A local analytic sector subtraction at NNLO

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Motivation for a new subtraction scheme

- Several schemes available for NNLO subtraction/slicing.
    - Slicing: qT [Catani, Grazzini, et al.], N-jettiness [Boughezal, Petriello, et al.], [Gaunt, Tackmann, et al.].
    - Subtraction: Antennae [DeRidder, Gehrmann, Glover, et al.], Stripper [Czakon, Mitov, et al.], nested soft-collinear [Caola, Melnikov, et al.], Colourful [Del Duca, Troscanyi, et al.], projection to Born [Salam, et al.], sector decomposition [Anastasiou, et al.], [Binoth, Heinrich, et al.], $\mathcal{E}$-prescription [Frixione, Grazzini], geometric [Herzog].
    - New developments: loop-tree duality [Rodrigo, et al.], FDR [Pittau, et al.].
    - Some methods already applied to N^{3}LO: projection to Born [Currie, et al., 1803.09973], qT [Cieri, et al., 1807.11501].
- Complexity in the subtraction increases a lot with respect to NLO, room for studies/improvements.

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  - Some methods already applied to N^{3}LO: projection to Born [Currie, et al., 1803.09973], qT [Cieri, et al., 1807.11501].
- Complexity in the subtraction increases a lot with respect to NLO, room for studies/improvements.
- Our motivation for studying a new scheme:
  - How much can one simplify subtraction and involved calculations?
  - What NLO properties/choices can be usefully exported to NNLO?
- In the following, still partial results on massless and final-state-only QCD partons.

NLO

Subtracted NLO cross sections

- NLO coefficient of the differential cross section with respect to X ($X =$ IRC safe, $X_i =$ observable computed with i-body kinematics, $\delta_i \equiv \delta(X - X_i)$):

$$\frac{d\sigma_{\rm NLO}}{dX} = \int d\Phi_n \, V \, \delta_n + \int d\Phi_{n+1} \, R \, \delta_{n+1} .$$

- Add and subtract local counterterm $\overline{K}$:

$$\int d\Phi_{n+1} \, \overline{K} \, \delta_n \, .$$

- $\overline{K}$ = same singularities as R, locally in phase space, but simple enough to be integrated analytically in $d \neq 4$.

- d-dimensional integrated counterterm:

$$I = \int d\Phi_{\rm rad} \, \overline{K} \, , \qquad d\Phi_{\rm rad} = d\Phi_{n+1} \, / \, d\Phi_n .$$

- Subtracted NLO coefficient

$$\frac{d\sigma_{\rm NLO}}{dX} = \int d\Phi_n \, (V + I) \, \delta_n + \int d\Phi_{n+1} (R \, \delta_{n+1} - \overline{K} \, \delta_n).$$

- Integrals $\int (V + I)$ and $\int (R - \overline{K})$ separately finite and evaluated numerically in $d = 4$.

NLO sectors (à la FKS) [Frixione, Kunszt, Signer, 9512328]

- Partition phase space Φ_{n+1} with sector functions $\mathcal{W}_{ij}$, (normalised as $\sum_{i,\,j\neq i} \mathcal{W}_{ij} = 1$), such that $R\,\mathcal{W}_{ij}$ is singular only in one soft ($\mathbf{S}_i$) and one collinear ($\mathbf{C}_{ij}$) configuration.

- Sum rules:

$$\mathbf{S}_i \sum_{k\neq i} \mathcal{W}_{ik} = 1\,, \qquad \mathbf{C}_{ij} \sum_{ab\,\in\,\mathrm{perm}(ij)} \mathcal{W}_{ab} = 1\,,$$

- Summing over all sectors sharing a given singularity, and taking that singular limit on the sum, the $\mathcal{W}$'s disappear. Key for simplifying analytic integration of $\overline{K}$.

- Example of sector functions ($s_{qi} = 2\,q_{\mathrm{cm}}\cdot k_i, \quad s_{ij} = 2\,k_i\cdot k_j$), very similar to those used in MadFKS [Frederix, et al., 0908.4272]:

$$\mathcal{W}_{ij} = \frac{\sigma_{ij}}{\sum\limits_{k,\,l\neq k} \sigma_{kl}}\,, \qquad \text{with} \qquad \sigma_{ij} = \frac{1}{e_i\,w_{ij}}\,, \qquad e_i = \frac{s_{qi}}{s}\,, \qquad w_{ij} = \frac{s\,s_{ij}}{s_{qi}\,s_{qj}}\,.$$

Structure of NLO subtraction

- Singularities of real matrix element in sector ij known in advance in terms of dot products s_{ab}, without parametrising the sector (at variance with FKS).

- $\mathbf{S}_i\,R$ = leading term in R as $k_i^\mu \to 0$.
  $\mathbf{C}_{ij}\,R$ = leading term in R as relative $k_\perp^\mu \to 0$.

$$\begin{aligned}
\mathbf{S}_i\,R\,(\{k\}) \quad &= \quad -\mathcal{N}_1 \sum_{l,\,m} \delta_{f_i g}\,\frac{s_{lm}}{s_{il}\,s_{im}}\,B_{lm}\big(\{k\}_{\not i}\big)\ , \\
\mathbf{C}_{ij}\,R\,(\{k\}) \quad &= \quad \frac{\mathcal{N}_1}{s_{ij}}\,P_{ij}^{\mu\nu}(s_{ir},s_{jr})\,B_{\mu\nu}\Big(\{k\}_{\not i \not j},k\Big)\ , \\
\mathbf{S}_i\,\mathbf{C}_{ij}\,R\,(\{k\}) \quad &= \quad 2\,\mathcal{N}_1\,C_{f_j}\,\delta_{f_i g}\,\frac{s_{jr}}{s_{ij}\,s_{ir}}\,B\big(\{k\}_{\not i}\big)\ .
\end{aligned}$$

- Candidate counterterm in sector ij: $K_{ij} = (\mathbf{S}_i + \mathbf{C}_{ij} - \mathbf{S}_i\,\mathbf{C}_{ij})R\,\mathcal{W}_{ij}$
  (limits applied to both R and $\mathcal{W}_{ij}$), limits commute.

- As minimal as FKS, but no parametrisation yet: freedom to be exploited to simplify analytic integration.

Mapping from NLO to Born kinematics (à la CS) [Catani, Seymour, 9605323]

- Need a momentum mapping $\{k_1, ..., k_{n+1}\} \to \{\bar{k}_1, ..., \bar{k}_n\}$ to factorise radiation phase space from Born phase-space, and integrate conuterterm in the latter.

- Catani-Seymour massless final-state mapping $\{k\} \to \{\bar{k}\}^{(abc)}$:

$$\bar{k}_i^{(abc)} = k_i, \quad \text{if } i \neq a, b, c,$$

$$\bar{k}_b^{(abc)} = k_a + k_b - \frac{s_{ab}}{s_{ac} + s_{bc}}\, k_c\,, \qquad \bar{k}_c^{(abc)} = \frac{s_{abc}}{s_{ac} + s_{bc}}\, k_c\,,$$

with $s_{abc} = s_{ab} + s_{ac} + s_{bc}$, and $\bar{k}_b^{(abc)} + \bar{k}_c^{(abc)} = k_a + k_b + k_c$.

Phase-space parametrisation (à la CS)

- Catani-Seymour variables $y, z \in [0,1]$ for mapping $\{k\} \to \{\bar{k}\}^{(abc)}$:

$$s_{ab} = y\, s_{abc}\,, \qquad s_{ac} = z(1-y)\, s_{abc}\,, \qquad s_{bc} = (1-z)(1-y)\, s_{abc}\,.$$

- Phase-space factorisation:

$$d\Phi_{n+1} = d\Phi_n^{(abc)}\, d\Phi_{\rm rad}^{(abc)}\,, \qquad d\Phi_{\rm rad}^{(abc)} \equiv d\Phi_{\rm rad}\left(\bar{s}_{bc}^{(abc)}; y, z, \phi\right)\,,$$

$$\int d\Phi_{\rm rad}\,(s; y, z, \phi) \equiv N(\epsilon)\, s^{1-\epsilon} \int_0^\pi d\phi\, \sin^{-2\epsilon}\phi \int_0^1 dy \int_0^1 dz \Big[y(1-y)^2\, z(1-z)\Big]^{-\epsilon}(1-y)\,,$$

$$N(\epsilon) \equiv \frac{(4\pi)^{\epsilon-2}}{\sqrt{\pi}\,\Gamma(1/2-\epsilon)}\,, \qquad \bar{s}_{bc}^{(abc)} \equiv 2\,\bar{k}_b^{(abc)} \cdot \bar{k}_c^{(abc)} = s_{abc}\,.$$

- ϕ = azimuth between $\vec{k}_a$ and an reference three-momentum ($\neq \vec{k}_b, \vec{k}_c$).

Local-counterterm definition

- Mapping $\{k\} \to \{\bar{k}\}^{(abc)}$: freedom to choose labels a, b, c as we want. Adapt the choice to the invariants appearing in the kernels.

- $\mathbf{C}_{ij}\,R$ features invariants s_{ij}, s_{ir}, and s_{jr}: choose $(abc) = (ijr)$. Each term in the eikonal sum in $\mathbf{S}_i\,R$ features s_{il}, s_{im}, and s_{lm}: choose $(abc) = (ilm)$.

- Remapped singular limits:

$$\begin{aligned}
\overline{\mathbf{S}}_i\,R\left(\{k\}\right) &= -\mathcal{N}_1 \sum_{l,\,m} \delta_{f_i g}\,\frac{s_{lm}}{s_{il}\,s_{im}}\,B_{lm}\Big(\{\bar{k}\}^{(ilm)}\Big)\,,\\
\overline{\mathbf{C}}_{ij}\,R\left(k\right) &= \frac{\mathcal{N}_1}{s_{ij}}\,P_{ij}^{\mu\nu}(s_{ir}, s_{jr})\,B_{\mu\nu}\Big(\{\bar{k}\}^{(ijr)}\Big)\,,\\
\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,R\left(\{k\}\right) &= 2\,\mathcal{N}_1\,C_{f_j}\,\delta_{f_i g}\,\frac{s_{jr}}{s_{ij}\,s_{ir}}\,B\Big(\{\bar{k}\}^{(ijr)}\Big)\,,
\end{aligned}$$

- Local-counterterm definition:

$$\overline{K}_{ij} \equiv \left(\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\right) R\,\mathcal{W}_{ij}\,, \qquad \overline{K} = \sum_{i,j\neq i} \overline{K}_{ij}\,,$$

where barred limits on $\mathcal{W}$'s act as unbarred ones.

NLO-counterterm integration (I)

$$\overline{K} \;=\; \sum_{i,j\neq i} \overline{K}_{ij} \;=\; \sum_{i} \overline{\mathbf{S}}_i\, R \;+\; \sum_{i,\,j>i} \overline{\mathbf{C}}_{ij}\, \left(1 - \overline{\mathbf{S}}_i - \overline{\mathbf{S}}_j\right) R\,.$$

sum rules ↗

- Explicit soft integrated counterterm (ς_k = symmetry factor of k-body phase space):

$$\begin{aligned} I^{\,\rm s} \;&=\; -\mathcal{N}_1\, \frac{\varsigma_{n+1}}{\varsigma_n} \sum_{i} \delta_{f_i g} \sum_{\substack{l\neq i\\ m\neq i}} B_{lm}\Big(\{\bar{k}\}^{(ilm)}\Big)\, \frac{1}{\bar{s}_{lm}^{(ilm)}} \int d\Phi_{\rm rad}\left(\bar{s}_{lm}^{(ilm)}; y, z, \phi\right) \frac{1-z}{yz} \\ &=\; -\mathcal{N}_1\, \frac{\varsigma_{n+1}}{\varsigma_n} \sum_{i} \delta_{f_i g} \sum_{\substack{l\neq i\\ m\neq i}} B_{lm}\Big(\{\bar{k}\}^{(ilm)}\Big)\, \frac{(4\pi)^{\epsilon-2}}{\big(\bar{s}_{lm}^{(ilm)}\big)^{\epsilon}}\, \frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon^2\,\Gamma(2-3\epsilon)}\,. \end{aligned}$$

NLO-counterterm integration (II)

- Full result, including hard-collinear

$$
\begin{aligned}
I(\{\bar{k}\}) \quad = \quad & -\mathcal{N}_1 \sum_{l,\, m\neq l} \frac{(4\pi)^{\epsilon-2}}{\bar{s}_{lm}^{\,\epsilon}} \, \frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon^2\,\Gamma(2-3\epsilon)} \; B_{lm}\big(\{\bar{k}\}\big) \\
& -\mathcal{N}_1 \sum_{p} \frac{(4\pi)^{\epsilon-2}}{\bar{s}_{pr}^{\,\epsilon}} \, \frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon\,\Gamma(2-3\epsilon)} \, \mathbb{C}\, B\big(\{\bar{k}\}\big)\,,
\end{aligned}
$$

with $\mathbb{C} = \frac{C_A+4\,T_R\,N_f}{2(3-2\epsilon)}\,\delta_{f_p g} + \frac{C_F}{2}\delta_{f_p \{q,\bar{q}\}}$.

- Result exact in ϵ. Not important per se, but a sign of simplicity.

- Virtual ϵ poles analytically reproduced in general.

- Finite parts checked differentially in a variety of cases.

NLO summary

- Partition functions and their sum rules are convenient tools, as in FKS.
- Adapt CS mappings to the involved invariants term by term $\Longrightarrow$ simplifications in analytic counterterm integration.
- Like a bridge between FKS and CS (sector approach, and minimal structure from FKS; Lorentz invariance, and mappings from CS).
- These features can be exported to NNLO.

NNLO

Subtracted NNLO cross sections

- NNLO coefficient of the differential cross section:

$$\frac{d\sigma_{\rm NNLO}}{dX} \;=\; \int d\Phi_n\, VV\, \delta_n + \int d\Phi_{n+1}\, RV\, \delta_{n+1} + \int d\Phi_{n+2}\, RR\, \delta_{n+2}.$$

- Add and subtract local counterterms:

$$\int d\Phi_{n+2}\, \overline{K}^{\,(\mathbf{1})}\, \delta_{n+1}\,, \qquad \int d\Phi_{n+2}\, (\overline{K}^{\,(\mathbf{2})} + \overline{K}^{\,(\mathbf{12})})\, \delta_n\,, \qquad \int d\Phi_{n+1}\, \overline{K}^{(\mathbf{RV})}\, \delta_n\,.$$

- d-dimensional integrated counterterms ($d\Phi_{{\rm rad},i} \;=\; d\Phi_{n+2} \,/\, d\Phi_{n+2-i}$):

$$I^{(\mathbf{i})} \;=\; \int d\Phi_{{\rm rad},i}\, \overline{K}^{(\mathbf{i})}\,, \qquad I^{\,(\mathbf{12})} \;=\; \int d\Phi_{{\rm rad},1}\, \overline{K}^{\,(\mathbf{12})}\,, \qquad I^{(\mathbf{RV})} \;=\; \int d\Phi_{\rm rad}\, \overline{K}^{(\mathbf{RV})}\,,$$

- Subtracted NNLO coefficient:

$$\begin{aligned}\frac{d\sigma_{\rm NNLO}}{dX} \;=\; & \int d\Phi_n \left(VV + I^{\,(\mathbf{2})} + I^{(\mathbf{RV})}\right) \delta_n \\ & + \int d\Phi_{n+1} \Big[\left(RV + I^{\,(\mathbf{1})}\right) \delta_{n+1} \;-\; \left(\overline{K}^{(\mathbf{RV})} - I^{\,(\mathbf{12})}\right) \delta_n\Big] \\ & + \int d\Phi_{n+2} \Big[RR\, \delta_{n+2} - \overline{K}^{\,(\mathbf{1})}\, \delta_{n+1} - \left(\overline{K}^{\,(\mathbf{2})} + \overline{K}^{\,(\mathbf{12})}\right) \delta_n\Big]\,.\end{aligned}$$

- Each line separately finite and evaluated numerically in $d = 4$.

NNLO sectors

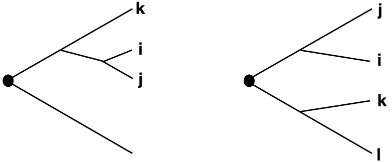
- Partition of Φ_{n+2} through sector functions $\mathcal{W}_{ijkl}$, (normalised as $\sum_{ijkl} \mathcal{W}_{ijkl} = 1$), to select as few singularities at a time as possible. Our choice:

$$\mathcal{W}_{ijkl} = \frac{\sigma_{ijkl}}{\sum\limits_{a,\, b\neq a} \sum\limits_{\substack{c\neq a \\ d\neq a,c}} \sigma_{abcd}} , \qquad \sigma_{ijkl} = \frac{1}{e_i^{\alpha} \, w_{ij}^{\beta}} \frac{1}{(e_k + \delta_{kj}\, e_i)\, w_{kl}} , \qquad \alpha > \beta > 1 .$$

- $RR\,\mathcal{W}_{abcd}$ is singular only in few kinematic configurations ($\mathbf{S}_{ab} = a\,b$ uniformly soft, $\mathbf{C}_{ijk} = j\,k$ uniformly collinear to i, and so on)

$$\begin{aligned} \mathcal{W}_{ijjk} &: \quad \mathbf{S}_i\,, \quad \mathbf{C}_{ij}\,, \quad \mathbf{S}_{ij}\,, \quad \mathbf{C}_{ijk}\,, \quad \mathbf{SC}_{ijk}\,; \\ \mathcal{W}_{ijkj} &: \quad \mathbf{S}_i\,, \quad \mathbf{C}_{ij}\,, \quad \mathbf{S}_{ik}\,, \quad \mathbf{C}_{ijk}\,, \quad \mathbf{SC}_{ijk}\,, \quad \mathbf{CS}_{ijk}\,; \\ \mathcal{W}_{ijkl} &: \quad \mathbf{S}_i\,, \quad \mathbf{C}_{ij}\,, \quad \mathbf{S}_{ik}\,, \quad \mathbf{C}_{ijkl}\,, \quad \mathbf{SC}_{ikl}\,, \quad \mathbf{CS}_{ijk}\,. \end{aligned}$$

- Roughly, sector functions select two topologies (left: $\mathcal{W}_{ijjk}$, $\mathcal{W}_{ijkj}$, right: $\mathcal{W}_{ijkl}$)



NNLO sectors: properties (I)

- Sum rules in double-unresolved limits: by summing over all sectors sharing the same singularity, and taking that singular limit on the sum, $\mathcal{W}$ functions disappear.

$$\mathbf{S}_{ik}\left(\sum_{b\neq i}\sum_{d\neq i,k}\mathcal{W}_{ibkd}+\sum_{b\neq k}\sum_{d\neq k,i}\mathcal{W}_{kbid}\right)=1\,,$$

$$\mathbf{C}_{ijk}\sum_{abc\,\in\,\mathrm{perm}(ijk)}\left(\mathcal{W}_{abbc}+\mathcal{W}_{abcb}\right)=1\,,\qquad \mathbf{C}_{ijkl}\sum_{\substack{ab\,\in\,\mathrm{perm}(ij)\\ cd\,\in\,\mathrm{perm}(kl)}}\left(\mathcal{W}_{abcd}+\mathcal{W}_{cdab}\right)=1\,,$$

$$\mathbf{SC}_{ikl}\sum_{b\neq i}\left(\mathcal{W}_{ibkl}+\mathcal{W}_{iblk}\right)=1\,,\qquad \mathbf{CS}_{ijk}\left(\sum_{d\neq i,k}\mathcal{W}_{ijkd}+\sum_{d\neq j,k}\mathcal{W}_{jikd}\right)=1\,.$$

- Key for simplifying analytic integration of double-unresolved counterterms.

NNLO sectors: properties (II)

- In the single-unresolved limits, NNLO sector functions factorise NLO sector functions. For example

$$\mathbf{C}_{ij}\,\mathcal{W}_{ijkl} \;=\; \mathcal{W}_{kl}\,\mathbf{C}_{ij}\,\mathcal{W}_{ij}^{(\alpha\beta)}\,, \qquad\qquad \mathbf{S}_{i}\,\mathcal{W}_{ijkl} \;=\; \mathcal{W}_{kl}\,\mathbf{S}_{i}\,\mathcal{W}_{ij}^{(\alpha\beta)}\,,$$

  where

$$\mathcal{W}_{ij}^{(\alpha\beta)} \;=\; \frac{\sigma_{ij}^{(\alpha\beta)}}{\sum\limits_{a,\,b\neq a}\sigma_{ab}^{(\alpha\beta)}}\,, \qquad\qquad \sigma_{ab}^{(\alpha\beta)} \;=\; \frac{1}{(e_a)^{\alpha}(w_{ab})^{\beta}}\,.$$

  with the same properties of NLO sector functions.

- Allows $(RV + I^{\,(\mathbf{1})})$ and $(K^{(\mathbf{RV})} - I^{\,(\mathbf{12})})$ to be finite in $d = 4$ NLO sector by NLO sector.

NNLO counterterms

- In each sector, candidate (i.e. not yet momentum-remapped) counterterms built collecting singular limits of $RR\,\mathcal{W}$, written in terms of dot products.

- Example for sector $\mathcal{W}_{ijkj}$ (where nonzero limits are $\mathbf{S}_i$, $\mathbf{C}_{ij}$, $\mathbf{S}_{ik}$, $\mathbf{C}_{ijk}$, $\mathbf{SC}_{ijk}$, $\mathbf{CS}_{ijk}$):

$$
\begin{aligned}
K^{(\mathbf{1})}_{ijkj} &= \Big[\mathbf{S}_i + \mathbf{C}_{ij}(1-\mathbf{S}_i)\Big]\,RR\,\mathcal{W}_{ijkj}\,, \\
K^{(\mathbf{2})}_{ijkj} &= \Big[\mathbf{S}_{ik} + \mathbf{C}_{ijk}(1-\mathbf{S}_{ik}) + \mathbf{SC}_{ijk}(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk}) \\
&\qquad + \mathbf{CS}_{ijk}(1-\mathbf{SC}_{ijk})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk})\Big]\,RR\,\mathcal{W}_{ijkj}\,, \\
K^{(\mathbf{12})}_{ijkj} &= -\Big[\mathbf{S}_i + \mathbf{C}_{ij}(1-\mathbf{S}_i)\Big]\Big[\mathbf{S}_{ik} + \mathbf{C}_{ijk}(1-\mathbf{S}_{ik}) + \mathbf{SC}_{ijk}(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk}) \\
&\qquad + \mathbf{CS}_{ijk}(1-\mathbf{SC}_{ijk})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk})\Big]\,RR\,\mathcal{W}_{ijkj}\,,
\end{aligned}
$$

  and analogously for sectors $\mathcal{W}_{ijjk}$ and $\mathcal{W}_{ijkl}$.

- $\mathbf{S}_{ij}\,RR$, $\mathbf{C}_{ikj}\,RR$, and $\mathbf{SC}_{ijk}\,RR$ are universal kernels [Catani, Grazzini, 9810389, 9908523], [Campbell, Glover, 9710255], [Berends, Giele, 1989].

- All limits commute on RR and $\mathcal{W}$ functions.

NNLO-counterterm simplifications

- Simplifications possible, thanks to idempotency relations

$$(1-\mathbf{S}_i)\,\mathbf{SC}_{icd}\,RR\,\mathcal{W}_{ibcd}=0\,, \qquad\qquad (1-\mathbf{C}_{ij})\,\mathbf{CS}_{ijk}\,RR\,\mathcal{W}_{ijkd}=0\,.$$

- Limits **SC** and **CS** disappear from $K^{(\mathbf{2})}+K^{(\mathbf{12})}$ (see also [Caola, Melnikov, Roentsch] about redundancy of **SC** in nested soft-collinear subtraction):

$$K^{(\mathbf{2})}_{ijkj}+K^{(\mathbf{12})}_{ijkj} \quad = \quad (1-\mathbf{S}_i)(1-\mathbf{C}_{ij})\Big[\mathbf{S}_{ik}+\mathbf{C}_{ijk}(1-\mathbf{S}_{ik})\Big]\,RR\,\mathcal{W}_{ijkj}\,,$$

  very simple structure!

- Still, since integrated $I^{(\mathbf{12})}$ and $I^{(\mathbf{2})}$ enter separately, they receive contributions from **SC** and **CS** (which however cancel in the sum).

Counterterm $\overline{K}^{\,(\mathbf{1})}$ and integrated $I^{\,(\mathbf{1})}$

- Use factorisation properties of $\mathcal{W}_{abcd}$, and sum rules of $\mathcal{W}_{ab}^{(\alpha\beta)}$:

$$\overline{K}^{(\mathbf{1})} = \sum_{k,l} \boxed{\overline{\mathcal{W}}_{kl}} \boxed{\left[\sum_{i,\,j>i} \overline{\mathbf{C}}_{ij} \left(1 - \overline{\mathbf{S}}_i - \overline{\mathbf{S}}_j\right) RR + \sum_i \overline{\mathbf{S}}_i \, RR \right]} = \sum_{k,l} \overline{K}_{kl}^{(\mathbf{1})} \,.$$

in each NLO sector

full structure of single-unres. singularities

- $I^{\,(\mathbf{1})}$ is the same integral as the NLO integrated counterterm I (known to all orders in ϵ):

$$\begin{aligned} I_{kl}^{(\mathbf{1})}(\{\bar{k}\}) \quad = \quad & -\mathcal{N}_1 \sum_{a,\,b\neq a} \frac{(4\pi)^{\epsilon-2}}{\bar{s}_{ab}^{\,\epsilon}} \, \frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon^2\,\Gamma(2-3\epsilon)} \; R_{ab}(\{\bar{k}\}) \; \overline{\mathcal{W}}_{kl}(\{\bar{k}\}) \\ & -\mathcal{N}_1 \sum_{p} \frac{(4\pi)^{\epsilon-2}}{\bar{s}_{pr}^{\,\epsilon}} \, \frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon\,\Gamma(2-3\epsilon)} \, \mathbb{C}\, R(\{\bar{k}\}) \; \overline{\mathcal{W}}_{kl}(\{\bar{k}\}) \ , \end{aligned}$$

- $RV\,\overline{\mathcal{W}}_{kl} + I_{kl}^{(\mathbf{1})}$ finite in $d=4$ sector by sector in the NLO phase space.

Integration of counterterm $\overline{K}^{(\mathbf{12})}$

- Use factorisation properties of $\mathcal{W}_{abcd}$, and sum rules of $\mathcal{W}_{ab}^{(\alpha\beta)}$:

$$
\begin{aligned}
I_{kl}^{(\mathbf{12})} = & \\
& -\mathcal{N}_1 \sum_{a,\, b\neq a} \frac{(4\pi)^{\epsilon-2}}{\bar{s}_{ab}^{\,\epsilon}} \, \frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon^2\,\Gamma(2-3\epsilon)} \Big[\overline{\mathbf{S}}_k + \overline{\mathbf{C}}_{kl}\left(1-\overline{\mathbf{S}}_k\right)\Big] R_{ab}\big(\{\bar{k}\}\big)\, \overline{\mathcal{W}}_{kl}\big(\{\bar{k}\}\big) \\
& -\mathcal{N}_1 \sum_{p} \frac{(4\pi)^{\epsilon-2}}{\bar{s}_{pr}^{\,\epsilon}} \, \frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon\,\Gamma(2-3\epsilon)} \, \mathbb{C}\Big[\overline{\mathbf{S}}_k + \overline{\mathbf{C}}_{kl}\left(1-\overline{\mathbf{S}}_k\right)\Big] R\big(\{\bar{k}\}\big)\, \overline{\mathcal{W}}_{kl}\big(\{\bar{k}\}\big)
\end{aligned}
$$

- $\overline{K}_{kl}^{(\mathbf{RV})} - I_{kl}^{(\mathbf{12})}$ finite in $d = 4$ sector by sector in the NLO phase space.

Counterterm $\overline{K}^{(\mathbf{2})}$

- Using sum rules, $\mathcal{W}$'s disappear from $\overline{K}^{(\mathbf{2})}$ and from its integral $I^{(\mathbf{2})}$. In the end:

$$
\begin{aligned}
\overline{K}^{(\mathbf{2})} = \sum_{i} \Bigg\{ & \sum_{j>i} \overline{\mathbf{S}}_{ij} + \sum_{j>i} \sum_{k>j} \overline{\mathbf{C}}_{ijk} \left(1 - \overline{\mathbf{S}}_{ij} - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{jk}\right) \\
& + \sum_{j>i} \sum_{\substack{k>i \\ k\neq j}} \sum_{\substack{l>k \\ l\neq j}} \overline{\mathbf{C}}_{ijkl} \left(1 - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{jk} - \overline{\mathbf{S}}_{il} - \overline{\mathbf{S}}_{jl}\right) \\
& + \sum_{j\neq i} \sum_{\substack{k\neq i \\ k>j}} \overline{\mathbf{SC}}_{ijk} \left(1 - \overline{\mathbf{S}}_{ij} - \overline{\mathbf{S}}_{ik}\right) \Bigg(1 - \overline{\mathbf{C}}_{ijk} - \sum_{l\neq i,j,k} \overline{\mathbf{C}}_{iljk}\Bigg) \\
& + \sum_{j>i} \sum_{k\neq i,j} \overline{\mathbf{CS}}_{ijk} \left(1 - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{jk}\right) \Bigg(1 - \overline{\mathbf{C}}_{ijk} - \sum_{l\neq i,j,k} \overline{\mathbf{C}}_{ijkl}\Bigg) \Bigg\} RR \,,
\end{aligned}
$$

- Analytic integration of a set of universal NNLO kernels with no $\mathcal{W}$ functions.

- As at NLO, different terms in the same kernel mapped differently to ease integration.

Mappings from NNLO to Born kinematics

- $\{k\} \to \{\bar{k}\}^{(abcd)}$ mapping example, $d\Phi_{n+2} \;=\; d\Phi_n^{(abcd)} \times d\Phi_{\mathrm{rad},2}^{(abcd)}$

$$\bar{k}_n^{(abcd)} = k_n\,, \qquad n \neq a,b,c,d\,,$$

$$\bar{k}_c^{(abcd)} = k_a + k_b + k_c - \frac{s_{abc}}{s_{ad}+s_{bd}+s_{cd}}\,k_d\,, \qquad \bar{k}_d^{(abcd)} = \frac{s_{abcd}}{s_{ad}+s_{bd}+s_{cd}}\,k_d\,,$$

with $s_{abcd} = s_{ab}+s_{ac}+s_{ad}+s_{bc}+s_{bd}+s_{cd}$ and $\bar{k}_c^{(abcd)} + \bar{k}_d^{(abcd)} = k_a+k_b+k_c+k_d$.

- This is used to define double-collinear $\overline{\mathbf{C}}_{ijk}\,RR$ and (part of) the double-soft $\overline{\mathbf{S}}_{ij}\,RR$ counterterms:

$$\overline{\mathbf{S}}_{ij}\,RR \;=\; \frac{\mathcal{N}_1^2}{2}\sum_{\substack{c\neq i,j\\ d\neq i,j,c}} \mathcal{I}_{cd}^{(ij)}\,B_{cd}\Big(\{\bar{k}\}^{(ijcd)}\Big) + \ldots,$$

$$\overline{\mathbf{C}}_{ijk}\,RR \;=\; \frac{\mathcal{N}_1^2}{s_{ijk}^2}\,P_{ijk}^{\mu\nu}\,B_{\mu\nu}\Big(\{\bar{k}\}^{(ijkr)}\Big)$$

Example integration of $\overline{K}^{(\mathbf{2})}$

- Example double-soft $q\bar{q}$: each term of the sum parametrised with $(abcd) = (ijlm)$.

$$\begin{aligned}
\int d\Phi_{\mathrm{rad},2}\, \overline{\mathbf{S}}_{ij}\, RR &= \mathcal{N}_1^2\, T_R \sum_{l,m=1}^{2} B_{lm}\Big(\{\bar{k}\}^{(ijlm)}\Big) \int d\Phi^{(ijlm)}_{\mathrm{rad},2}\, \frac{s_{il}s_{jm} + s_{im}s_{jl} - s_{ij}s_{lm}}{s_{ij}^2\,(s_{il}+s_{jl})\,(s_{im}+s_{jm})} \\
&= \mathcal{N}_1^2\, B\, T_R\, C_F\, \frac{8}{s^2} \int d\Phi_{\mathrm{rad},2}\,(s; y, z, \phi, y', z', x')\, \frac{z'\,(1-z')}{y^2 y'^2}\, \frac{y'\,(1-z)}{y'\,(1-z)+z} \\
&= B \left(\frac{\alpha_{\mathrm{S}}}{2\pi}\right)^2 T_R C_F \left(\frac{\mu^2}{s}\right)^{2\epsilon} \left[-\frac{1}{3\epsilon^3} - \frac{17}{9\epsilon^2} + \frac{1}{\epsilon}\left(\frac{7}{18}\pi^2 - \frac{232}{27}\right) + \left(\frac{38}{9}\zeta_3 + \frac{131}{54}\pi^2 - \frac{2948}{81}\right)\right] + \mathcal{O}(\epsilon).
\end{aligned}$$

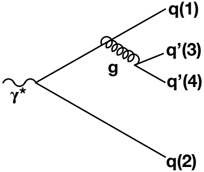
- Double-collinear $q \to q q' \bar{q}'$, parametrised with $(abcd) = (ijkr)$.

$$\begin{aligned}
\int d\Phi^{(ijkr)}_{\mathrm{rad},2}\, \overline{\mathbf{C}}_{ijk} RR \;&=\; \mathcal{N}_1^2 T_R C_F B \int d\Phi^{(ijkr)}_{\mathrm{rad},2}\, \frac{1}{2 s_{ijk} s_{ik}} \left[- \frac{t_{ik,j}^2}{s_{ik} s_{ikj}} + \frac{4z_j + (z_i - z_k)^2}{z_i + z_k} + (1-2\epsilon)\Big(z_i + z_k - \frac{s_{ik}}{s_{ikj}}\Big)\right] \\
&= B \left(\frac{\alpha_{\mathrm{S}}}{2\pi}\right)^2 T_R C_F \left(\frac{\mu^2}{s}\right)^{2\epsilon} \left[-\frac{1}{3\epsilon^3} - \frac{31}{18\epsilon^2} + \frac{1}{\epsilon}\left(\frac{1}{2}\pi^2 - \frac{889}{108}\right) + \left(\frac{80}{9}\zeta_3 + \frac{31}{12}\pi^2 - \frac{23941}{648}\right)\right] + \mathcal{O}(\epsilon)\,.
\end{aligned}$$

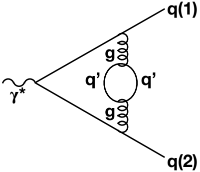
- Other kernels more complicated, but manageable analytically in the massless case (ongoing).

Proof-of-concept example

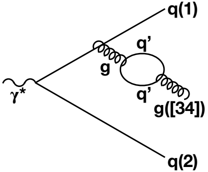
- $T_R C_F$ contribution to $\sigma_{\mathrm{NNLO}}(e^+e^- \to q\bar{q})$



RR



VV



RV

- Finiteness in the n-body phase space:

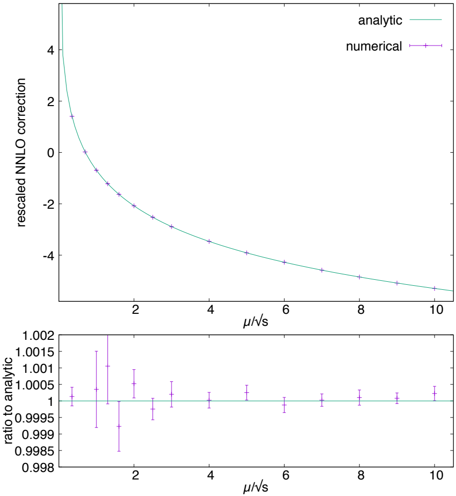
$$VV + I^{\,(\mathbf{2})} + I^{(\mathbf{RV})} \quad = \quad B\left(\frac{\alpha_{\mathrm{S}}}{2\pi}\right)^2 T_R C_F \left(\frac{8}{3}\zeta_3 - \frac{1}{9}\pi^2 - \frac{44}{9} - \frac{4}{3}\ln\frac{\mu^2}{s}\right).$$

- Finiteness in the $(n+1)$-body phase space, sector by sector:

$$RV\,\overline{\mathcal{W}}_{hq} + I^{(\mathbf{1})}_{hq} \quad = \quad -\frac{\alpha_{\mathrm{S}}}{2\pi}\,\frac{2}{3}\,T_R\left(\ln\frac{\mu^2}{\bar{s}_{[34]r}} + \frac{8}{3}\right) R\,\overline{\mathcal{W}}_{hq}.$$

$$\overline{K}^{(\mathbf{RV})}_{hq} - I^{(\mathbf{12})}_{hq} \quad = \quad -\frac{\alpha_{\mathrm{S}}}{2\pi}\,\frac{2}{3}\,T_R\left(\ln\frac{\mu^2}{\bar{s}_{[34]r}} + \frac{8}{3}\right)\Big[\bar{\mathbf{S}}_h + \overline{\mathbf{C}}_{hq}\left(1 - \bar{\mathbf{S}}_h\right)\Big] R\,\overline{\mathcal{W}}_{hq}\,.$$

Total NNLO cross section



- Example for $\mu/\sqrt{s} = 0.35$.
- Analytic:
  $\sigma_{\mathrm{NNLO}} = \sigma_{\mathrm{LO}}\, k \times 1.40787186$
- Subtraction method:
  $\sigma_{\mathrm{NNLO}} = \sigma_{\mathrm{LO}}\, k \times (1.40806 \pm 0.00040)$
- $k = \left(\frac{\alpha_{\mathrm{S}}}{2\pi}\right)^2 T_R C_F$

NNLO summary

- Sector functions at NNLO engineered to factorise NLO-sector structure.
- Sector-function sum rules to simplify as much as possible double-unresolved integrands $\overline{K}^{(\mathbf{2})}$: only sums of universal kernels.
- Every time one has to analytically integrate, sector functions are not there.
- Exploit full freedom in mapping and parametrisation of each contribution separately.

Status

- Method for the moment applied to FSR only and massless.
- Analytic integration of $\overline{K}^{(\mathbf{2})}$ to be finished. Most probably possible without IBP methods for the massless case.
- Real-virtual counterterms to be integrated in general (simpler than $\overline{K}^{(\mathbf{2})}$).
- Ongoing implementation in a differential code.
- Planned extensions to initial-state radiation, masses, ...

Thank you

Backup

Soft/collinear commutation at NLO

- Soft limit $\mathbf{S}_i$ $(k_i^\mu \to 0)$: $s_{ia}/s_{ib} \to$ constant, $s_{ia}/s_{bc} \to 0, \forall\, a, b, c \neq i$.
- Collinear limit $\mathbf{C}_{ij}$ $(k_\perp \to 0)$: $s_{ij}/s_{ia} \to 0$, $s_{ij}/s_{jb} \to 0$, $s_{ij}/s_{ab} \to 0, \forall\, a, b \neq i, j$. $s_{ia}/s_{ja} \to$ independent of a.
- Commutation in case $i =$ gluon and $j =$ quark.
- Altarelli-Parisi collinear kernel involved is $P_{ij}(x_i) = [1 + (1 - x_i)^2]/x_i$, with $x_i = s_{ir}/(s_{ir} + s_{jr})$, with arbitrary $r \neq i, j$.

$$
\begin{aligned}
\mathbf{S}_i\, R \quad &= \quad -\mathcal{N}_1 \sum_{\substack{l \neq i \\ m \neq i}} \frac{s_{lm}}{s_{il} s_{im}} B_{lm} \\
\Longrightarrow \quad \mathbf{C}_{ij}\, \mathbf{S}_i\, R \quad &= \quad -2\, \mathcal{N}_1 \sum_{l \neq i,j} \frac{s_{jl}}{s_{il} s_{ij}} B_{lj} \quad = \quad -\, 2\, \mathcal{N}_1 \frac{s_{jr}}{s_{ir} s_{ij}} (-C_{f_j} B), \\
\mathbf{C}_{ij}\, R \quad &= \quad \mathcal{N}_1 \frac{1}{s_{ij}} C_{f_j} B \frac{1 + [1 - s_{ir}/(s_{ir} + s_{jr})]^2}{s_{ir}/(s_{ir} + s_{jr})} \\
\Longrightarrow \quad \mathbf{S}_i \mathbf{C}_{ij} R \quad &= \quad -2\, \mathcal{N}_1 \frac{s_{jr}}{s_{ir} s_{ij}} (-C_{f_j} B).
\end{aligned}
$$

Soft counterterm in FKS

- The soft FKS counterterm does not feature gluon energy, thus it reduces to an angular integral:

$$I^{\mathrm{s}}_{\mathrm{FKS}} \propto \sum_{lm} \int d\cos\theta\, d\phi (\sin\phi \sin\theta)^{-2\epsilon} \frac{1-\cos\theta_{lm}}{(1-\cos\theta_{li})(1-\cos\theta_{mi})}.$$

- Doable (actually relevant to angular-ordering), but not maximally easy: relations among θ_{lm}, θ_{li} and θ_{mi} are non-trivial in terms of integration variables.

- Analogous features at NNLO may be much more severe.

Cancellation of virtual NLO poles

- Integrated counterterm I computed at all orders in ϵ.
- ϵ expansion:

$$
\begin{aligned}
I(\{\bar{k}\}) \;=\; \frac{\alpha_S}{2\pi}\left(\frac{\mu^2}{s}\right)^{\epsilon} \Bigg\{ & \Bigg[B(\{\bar{k}\}) \sum_k \left(\frac{C_{f_k}}{\epsilon^2} + \frac{\gamma_k}{\epsilon} \right) + \sum_{k,\, l\neq k} B_{kl}(\{\bar{k}\})\, \frac{1}{\epsilon} \ln \bar{\eta}_{kl} \Bigg] \\
& + \Bigg[B(\{\bar{k}\}) \sum_k \Bigg(\delta_{f_k g}\, \frac{C_A + 4\, T_R\, N_f}{6} \left(\ln \bar{\eta}_{kr} - \frac{8}{3} \right) \\
& + \delta_{f_k g}\, C_A \left(6 - \frac{7}{2}\zeta_2 \right) + \delta_{f_k \{q,\bar{q}\}}\, \frac{C_F}{2} \left(10 - 7\zeta_2 + \ln \bar{\eta}_{kr} \right) \Bigg) \\
& + \sum_{k,\, l\neq k} B_{kl}\left(\{\bar{k}\}\right) \ln \bar{\eta}_{kl} \left(2 - \frac{1}{2} \ln \bar{\eta}_{kl} \right) \Bigg] \Bigg\}.
\end{aligned}
$$

- $\bar{\eta}_{ab} = \bar{s}_{ab}/s$, and $\quad \gamma_k \;=\; \delta_{f_k g}\, \frac{11 C_A - 4\, T_R N_f}{6} + \delta_{f_k \{q,\bar{q}\}}\, \frac{3}{2} C_F$.
- Same structure of ϵ singularities as V (up to a sign).

NNLO singularity-cancellation pattern

- $RR - \overline{K}^{\,(\mathbf{1})} - (\overline{K}^{\,(\mathbf{2})} + \overline{K}^{\,(\mathbf{12})})$ finite in $d = 4$, and in Φ_{n+2}.

- $RV + I^{\,(\mathbf{1})}$ finite in $d = 4$, but singular in Φ_{n+1}.

- $\overline{K}^{(\mathbf{RV})} - I^{\,(\mathbf{12})}$ finite in $d = 4$, but singular in Φ_{n+1}.

- $RV + I^{\,(\mathbf{1})} - (\overline{K}^{(\mathbf{RV})} - I^{\,(\mathbf{12})})$ finite in $d = 4$, and in Φ_{n+1}.

- $VV + I^{\,(\mathbf{2})} + I^{(\mathbf{RV})}$ finite in $d = 4$, and in Φ_n.

NNLO sector-function sum rules for composite limits

$$\mathbf{S}_i\,\mathbf{C}_{ijk}\,\left(\mathcal{W}_{ij}^{(\alpha\beta)}+\mathcal{W}_{ik}^{(\alpha\beta)}\right)\,=\,1\,,$$

$$\mathbf{S}_{ij}\,\mathbf{C}_{ijk}\sum_{ab\,\in\,\mathrm{perm}(ij)}\left(\mathcal{W}_{abbk}+\mathcal{W}_{akbk}\right)\,=\,1\,,\qquad\qquad \mathbf{S}_{ik}\,\mathbf{C}_{ijkl}\,\left(\mathcal{W}_{ijkl}+\mathcal{W}_{klij}\right)\,=\,1\,.$$

$$\mathbf{SC}_{ijk}\,\mathbf{S}_{ij}\sum_{b\neq i}\mathcal{W}_{ibjk}\,=\,1\,,\qquad\qquad \mathbf{CS}_{ijk}\,\mathbf{S}_{ik}\sum_{d\neq i,k}\mathcal{W}_{ijkd}\,=\,1\,,$$

$$\mathbf{CS}_{ijk}\,\mathbf{C}_{ijk}\,\left(\mathcal{W}_{ijkj}+\mathcal{W}_{jiki}\right)\,=\,1\,,\qquad\qquad \mathbf{CS}_{ijk}\,\mathbf{C}_{ijkl}\,\left(\mathcal{W}_{ijkl}+\mathcal{W}_{jikl}\right)\,=\,1\,,$$

$$\mathbf{CS}_{ijk}\,\mathbf{C}_{ijk}\,\mathbf{S}_{ik}\,\mathcal{W}_{ijkj}\,=\,1\,,\qquad\qquad \mathbf{CS}_{ijk}\,\mathbf{C}_{ijkl}\,\mathbf{S}_{ik}\,\mathcal{W}_{ijkl}\,=\,1\,,$$

$$\mathbf{SC}_{ijk}\,\mathbf{C}_{ijk}\sum_{ab\,\in\,\mathrm{perm}(jk)}\left(\mathcal{W}_{iaab}+\mathcal{W}_{iaba}\right)\,=\,1\,,\qquad\qquad \mathbf{SC}_{ikl}\,\mathbf{C}_{ijkl}\,\left(\mathcal{W}_{ijkl}+\mathcal{W}_{ijlk}\right)\,=\,1\,,$$

$$\mathbf{SC}_{ijk}\,\mathbf{C}_{ijk}\,\mathbf{S}_{ik}\,\left(\mathcal{W}_{ijkj}+\mathcal{W}_{ikkj}\right)\,=\,1\,,\qquad\qquad \mathbf{SC}_{ijk}\,\mathbf{C}_{ijkl}\,\mathbf{S}_{ik}\,\mathcal{W}_{ijkl}\,=\,1\,.$$

Double-radiation phase space

- Catani-Seymour variables $y, z, y', z', x' \in [0,1]$ for mapping $\{k\} \to \{\bar{k}\}^{(abcd)}$:

$$
\begin{aligned}
s_{ab} &= y'\, y\, s_{abcd}\,, \qquad\qquad s_{cd} = \left(1-y'\right)(1-y)\,(1-z)\; s_{abcd}\,,\\
s_{ac} &= z'\left(1-y'\right)\; y\, s_{abcd}\,, \qquad\qquad s_{bc} = \left(1-y'\right)\left(1-z'\right)\; y\, s_{abcd}\,,\\
s_{ad} &= (1-y)\left[y'\left(1-z'\right)(1-z) + z'z - 2\left(1-2x'\right)\sqrt{y'z'\left(1-z'\right)\,z\,(1-z)}\right] s_{abcd}\,,\\
s_{bd} &= (1-y)\left[y'z'\,(1-z) + \left(1-z'\right)z + 2\left(1-2x'\right)\sqrt{y'z'\left(1-z'\right)\,z\,(1-z)}\right] s_{abcd}\,,
\end{aligned}
$$

- Phase-space factorisation:

$$
d\Phi_{n+2} \quad = \quad d\Phi_n^{(abcd)}\, d\Phi_{\mathrm{rad},2}^{(abcd)}\,,
$$

$$
\begin{aligned}
\int d\Phi_{\mathrm{rad},2}^{(abcd)} \quad &= \quad \int d\Phi_{\mathrm{rad},2}\left(s_{abcd};\, y, z, \phi, y', z', x'\right)\\
&= \quad N^2(\epsilon)\,(s_{abcd})^{2-2\epsilon}\int_0^1 dx' \int_0^1 dy' \int_0^1 dz' \int_0^\pi d\phi\, \left(\sin\phi\right)^{-2\epsilon} \int_0^1 dy \int_0^1 dz\\
&\quad\; \times \left[4\, x'\left(1-x'\right)\; y'\left(1-y'\right)^2\; z'\left(1-z'\right)\; y^2(1-y)^2\, z\,(1-z)\right]^{-\epsilon}\\
&\quad\; \times \left[x'\left(1-x'\right)\right]^{-1/2}\left(1-y'\right)\; y\,(1-y)\;.
\end{aligned}
$$

Matrix elements for the $T_R C_F$ contrib. to $e^+e^- \to q\bar{q}$ at NNLO

- Analytic matrix elements from [Hamberg, van Neerven, Matsuura, 1991], [Gehrmann De Ridder, Gehrmann, Glover, 0403057], [Ellis, Ross, Terrano, 1980]

$$
\begin{aligned}
VV \;&=\; B\left(\frac{\alpha_{\rm S}}{2\pi}\right)^2 T_R\, C_F \left\{ \left(\frac{\mu^2}{s}\right)^{2\epsilon}\left[\frac{1}{3\epsilon^3}+\frac{14}{9\epsilon^2}+\frac{1}{\epsilon}\left(-\frac{11}{18}\pi^2+\frac{353}{54}\right)+\left(-\frac{26}{9}\zeta_3-\frac{77}{27}\pi^2+\frac{7541}{324}\right)\right]\right. \\
&\qquad\qquad \left. +\left(\frac{\mu^2}{s}\right)^{\epsilon}\left[-\frac{4}{3\epsilon^3}-\frac{2}{\epsilon^2}+\frac{1}{\epsilon}\left(\frac{7}{9}\pi^2-\frac{16}{3}\right)+\left(\frac{28}{9}\zeta_3+\frac{7}{6}\pi^2-\frac{32}{3}\right)\right]\right\}, \\
\int d\Phi_{\rm rad}\, RV \;&=\; \frac{\alpha_{\rm S}}{2\pi}\,\frac{1}{\epsilon}\,\frac{2}{3}\,T_R \int d\Phi_{\rm rad}\, R \\
&=\; B\left(\frac{\alpha_{\rm S}}{2\pi}\right)^2 T_R\, C_F \left(\frac{\mu^2}{s}\right)^{\epsilon}\left[\frac{4}{3\epsilon^3}+\frac{2}{\epsilon^2}+\frac{1}{\epsilon}\left(-\frac{7}{9}\pi^2+\frac{19}{3}\right)+\left(-\frac{100}{9}\zeta_3-\frac{7}{6}\pi^2+\frac{109}{6}\right)\right], \\
\int d\Phi_{\rm rad,2}\, RR \;&=\; B\left(\frac{\alpha_{\rm S}}{2\pi}\right)^2 T_R\, C_F \left(\frac{\mu^2}{s}\right)^{2\epsilon}\left[-\frac{1}{3\epsilon^3}-\frac{14}{9\epsilon^2}+\frac{1}{\epsilon}\left(\frac{11}{18}\pi^2-\frac{407}{54}\right)+\left(\frac{134}{9}\zeta_3+\frac{77}{27}\pi^2-\frac{11753}{324}\right)\right].
\end{aligned}
$$

Integrated counterterms in the T_RC_F contrib. to $e^+e^- \to q\bar{q}$ at NNLO

$$
\begin{aligned}
I^{(\mathbf{2})} &= \int d\Phi_{\mathrm{rad},2}\left[\overline{\mathbf{S}}_{34} + \overline{\mathbf{C}}_{134}\,\left(1-\overline{\mathbf{S}}_{34}\right) + \overline{\mathbf{C}}_{234}\,\left(1-\overline{\mathbf{S}}_{34}\right)\right]RR \\
&= B\left(\frac{\alpha_{\mathrm{S}}}{2\pi}\right)^2 T_RC_F\left(\frac{\mu^2}{s}\right)^{2\epsilon}\left[-\frac{1}{3\epsilon^3} - \frac{14}{9\epsilon^2} + \frac{1}{\epsilon}\left(\frac{11}{18}\pi^2 - \frac{425}{54}\right)\right. \\
&\qquad\qquad\left. + \left(\frac{122}{9}\zeta_3 + \frac{74}{27}\pi^2 - \frac{12149}{324}\right)\right] + \mathcal{O}(\epsilon).
\end{aligned}
$$

$$
I_{hq}^{(\mathbf{1})} = -\frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{\mu^2}{s}\right)^{\epsilon}\frac{2}{3}\,T_R\left(\frac{1}{\epsilon} - \ln\bar{\eta}_{\,[34]r} + \frac{8}{3}\right)R\,\overline{\mathcal{W}}_{hq} + \mathcal{O}(\epsilon)\,,
$$

$$
I_{hq}^{(\mathbf{12})} = \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{\mu^2}{s}\right)^{\epsilon}\frac{2}{3}\,T_R\left(\frac{1}{\epsilon} - \ln\bar{\eta}_{\,[34]r} + \frac{8}{3}\right)\left[\bar{\mathbf{S}}_h + \overline{\mathbf{C}}_{hq}\,\left(1-\bar{\mathbf{S}}_h\right)\right]R\,\overline{\mathcal{W}}_{hq} + \mathcal{O}(\epsilon)\,.
$$

$$
\begin{aligned}
I^{(\mathbf{RV})} &= \frac{\alpha_{\mathrm{S}}}{2\pi}\,\frac{2}{3}\,\frac{1}{\epsilon}\,T_R\int d\Phi_{\mathrm{rad}}\left[\overline{\mathbf{S}}_{[34]} + \overline{\mathbf{C}}_{1[34]}\,\left(1-\overline{\mathbf{S}}_{[34]}\right) + \overline{\mathbf{C}}_{2[34]}\,\left(1-\overline{\mathbf{S}}_{[34]}\right)\right]R \\
&= B\left(\frac{\alpha_{\mathrm{S}}}{2\pi}\right)^2 T_R\,C_F\left(\frac{\mu^2}{s}\right)^{\epsilon}\left[\frac{4}{3\epsilon^3} + \frac{2}{\epsilon^2} - \frac{1}{\epsilon}\left(\frac{7}{9}\pi^2 - \frac{20}{3}\right) - \left(\frac{100}{9}\zeta_3 + \frac{7}{6}\pi^2 - 20\right)\right] + \mathcal{O}(\epsilon)\,,
\end{aligned}
$$