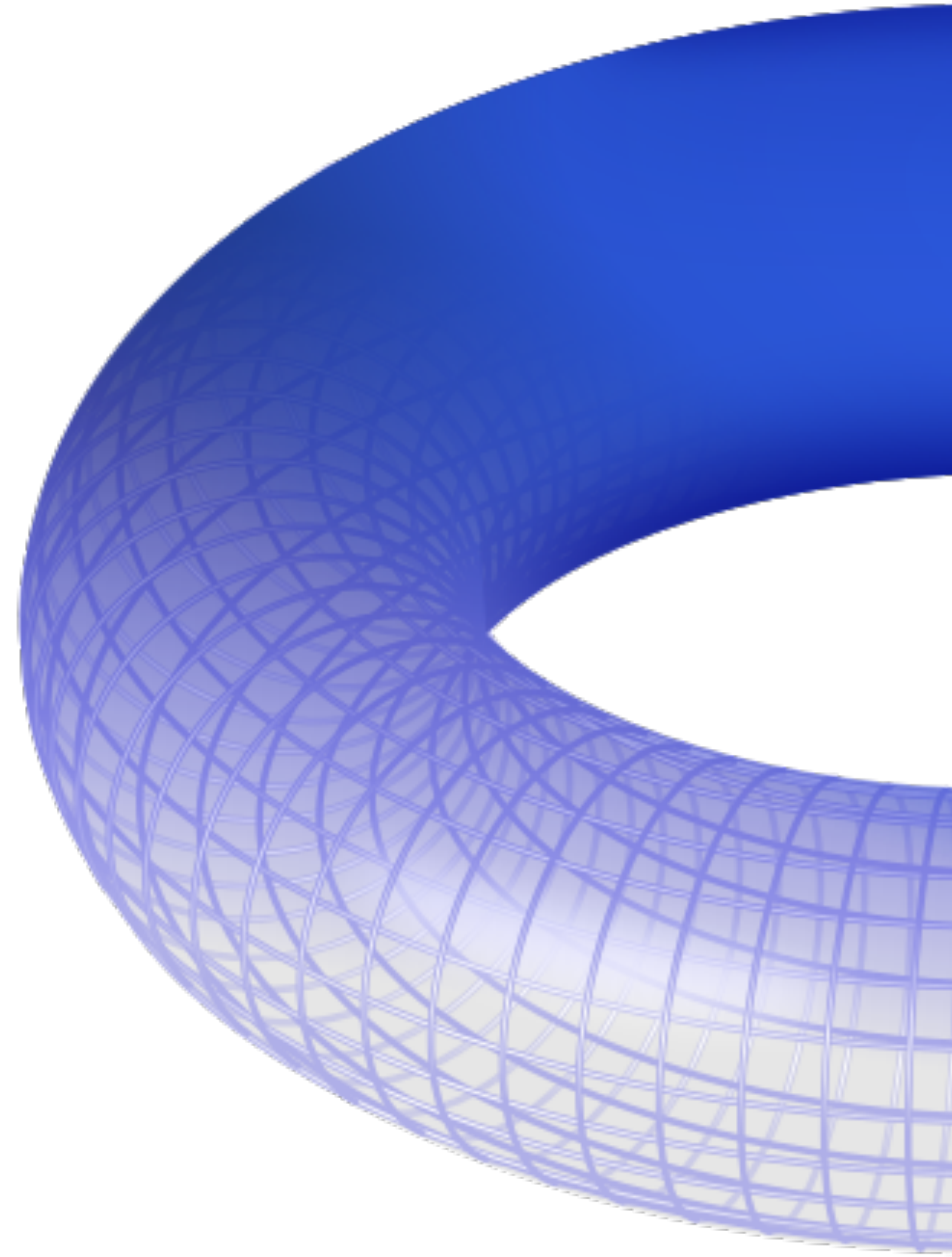
FEYNMAN INTEGRALS AND ELLIPTIC POLYLOGARITHMS

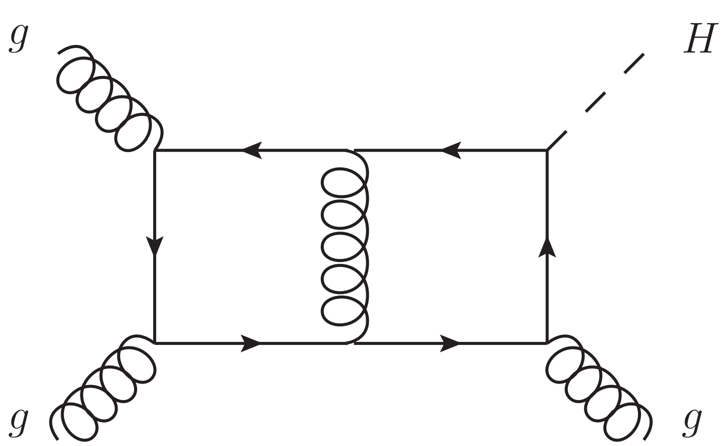


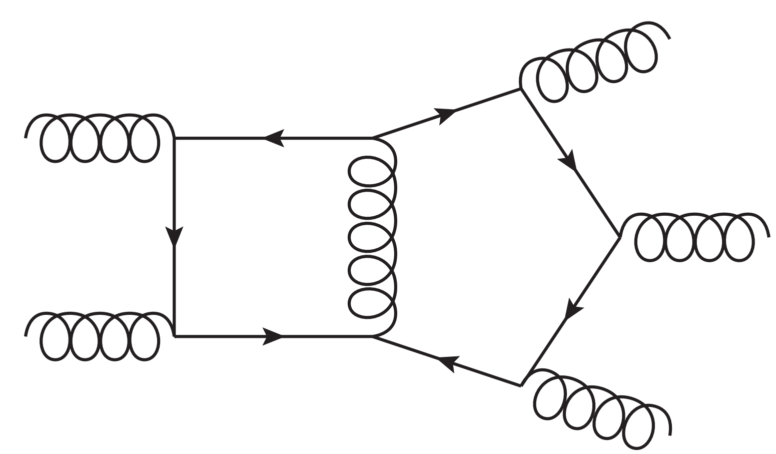
High Precision for Hard Processes
HP2 2018 - Freiburg 01/10/2018

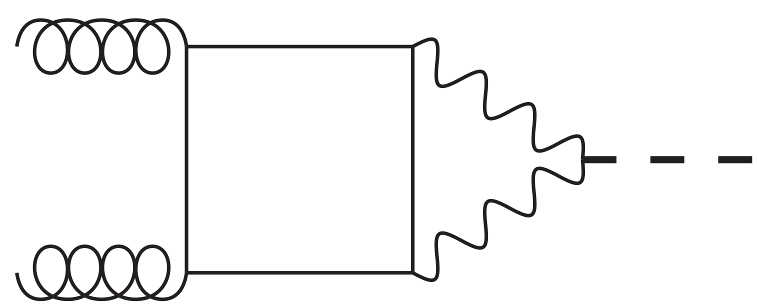
Lorenzo Tancredi - CERN TH

INTRODUCTION

High precision phenomenology @ the LHC typically requires the calculation of **two-loop Feynman integrals**







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INTRODUCTION

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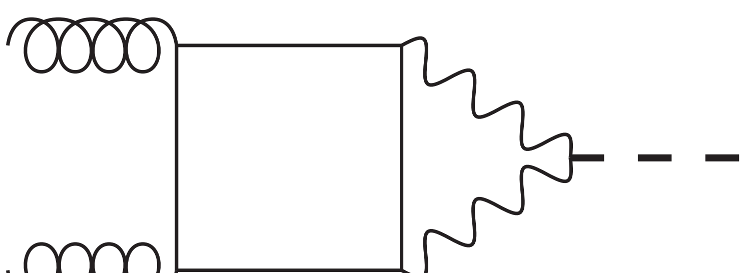
Rule of thumb: **Masses** and **scales in general** ***are difficult***.

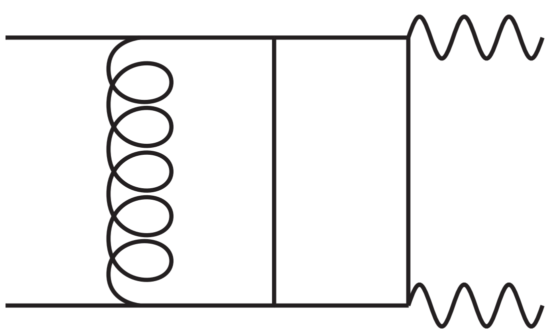
We have good control on:

$2 \to 1$



Different internal and external particles

$2 \to 2$



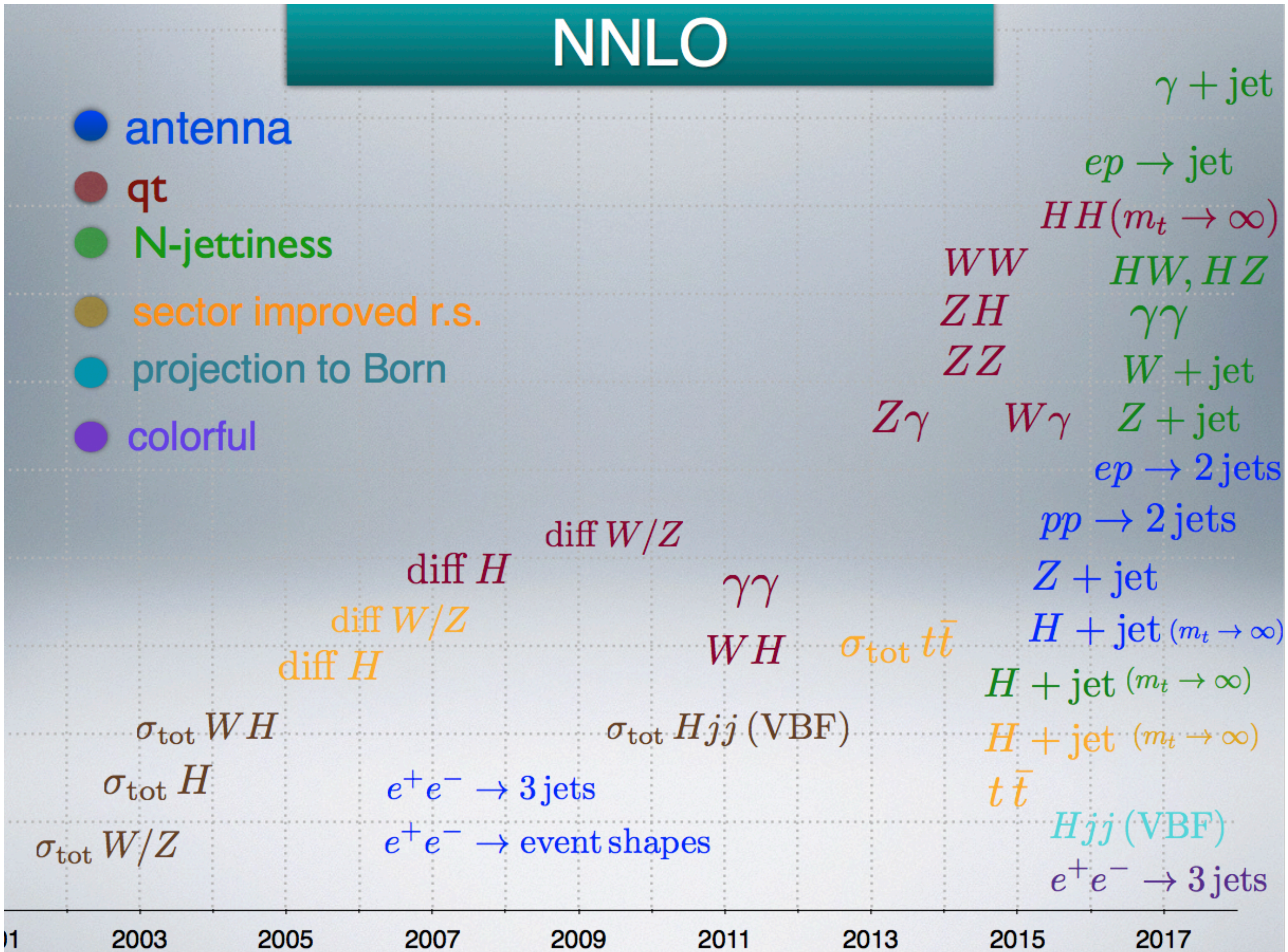
Mainly ***massless internal particles***. Up to ***2 massive external particles (HH, WZ,…)***

HOW DID WE GET THERE

- Development of the **differential equation method**
  [Kotikov '90; Remiddi '97; Gehrmann Remiddi '00]
- Definition of the **Harmonic Polylogarithms (HPLs)** and discovery of their *large applicability in high energy physics* [Remiddi, Vermaseren '99]
- Generalization to **Multiple Polylogarithms (MPLs)**, well known to the mathematicians [Kummer 1840; Lappo-Danilevsky 1954; Gehrmann, Remiddi '01]
- Development of routines for their numerical evaluation [Vollinga, Weinzierl '04]
- Study of their **analytical and algebraic properties** (*Symbols and Co-action* for MPLs and Feynman integrals) [Goncharov '01,…, Duhr, Gangl, Rhodes '13,…]
- Finally, discovery of **Canonical Bases**. Close the circle with the differential equations method. FIs that fulfil DEs in canonical basis can be straightforwardly solved in terms of MPLs… ***well not quite, but often…***
  [Henn '13]

WHAT CAN WE DO

While most of these developments appear formal and even purely mathematical, LHC phenomenology had a lot to gain from them!

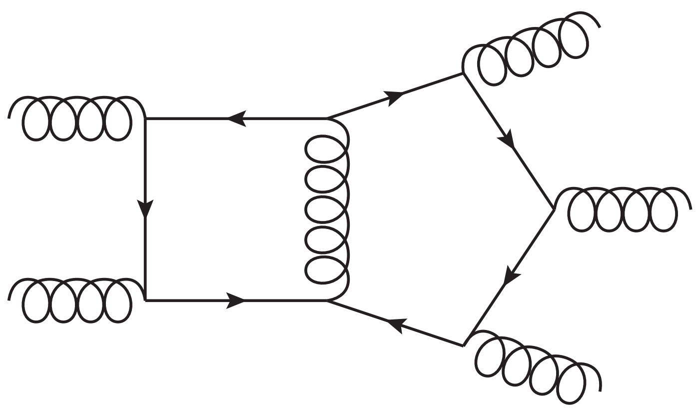


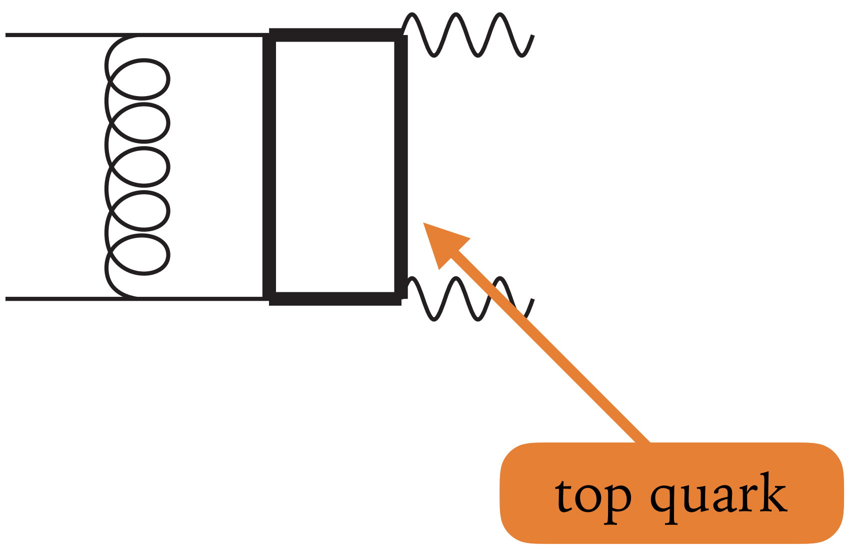
[Slide by G. Heinrich, '17]

WHERE DO WE STAND NOW

There is a lot that we can (could?) do, but we are not quite there yet…

Properly modelling LHC processes with high precision requires ***more external particles*** and ***massive internal states***





What are we fighting against?

Algebraic complexity

Analytical complexity

New mathematical insight needed to tame these processes!

THE SIMPLICITY OF MPLS: INTEGRATING ON THE RIEMANN SPHERE

Most of these processes expressible **in terms of MPLs**

$$G(c_1, c_2, ..., c_n, x) = \int_0^x \frac{dt_1}{t_1 - c_1} G(c_2, ..., c_n, t_1)$$
$$= \int_0^x \frac{dt_1}{t_1 - c_1} \int_0^{t_1} \frac{dt_2}{t_2 - c_2} ... \int_0^{t_{n-1}} \frac{dt_n}{t_n - c_n}$$

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The singularities are generically **complex numbers**!

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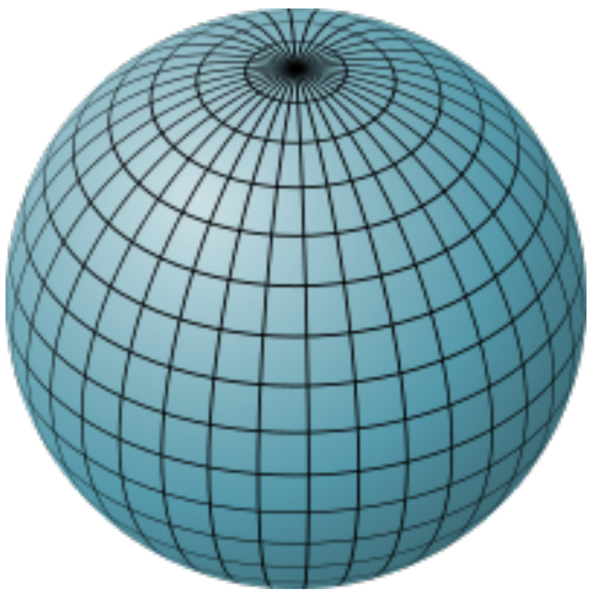

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We integrate **rational functions**

Riemann sphere ~ complex plane through stereographic projection

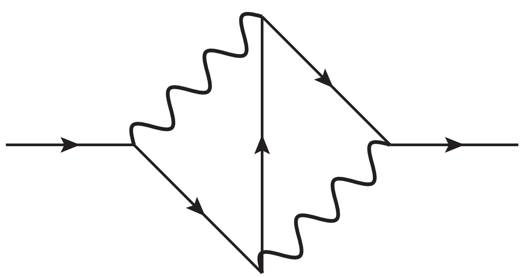
The singularities are generically **complex numbers**!

Rational functions have no ***non-trivial branch-cut structure***, only ***poles in the complex plane***

THE ELLIPTIC WORLD: THE SUNRISE AND HIS FRIENDS

At **two loops**, MPLs with there beautiful properties are not enough.

Electron self-energy in QED @ 2 loops *(computation attempted in 1961 by A. Sabry!)*



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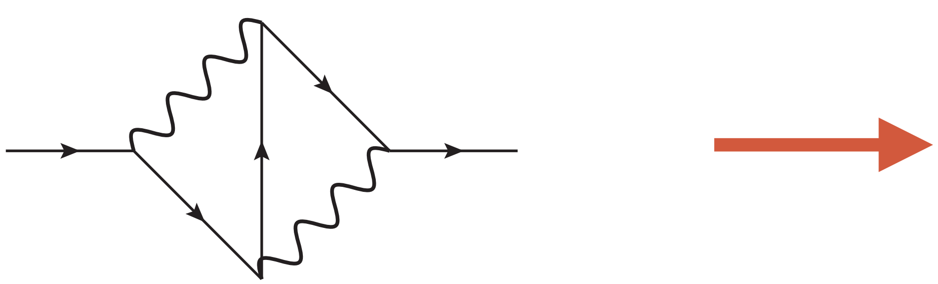
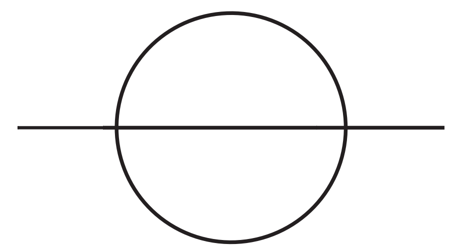
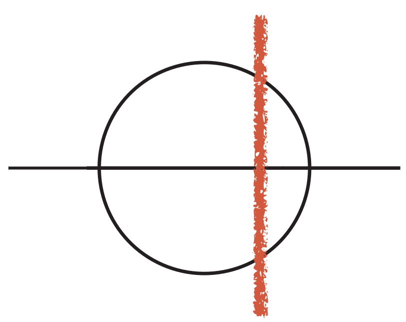




The sunrise integral



$$= \frac{1}{\sqrt{(3m-\sqrt{s})(\sqrt{s}+m)^3}}\,\mathrm{K}\left(\frac{16m^3\sqrt{s}}{(3m-\sqrt{s})(\sqrt{s}+m)^3}\right)$$

Elliptic Integral of the first kind:

$$\mathrm{K}(x) = \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-x\,z^2)}}$$

A NEW GEOMETRY

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These integrals "live" in a new geometry:

Consider the function

$$y(z) = \sqrt{(1-z^2)(k^2-z^2)} \quad \text{with} \quad k^2 = \frac{1}{x} > 1$$

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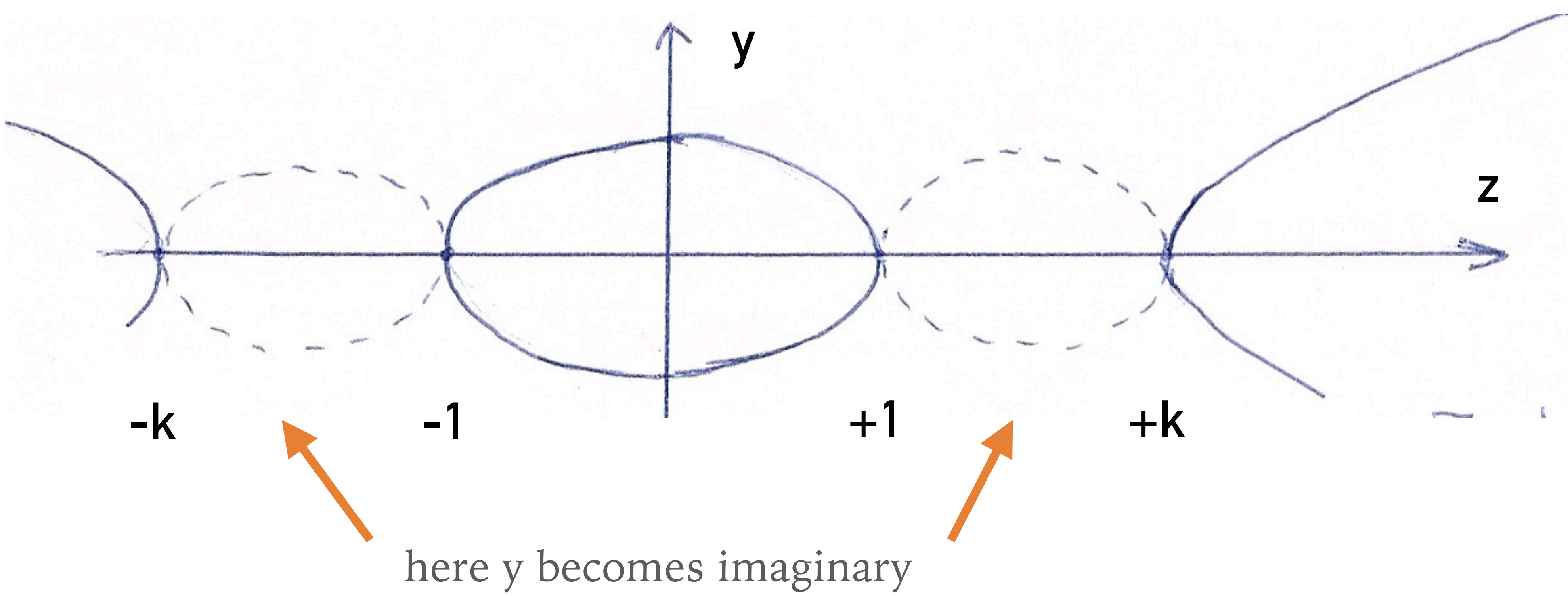
This algebraic equation defines geometrically ***an elliptic curve***



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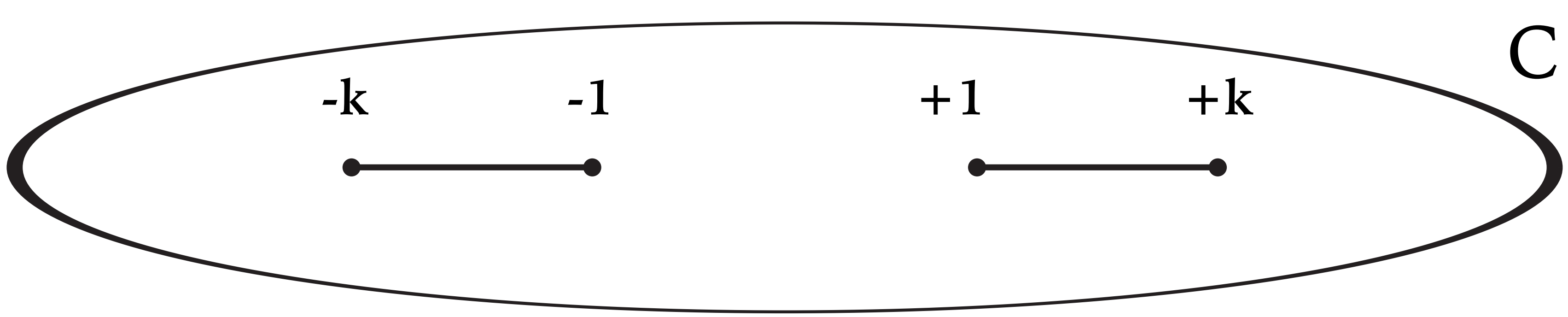
The usual trick is: restrict the function to be valid on its own graph -> **"Riemann Surface"**

$$T = \{(z,y) \in \mathbb{C}^2 \,|\, y^2 = (1-z^2)(z^2-k^2)\}$$

The 4 points **z={-1,+1,-k,+k},** are branching points for y(z) (the argument changes sign). In order to get a continuous branch, we need to cut the complex plane!



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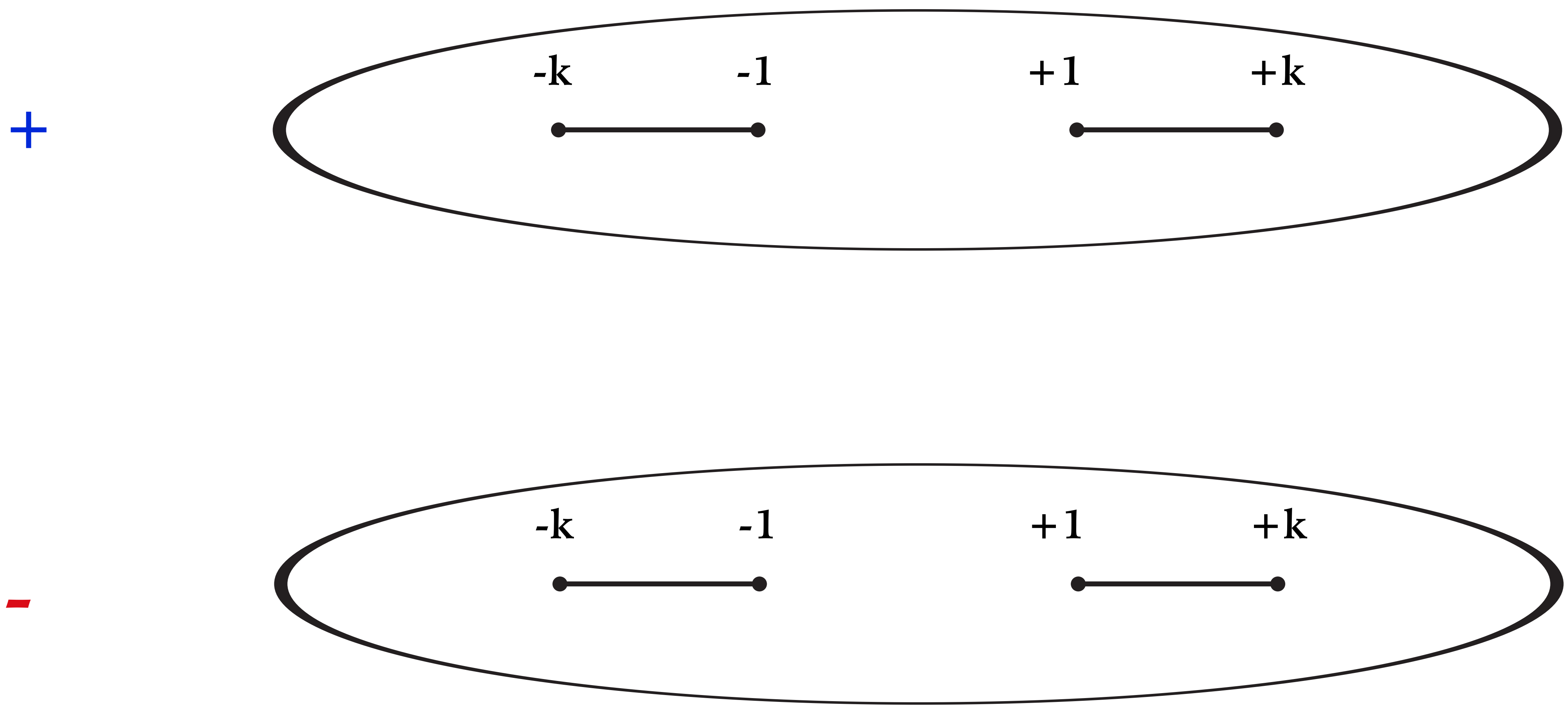
But one copy of the complex plane is not enough. **For every z,** except the branching points, **there are two choices for y** (two signs of the square root!).

Plus ***two signs on each side of the branches!***



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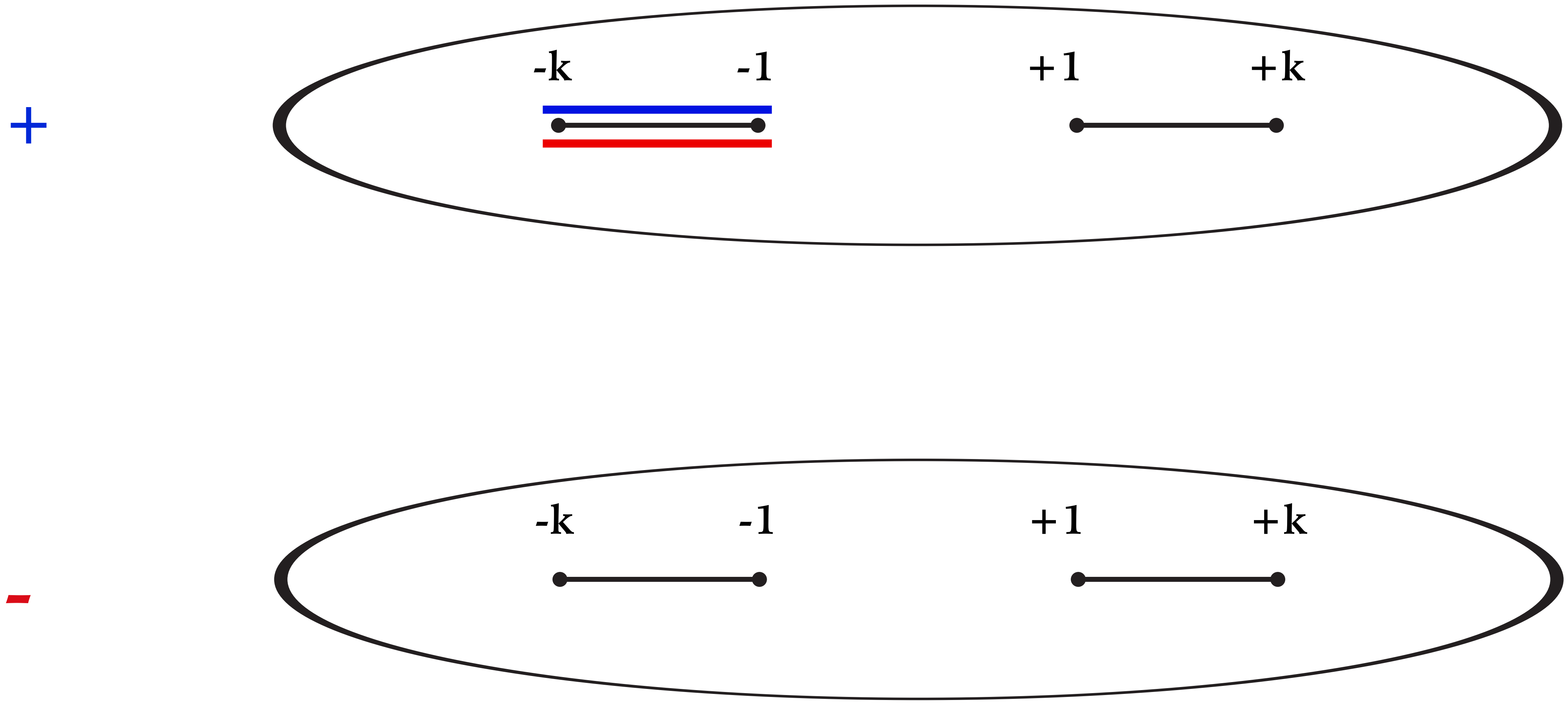

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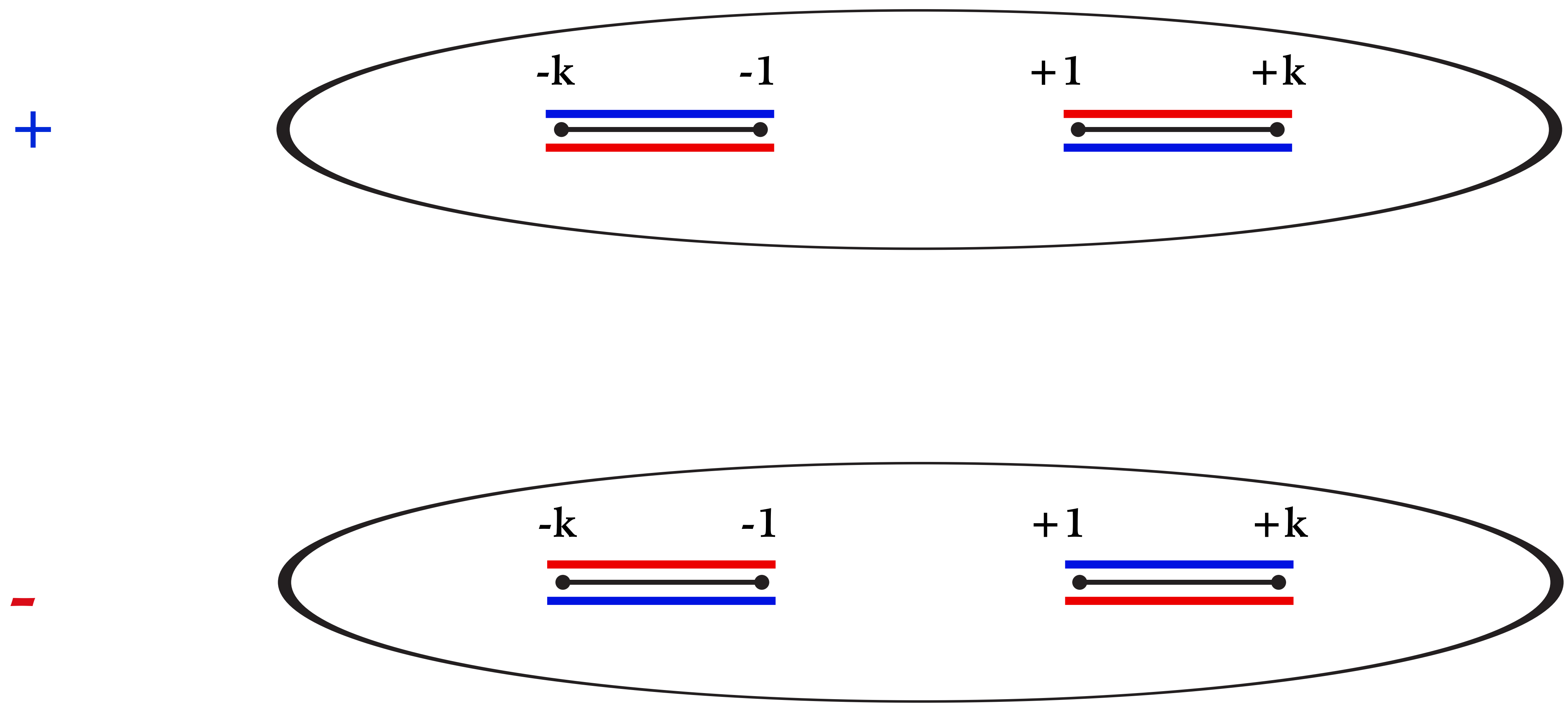

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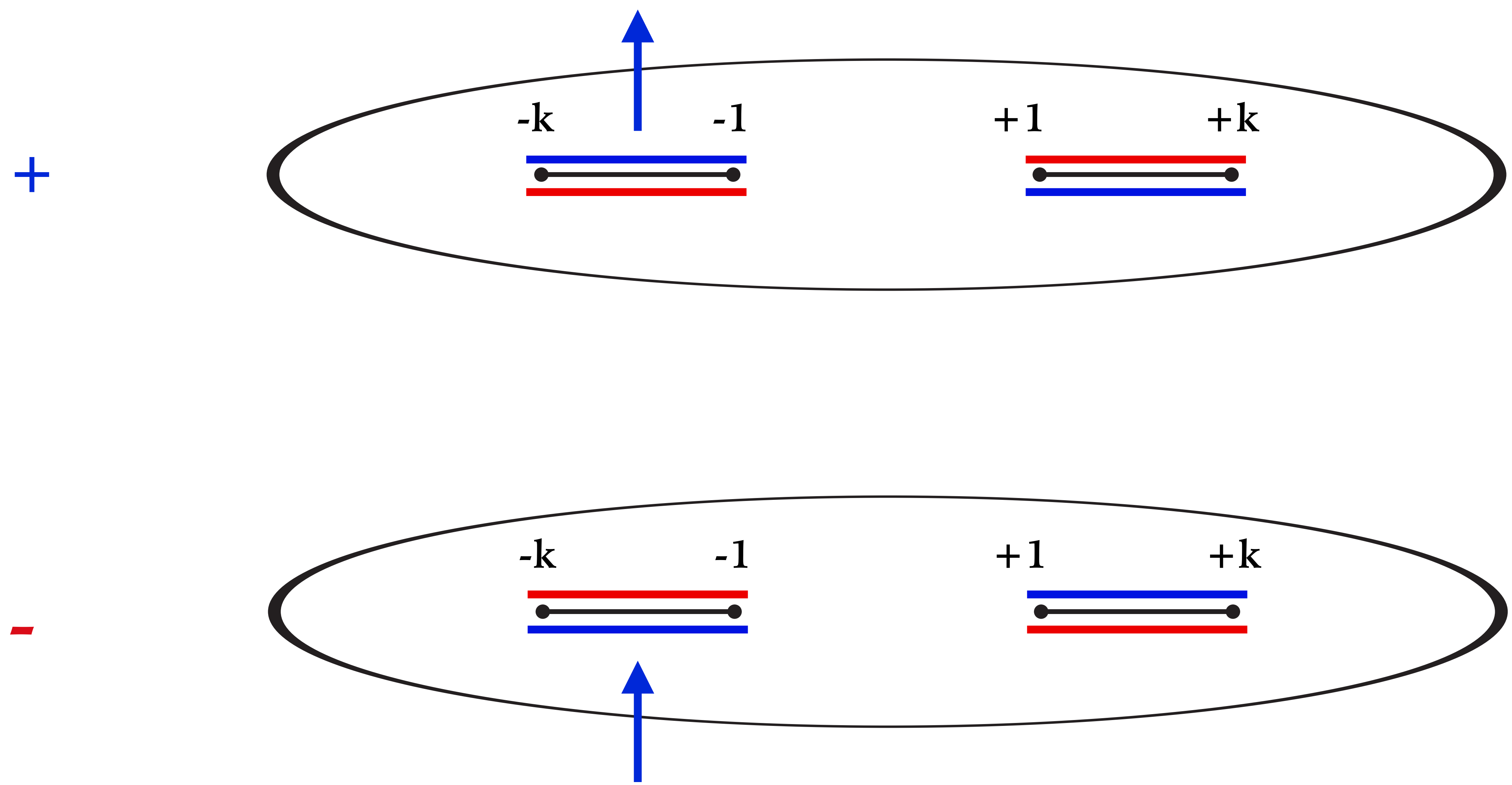

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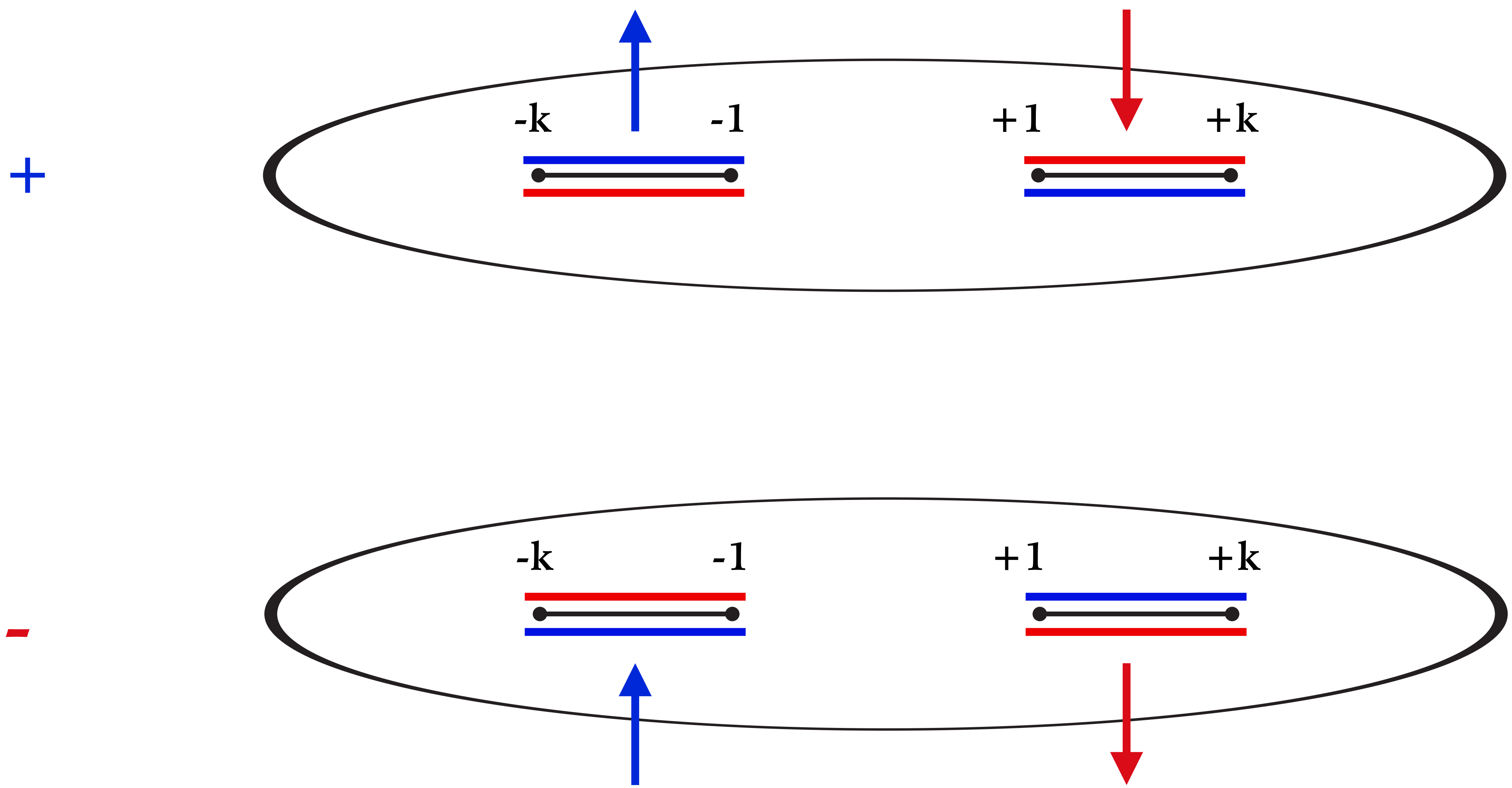

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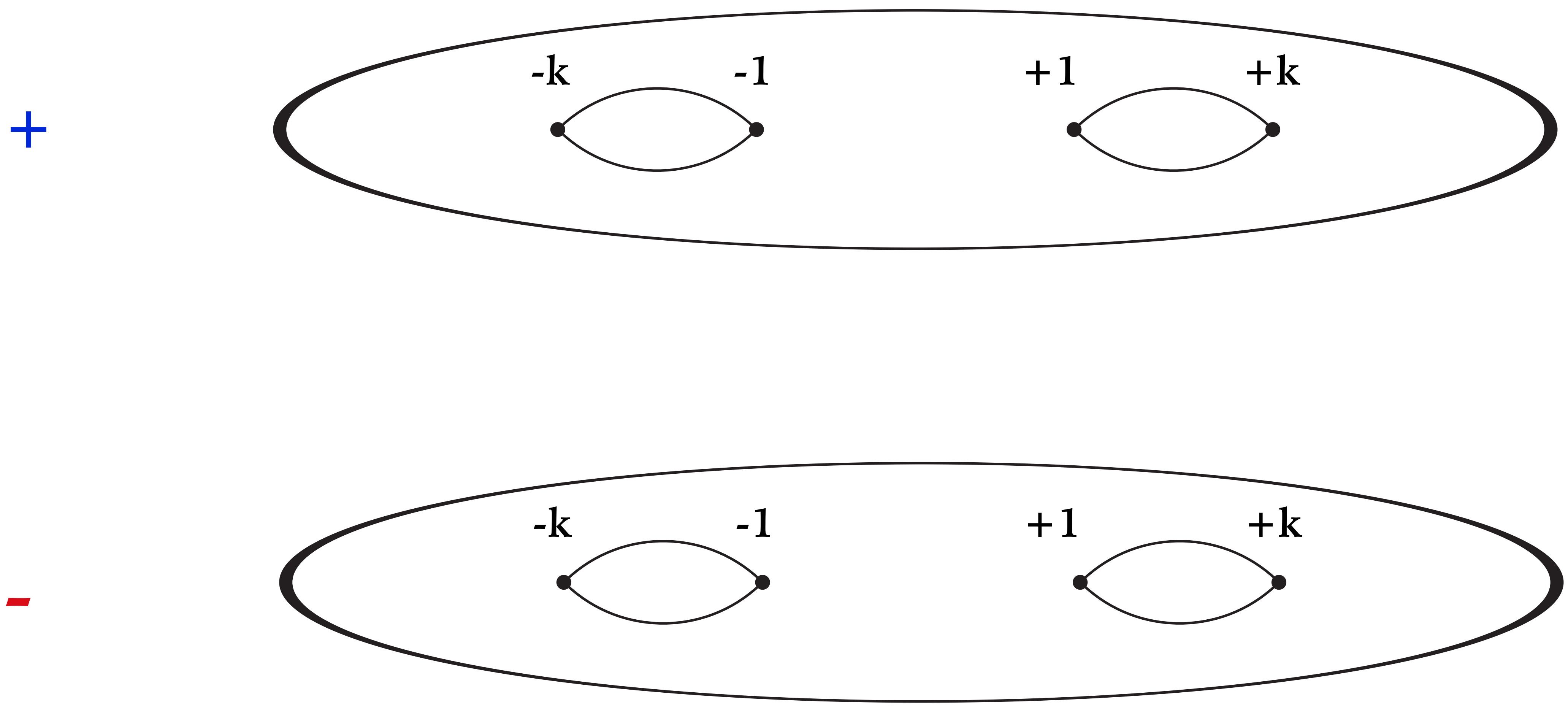
To obtain a ***continuous determination of y***, I need to glue together the two planes.

Flip one plane along x axis, and ***glue + with +, and - with -***



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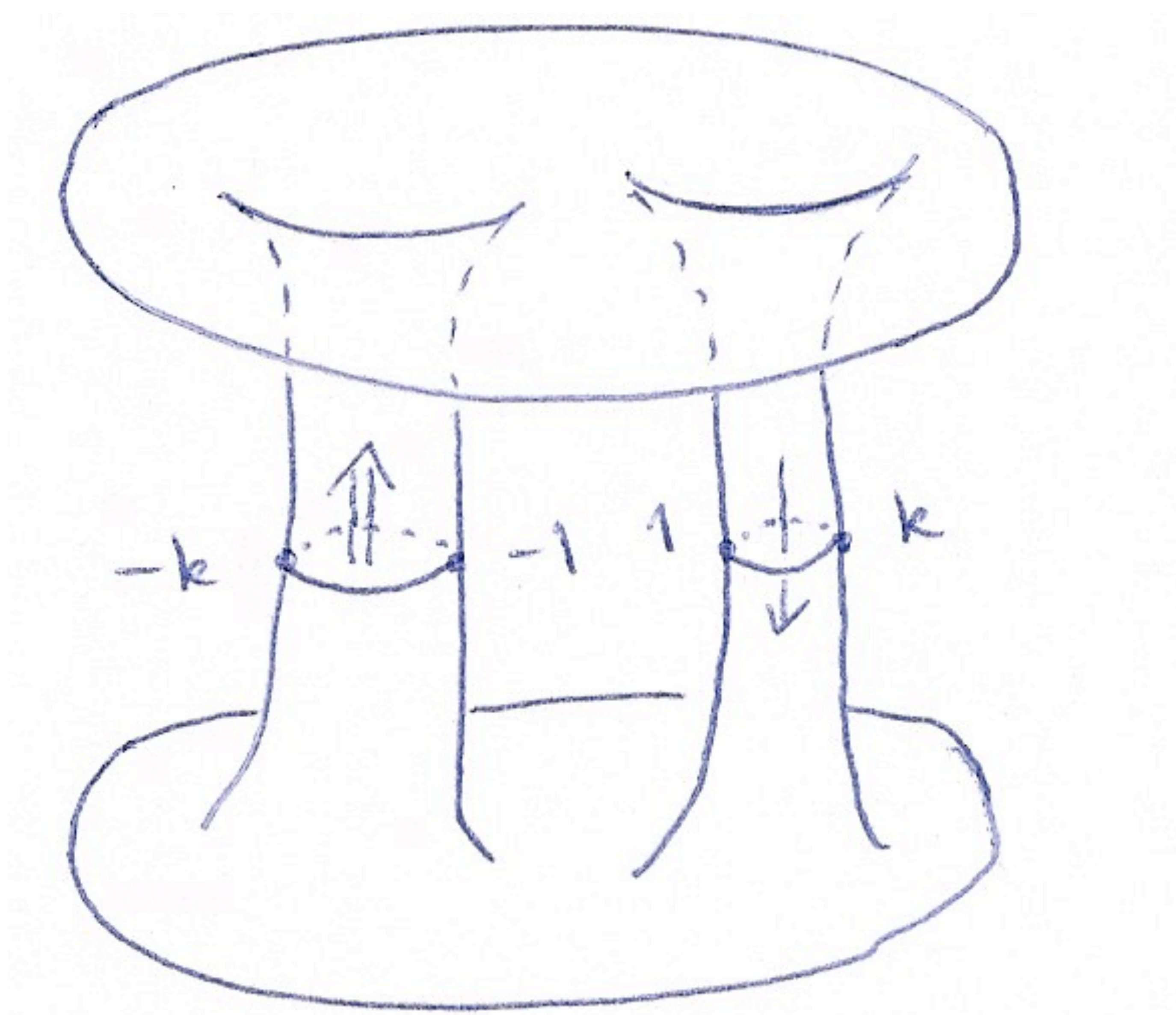

[Drawings by C. Teleman, Riemann Surfaces]

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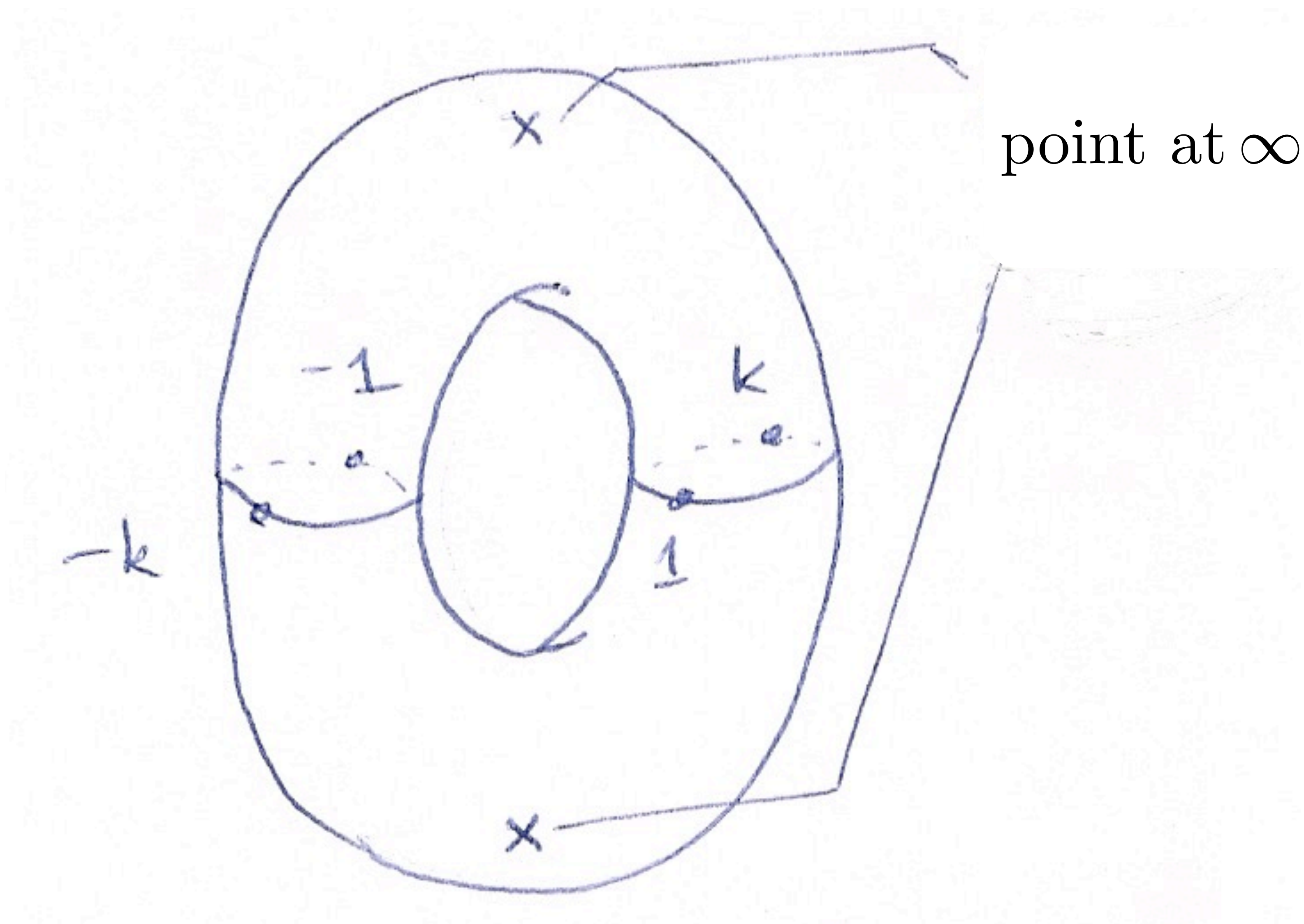

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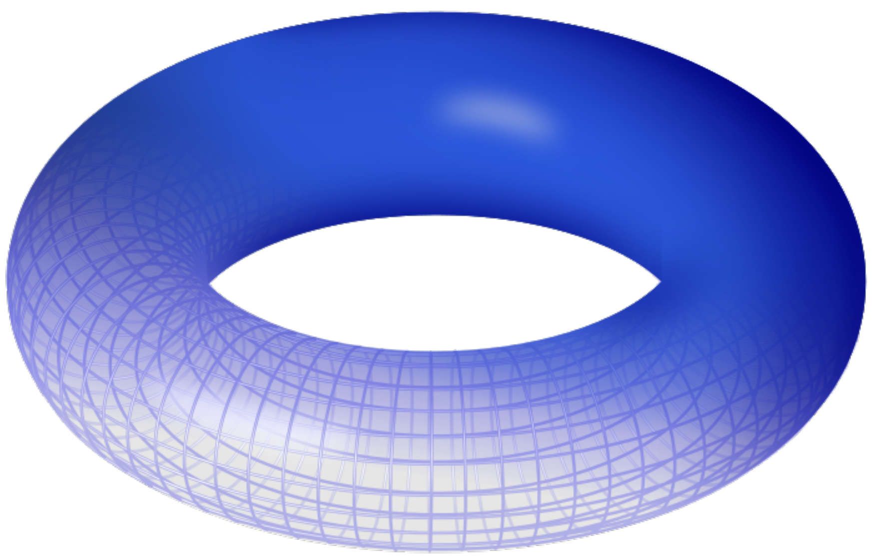
In conclusion, the **Riemann surface** associated to the algebraic equation that defines an **elliptic curve** is a **complex Torus!**



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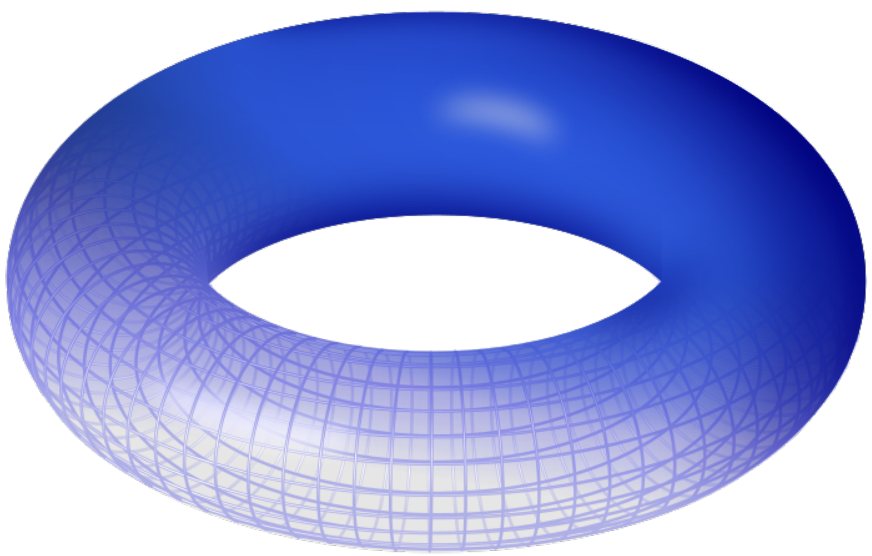

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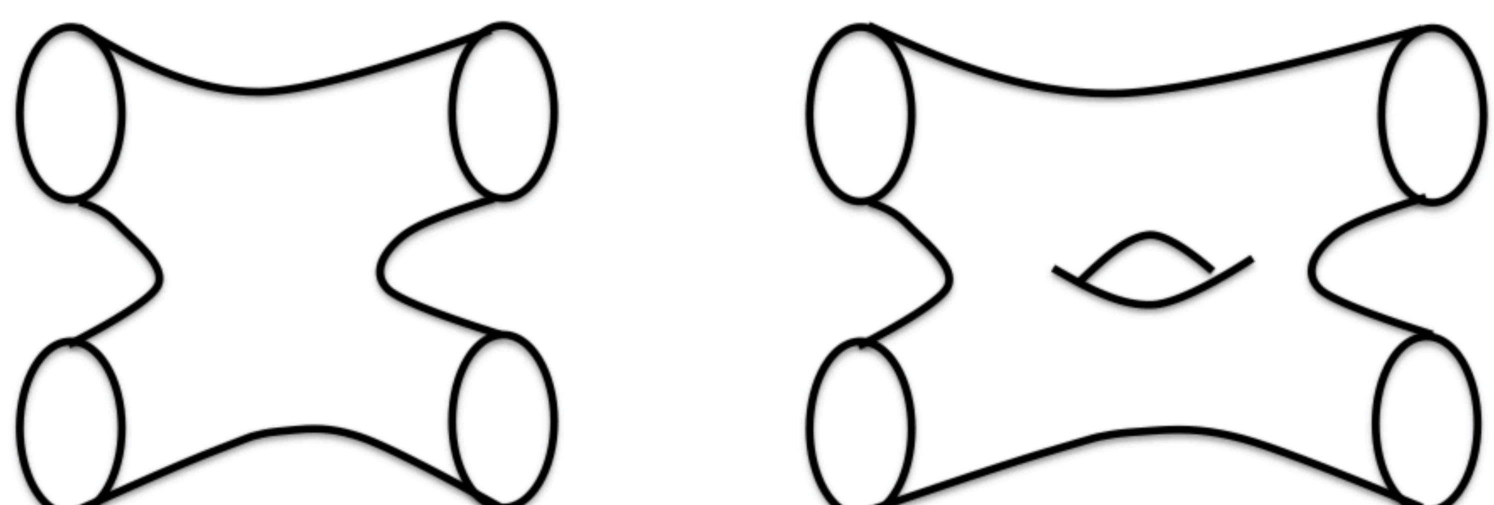
Easy to see why the same structures appear in one-loop string theory amplitudes!



ELLIPTIC MULTIPLE POLYLOGARITHMS

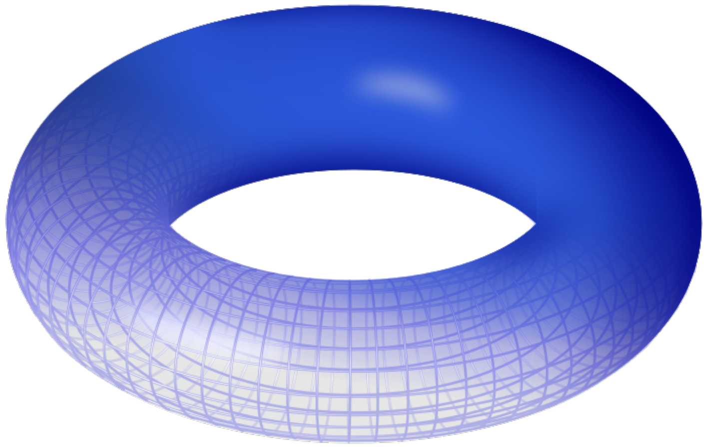
Some definitions. Take a completely general elliptic curve:

$$y^2 = (x-a_1)(x-a_2)(x-a_3)(x-a_4)$$

We define the two periods as

$$\omega_1 = 2c_4 \int_{a_2}^{a_3} \frac{dx}{y} = 2\mathrm{K}(\lambda)\,, \qquad \omega_2 = 2c_4 \int_{a_1}^{a_2} \frac{dx}{y} = 2\,i\,\mathrm{K}(1-\lambda)$$

$$\lambda = \frac{(a_1-a_4)(a_2-a_3)}{(a_1-a_3)(a_2-a_4)}\,, \qquad c_4 = \frac{1}{2}\sqrt{(a_1-a_3)(a_2-a_4)}$$



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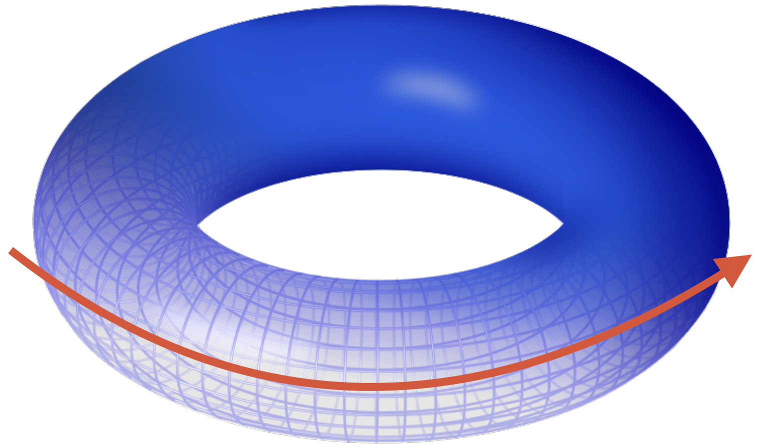

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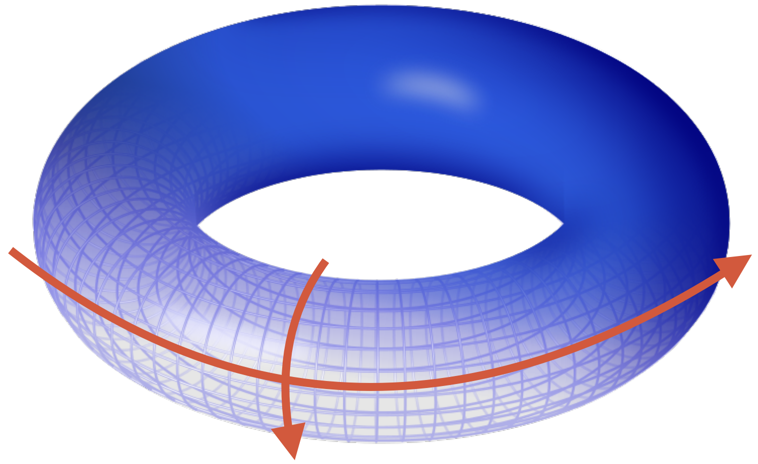
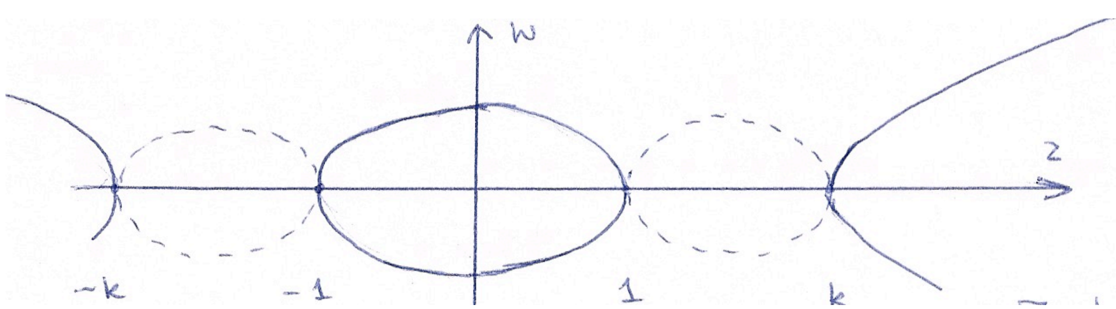


ELLIPTIC MULTIPLE POLYLOGARITHMS

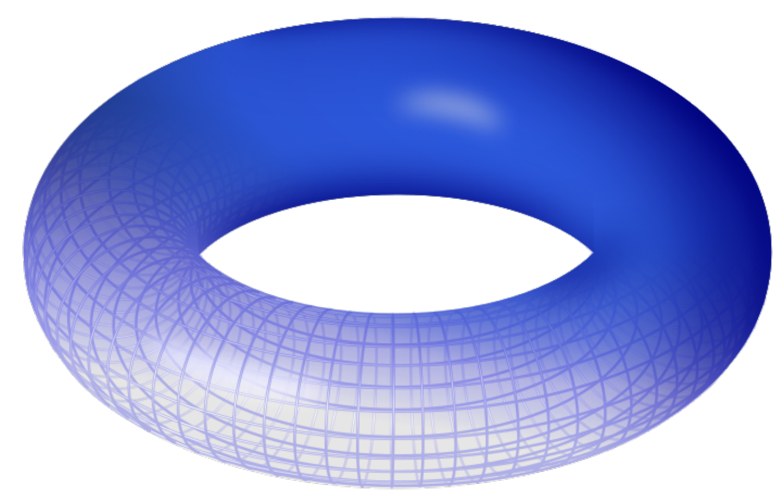
Dual description of the same problem

Elliptic curve as algebraic equation

$$y^2 = (x - a_1)(x - a_2)(x - a_3)(x - a_4)$$



Genus one complex surface - Torus



Move between the two using Abel's Map

$$z_x = \frac{c_4}{\omega_1} \int_{a_1}^{x} \frac{dt}{y(t)}$$

ELLIPTIC MULTIPLE POLYLOGARITHMS

Iterated integrals on the Torus have been defined and studied by mathematicians

[Brown, Levin '11]

The elliptic curve representation is easier to relate directly to Feynman diagrams!
So can construct elliptic polylogarithms on the elliptic curve?

What are rational functions on the elliptic curve?

A rational function on the elliptic curve is a function $R(x, y)$ subject to the constraint $y = \sqrt{P(x)}$

$$R(x, y) = \frac{p_1(x) + p_2(x)\, y}{q_1(x) + q_2(x)\, y} = \frac{p_1(x) + p_2(x)\sqrt{P(x)}}{q_1(x) + q_2(x)\sqrt{P(x)}} = R_1(x) + \frac{1}{\sqrt{P(x)}} R_2(x)$$

ELLIPTIC MULTIPLE POLYLOGARITHMS

So we need to study iterated integrals of this form

$$\int dx \left(R_1(x) + \frac{1}{\sqrt{P(x)}} R_2(x) \right) = \ ?$$

ELLIPTIC MULTIPLE POLYLOGARITHMS

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Integrals of the kind:

$\int \frac{dx}{(x-c_i)^k}$	from $R_1(x)$
$\int \frac{dx}{y} x^k \, , \quad \int \frac{dx}{y(x-c_i)^k} \, ,$	from $\frac{1}{y} R_2(x)$

Partial fractioning reduces everything to

$$\int \frac{dx}{x-c}\, , \quad \int \frac{dx}{y}\, , \quad \int \frac{x\, dx}{y}\, , \quad \int \frac{x^2\, dx}{y}\, , \quad \int \frac{dx}{y(x-c)}$$

ELLIPTIC MULTIPLE POLYLOGARITHMS

Still something is not optimal. MPLs defined as iterated integrals over kernels with simple poles —> logarithmic singularities in scattering amplitudes!

$$\int \frac{dx}{x-c}\,, \quad \int \frac{dx}{y}\,, \quad \int \frac{x\,dx}{y}\,, \quad \int \frac{x^2\,dx}{y}\,, \quad \int \frac{dx}{y(x-c)}$$

This guy here has a double pole at infinity!!

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This guy here has a double pole at infinity!!

Only way to get a full set of kernels with simple poles is to introduce a transcendental kernel! **Choose its primitive.**

$$Z_4(x) \sim \int^x \frac{x^2\,dx}{y}$$

ELLIPTIC MULTIPLE POLYLOGARITHMS

Fundamental differences with MPLs:

- Impossible to find basis of kernels which are **algebraic** and only with **simple poles**.

- We need **infinite tower** of integration kernels to span the whole space!

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$$\begin{aligned}
\psi_0(0,x) &= \frac{c_4}{y}\,, \\
\psi_1(c,x) &= \frac{1}{x-c}\,, \qquad \psi_{-1}(c,x) = \frac{y_c}{y(x-c)}\,, \\
\psi_1(\infty,x) &= \frac{c_4}{y}\,Z_4(x)\,, \qquad \psi_{-1}(\infty,x) = \frac{x}{y}\,, \\
\psi_{-n}(\infty,x) &= \frac{x}{y}\,Z_4^{(n-1)}(x) - \frac{\delta_{n2}}{c_4}\,, \\
\psi_n(c,x) &= \frac{1}{x-c}\,Z_4^{(n-1)}(x) - \delta_{n2}\,\Phi_4(x)\,, \\
\psi_n(\infty,x) &= \frac{c_4}{y}\,Z_4^{(n)}(x)\,, \qquad \psi_{-n}(c,x) = \frac{y_c}{y(x-c)}\,Z_4^{(n-1)}(x)\,,
\end{aligned}$$

$$\mathrm{E}_4\left(\begin{smallmatrix} n_1 & \dots & n_k \\ c_1 & \dots & c_k \end{smallmatrix};x\right) = \int_0^x dt\,\psi_{n_1}(c_1,t)\,\mathrm{E}_4\left(\begin{smallmatrix} n_2 & \dots & n_k \\ c_2 & \dots & c_k \end{smallmatrix};t\right)$$

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- We need **infinite tower** of integration kernels to span the whole space!

$$\psi_0(0,x) = \frac{c_4}{y}\,,$$

$$\psi_1(c,x) = \frac{1}{x-c}\,, \qquad \psi_{-1}(c,x) = \frac{y_c}{y(x-c)}\,,$$

$$\psi_1(\infty,x) = \frac{c_4}{y}\,Z_4(x)\,, \qquad \psi_{-1}(\infty,x) = \frac{x}{y}\,,$$

$$\psi_{-n}(\infty,x) = \frac{x}{y}\,Z_4^{(n-1)}(x) - \frac{\delta_{n2}}{c_4}\,,$$

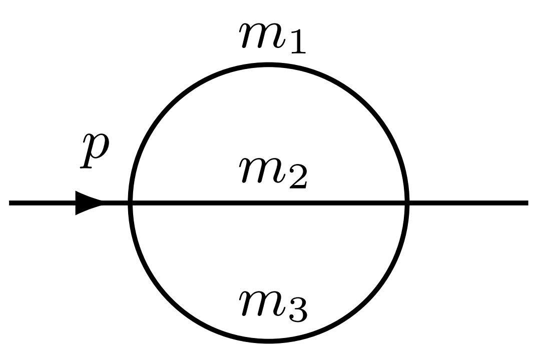
$$\psi_n(c,x) = \frac{1}{x-c}\,Z_4^{(n-1)}(x) - \delta_{n2}\,\Phi_4(x)\,,$$

$$\psi_n(\infty,x) = \frac{c_4}{y}\,Z_4^{(n)}(x)\,, \qquad \psi_{-n}(c,x) = \frac{y_c}{y(x-c)}\,Z_4^{(n-1)}(x)\,,$$

$$\mathrm{E}_4\left(\begin{smallmatrix} n_1 & \dots & n_k \\ c_1 & \dots & c_k \end{smallmatrix}; x\right) = \int_0^x dt\,\psi_{n_1}(c_1,t)\,\mathrm{E}_4\left(\begin{smallmatrix} n_2 & \dots & n_k \\ c_2 & \dots & c_k \end{smallmatrix}; t\right)$$

ARE THEY REALLY USEFUL?

The Sunrise graph



$p^2 = -S$

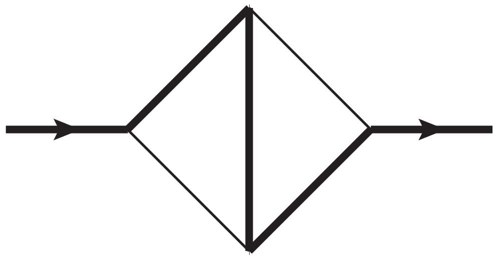
In the **equal-mass case**, particularly compact expression

$$S_{111}(S, m^2)\Big|_{\epsilon^0} = \frac{1}{(m^2+S)c_4}\Bigg[\frac{1}{c_4}\,\mathrm{E}_4\left(\begin{smallmatrix}0 & 0\\ 0 & 0\end{smallmatrix};1\right) - 2\,\mathrm{E}_4\left(\begin{smallmatrix}0 & -1\\ 0 & \infty\end{smallmatrix};1\right) - \mathrm{E}_4\left(\begin{smallmatrix}0 & -1\\ 0 & 0\end{smallmatrix};1\right)$$
$$- \mathrm{E}_4\left(\begin{smallmatrix}0 & -1\\ 0 & 1\end{smallmatrix};1\right) - \mathrm{E}_4\left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix};1\right)\Bigg].$$

Different mass case can also be expressed in terms of the same functions

ARE THEY REALLY USEFUL?

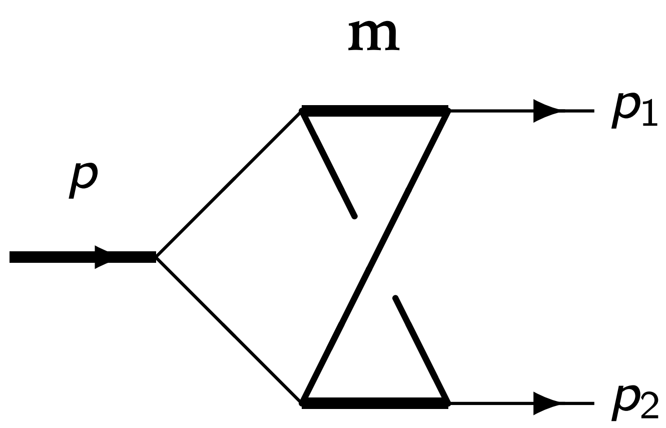
A first generalisation: the Kite integral

$K_{11111}(p^2, m^2) =$  $\qquad z = \frac{p^2}{m^2}$

$$= \frac{1}{z}\Big[2\pi^2 G(0,z) - 2\pi^2 G(1,z) + 3G(0,0,0,z) - 6G(0,1,0,z) - 24\zeta(3)$$
$$+12G(0,1,1,z) - 3G(1,0,0,z) - 6G(1,0,1,z) + 6G(1,1,0,z) + \ldots]$$
$$+ \frac{1+z}{(a_1-a_3)^2(1-z)z}\left[\mathrm{E}_4\left(\begin{smallmatrix}0&0&1\\0&0&1\end{smallmatrix};1\right) + \mathrm{E}_4\left(\begin{smallmatrix}0&1&0\\0&1&0\end{smallmatrix};1\right) - \mathrm{E}_4\left(\begin{smallmatrix}1&0&0\\0&0&0\end{smallmatrix};1\right)\right]$$
$$+ \frac{1+z}{(a_1-a_3)(1-z)z}\Big[\mathrm{E}_4\left(\begin{smallmatrix}0&-1&1\\0&0&1\end{smallmatrix};1\right) + \mathrm{E}_4\left(\begin{smallmatrix}0&-1&1\\0&1&1\end{smallmatrix};1\right)$$
$$+2\mathrm{E}_4\left(\begin{smallmatrix}0&-1&1\\0&\infty&1\end{smallmatrix};1\right) + \mathrm{E}_4\left(\begin{smallmatrix}0&1&1\\0&0&1\end{smallmatrix};1\right) + \ldots\Big] + 79 \text{ more } \mathrm{E}_4 s$$

ARE THEY REALLY USEFUL?

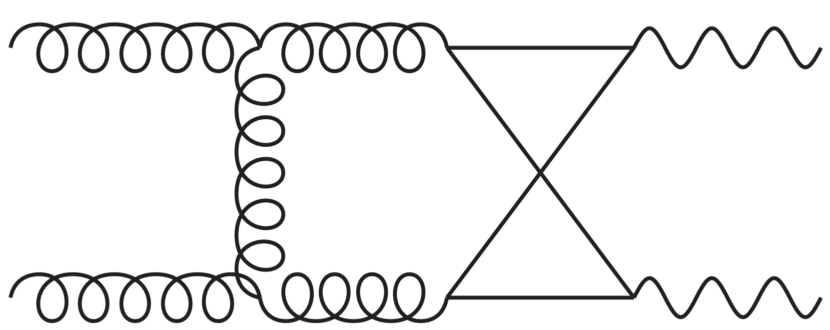
A three point function



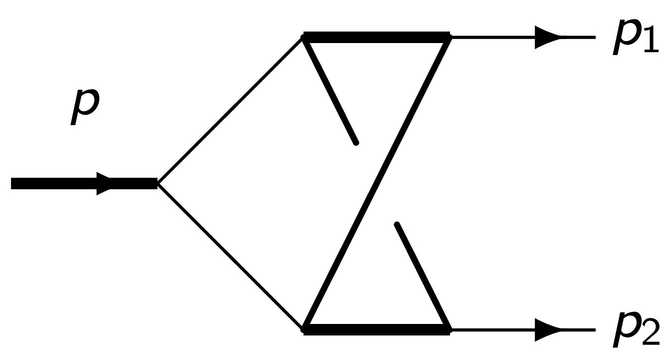
$$a = \frac{p^2}{m^2}$$

$$r_{-\pm} = \frac{1}{2}(1 - \sqrt{1 \pm 4a}), \quad r_{+\pm} = 1 - r_{-\pm} \ .$$

Relevant for ttb, gg, 2jet, HH, Hj production



ARE THEY REALLY USEFUL?



$$a = \frac{p^2}{m^2}$$

$$r_{-\pm} = \frac{1}{2}(1 - \sqrt{1 \pm 4a}), \quad r_{+\pm} = 1 - r_{-\pm}\ .$$

$$\begin{aligned}
I = \frac{2a^2}{c_4^2}\Bigg[& 5\mathrm{E}_4\left(\begin{smallmatrix}0&0&1&1\\0&0&0&r_{-+}\end{smallmatrix};1\right) + 5\mathrm{E}_4\left(\begin{smallmatrix}0&0&1&1\\0&0&0&r_{++}\end{smallmatrix};1\right) + 5\mathrm{E}_4\left(\begin{smallmatrix}0&0&1&1\\0&0&1&r_{-+}\end{smallmatrix};1\right) + 5\mathrm{E}_4\left(\begin{smallmatrix}0&0&1&1\\0&0&1&r_{++}\end{smallmatrix};1\right) \\
& - 3\mathrm{E}_4\left(\begin{smallmatrix}0&0&1&1\\0&0&r_{--}&0\end{smallmatrix};1\right) - 3\mathrm{E}_4\left(\begin{smallmatrix}0&0&1&1\\0&0&r_{--}&1\end{smallmatrix};1\right) - 3\mathrm{E}_4\left(\begin{smallmatrix}0&0&1&1\\0&0&r_{+-}&0\end{smallmatrix};1\right) - 3\mathrm{E}_4\left(\begin{smallmatrix}0&0&1&1\\0&0&r_{+-}&1\end{smallmatrix};1\right) \\
& + 3\log a\,\left(\mathrm{E}_4\left(\begin{smallmatrix}0&0&1\\0&0&r_{--}\end{smallmatrix};1\right) + \mathrm{E}_4\left(\begin{smallmatrix}0&0&1\\0&0&r_{+-}\end{smallmatrix};1\right)\right)\Bigg] \\
- \frac{4a^2}{c_4}\Bigg[& 5\mathrm{E}_4\left(\begin{smallmatrix}0&-1&1&1\\0&\infty&0&r_{-+}\end{smallmatrix};1\right) + 5\mathrm{E}_4\left(\begin{smallmatrix}0&-1&1&1\\0&\infty&0&r_{++}\end{smallmatrix};1\right) + 5\mathrm{E}_4\left(\begin{smallmatrix}0&-1&1&1\\0&\infty&1&r_{-+}\end{smallmatrix};1\right) + 5\mathrm{E}_4\left(\begin{smallmatrix}0&-1&1&1\\0&\infty&1&r_{++}\end{smallmatrix};1\right) \\
& - 3\mathrm{E}_4\left(\begin{smallmatrix}0&-1&1&1\\0&\infty&r_{--}&0\end{smallmatrix};1\right) - 3\mathrm{E}_4\left(\begin{smallmatrix}0&-1&1&1\\0&\infty&r_{--}&1\end{smallmatrix};1\right) - 3\mathrm{E}_4\left(\begin{smallmatrix}0&-1&1&1\\0&\infty&r_{+-}&0\end{smallmatrix};1\right) - 3\mathrm{E}_4\left(\begin{smallmatrix}0&-1&1&1\\0&\infty&r_{+-}&1\end{smallmatrix};1\right) \\
& + 3\log a\,\left(\mathrm{E}_4\left(\begin{smallmatrix}0&-1&1\\0&\infty&r_{--}\end{smallmatrix};1\right) + \mathrm{E}_4\left(\begin{smallmatrix}0&-1&1\\0&\infty&r_{+-}\end{smallmatrix};1\right)\right)\Bigg]\,,
\end{aligned}$$

ARE THEY REALLY USEFUL?

Soon more examples to come, in particular also four point functions!

The properties of these functions are currently under study, many recent developments:

- Study of the algebra generated by this function following [Brown '14]
- We developed a set of tools which works very well for direct integration of FIs *(like before 1999 for MPLs and FIs…)* [Gehrmann, Remiddi '00]
- Connection to the differential equation method is non-trivial. Understood for the special case of ***Iterated integrals over modular forms***
- **Today** "concept of **purity**". Rotation in the basis of E4 functions to obtain a class of functions of "pure transcendental weight"

CONCLUSIONS

- We have built a class of functions which span the space generated by repeated integrations of rational functions on an elliptic curve
- The functions span the same space of the eMPLs defined by mathematicians and string theorists on the Torus. We have highlighted the connection between the two formalisms
- We have showed how many physically relevant FIs can be expressed in terms of these functions
- We can associate to them a concept of transcendental weight and we recover an idea of "purity" associated to some classes of FIs

Still to do:

- Understand the connection with the differential equations method
- Study general algorithm for their numerical evaluation
- Finally, use them to do some real physics!

Stay tuned!

THANK YOU!

BACK UP

PURE ELLIPTIC MULTIPLE POLYLOGARITHMS

$$\mathcal{E}_4\left(\begin{smallmatrix} n_1 & \dots & n_k \\ c_1 & \dots & c_k \end{smallmatrix}; x, \vec{a}\right) = \int_0^x dt\, \Psi_{n_1}(c_1, t, \vec{a})\, \mathcal{E}_4\left(\begin{smallmatrix} n_2 & \dots & n_k \\ c_2 & \dots & c_k \end{smallmatrix}; t, \vec{a}\right)$$

$$\Psi_1(c, x, \vec{a}) = \frac{1}{x - c} \qquad \Psi_0(0, x, \vec{a}) = \frac{1}{\omega_1}\, \psi_0(0, x, \vec{a}) = \frac{c_4}{\omega_1\, y}$$

$$\Psi_{-1}(c, x, \vec{a}) = \psi_{-1}(c, x, \vec{a}) + Z_4(c, \vec{a})\, \psi_0(0, x, \vec{a}) = \frac{y_c}{y(x - c)} + Z_4(c, \vec{a})\, \frac{c_4}{y}\,,$$

$$\Psi_1(\infty, x, \vec{a}) = -\psi_1(\infty, x, \vec{a}) = -Z_4(x, \vec{a})\, \frac{c_4}{y}\,,$$

A PURE VERSION OF THE SUNRISE GRAPH

$$S_1(p^2,m^2) = -\frac{\omega_1}{(p^2+m^2)\,c_4}\,T_1(p^2,m^2)\,,$$

$$\begin{aligned}
T_1^{(0)} &= 2\mathcal{E}_4\left(\begin{smallmatrix}0 & -1\\ 0 & \infty\end{smallmatrix};1,\vec{a}\right) + \mathcal{E}_4\left(\begin{smallmatrix}0 & -1\\ 0 & 0\end{smallmatrix};1,\vec{a}\right) + \mathcal{E}_4\left(\begin{smallmatrix}0 & -1\\ 0 & 1\end{smallmatrix};1,\vec{a}\right)\;,\\
T_1^{(1)} &= -4\mathcal{E}_4\left(\begin{smallmatrix}0 & 1 & -1\\ 0 & a_3 & \infty\end{smallmatrix};1,\vec{a}\right) - 4\mathcal{E}_4\left(\begin{smallmatrix}0 & 1 & -1\\ 0 & a_1 & \infty\end{smallmatrix};1,\vec{a}\right) - 4\mathcal{E}_4\left(\begin{smallmatrix}0 & 1 & -1\\ 0 & a_4 & \infty\end{smallmatrix};1,\vec{a}\right) - 4\mathcal{E}_4\left(\begin{smallmatrix}0 & 1 & -1\\ 0 & a_2 & \infty\end{smallmatrix};1,\vec{a}\right)\\
&\quad - 2\mathcal{E}_4\left(\begin{smallmatrix}0 & 1 & -1\\ 0 & a_3 & 0\end{smallmatrix};1,\vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix}0 & 1 & -1\\ 0 & a_3 & 1\end{smallmatrix};1,\vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix}0 & 1 & -1\\ 0 & a_1 & 0\end{smallmatrix};1,\vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix}0 & 1 & -1\\ 0 & a_1 & 1\end{smallmatrix};1,\vec{a}\right)\\
&\quad - 2\mathcal{E}_4\left(\begin{smallmatrix}0 & 1 & -1\\ 0 & a_4 & 0\end{smallmatrix};1,\vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix}0 & 1 & -1\\ 0 & a_4 & 1\end{smallmatrix};1,\vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix}0 & 1 & -1\\ 0 & a_2 & 0\end{smallmatrix};1,\vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix}0 & 1 & -1\\ 0 & a_2 & 1\end{smallmatrix};1,\vec{a}\right)\\
&\quad + 2\mathcal{E}_4\left(\begin{smallmatrix}0 & -1 & 1\\ 0 & \infty & 0\end{smallmatrix};1,\vec{a}\right) + 2\mathcal{E}_4\left(\begin{smallmatrix}0 & -1 & 1\\ 0 & \infty & 1\end{smallmatrix};1,\vec{a}\right) + 6\mathcal{E}_4\left(\begin{smallmatrix}0 & 1 & -1\\ 0 & 0 & \infty\end{smallmatrix};1,\vec{a}\right) + 6\mathcal{E}_4\left(\begin{smallmatrix}0 & 1 & -1\\ 0 & 1 & \infty\end{smallmatrix};1,\vec{a}\right)\\
&\quad - 2\mathcal{E}_4\left(\begin{smallmatrix}0 & -1 & 1\\ 0 & 0 & 0\end{smallmatrix};1,\vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix}0 & -1 & 1\\ 0 & 0 & 1\end{smallmatrix};1,\vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix}0 & -1 & 1\\ 0 & 1 & 0\end{smallmatrix};1,\vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix}0 & -1 & 1\\ 0 & 1 & 1\end{smallmatrix};1,\vec{a}\right)\\
&\quad + 6i\pi\mathcal{E}_4\left(\begin{smallmatrix}0 & 0 & 1\\ 0 & 0 & 0\end{smallmatrix};1,\vec{a}\right) + 6i\pi\mathcal{E}_4\left(\begin{smallmatrix}0 & 0 & 1\\ 0 & 0 & 1\end{smallmatrix};1,\vec{a}\right) + 3\mathcal{E}_4\left(\begin{smallmatrix}0 & 1 & -1\\ 0 & 0 & 0\end{smallmatrix};1,\vec{a}\right) + 3\mathcal{E}_4\left(\begin{smallmatrix}0 & 1 & -1\\ 0 & 0 & 1\end{smallmatrix};1,\vec{a}\right)\\
&\quad + 3\mathcal{E}_4\left(\begin{smallmatrix}0 & 1 & -1\\ 0 & 1 & 0\end{smallmatrix};1,\vec{a}\right) + 3\mathcal{E}_4\left(\begin{smallmatrix}0 & 1 & -1\\ 0 & 1 & 1\end{smallmatrix};1,\vec{a}\right) + \zeta_2\,\mathcal{E}_4\left(\begin{smallmatrix}0\\ 0\end{smallmatrix};1,\vec{a}\right)\;.
\end{aligned}$$

A PURE VERSION OF THE TRIANGLE

$$I = \frac{32\omega_1}{q^2(1+\sqrt{1-16a})}\left[T_0(a) + 3T_-(a) + 5T_+(a) + \mathcal{O}(\epsilon)\right]$$

$$\begin{aligned}
T_a \;=\; & -\mathcal{E}_4\big(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 0 & 0 \end{smallmatrix};1\big) - \mathcal{E}_4\big(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 0 & 1 \end{smallmatrix};1\big) - \mathcal{E}_4\big(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 1 & 0 \end{smallmatrix};1\big) - \mathcal{E}_4\big(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 1 & 1 \end{smallmatrix};1\big) + \\
& \log(a)\big[\mathcal{E}_4\big(\begin{smallmatrix} 0 & -1 & 1 \\ 0 & \infty & 0 \end{smallmatrix};1\big) + \mathcal{E}_4\big(\begin{smallmatrix} 0 & -1 & 1 \\ 0 & \infty & 1 \end{smallmatrix};1\big)\big] + \frac{1}{2}\mathcal{E}_4\big(\begin{smallmatrix} 0 & -1 \\ 0 & \infty \end{smallmatrix};1\big)\left(\zeta_2 - \log^2(a)\right)
\end{aligned}$$

$$\begin{aligned}
T_- \;=\; & -\frac{3}{2}\zeta_2\,\mathcal{E}_4\big(\begin{smallmatrix} -1 \\ \infty \end{smallmatrix};r_{--}\big) + \zeta_2\mathcal{E}_4\big(\begin{smallmatrix} -1 & 0 \\ \infty & 0 \end{smallmatrix};r_{--}\big) - 2\mathcal{E}_4\big(\begin{smallmatrix} -1 & -1 \\ \infty & \infty \end{smallmatrix};r_{--}\big)\,\mathcal{E}_4\big(\begin{smallmatrix} 0 & -1 \\ 0 & \infty \end{smallmatrix};1\big) \\
& + \mathcal{E}_4\Big(\begin{smallmatrix} -1 & 0 & 1 & 1 \\ \infty & 0 & 0,0 & \end{smallmatrix};r_{--}\Big) + \mathcal{E}_4\big(\begin{smallmatrix} -1 & 0 & 1 & 1 \\ \infty & 0 & 0 & 1 \end{smallmatrix};r_{--}\big) - \mathcal{E}_4\big(\begin{smallmatrix} -1 & 0 & 1 & 1 \\ \infty & 0 & 1 & 0 \end{smallmatrix};r_{--}\big) - \mathcal{E}_4\big(\begin{smallmatrix} -1 & 0 & 1 & 1 \\ \infty & 0 & 1 & 1 \end{smallmatrix};r_{--}\big) \\
& + \mathcal{E}_4\big(\begin{smallmatrix} -1 & 1 & 0 & 1 \\ \infty & 0 & 0 & 1 \end{smallmatrix};r_{--}\big) - \mathcal{E}_4\big(\begin{smallmatrix} -1 & 1 & 0 & 1 \\ \infty & 1 & 0 & 0 \end{smallmatrix};r_{--}\big) + \mathcal{E}_4\big(\begin{smallmatrix} 1 & -1 & 0 & 1 \\ 0 & \infty & 0 & 1 \end{smallmatrix};r_{--}\big) - \mathcal{E}_4\big(\begin{smallmatrix} 1 & -1 & 0 & 1 \\ 1 & \infty & 0 & 0 \end{smallmatrix};r_{--}\big) \\
& - \mathcal{E}_4\big(\begin{smallmatrix} -1 & 0 & 1 \\ \infty & 0 & 1 \end{smallmatrix};r_{--}\big)\log(r_{--}) + \mathcal{E}_4\big(\begin{smallmatrix} -1 & 0 & 1 \\ \infty & 0 & 0 \end{smallmatrix};r_{--}\big)\log(1-r_{--})
\end{aligned}$$

$$\begin{aligned}
T_+ \;=\; & \frac{i\pi}{4}\Big(\mathcal{E}_4\big(\begin{smallmatrix} 1 & -1 \\ 0 & \infty \end{smallmatrix};r_{-+}\big) + \mathcal{E}_4\big(\begin{smallmatrix} 1 & -1 \\ 1 & \infty \end{smallmatrix};r_{-+}\big) - 4\big(\mathcal{E}_4\big(\begin{smallmatrix} 1 & -1 & 0 \\ 0 & \infty & 0 \end{smallmatrix};r_{-+}\big) + \mathcal{E}_4\big(\begin{smallmatrix} 1 & -1 & 0 \\ 1 & \infty & 0 \end{smallmatrix};r_{-+}\big)\big)\Big) \\
& - \mathcal{E}_4\big(\begin{smallmatrix} 1 & -1 & 0 & 1 \\ 0 & \infty & 1 & 0 \end{smallmatrix};r_{-+}\big) + \mathcal{E}_4\big(\begin{smallmatrix} 1 & -1 & 0 & 1 \\ 0 & \infty & 0 & 1 \end{smallmatrix};r_{-+}\big) - \mathcal{E}_4\big(\begin{smallmatrix} 1 & -1 & 0 & 1 \\ 1 & \infty & 1 & 0 \end{smallmatrix};r_{-+}\big) + \mathcal{E}_4\big(\begin{smallmatrix} 1 & -1 & 0 & 1 \\ 1 & \infty & 0 & 1 \end{smallmatrix};r_{-+}\big)
\end{aligned}$$