Two-loop integrals with masses

Stefan Weinzierl

Institut für Physik, Universität Mainz

0:		**Introduction**
I:	**Within multiple polylogs**:	**Rationalising roots**
II:	**Beyond multiple polylogs**:	**Iterated integrals of modular forms**
IV:	**Beyond multiple polylogs**:	**Several elliptic curves**

Motivation

- Precision physics for heavy particles (e.g. Higgs, top, W/Z-bosons).
- If we allow masses, already two-loop two-point functions go beyond multiple polylogarithms.
- We want to learn what comes beyond multiple polylogarithms!
- Now an active field of research

  Laporta, Remiddi, Müller-Stach, Zayadeh, Adams, Bogner, Bloch, Kerr, Vanhove, Søgaard, Zhang, von Manteuffel, Tancredi, Primo, Bonciani, Del Duca, Frellesvig, Henn, Hidding, Moriello, V. Smirnov, A. Smirnov, Lee, Schweitzer, Chaubey, Broedel, Duhr, Dulat, Penante, Ablinger, Blümlein, De Freitas, van Hoeij, Imamoglu, Raab, Schneider, Bourjaily, McLeod, Spradlin, von Hippel, Wilhelm, S.W.

Differential equations

Let x_k be a kinematic variable. Let $I_i \in \{I_1, ..., I_{N_{\mathrm{master}}}\}$ be a master integral. Carrying out the derivative

$$\frac{\partial}{\partial x_k} I_i$$

under the integral sign and using integration-by-parts identities allows us to express the derivative as a linear combination of the master integrals.

$$\frac{\partial}{\partial x_k} I_i = \sum_{j=1}^{N_{\mathrm{master}}} a_{ij} I_j$$

(Kotikov ’90, Remiddi ’97, Gehrmann and Remiddi ’99)

Differential equations

Let us formalise this:

$\vec{I} = (I_1, ..., I_{N_{\mathrm{master}}})$, set of master integrals,

$\vec{x} = (x_1, ..., x_{N_B})$, set of kinematic variables the master integrals depend on.

We obtain a system of differential equations of Fuchsian type

$$d\vec{I} + A\vec{I} = 0,$$

where A is a matrix-valued one-form

$$A = \sum_{i=1}^{N_B} A_i dx_i.$$

The matrix-valued one-form A satisfies the integrability condition

$$dA + A \wedge A = 0$$

Computation of Feynman integrals reduced to solving differential equations!

Iterated integrals

For ω_1, ..., ω_k differential 1-forms on a manifold M and $\gamma : [0,1] \to M$ a path, write for the pull-back of ω_j to the interval $[0,1]$

$$f_j(\lambda)\, d\lambda \quad = \quad \gamma^* \omega_j.$$

The iterated integral is defined by (Chen '77)

$$I_\gamma(\omega_1, \ldots, \omega_k; \lambda) \quad = \quad \int_0^\lambda d\lambda_1 f_1(\lambda_1) \int_0^{\lambda_1} d\lambda_2 f_2(\lambda_2) \ldots \int_0^{\lambda_{k-1}} d\lambda_k f_k(\lambda_k).$$

Example 1: Multiple polylogarithms (Goncharov '98)

$$\omega_j \quad = \quad \frac{d\lambda}{\lambda - z_j}.$$

Example 2: Iterated integrals of modular forms (Brown '14): $f_j(\tau)$ a modular form,

$$\omega_j \quad = \quad 2\pi i\, f_j(\tau)\; d\tau.$$

The ε-form of the differential equation

If we change the basis of the master integrals $\vec{J} = U\vec{I}$, the differential equation becomes

$$(d + A')\vec{J} = 0, \qquad A' = UAU^{-1} + U\,dU^{-1}$$

Suppose one finds a transformation matrix U, such that

$$A' \quad = \quad \varepsilon \sum_j C_j \, d\ln p_j(\vec{x}),$$

where

- ε appears only as prefactor,
- C_j are matrices with constant entries,
- $p_j(\vec{x})$ are polynomials in the external variables,

then the system of differential equations is easily solved in terms of multiple polylogarithms.

Henn '13

Transformation to the ε-form

We may

- change the basis of the master integrals

$$\vec{I} \quad \rightarrow \quad U \vec{I},$$

where U is rational in the kinematic variables.

Henn '13; Gehrmann, von Manteuffel, Tancredi, Weihs '14; Argeri et al. '14; Lee '14; Meyer '16, '17; Prausa '17; Gituliar, Magerya '17; Lee, Pomeransky '17;

- perform a rational / algebraic transformation on the kinematic variables

$$(x_1, \ldots, x_{N_B}) \quad \rightarrow \quad (x'_1, \ldots, x'_{N_B}),$$

often done to absorb square roots.

Becchetti, Bonciani '17, Bourjaily, McLeod, von Hippel, Wilhelm, '18, Besier, van Straten, S.W., '18

Part I

Rationalising roots

Example

The one-loop massive bubble in pre-canonical form:

$$\frac{d}{dx}\vec{I} = \begin{pmatrix} 0 & 0 \\ \frac{\varepsilon}{4x} - \frac{\varepsilon}{4(x-4)} & -\frac{1}{2x} - \frac{1+2\varepsilon}{2(x-4)} \end{pmatrix} \vec{I}$$

Change the basis of master integrals $\vec{J} = U\vec{I}$ to obtain an ε-form:

$$\frac{d}{dx}\vec{J} = \varepsilon \begin{pmatrix} 0 & 0 \\ -\frac{1}{\sqrt{-x(4-x)}} & -\frac{1}{x-4} \end{pmatrix} \vec{J}$$

Change the kinematic variable $x = -\frac{(1-y)^2}{y}$ to rationalise the square root:

(Fleischer, Kotikov, Veretin, '98)

$$\frac{d}{dy}\vec{J} = \varepsilon \begin{pmatrix} 0 & 0 \\ -\frac{1}{y} & \frac{1}{y} - \frac{2}{y+1} \end{pmatrix} \vec{J}$$

Is there a systematic way to find a transformation which rationalises the square root?

The algorithm

Preparation: Associate to a square root a hypersurface:

$$v = \sqrt{-x(4-x)} \quad \Rightarrow \quad v^2 + x(4-x) = 0$$

Definition: A point p of a hypersurface $f(x_1, \ldots, x_{N_B+1}) = 0$ is said to be of multiplicity r, if there is at least one non-vanishing r-th partial derivative of f at p and, at the same time, all partial derivatives of lower order vanish at p.

Theorem: Let $d = \deg f$. If the hypersurface has a point of multiplicity $(d-1)$, we may algorithmically rationalise the square root.

Works for any number of variables N_B, works for any degree d. Works also iteratively, as long as there is always a point of multiplicity $(d-1)$.

Example

Hexagon functions:

$$\Delta_6^{\{123456\}} = \sqrt{(1-u_1-u_2-u_3)^2 - 4u_1u_2u_3}$$

Associated hypersurface:

$$f(u,u_1,u_2,u_3) = (1-u_1-u_2-u_3)^2 - 4u_1u_2u_3 - u^2$$

Points of multiplicity 2:

$$\{(0,0,0,1),(0,0,1,0),(0,1,0,0),(0,1,1,1)\}$$

Pick the first one. One obtains

$$u_1 = \frac{(t_1+t_2+t_3)^2 - 4t_1t_2 - 1}{4t_2t_3}, \quad u_2 = \frac{(t_1+t_2+t_3)^2 - 4t_1t_2 - 1}{4t_1t_3}, \quad u_3 = \frac{(t_1+t_2+t_3)^2 - 1}{4t_1t_2}$$

Part II

Feynman integrals depending on a single scale and associated to a single elliptic curve

(Iterated integrals of modular forms)

Modular forms

Denote by $\mathbb{H}$ the complex upper half plane. A meromorphic function $f : \mathbb{H} \to \mathbb{C}$ is a modular form of modular weight k for $\mathrm{SL}_2(\mathbb{Z})$ if

(i) f transforms under Möbius transformations as

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \cdot f(\tau) \qquad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

(ii) f is holomorphic on $\mathbb{H}$,

(iii) f is holomorphic at ∞.

Congruence subgroups

Apart from $\mathrm{SL}_2(2,\mathbb{Z})$ we may also look at congruence subgroups, for example

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}_2(\mathbb{Z}) : c \equiv 0 \text{ mod } N \right\}$$

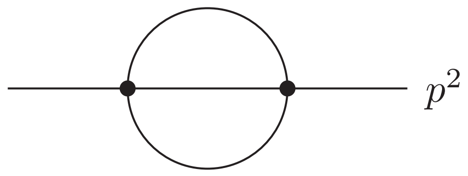
$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}_2(\mathbb{Z}) : a,d \equiv 1 \text{ mod } N,\ c \equiv 0 \text{ mod } N \right\}$$

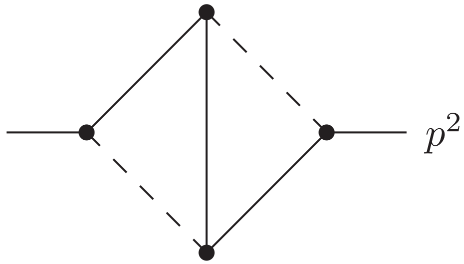
$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}_2(\mathbb{Z}) : a,d \equiv 1 \text{ mod } N,\ b,c \equiv 0 \text{ mod } N \right\}$$

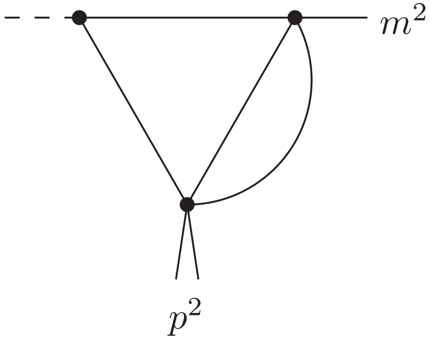
Modular forms for congruence subgroups: Require “nice” transformation properties only for subgroup Γ (plus holomorphicity on $\mathbb{H}$ and at the cusps).

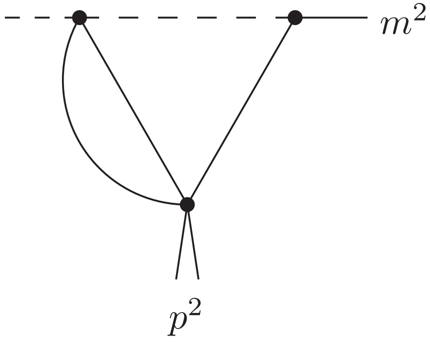
Single-scale Feynman integrals beyond multiple polylogarithms

Not all Feynman integrals are expressible in terms of multiple polylogarithms!









The Picard-Fuchs operator

Let I be one of the master integrals $\{I_1, ..., I_{N_{\mathrm{master}}}\}$. Choose a path $\gamma : [0,1] \to M$ and study the integral I as a function of the path parameter λ.

Instead of a system of N_{master} first-order differential equations

$$(d+A)\vec{I} \;=\; 0,$$

we may equivalently study a single differential equation of order N_{master}

$$\sum_{j=0}^{N_{\mathrm{master}}} p_j(\lambda) \frac{d^j}{d\lambda^j} I \;=\; 0.$$

We may work modulo sub-topologies and ε-corrections:

$$L = \sum_{j=0}^{r} p_j(\lambda) \frac{d^j}{d\lambda^j} : \qquad L\, I = 0 \quad \mathrm{mod}\ (\text{sub-topologies, } \varepsilon\text{-corrections})$$

Factorisation of the Picard-Fuchs operator

- Feynman integral evaluates to multiple polylogarithms
  $\Rightarrow$ Picard-Fuchs operator factorises into linear factors.

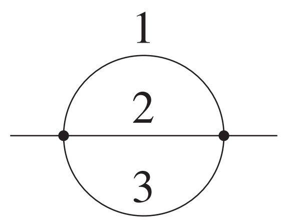
- Picard-Fuchs operator does not factorise into linear factors
  $\Rightarrow$ Feynman integral does not evaluate to multiple polylogarithms.

The next more complicate case:
The differential operator contains one irreducible second-order differential operator

$$a_j(\lambda)\frac{d^2}{d\lambda^2} + b_j(\lambda)\frac{d}{d\lambda} + c_j(\lambda)$$

An example from physics: The two-loop sunrise integral

$$S_{\nu_1\nu_2\nu_3}(D,x) \quad =$$



Picard-Fuchs operator for $S_{111}(2,x)$:

$$L \quad = \quad x(x-1)(x-9)\frac{d^2}{dx^2} + \left(3x^2 - 20x + 9\right)\frac{d}{dx} + (x-3)$$

(Broadhurst, Fleischer, Tarasov ’93)

Irreducible second-order differential operator.

Picard-Fuchs operator for the periods of a family of elliptic curves.

The elliptic curve

How to get the elliptic curve?

- From the Feynman graph polynomial:

$$-x_1 x_2 x_3 x + (x_1 + x_2 + x_3)(x_1 x_2 + x_2 x_3 + x_3 x_1) = 0$$

- From the maximal cut:

$$v^2 - (u-x)(u-x+4)\left(u^2+2u+1-4x\right) = 0$$

Baikov '96; Lee '10; Kosower, Larsen, '11; Caron-Huot, Larsen, '12; Frellesvig, Papadopoulos, '17; Bosma, Sogaard, Zhang, '17; Harley, Moriello, Schabinger, '17

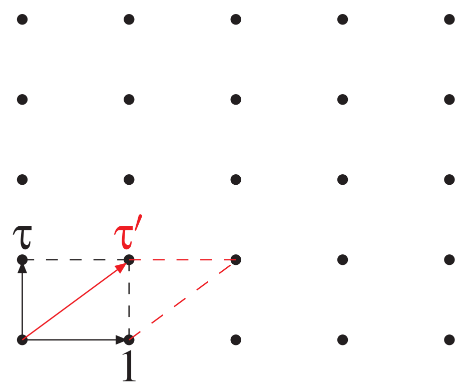
The periods ψ_1, ψ_2 of the elliptic curve are solutions of the homogeneous differential equation.

Adams, Bogner, S.W., '13; Primo, Tancredi, '16

$$\text{Set} \quad \tau = \frac{\psi_2}{\psi_1}, \qquad q = e^{2i\pi\tau}.$$

Bases of lattices

The periods ψ_1 and ψ_2 generate a lattice. Any other basis as good as (ψ_2, ψ_1).
Convention: Normalise $(\psi_2, \psi_1) \to (\tau, 1)$ where $\tau = \psi_2/\psi_1$.



Change of basis: $$\begin{pmatrix} \psi_2' \\ \psi_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix},$$

Transformation should be invertible: $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}),$$

In terms of τ and τ': $$\tau' = \frac{a\tau + b}{c\tau + d}$$

The ε-form of the differential equation for the sunrise

It is not possible to obtain an ε-form by a rational/algebraic change of variables and/or a rational/algebraic transformation of the basis of master integrals.

However by factoring off the (non-algebraic) expression ψ_1/π from the master integrals in the sunrise sector one obtains an ε-form:

$$I_1 = 4\varepsilon^2 S_{110}(2-2\varepsilon,x), \quad I_2 = -\varepsilon^2 \frac{\pi}{\psi_1} S_{111}(2-2\varepsilon,x), \quad I_3 = \frac{1}{\varepsilon}\frac{1}{2\pi i}\frac{d}{d\tau} I_2 + \frac{1}{24}\left(3x^2-10x-9\right)\frac{\psi_1^2}{\pi^2} I_2.$$

If in addition one makes a (non-algebraic) change of variables from x to τ, one obtains

$$\frac{d}{d\tau}\vec{I} \quad = \quad \varepsilon\, A(\tau)\, \vec{I},$$

where $A(\tau)$ is an ε-independent 3×3-matrix whose entries are modular forms.

The ε-form of the differential equation for the sunrise

The matrix $A(\tau)$ is given by

$$A(\tau) \quad = \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & -f_2(\tau) & 1 \\ \frac{1}{4}f_3(\tau) & f_4(\tau) & -f_2(\tau) \end{pmatrix},$$

where f_2, f_3 and f_4 are modular forms of $\Gamma_1(6)$ of modular weight 2, 3 and 4, respectively.

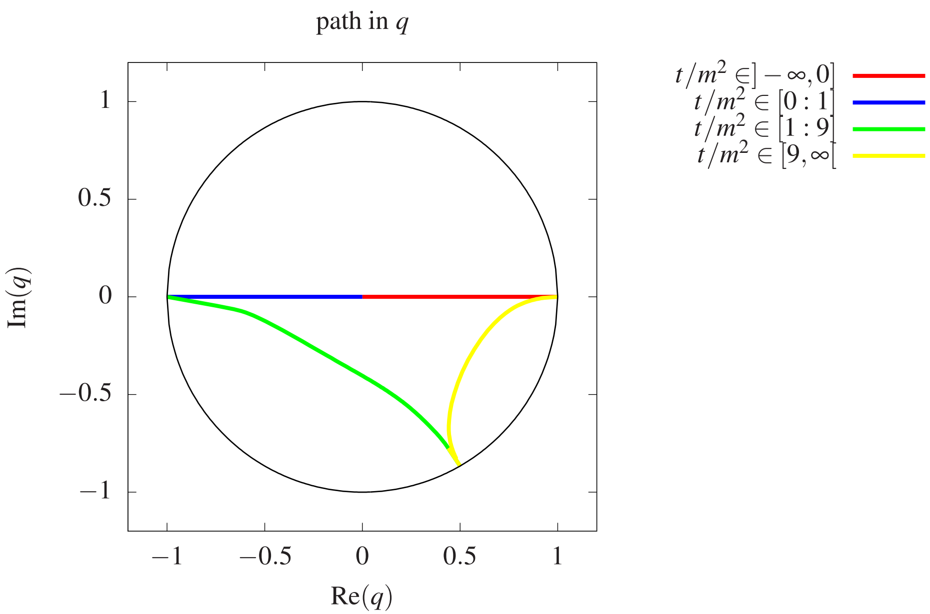
I_1, I_2 and I_3 are expressed as iterated integrals of modular forms to all orders in ε.

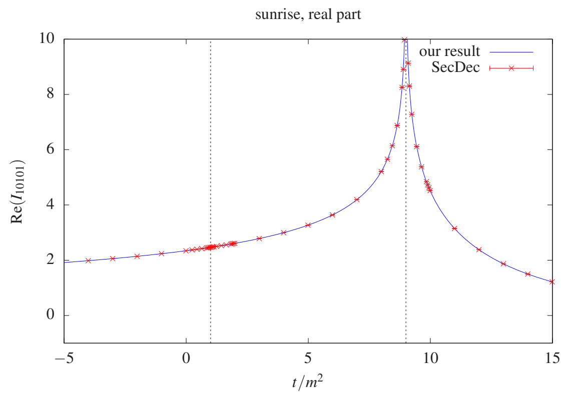
Adams, S.W., ’17, ’18

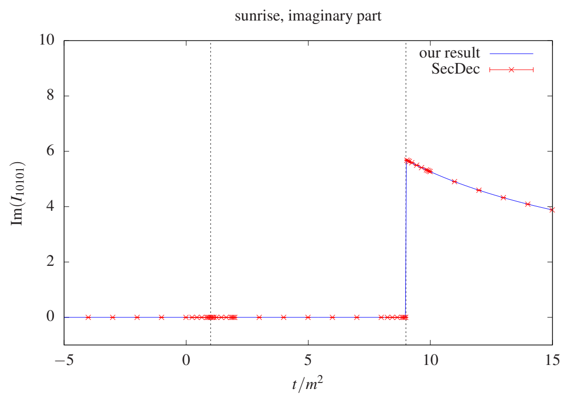
Numerical evaluations

Complete elliptic integrals efficiently computed from arithmetic-geometric mean.

q-series converges for all $t \in \mathbb{R} \backslash \{m^2, 9m^2, \infty\}$.



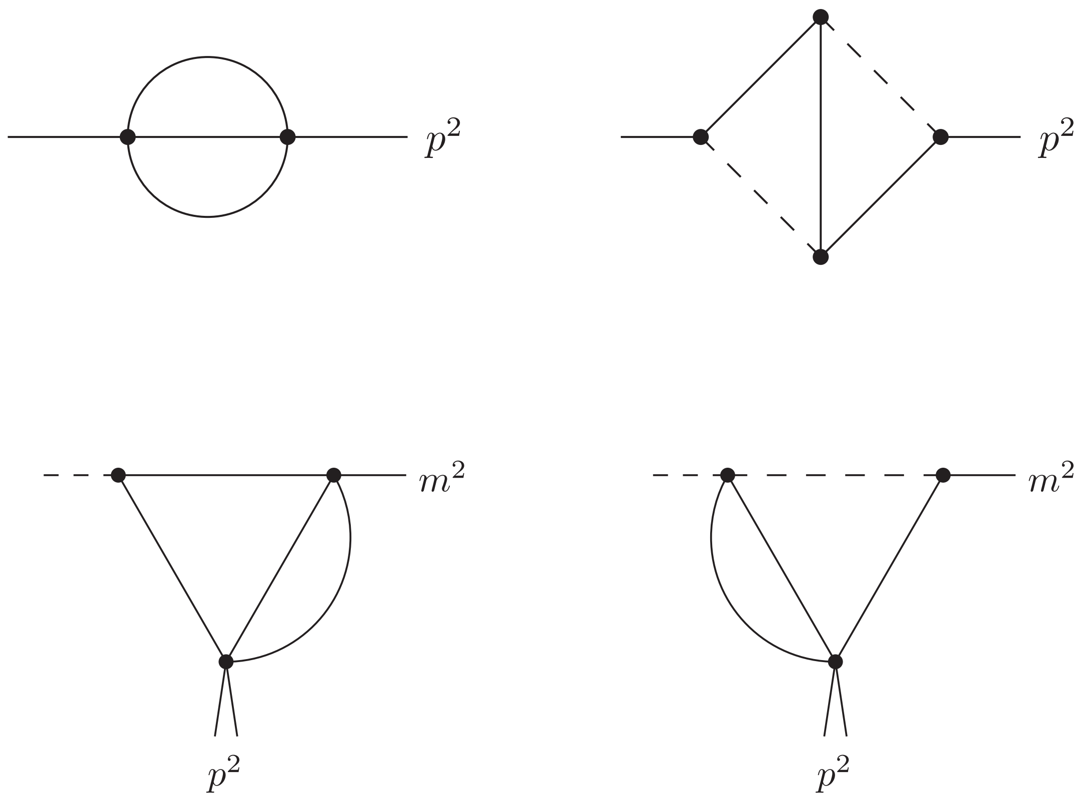




Bogner, Schweitzer, S.W., ’17

Feynman integrals evaluating to iterated integrals of modular forms

This applies to a wider class of Feynman integrals:



Part III

Feynman integrals depending on multiple scales and associated to a single elliptic curve

(see Lorenzo's talk)

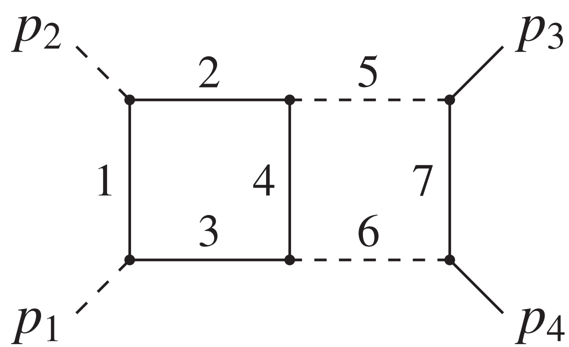
Part IV

Feynman integrals depending on multiple scales and associated to several elliptic curves

(An example from top-pair production)

Kinematics

$$I_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7}\left(D,\frac{s}{m^2},\frac{t}{m^2}\right) = \left(m^2\right)^{\sum\limits_{j=1}^{7}\nu_j - D} \int \frac{d^D k_1}{(2\pi)^D}\frac{d^D k_2}{(2\pi)^D}\prod_{j=1}^{7}\frac{1}{P_j^{\nu_j}},$$



$$p_1^2 = p_2^2 = 0, \qquad p_3^2 = p_4^2 = m^2,$$

$$s = (p_1 + p_2)^2, \qquad t = (p_2 + p_3)^2.$$

Picard-Fuchs operator of elliptic curves

- Sunrise integral: An elliptic curve can be obtained either from
  - Feynman graph polynomial
  - maximal cut

  The periods ψ_1, ψ_2 are the solutions of the homogeneous differential equations.

  Adams, Bogner, S.W., '13, '14

- In general: The maximal cuts are solutions of the homogeneous differential equations.

  Primo, Tancredi, '16

Search for Feynman integrals, whose maximal cuts are periods of an elliptic curve.

Maximal cuts

Maximal cut: For a Feynman integral

$$I_{\nu_1\nu_2\ldots\nu_n} = \left(\mu^2\right)^{\nu - lD/2} \int \frac{d^Dk_1}{(2\pi)^D} \cdots \frac{d^Dk_l}{(2\pi)^D} \prod_{j=1}^{n} \frac{1}{P_j^{\nu_j}}$$

take the n-fold residue at

$$P_1 = \ldots = P_n = 0$$

of the integrand and integrate over the remaining $(lD - n)$ variables along a contour $\mathcal{C}$.

Maximal cuts

Sunrise :

$$\mathrm{MaxCut}_{\mathcal{C}}\, I_{1001001}\,(2-2\varepsilon) =$$

$$\frac{um^2}{\pi^2}\int_{\mathcal{C}} \frac{dP}{(P-t)^{\frac{1}{2}}\,(P-t+4m^2)^{\frac{1}{2}}\,(P^2+2m^2P-4m^2t+m^4)^{\frac{1}{2}}} + O(\varepsilon)\,.$$

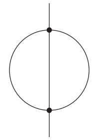
Double box :

$$\mathrm{MaxCut}_{\mathcal{C}}\, I_{1111111}\,(4-2\varepsilon) =$$

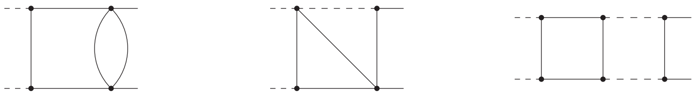
$$\frac{um^6}{4\pi^4 s^2}\int_{\mathcal{C}} \frac{dP}{(P-t)^{\frac{1}{2}}\,(P-t+4m^2)^{\frac{1}{2}}\left(P^2+2m^2P-4m^2t+m^4-\frac{4m^2\left(m^2-t\right)^2}{s}\right)^{\frac{1}{2}}} + O(\varepsilon)\,.$$

Three elliptic curves

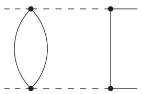
$$E^{(a)} \quad : \quad w^2 = (z-t)\left(z-t+4m^2\right)\left(z^2+2m^2z-4m^2t+m^4\right)$$



$$E^{(b)} \quad : \quad w^2 = (z-t)\left(z-t+4m^2\right)\left(z^2+2m^2z-4m^2t+m^4-\frac{4m^2\left(m^2-t\right)^2}{s}\right)$$



$$E^{(c)} \quad : \quad w^2 = (z-t)\left(z-t+4m^2\right)\left(z^2+\frac{2m^2\left(s+4t\right)}{\left(s-4m^2\right)}z+\frac{sm^2\left(m^2-4t\right)-4m^2t^2}{s-4m^2}\right)$$



Remarks

- $E^{(a)}$ gives rise to iterated integrals of modular forms of $\Gamma_1(6)$.
- For $s \to \infty$ the curves $E^{(b)}$ and $E^{(c)}$ degenerate to $E^{(a)}$.
- If we would have only one curve, we expect that the result can be written in elliptic polylogarithms.
- We have three elliptic curves.

Results

The differential equation for the master integrals can be brought into the form

$$d\vec{I} \quad = \quad \varepsilon A \vec{I},$$

where A is independent of ε.

The Laurent expansion in ε of all master integrals can be computed systematically to all orders in ε in terms of iterated integrals.

The solution

- reduces to multiple polylogarithms for $t = m^2$ and
- reduces to iterated integrals of modular forms of $\Gamma_1(6)$ for $s = \infty$.

Adams, Chaubey, S.W., '18

Conclusions

- Loop integrals with masses important for top, W/Z- and H-physics.
- May involve elliptic sectors from two loops onwards.
- There is a class of Feynman integrals evaluating to iterated integrals of modular forms.
- The planar double box integral relavant to $t\bar{t}$-production with a closed top loop depends on two variables and involves several elliptic sub-sectors. More than one elliptic curve occurs. Results expressed in terms of Chen's iterated integrals.
- We may expect more results in the near future.