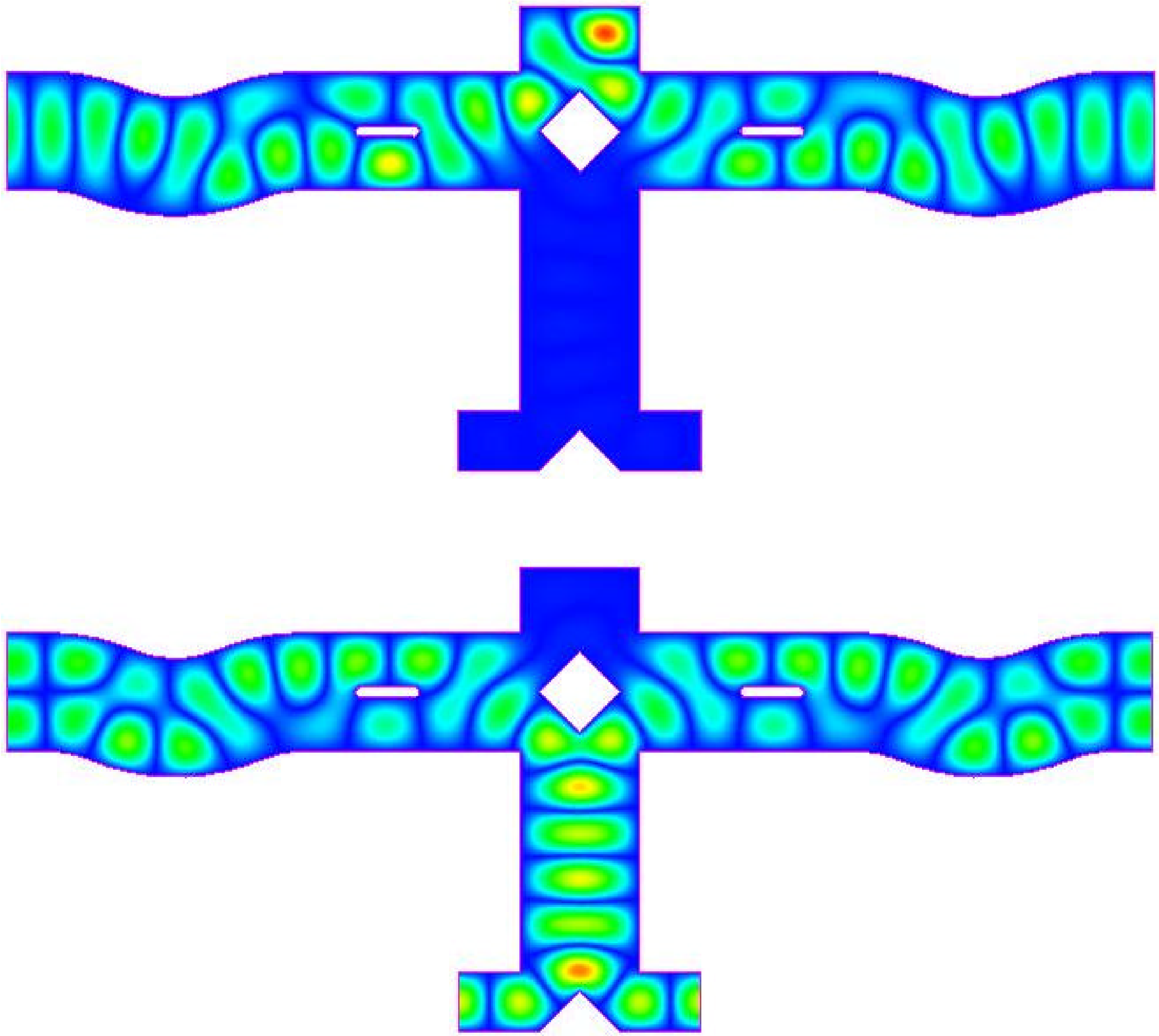
Manipulating Modes



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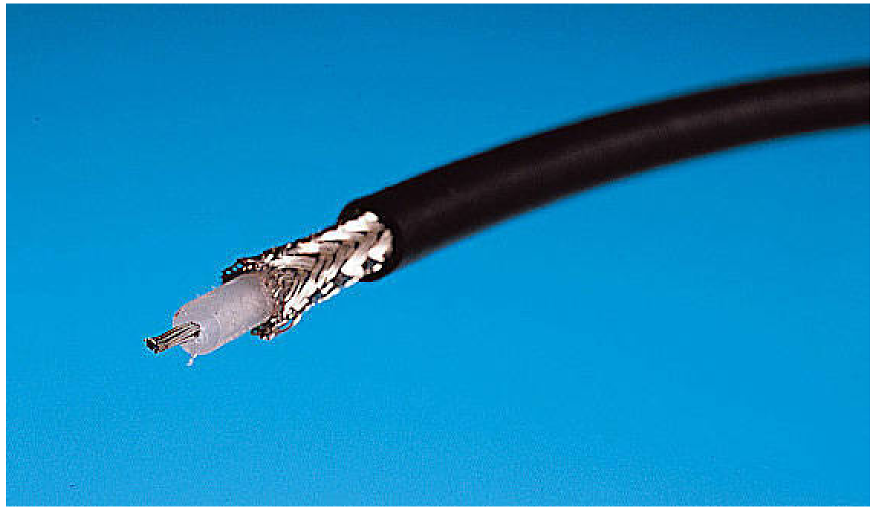


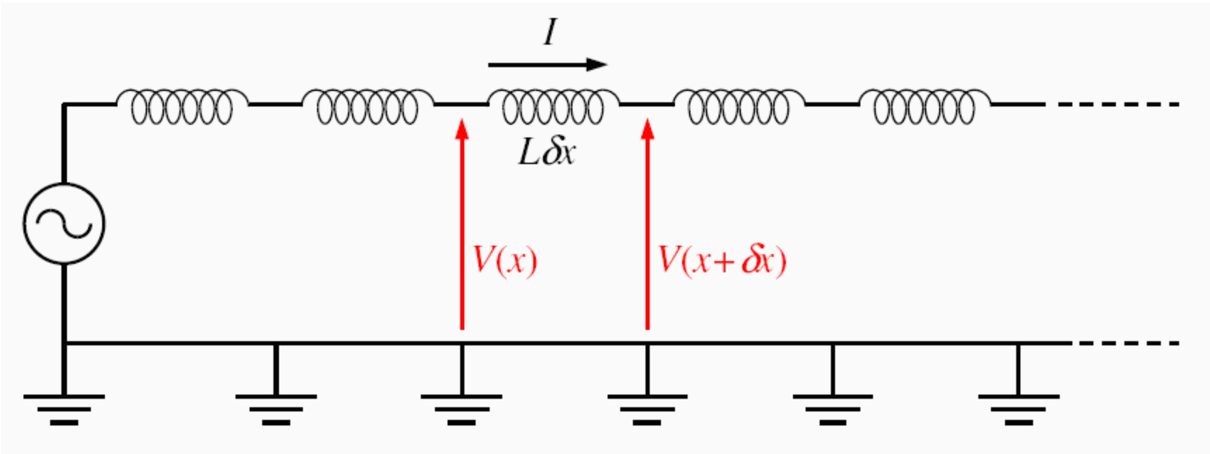
Transmission lines are used (as are waveguides) to guide electromagnetic waves from one place to another. A coaxial cable (used, for example, to connect a radio or television to an aerial) is an example of a transmission line.

Transmission lines may be less bulky and less expensive than waveguides; but they generally have higher losses, so are more appropriate for carrying low-power signals over short distances.

To complete these lectures, we shall consider:

- a simple LC model of a general transmission line;
- the speed of propagation of a wave in a transmission line;
- the characteristic impedance of a transmission line;
- impedance matching at the termination of a transmission line;
- practical transmission lines (parallel wires; coaxial cable).

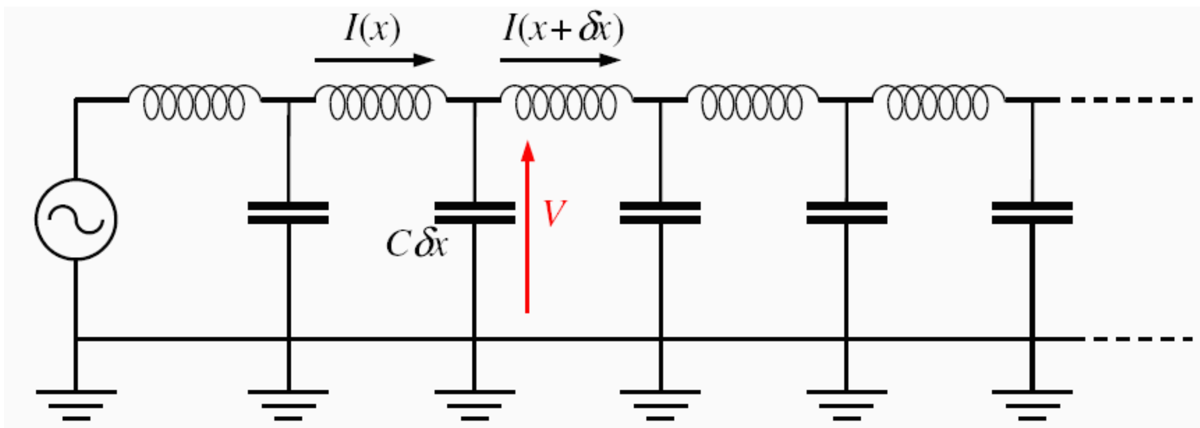
Consider an infinitely long, parallel wire with zero resistance. In general, the wire will have some inductance per unit length, L, which means that when an alternating current I flows in the wire, there will be a potential difference between different points along the wire.



If V is the potential at some point along the wire with respect to earth, then the potential difference between two points along the wire is given by:

$$\Delta V = \frac{\partial V}{\partial x}\delta x = -L\delta x\frac{\partial I}{\partial t} \tag{37}$$

In general, as well as the inductance, there will also be some capacitance per unit length, C, between the wire and earth.



This means that the current in the wire can vary with position:

$$\frac{\partial I}{\partial x}\delta x = -C\delta x\frac{\partial V}{\partial t} \tag{38}$$

Let us take Equations (37) and (38) above:

$$\frac{\partial V}{\partial x} = -L\frac{\partial I}{\partial t} \tag{39}$$

$$\frac{\partial I}{\partial x} = -C\frac{\partial V}{\partial t} \tag{40}$$

Differentiate (39) with respect to t:

$$\frac{\partial^2 V}{\partial x \partial t} = -L\frac{\partial^2 I}{\partial t^2} \tag{41}$$

and (40) with respect to x:

$$\frac{\partial^2 I}{\partial x^2} = -C\frac{\partial^2 V}{\partial t \partial x} \tag{42}$$

Hence:

$$\frac{\partial^2 I}{\partial x^2} = LC\frac{\partial^2 I}{\partial t^2} \tag{43}$$

Similarly (by differentiating (39) with respect to x and (40) with respect to t), we find:

$$\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2} \tag{44}$$

Equations (43) and (44) are wave equations for the current in the wire, and the voltage between the wire and earth. The waves travel with speed v, given by:

$$v = \frac{1}{\sqrt{LC}} \tag{45}$$

The solutions to the wave equations may be written:

$$V = V_0 e^{i(kx-\omega t)} \tag{46}$$

$$I = I_0 e^{i(kx-\omega t)} \tag{47}$$

where the phase velocity is:

$$v = \frac{\omega}{k} = \frac{1}{\sqrt{LC}} \tag{48}$$

Note that the inductance per unit length L and the capacitance per unit length C are real and positive. Therefore, if the frequency ω is real, the wave number k will also be real: this implies that waves propagate along the transmission line with constant amplitude. This result is expected, given our assumption about the line having zero resistance.

The solutions must also satisfy the first-order equations (37) and (38). Substituting the above solutions into these equations, we find:

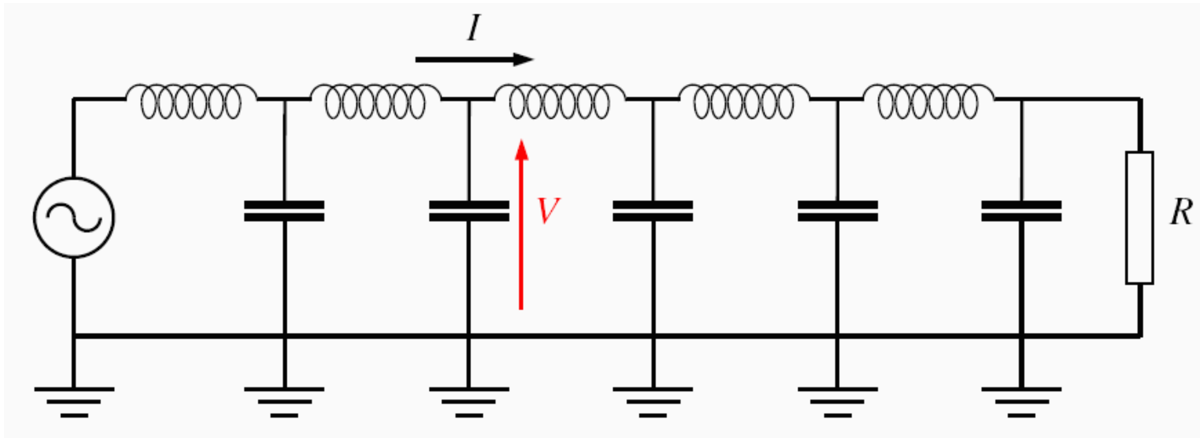
$$kV_0 = \omega L I_0 \tag{49}$$

$$kI_0 = \omega C V_0 \tag{50}$$

Hence:

$$\frac{V_0}{I_0} = \sqrt{\frac{L}{C}} = Z \tag{51}$$

The ratio of the voltage to the current is called the *characteristic impedance*, Z, of the transmission line. Z is measured in ohms, Ω. Note that, since L and C are real and positive, the impedance is a real number: this means that the voltage and current are in phase.



So far, we have assumed that the transmission line has infinite length.

Obviously, this cannot be achieved in practice. We can terminate the transmission line using a "load" that dissipates the energy in the wave while maintaining the same ratio of voltage to current as exists all along the transmission line.

In that case, our above analysis for the infinite line will remain valid for the finite line, and we say that the impedances of the line and the load are properly *matched*.

What happens if the impedance of the load, Z_L, is not properly matched to the characteristic impedance of the transmission line, Z?

In that case, we need to consider a solution consisting of a superposition of waves travelling in opposite directions:

$$V = V_0 e^{i(kx - \omega t)} + K V_0 e^{i(-kx - \omega t)} \tag{52}$$

The corresponding current is given by:

$$I = \frac{V_0}{Z} e^{i(kx - \omega t)} - K \frac{V_0}{Z} e^{i(-kx - \omega t)} \tag{53}$$

Note the minus sign in the second term in the expression for the current: this comes from equations (39) and (40).

Let us take the end of the transmission line, where the load is located, to be at $x = 0$. At this position, we have:

$$V = V_0 e^{-i\omega t}(1 + K) \tag{54}$$

$$I = \frac{V_0}{Z} e^{-i\omega t}(1 - K) \tag{55}$$

If the impedance of the load is Z_L, then:

$$Z_L = \frac{V}{I} = Z\frac{1+K}{1-K} \tag{56}$$

Solving this equation for K (which gives the relative amplitude and phase of the "reflected" wave), we find:

$$K = \frac{Z_L/Z - 1}{Z_L/Z + 1} = \frac{Z_L - Z}{Z_L + Z} \tag{57}$$

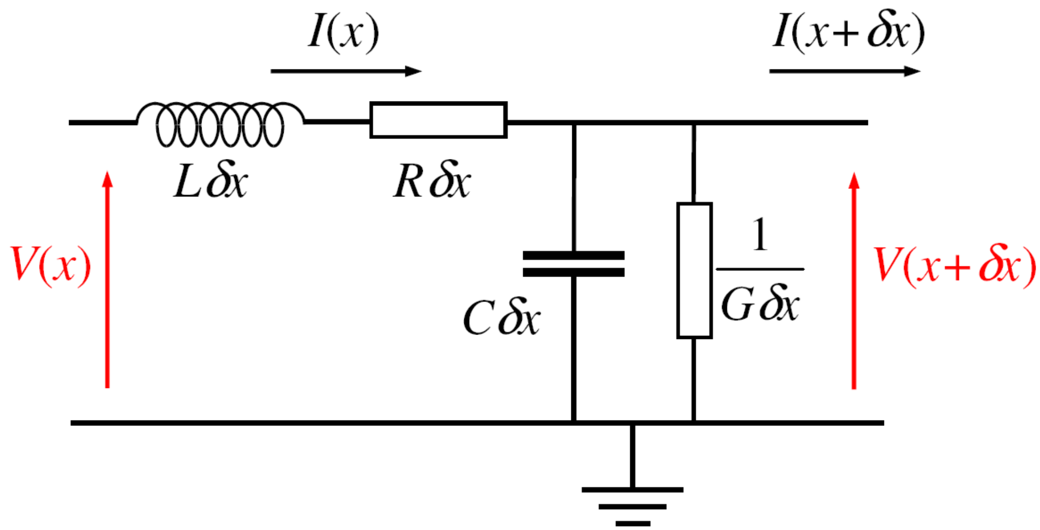
Note that if $Z_L = Z$, there is no reflected wave.

So far, we have assumed that the conductors in the transmission line have zero resistance, and are separated by a perfect insulator.

Usually, though, the conductors will have finite conductivity; and the insulator will have some finite resistance.

To understand the impact that this has, we need to modify our transmission line model to include:

- a resistance per unit length R in series with the inductance;
- a conductance per unit length G in parallel with the capacitance.



The equations for the current and voltage are then:

$$\frac{\partial V}{\partial x} = -L\frac{\partial I}{\partial t} - RI \qquad (58)$$

$$\frac{\partial I}{\partial x} = -C\frac{\partial V}{\partial t} - GV \qquad (59)$$

We can find solutions to the equations (58) and (59) for the voltage and current in the lossy transmission line by considering the case that we propagate a wave with a single, well-defined frequency ω.

In that case, we can replace each time derivative by a factor $-i\omega$. The equations become:

$$\frac{\partial V}{\partial x} = i\omega LI - RI = -\tilde{L}\frac{\partial I}{\partial t} \tag{60}$$

$$\frac{\partial I}{\partial x} = i\omega CV - GV = -\tilde{C}\frac{\partial V}{\partial t} \tag{61}$$

where

$$\tilde{L} = L - \frac{R}{i\omega} \qquad \text{and} \qquad \tilde{C} = C - \frac{G}{i\omega} \tag{62}$$

The new equations (60) and (61) for the lossy transmission line look exactly like the original equations (39) and (40) for a lossless transmission line, but with the capacitance C and inductance L replaced by (complex) quantities $\tilde{C}$ and $\tilde{L}$. The imaginary parts of $\tilde{C}$ and $\tilde{L}$ characterise the losses in the lossy transmission line.

Mathematically, we can solve the equations for a lossy transmission line in exactly the same way as we did for the lossless line. In particular, we find for the phase velocity:

$$v = \frac{1}{\sqrt{\tilde{L}\tilde{C}}} = \frac{1}{\sqrt{\left(L + i\frac{R}{\omega}\right)\left(C + i\frac{G}{\omega}\right)}} \tag{63}$$

and for the impedance:

$$Z = \sqrt{\frac{\tilde{L}}{\tilde{C}}} = \sqrt{\frac{L + i\frac{R}{\omega}}{C + i\frac{G}{\omega}}} \tag{64}$$

Since the impedance (64) is now a complex number, there will be a phase difference (given by the complex phase of the impedance) between the current and voltage in the transmission line.

Note that the phase velocity (63) depends explicitly on the frequency. That means that a lossy transmission line will exhibit dispersion: waves of different frequencies will travel at different speeds, and a the shape of a wave "pulse" composed of different frequencies will change as it travels along the transmission line.

This is one reason why it is important to keep losses in a transmission line as small as possible (for example, by using high-quality materials). The other reason is that in a lossy transmission line, the wave amplitude will attenuate, much like an electromagnetic wave propagating in a conductor.

Recall that we can write the phase velocity:

$$v = \frac{\omega}{k} \tag{65}$$

where k is the wave number appearing in the solution to the wave equation:

$$V = V_0 e^{i(kx - \omega t)} \tag{66}$$

(and similarly for the current I). Using equation (63) for the phase velocity, we have:

$$k = \omega\sqrt{LC}\sqrt{\left(1 + i\frac{R}{\omega L}\right)\left(1 + i\frac{G}{\omega C}\right)} \tag{67}$$

Let us assume $R \ll \omega L$ (i.e. good conductivity along the transmission line) and $G \ll \omega C$ (i.e. poor conductivity between the lines); then we can make a Taylor series expansion, to find:

$$k \approx \omega\sqrt{LC}\left[1 + \frac{i}{2\omega}\left(\frac{R}{L} + \frac{G}{C}\right)\right] \tag{68}$$

Finally, we write:

$$k = \alpha + i\beta \tag{69}$$

and equate real and imaginary parts in equation (68) to give:

$$\alpha \approx \omega\sqrt{LC} \tag{70}$$

and:

$$\beta \approx \frac{1}{2}\left(\frac{R}{Z_0} + GZ_0\right) \tag{71}$$

where $Z_0 = \sqrt{L/C}$ is the impedance with $R = G = 0$ (not to be confused with the impedance of free space). Note that since:

$$V = V_0 e^{i(kx-\omega t)} = V_0 e^{-\beta x} e^{i(\alpha x - \omega t)} \tag{72}$$

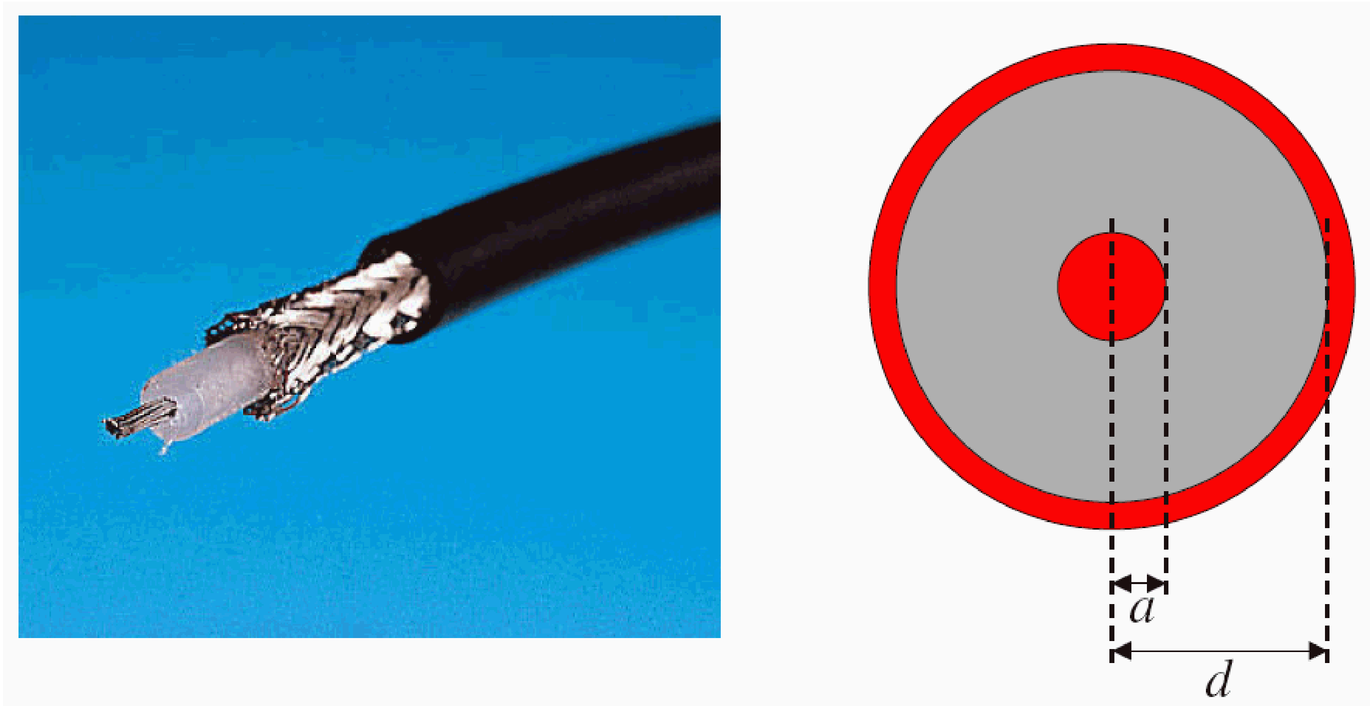
the value of α gives the wavelength $\lambda = 1/\alpha$, and the value of β gives the attenuation length $\delta = 1/\beta$.

A lossless transmission line has two key properties: the phase velocity v, and the characteristic impedance Z. These are given in terms of the inductance per unit length L, and the capacitance per unit length C:

$$v = \frac{1}{\sqrt{LC}} \qquad Z = \sqrt{\frac{L}{C}}$$

The problem, when designing or analysing a transmission line, is to calculate the values of L and C. These are determined by the geometry of the transmission line, and are calculated by solving Maxwell's equations.

Coaxial cable transmission line



As an example, we consider a coaxial cable transmission line, consisting of a central wire of radius a, surrounded by a conducting "sheath" of internal radius d. The central wire and surrounding sheath are separated by a dielectric of permittivity ε and permeability μ.

Suppose that the central wire carries line charge $+\lambda$, and the surrounding sheath carries line charge $-\lambda$ (so that the sheath is at zero potential). We can apply Maxwell's equations as before, to find that the electric field in the dielectric is given by:

$$|\vec{E}| = \frac{\lambda}{2\pi\varepsilon r} \tag{73}$$

where r is the radial distance from the axis. The potential between the conductors is given by:

$$V = \int_a^d \vec{E} \cdot d\vec{r} = \frac{\lambda}{2\pi\varepsilon} \ln\left(\frac{d}{a}\right) \tag{74}$$

Hence the capacitance per unit length of the coaxial cable is:

$$C = \frac{\lambda}{V} = \frac{2\pi\varepsilon}{\ln(d/a)} \tag{75}$$

To find the inductance per unit length, we consider a length l of the cable. If the central wire carries a current I, then the magnetic field at a radius r from the axis is given by:

$$|\vec{B}| = \frac{\mu I}{2\pi r} \tag{76}$$

The flux through the shaded area shown in the diagram is given by:

$$\Phi = l \int_a^d |\vec{B}| dr = \frac{\mu l I}{2\pi} \ln\left(\frac{d}{a}\right) \tag{77}$$

Hence the inductance per unit length is given by:

$$L = \frac{\Phi}{lI} = \frac{\mu}{2\pi} \ln\left(\frac{d}{a}\right) \tag{78}$$

With the expression (75) for the capacitance per unit length:

$$C = \frac{2\pi\varepsilon}{\ln(d/a)} \tag{79}$$

the phase velocity of waves along the coaxial cable is given by:

$$v = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu\varepsilon}} \tag{80}$$

and the characteristic impedance of the cable is given by:

$$Z = \sqrt{\frac{L}{C}} = \frac{1}{2\pi} \ln\left(\frac{d}{a}\right) \sqrt{\frac{\mu}{\varepsilon}} \tag{81}$$

Summary of Part III: Travelling Waves

We have shown that:

- Electromagnetic fields in waveguides can be described in terms of modes, in much the same way as fields in cavities.
- In a waveguide, a given mode can support a range of different frequencies of oscillation for the fields in the waveguide.
- For any given mode, there is a "cut-off" frequency, below which waves will not propagate along the waveguide. The group velocity increases with frequency, from zero (at the cut-off frequency) to the speed of light (in the limit of high frequency).
- A simple LC model of a transmission line leads to wave equations for the current and voltage in the line.
- The phase velocity and characteristic impedance are determined by the inductance and capacitance per unit length, which are in turn determined by the geometry (and materials) in the transmission line.

- A coaxial cable has an inner core with diameter 2 mm. The dielectric between the conductors has relative permeability 1, and relative permittivity 1.2. Calculate the diameter of the outer conductor to give a characteristic impedance equal to 75 Ω.

  If the coaxial cable is terminated with an impedance of 50 Ω, calculate the fraction of power reflected from the termination.