Five-Parton Two-Loop Amplitudes from Numerical Unitarity

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Based on [arXiv:1809.09067]

Outline

Introduction

Motivation

Precision era at the LHC

- No direct detection of new physics $\implies$ precision measurements
- Rule of thumb: to achieve accuracy of theory predictions at a level of *a few percent*, NNLO computations are required
- Physical case for $3j$, $H + 2j$, $V + 2j$, $t\bar{t} + j$, $VV'j$ processes

State of the art

- Many $2 \to 2$ processes are available at NNLO QCD, next frontier is $2 \to 3$
- Handling IR divergences for $2 \to 3$ processes is very challenging
- Huge effort towards computation of multi-scale Feynman integrals
- $2 \to 3$ two-loop amplitudes are being actively attacked and some first results have been recently achieved (talks by Christian Brønnum-Hansen and Herschel Chawdhry)

> We focus on computation of two-loop amplitudes with the numerical unitarity method.

Two-loop Numerical Unitarity

The “Standard” Approach to General Two-loop Amplitudes

Feynman diagrams

Tensor reduction [Passarino, Veltman ’79]
IBPs [Tkachov, Chetyrkin ’81]

Sum of master integrals

$$A = \sum_{\Gamma \in \Delta} \sum_{i \in M_\Gamma} c_{\Gamma,i} I_{\Gamma,i}$$

Master integrals

direct computation, differential equations,

numerical integration,

etc.

Integrated amplitude

Challenges

- Large expressions
- Solving IBP relations (producing reduction tables) is particularly difficult

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try to avoid these by

Two-loop numerical unitarity

- Only a *targeted set of IBP relations* is required for each topology
- *Implicit numerical reduction* to master integrals
- Fully *numerical framework* mitigates sensitivity to additional scales

Ansatz for integrand

$$\mathcal{A}(\ell_l) = \sum_{\text{topologies }\Gamma} \; \sum_{i \in M_\Gamma \cup S_\Gamma} \frac{c_{\Gamma,i}\, m_{\Gamma,i}(\ell_l)}{\prod_{\text{props }j} \rho_{\Gamma,j}}$$

Based on unitarity approach
[Bern, Dixon, Kosower, Dunbar '94, '95] [Britto, Feng, Cachazo '05] [Ossola, Papadopoulos, Pittau '07] [Ellis, Giele, Kunszt '08] [Giele, Kunszt, Melnikov '08]

Recent work
[Badger, Frellesvig, Zhang '12] [Zhang '12] [Mastrolia, Mirabella, Ossola, Peraro '13] [Mastrolia, Peraro, Primo '16] [Abreu, Febres Cordero, Ita, Jaquier, Page, Zeng '17]

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Topologies

Master/Surface decomposition

Unitarity compatible IBPs [Ita '15]

Kinematics

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Two-Loop Amplitudes with (Numerical) Unitarity

[Abreu, Febres Cordero, Ita, Jaquier, Page, Zeng '17]

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- Factorization on on-shell phase-space
- Cut equations

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$$m_{i \in S_\Gamma} \to 0$$

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Master Integrals

Integrated amplitude

Master/Surface Decomposition

1. Surface terms naturally produced by IBPs

$$0 = \int \prod_{l=1,2} d^D\ell_l \frac{\partial}{\partial \ell_j^\nu}\left[\frac{u_j^\nu}{\prod_{\text{props } k}\rho_k}\right]$$

- Unitarity compatible IBP relations for a topology Γ by controlling powers of ρ_j [Gluza, Kadja, Kosower '11]

$$u_i^\nu \frac{\partial}{\partial \ell_i^\nu}\rho_j = f_j \rho_j$$

- Compute IBP-generating vectors using SINGULAR [Abreu, Febres Cordero, Ita, Page, Zeng '17]
  (Related [Ita '15] [Larsen, Zhang '15][Georgoudis, Larsen, Zhang '16])

$$\left(u_{ka}^{\text{loop}}(\alpha,\rho)\ell_a^\nu + u_{kb}^{\text{ext}}(\alpha,\rho)p_b^\nu\right)\frac{\partial}{\partial \ell_k^\nu}\begin{pmatrix}\rho_{j(1)}\\ \rho_{j(2)}\\ \vdots\\ \rho_{j(|\Gamma|)}\end{pmatrix} - \begin{pmatrix}f_{j(1)}\rho_{j(1)}\\ f_{j(2)}\rho_{j(2)}\\ \vdots\\ f_{j(|\Gamma|)}\rho_{j(|\Gamma|)}\end{pmatrix} = 0$$

2. Complete remaining function space with master integrands

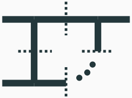
Coefficients from Numerical Unitarity

For each topology

1. Construct on-shell loop momenta from $\rho_{\Gamma,j}(\ell_l) = 0$

2. Build cut equations (linear systems) for master/surface coefficients $c_{\Gamma,i}$ by sampling ℓ_l

product of tree amplitudes

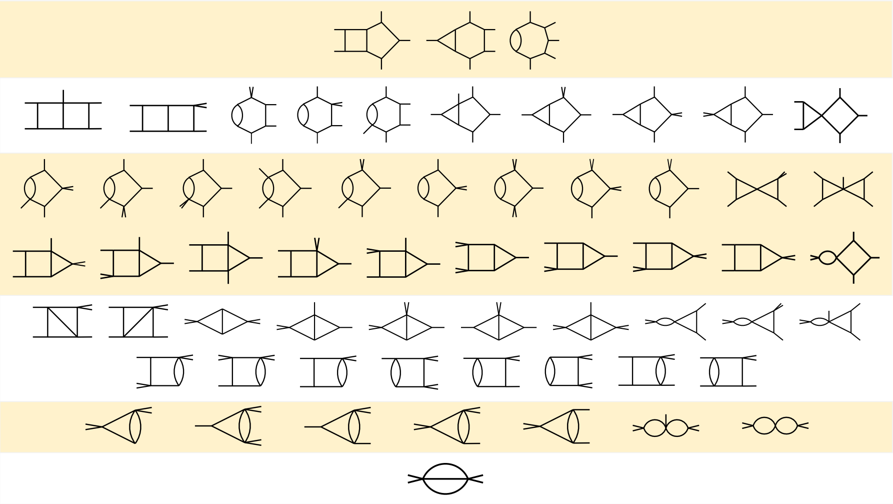
$$\sum_{i \in M_\Gamma \cup S_\Gamma} c_{\Gamma,i}(D, D_s)\, m_{\Gamma,i}(\ell_l^\Gamma) =$$  $$- \sum_{\text{ancestors } \Gamma'} \frac{N(\Gamma', \ell_l^\Gamma)}{\prod_{k \in P_{\Gamma'} \setminus P_\Gamma} \rho_k(\ell_l^\Gamma)}$$

topologies with more propagators

Factorization is guaranteed by Unitarity.

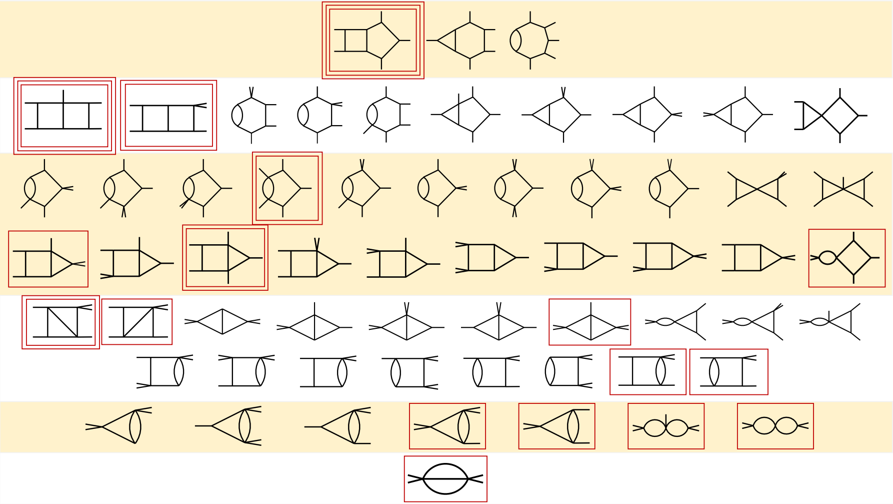
3. Invert linear systems for coefficients $c_{\Gamma,i}$
   - Solve for (a set of) given values of dimensional regularization parameters D and D_s
   - Reconstruct (exactly) rational functions $c_{\Gamma,i}(D, D_s)$

Topologies For 5-Parton Amplitudes



Full hierarchy

Topologies For 5-Parton Amplitudes



Master integrals

[Gehrmann, Remiddi '00] [Papdopoulos et al '15] [Gehrmann, Henn, Presti '18]

New Developments for Fermions

Dimensional Regularization and Numerical Unitarity

Regularization

Compute in $D = 4 - 2\epsilon$ dimensions including tensor and spinor representation.

Numerical framework

Explicit finite-dimensional representations of all objects.

Finite-dimensional formulation

Understand how relevant information can be extracted from (a set of) computations in integer dimensions

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General Strategy

- Work in 't Hooft-Veltman (HV) scheme $\Longrightarrow$ many simplifications
- Separate dimensional regularization parameter in
  - D — loop momenta, enters through IBPs
  - D_s — tensor and spinor representations
- Compute tree amplitudes in $D_s = \{6, 8, 10\}$ dimensions to reconstruct *(quadratic)* D_s-dependence (aka *dimensional reconstruction* [Giele, Kunszt, Melnikov '08] [Abreu, Febres Cordero, Ita, Jaquier, Page, Zeng '17])

Dimensional Regularization and Numerical Unitarity

Issues specific to treatment of quarks

- Relation of the infinite-dimensional Clifford algebra to finite at any given D_s
- Consistent embedding of external states
  - trivial for vector particles
  - non-trivial for fermions
- Definition of dimensionally regularized helicity amplitudes with external quarks is ambiguous
- Colour treatment for quarks is more involved

Some of these issues are relevant already at one loop for massive quarks [Anger, VS '18]

In two-loop computations they become relevant for massless quarks as well!

Clifford Algebra in D_s Dimensions

Clifford algebra in D_s dimensions is defined by

$$\{\gamma^{\mu}_{[D_s]}, \gamma^{\nu}_{[D_s]}\} = 2g^{\mu\nu}_{[D_s]} \mathbb{1}_{[D_s]}$$

Useful to expose tensor product structure:

$$(\gamma^{\mu}_{[D_s]})^{b\lambda}_{a\kappa} = \begin{cases} \left(\gamma^{\mu}_{[4]}\right)_a^{\ b} \delta^{\lambda}_{\kappa}\,, & 0 \leq \mu \leq 3\,, \\ \left(\tilde{\gamma}_{[4]}\right)_a^{\ b} \left(\gamma^{(\mu-4)}_{[D_s-4]}\right)_{\kappa}^{\ \lambda}\,, & \mu > 3\,, \end{cases}$$

$$\tilde{\gamma}_{[4]} \equiv i(\gamma^0\gamma^1\gamma^2\gamma^3)$$
$$(\tilde{\gamma}_{[4]})^2 = \mathbb{1}_{[4]}$$

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Example

$\boxed{\le 3}$ $\boxed{> 3}$

$$\left(\gamma^{\mu_1}_{[D_s]}\gamma^{\hat{\mu}_2}_{[D_s]}\gamma^{\mu_3}_{[D_s]}\gamma^{\hat{\mu}_4}_{[D_s]}\right)_{a\kappa}^{b\lambda} = -\left(\gamma^{\mu_1}_{[D_s]}\gamma^{\mu_3}_{[D_s]}\gamma^{\hat{\mu}_2}_{[D_s]}\gamma^{\hat{\mu}_4}_{[D_s]}\right)_{a\kappa}^{b\lambda}$$
$$= -\left(\gamma^{\mu_1}_{[4]}\gamma^{\mu_3}_{[4]}\right)_a^{\ b} \left(\gamma^{(\hat{\mu}_2-4)}_{[D_s-4]}\gamma^{(\hat{\mu}_4-4)}_{[D_s-4]}\right)_{\kappa}^{\lambda}$$

Embedding of Fermionic States in Dimensional Regularization

Spinors for external states (*4-dim momenta*) from tensor product representation:

$$\psi_{s,a\kappa} = (u_h)_a (\eta^i)_\kappa \qquad (\eta^i)_\kappa = \delta^i_\kappa$$

$$\bar{\psi}^{a\kappa}_s = (\bar{u}_h)^a (\bar{\eta}_i)^\kappa \qquad (\bar{\eta}_i)^\kappa (\eta^j)_\kappa = \delta^j_i$$

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Example

$$\bar{u}^a \left(\gamma^{\mu_1}_{[D_s]}\gamma^{\hat{\mu}_2}_{[D_s]}\gamma^{\mu_3}_{[D_s]}\gamma^{\hat{\mu}_4}_{[D_s]}\right)^{b\lambda}_{a\kappa} u_b = -\,\bar{u}\gamma^{\mu_1}_{[4]}\gamma^{\mu_3}_{[4]}u \cdot \left(\gamma^{(\hat{\mu}_2-4)}_{[D_s-4]}\gamma^{(\hat{\mu}_4-4)}_{[D_s-4]}\right)^{\lambda}_{\kappa}$$

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Example

result in D_s=4

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Question: can one define helicity amplitudes unambiguously?

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$\downarrow$

Classify tensor structures of spinor indices in (D_s-4) dimensions!

Tensor Decomposition of Helicity Amplitudes

Inspired by [Glover '04]

Consider a helicity amplitude $M^{(k)}$ in D_s dimensions. Introduce a *basis* $\{v_n\}$ for the tensor space in spinor indices beyond $D_s = 4$:

$$M^{(k)} = \sum_n v_n \, M_n^{(k)}$$

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Example (a pair $q\bar{q}$ of external quarks)

$$M^{(k)}(q, \bar{q}, g, \ldots, g) = w_0 \, M_0^{(k)} \, , \qquad (w_0)^\lambda_\kappa := \delta^\lambda_\kappa \, .$$

to any loop order.

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to any loop order.

Example ($q\bar{q}\ Q\bar{Q}$ @ tree level)

$$\left(M^{(0)}\right)^{\lambda_1 \lambda_2}_{\kappa_1 \kappa_2} \sim \bar{u}^{a_1} \left(\gamma^{\mu}_{[D_s]}\right)^{b_1 \lambda_1}_{a_1 \kappa_1} u_{b_1} \cdot \bar{u}^{a_2} \left(\gamma_{[D_s]\mu}\right)^{b_2 \lambda_2}_{a_2 \kappa_2} u_{b_2} \to$$

$$\begin{cases} M_0^{(0)} \, \delta^{\lambda_1}_{\kappa_1} \delta^{\lambda_2}_{\kappa_2}, & \leftarrow \text{ in HV} \\ M_0^{(0)} \, \delta^{\lambda_1}_{\kappa_1} \delta^{\lambda_2}_{\kappa_2} + M_1^{(0)} \left(\gamma^{\mu}_{[D_s-4]}\right)^{\lambda_1}_{\kappa_1} \left(\gamma_{[D_s-4]\mu}\right)^{\lambda_2}_{\kappa_2} & \leftarrow \text{ in CDR} \end{cases}$$

Tensor Decomposition of Helicity Amplitudes

Example (two quark pairs of different flavors, $q\bar{q}$ and $Q\bar{Q}$)

$$
\begin{aligned}
(v_0)^{\lambda_1\lambda_2}_{\kappa_1\kappa_2} =& \delta^{\lambda_1}_{\kappa_1}\delta^{\lambda_2}_{\kappa_2}\,, \\
(v_1)^{\lambda_1\lambda_2}_{\kappa_1\kappa_2} =& (\gamma^{\mu_1}_{[D_s-4]})^{\lambda_1}_{\kappa_1}(\gamma_{[D_s-4]\mu_1})^{\lambda_2}_{\kappa_2}\,, \\
& \vdots \\
(v_m)^{\lambda_1\lambda_2}_{\kappa_1\kappa_2} =& (\gamma^{\mu_1\ldots\mu_m}_{[D_s-4]})^{\lambda_1}_{\kappa_1}(\gamma_{[D_s-4]\mu_1\ldots\mu_m})^{\lambda_2}_{\kappa_2}\,, \\
& \vdots
\end{aligned}
$$

infinite-dimensional for $D_s = 4 - 2\epsilon$, but finite to any (finite) loop order!

$$\boxed{M^{(k)}(q,\bar{q},Q,\bar{Q},g,\ldots,g) = \sum_{n=0}^{n_k} v_n M_n^{(k)}}$$

tree (k=0)	$n_0 = 0$
one loop (k=1)	$n_1 = 3$
two loop (k=2)	$n_2 = 5$

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Note

- Independent of number of external gluons in HV
- Integer D_s values can be used to represent the appropriate limit of the calculation in generic D_s

Relevant Coefficients

Extraction of tensor coefficients

We constructed the basis $\{v_n\}$ such that

$$v_n^\dagger \cdot v_m = c_n \delta^n_m \,, \quad c_0 = 1 \quad \text{and} \quad c_{n>0}(D_s) = \mathcal{O}(\epsilon)$$

$$M_n^{(k)} = \frac{1}{c_n} v_n^\dagger \cdot M^{(k)}$$

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Which tensor structures are (phenomenologically) relevant?

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Which tensor structures are (phenomenologically) relevant?

- In HV *tree amplitudes* are chosen to contain no singular gluons $\implies$ only v_0 contributes
- The tree amplitude acts as a projector on v_0:

$$\left(M^{(0)}\right)^\dagger M^{(2)} = \left(M_0^{(0)}\right)^\dagger M_0^{(2)}$$

- It is thus sufficient to compute

$$\boxed{A^{(2)}(q, \bar{q}, Q, \bar{Q}, g, \ldots, g) \equiv M_0^{(2)} = v_0^\dagger \cdot M^{(2)}(q, \bar{q}, Q, \bar{Q}, g, \ldots, g)}$$

Note: this definition of HV helicity amplitudes agrees with the one demonstrated in [Anger, VS '18]

Status and Results

Status

1. Our $\mathrm{C}{+}{+}$ framework for D-dimensional multi-loop numerical unitarity is now extended to include fermionic particles

2. Our implementation is compact and suitable for running on clusters

3. Finished implementation of all processes relevant for $pp \to 3j$ production through NNLO QCD at *leading color*:
   - 5g amplitudes with N_f^0, N_f^1, N_f^2 contributions
   - 2q3g amplitudes with N_f^0, N_f^1, N_f^2 contributions
   - 4q1g amplitudes with N_f^0, N_f^1, N_f^2 contributions

4. Evaluation using multi-precision floating point arithmetics as well as using finite fields (lifted to rational numbers) is available:
   - benchmarking and testing benefits considerably from evaluating coefficients of master integrals exactly
   - sampling over phase-space ($\Longrightarrow$ integrated cross-section) can be done with floating point evaluations
   - independent numerical setup provides a good consistency check

Validation

✓ Internal consistency checks:

- surface terms validated against FIRE [arXiv:1408.2372]
- unitarity (successful fit of ansatz for all topologies, aka "N=N test")
- master coefficients consistency between floating point and exact evaluation

✓ Known analytic results:

$4g$	[Bern, de Freitas, Dixon '02]
$2q2g$	[Bern, de Freitas, Dixon '03]
$4q$	[Glover '04], [Bern, de Freitas '04]
$5g$(only "$+++++$")	[Gehrmann, Henn, Presti '15], [Dunbar, Perkins '16]

✓ Pole structure⋆ [Catani '98] [Sterman, Tejeda-Yeomans '03] [Becher, Neubert '09] [Gardi, Magnea '09]

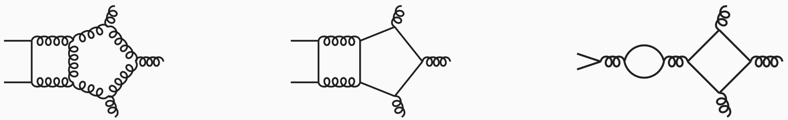
✓ $5g$ benchmark results of [Badger, Brønnum-Hansen, Hartanto, Peraro '17]

✓ Recent computation of N_f^0 contributions for $2q3g$ and $4q1g$ amplitudes [Badger, Brønnum-Hansen, Gehrmann, et al. '18] *(the latest revision)*

✓ N=4 SUSY Ward identities (for some amplitudes)

⋆ using one-loop amplitudes computed to $\mathcal{O}(\epsilon^2)$ with in-house implementation

Results for $2q3g$ Amplitudes



$$A(1_q, 2_{\bar{q}}, 3_g, 4_g, 5_g)\Big|_{\text{leading color}} = \sum_{\sigma \in S_3} (T^{a_{\sigma(3)}} T^{a_{\sigma(4)}} T^{a_{\sigma(5)}})_{i_1}^{\bar{\imath}_2} \times \mathcal{A}(1_q, 2_{\bar{q}}, \sigma(3)_g, \sigma(4)_g, \sigma(5)_g)\,,$$

Each partial $\mathcal{A}$ is perturbatively expanded as

$$\mathcal{A} = g_0^3 \left(\mathcal{A}^{(0)} + \frac{\alpha_0}{4\pi} N_c \mathcal{A}^{(1)} + \left(\frac{\alpha_0}{4\pi}\right)^2 {N_c}^2 \mathcal{A}^{(2)} + \mathcal{O}(\alpha_0^3) \right) \qquad \Bigg| \qquad \alpha_0 = g_0^2/(4\pi)$$

and further decomposed into

$$\mathcal{A}^{(2)} = \mathcal{A}^{(2)[N_f^0]} + \frac{N_f}{N_c} \mathcal{A}^{(2)[N_f^1]} + \left(\frac{N_f}{N_c}\right)^2 \mathcal{A}^{(2)[N_f^2]}$$

Results for $2q3g$ Amplitudes

$\mathcal{A}^{(2)[N_f^0]}/\mathcal{A}^{(\text{norm})}$	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
$(1_q^+, 2_{\bar{q}}^-, 3_g^+, 4_g^+, 5_g^+)$	0	0	-4.000000000	-33.66432052	-117.5792214
$(1_q^+, 2_{\bar{q}}^-, 3_g^+, 4_g^+, 5_g^-)$	8.000000000	51.38308777	127.3357346	55.24748112	-511.9128286
$(1_q^+, 2_{\bar{q}}^-, 3_g^+, 4_g^-, 5_g^+)$	8.000000000	51.38308777	137.2047686	143.1002284	-154.2224796
$(1_q^+, 2_{\bar{q}}^-, 3_g^-, 4_g^+, 5_g^+)$	8.000000000	51.38308777	133.2453937	110.9941406	-263.9507190
$\mathcal{A}^{(2)[N_f^1]}/\mathcal{A}^{(\text{norm})}$	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
$(1_q^+, 2_{\bar{q}}^-, 3_g^+, 4_g^+, 5_g^+)$	0	0	1.416882412	11.98234731	38.78056708
$(1_q^+, 2_{\bar{q}}^-, 3_g^+, 4_g^+, 5_g^-)$	0	0.6666666667	7.912904946	38.94492002	78.45710970
$(1_q^+, 2_{\bar{q}}^-, 3_g^+, 4_g^-, 5_g^+)$	0	0.6666666667	5.701796856	20.47669656	20.24036826
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$(1_q^+, 2_{\bar{q}}^-, 3_g^+, 4_g^+, 5_g^+)$	0	0	0	0.2361470687	2.541010053
$(1_q^+, 2_{\bar{q}}^-, 3_g^+, 4_g^+, 5_g^-)$	0	0	0	0.3690523831	3.782474720
$(1_q^+, 2_{\bar{q}}^-, 3_g^+, 4_g^-, 5_g^+)$	0	0	0	0.0005343680110	0.004830824685
$(1_q^+, 2_{\bar{q}}^-, 3_g^-, 4_g^+, 5_g^+)$	0	0	0	0.03001269961	0.3139119453

Normalization

$$\mathcal{A}^{(\text{norm})} = \begin{cases} \mathcal{A}^{(0)}, & \mathcal{A}^{(0)} \neq 0, \\ \mathcal{A}^{(1)[N_f^0]}(\epsilon = 0), & \mathcal{A}^{(0)} = 0 \end{cases}$$

Phase space point

$$s_{12} = -1, \quad s_{23} = -8/13, \quad s_{34} = -1094/2431,$$
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Results for $2q3g$ Amplitudes

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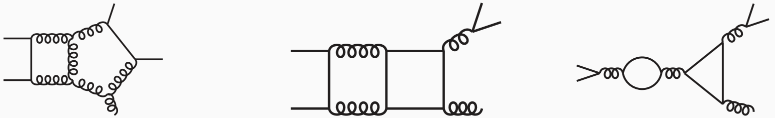
Results for $4q1g$ Amplitudes



$$A(1_q, 2_{\bar{q}}, 3_Q, 4_{\bar{Q}}, 5_g)\Big|_{\text{leading color}} =$$
$$(T^{a_5})_{i_3}^{\;\bar{\imath}_2}\delta_{i_1}^{\;\bar{\imath}_4}\,\mathcal{A}(1_q, 2_{\bar{q}}, 5_g, 3_Q, 4_{\bar{Q}}) \;+\; (T^{a_5})_{i_1}^{\;\bar{\imath}_4}\delta_{i_3}^{\;\bar{\imath}_2}\,\mathcal{A}(1_q, 2_{\bar{q}}, 3_Q, 4_{\bar{Q}}, 5_g)$$

- Partials expanded and decomposed as for $2q3g$
- Projected on the v_0 tensor structure

Results for $4q1g$ Amplitudes

$\mathcal{A}^{(2)[N_f^0]}/\mathcal{A}^{(\text{norm})}$	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
$(1_q^+, 2_{\bar{q}}^-, 3_Q^+, 4_{\bar{Q}}^-, 5_g^+)$	4.500000000	23.78050411	33.01035431	−76.65528489	−305.7123751
$(1_q^+, 2_{\bar{q}}^-, 3_Q^-, 4_{\bar{Q}}^+, 5_g^+)$	4.500000000	23.78050411	25.33119767	−122.8050519	−400.0885233
$(1_q^+, 2_{\bar{q}}^-, 3_Q^+, 4_{\bar{Q}}^-, 5_g^-)$	4.500000000	23.78050411	25.00917906	16.91995611	579.1225796
$(1_q^+, 2_{\bar{q}}^-, 3_Q^-, 4_{\bar{Q}}^+, 5_g^-)$	4.500000000	23.78050411	−1009.208812	−4797.768367	4827.790534
$\mathcal{A}^{(2)[N_f^1]}/\mathcal{A}^{(\text{norm})}$	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
$(1_q^+, 2_{\bar{q}}^-, 3_Q^+, 4_{\bar{Q}}^-, 5_g^+)$	0	2.500000000	17.25407596	48.27686582	11.71960460
$(1_q^+, 2_{\bar{q}}^-, 3_Q^-, 4_{\bar{Q}}^+, 5_g^+)$	0	2.500000000	17.27259645	44.99884204	−15.14666233
$(1_q^+, 2_{\bar{q}}^-, 3_Q^+, 4_{\bar{Q}}^-, 5_g^-)$	0	2.500000000	3.980556493	−29.18374008	−149.0347042
$(1_q^+, 2_{\bar{q}}^-, 3_Q^-, 4_{\bar{Q}}^+, 5_g^-)$	0	2.500000000	180.9505853	624.1255757	−2759.824817
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$(1_q^+, 2_{\bar{q}}^-, 3_Q^+, 4_{\bar{Q}}^-, 5_g^+)$	0	0	0.4444444444	3.910872659	18.01752271
$(1_q^+, 2_{\bar{q}}^-, 3_Q^-, 4_{\bar{Q}}^+, 5_g^+)$	0	0	0.4444444444	3.919103985	18.09637714
$(1_q^+, 2_{\bar{q}}^-, 3_Q^+, 4_{\bar{Q}}^-, 5_g^-)$	0	0	0.4444444444	−1.988469328	−28.36258323
$(1_q^+, 2_{\bar{q}}^-, 3_Q^-, 4_{\bar{Q}}^+, 5_g^-)$	0	0	0.4444444444	76.66487683	646.7253090

Normalization

$$\mathcal{A}^{(\text{norm})} = \mathcal{A}^{(0)}$$

Phase space point

$$s_{12} = -1, \quad s_{23} = -8/13, \quad s_{34} = -1094/2431,$$
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Outlook

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What's next?

- Non-planar topologies
  Sub-leading-color contributions, QCD corrections to electroweak production processes, etc.

- Beyond pure QCD
  - Processes with W/Z bosons, H and jets in the final state

- Towards integrated cross-section and automation of matrix element generation at NNLO

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Stay tuned!

Backup Slides

Finite Remainders

Inspired by [Weinzierl '11] [de Freitas, Bern '04]

Recall the basis of tensor structures

$$v_n^\dagger \cdot v_m = c_n \delta_m^n \,, \quad c_0 = 1 \quad \text{and} \quad c_{n>0}(D_s) = \mathcal{O}(\epsilon)$$

$$M_n^{(k)} = v^n \cdot M^{(k)} \qquad \Big| \qquad v^n \equiv \frac{1}{c_n} v_n^\dagger$$

Infrared poles of a *renormalized* QCD amplitude M_R have a universal structure

$$M_R^{(2)} = \mathbf{I}^{(2)} M_R^{(0)} + \mathbf{I}^{(1)} M_R^{(1)} + \mathcal{F}^{(2)}$$

where $\mathbf{I}^{(1)}$ and $\mathbf{I}^{(2)}$ are Catani operators in color space.

Since $\mathcal{F}^{(2)}$ is finite and $c_{n>0} = \mathcal{O}(\epsilon)$, we have

$$v^0 \cdot \mathcal{F}^{(2)} = \tilde{v}^0 \cdot \mathcal{F}^{(2)} + \mathcal{O}(\epsilon) \qquad \Big| \qquad \tilde{v}^0 = v^0 + \sum_i \alpha_i v^i$$

Tensor Decomposition for Identical Quarks

Requires an enlarged basis compared to the distinct-quark case.

1. Define the tensors $\{\tilde{v}_n\}$ as

$$(\tilde{v}_n)^{\lambda_1\lambda_2}_{\kappa_1\kappa_2} = (v_n)^{\lambda_2\lambda_1}_{\kappa_1\kappa_2}\,,$$

2. Decomposition is now over the sets $\{v_n\}$ and $\{\tilde{v}_n\}$. The basis tensors satisfy

$$v_n v^m = \delta_n^m\,, \qquad \tilde{v}_n \tilde{v}^m = \delta_n^m\,, \qquad v_n \tilde{v}^m = \delta_0^m\,\delta_{n,0} + \mathcal{O}(\epsilon)\,,$$

   where the set $\{\tilde{v}^n\}$ is constructed to be dual to $\{\tilde{v}_n\}$

Consider interference of the tree-level amplitude with the finite remainder, i.e.

$$\left(M^{(0)} - \tilde{M}^{(0)}\right)\cdot\left(\mathcal{F}^{(2)} - \tilde{\mathcal{F}}^{(2)}\right) = \left(M_0^{(0)} - \tilde{M}_0^{(0)}\right)\left(v_0\cdot\mathcal{F}^{(2)} - \tilde{v}_0\cdot\tilde{\mathcal{F}}^{(2)}\right) + \mathcal{O}(\epsilon),$$

Note

RHS only contains terms that can be computed through the “double trace” prescription

Derivation of c_n

Consider a Clifford algebra in d dimensions. Let $d_t = \mathrm{Tr}(\mathbb{1}_{[d]}) = 2^{d/2}$. We choose the basis

$$\gamma_{[d]}^{\mu_1\cdots\mu_n} = \frac{1}{n!}\sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)\gamma_{[d]}^{\mu_{\sigma(1)}}\ldots\gamma_{[d]}^{\mu_{\sigma n}},$$

We require the following traces of antisymmetric γ-matrix chains,

$$\mathrm{Tr}(\gamma_{[d]}^{\mu_1\cdots\mu_n}\gamma_{[d]\,\nu_m\ldots\nu_1}) = \begin{cases} d_t\sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)\delta_{\nu_1}^{\mu_{\sigma(1)}}\cdots\delta_{\nu_n}^{\mu_{\sigma(n)}} & m=n \\ 0 & m\neq n \end{cases},$$

$$\begin{aligned} c_n = v_n^\dagger\cdot v_n =& \mathrm{Tr}(\gamma_{[d]\,\mu_n\ldots\mu_1}\gamma_{[d]}^{\nu_1\cdots\nu_n})\,\mathrm{Tr}(\gamma_{[d]}^{\mu_n\cdots\mu_1}\gamma_{[d]\,\nu_1\ldots\nu_n}) \\ =& d_t^2\sum_{\sigma\in S_n}\sum_{\mu_1,\ldots,\mu_n}\sum_{\tilde\sigma\in S_n}\sum_{\nu_1,\ldots,\nu_n}\mathrm{sgn}(\sigma)\mathrm{sgn}(\tilde\sigma)\delta_{\nu_1}^{\mu_{\sigma(n)}}\cdots\delta_{\nu_n}^{\mu_{\sigma(1)}}\delta_{\mu_{\tilde\sigma(n)}}^{\nu_1}\cdots\delta_{\mu_{\tilde\sigma(1)}}^{\nu_n} \\ =& d_t^2\sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)\left(\sum_{\mu_1,\ldots,\mu_n}\sum_{\tilde\sigma\in S_n}\mathrm{sgn}(\tilde\sigma)\delta_{\mu_{\tilde\sigma(n)}}^{\mu_{\sigma(n)}}\cdots\delta_{\mu_{\tilde\sigma(1)}}^{\mu_{\sigma(1)}}\right) \\ =& d_t^2\sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)^2\frac{d!}{(d-n)!} = d_t^2\frac{d!\,n!}{(d-n)!} \end{aligned}$$

Finite Fields and Spinors

The unitarity approach can be formulated to employ only operations defined in any algebraic field $\mathbb{F}$ [Peraro'16]. In particular the finite field $\mathbb{Z}_p$ of all integers modulo a prime number p is useful.

We write the two-loop momenta as

$$\ell_1 = (\ell_{1,[4]}, \vec{\mu}_1)\,, \qquad \ell_2 = (\ell_{2,[4]}, \vec{\mu}_2)\,,$$

We choose an orthonormal basis $\vec{n}_i$ of the $(D-4)$-dimensional space

$$\vec{\mu}_1 = r_1 \vec{n}_1, \quad \vec{\mu}_2 = \frac{\mu_{12}}{\mu_{11}} r_1 \vec{n}_1 + r_2 \vec{n}_2 \quad \text{where} \quad r_1 = \sqrt{\mu_{11}}, \quad r_2 = \sqrt{\mu_{22} - \mu_{12}^2/\mu_{11}},$$

with $\mu_{ij} = \vec{\mu}_i \cdot \vec{\mu}_j$.

Berends-Giele currents are in general of the form — not $\mathbb{F}$-valued!

$$a_{00} + a_{10} r_1 + a_{01} r_2 + a_{11} r_1 r_2,$$

We consider the algebra $\mathbb{V}$ over the field $\mathbb{F}$, with $\mathbb{V}$ the vector space spanned by the basis $\{r_0 = 1, r_1, r_2, r_1 r_2\}$

Note

Coefficients of r_i cancel when amplitude is projected onto tensors v_n!

Basis of Numerator Functions

Recall the ansatz for master/surface decomposition:

$$\mathcal{A}(\ell_l) = \sum_{\text{Topologies } \Gamma} \sum_{i \in M_\Gamma \cup S_\Gamma} \frac{c_{\Gamma,i}\; m_{\Gamma,i}(\ell_l)}{\prod_{\text{props } j} \rho_j}$$

Loop momenta parameterization:

$$\ell_1 = (\ell_{1[4]}, \vec{\mu}_1), \quad \ell_2 = (\ell_{2[4]}, \vec{\mu}_2),$$

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- how numerator functions $m_{\Gamma,i}(\ell_l)$ depended on $\vec{\mu}_i$?
- can one use the same function basis (including only μ_{ij}) as for pure gluon amplitudes?

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- The issue was described in the literature for one-loop amplitudes with *massive* quarks [Fazio, Mastrolia, Mirabella, Bobadilla '14] [Badger, Brønnum-Hansen, Buciuni, O'Connell '17] [Anger, VS '18]
- With external quarks if one chooses a particular embedding of helicity spinors, numerators schematically are

Depends only on μ_{ij}

$$M_k(\ell_l) = \sum_{n,m} f_k^{\rho_1\cdots\rho_n,\sigma_1\cdots\sigma_m} \left(\prod_{i=1}^{n} \vec{\mu}_{1\,\rho_i}\right)\left(\prod_{j=1}^{m} \vec{\mu}_{2\,\sigma_i}\right)$$

$(D_s - 4)$ spinor indies

- For our definition of helicity amplitudes Lorentz invariance under rotations in $(D_s - 4)$ dimensions is explicit $\implies$ the same basis of numerator functions can be used

Basis Of Clifford Algebra

We construct $\{v_n\}$ starting from choosing a basis of Clifford algebra in $(D_s - 4)$:

anti-symmetrized

$$\gamma^{\mu_1 \ldots \mu_n}_{[D_s-4]} = \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \gamma^{\mu_{\sigma(1)}}_{[D_s-4]} \cdots \gamma^{\mu_{\sigma(n)}}_{[D_s-4]}$$

Note the following identities (k_μ — any 4-dim vector):

$$k_\mu \left(\gamma^\mu_{[D_s]}\right)^{b\lambda}_{a\kappa} = k_\mu \left(\gamma^\mu_{[4]}\right)^{b}_{a} \delta^\lambda_\kappa \qquad \Bigg| \qquad \left(\gamma^\mu_{[D_s]}\right)^{b_1 \lambda_1}_{a\kappa} \left(\gamma_{[D_s]\mu}\right)^{b\lambda}_{b_1 \lambda_1} = D_s \, \delta^b_a \delta^\lambda_\kappa$$

Particularly useful in HV! (external gluon states)