Scattering amplitudes from (super)conformal symmetry

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Based on

JHEP (2018) 82, D. Chicherin and E. Sokatchev

PRL 121 (2018) 021601, JMH, D. Chicherin and E. Sokatchev

and work in progress with D. Chicherin, E. Sokatchev and S. Zoia

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The team

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Emery Sokatchev (Annecy)

Simone Zoia (MPP)

Introduction

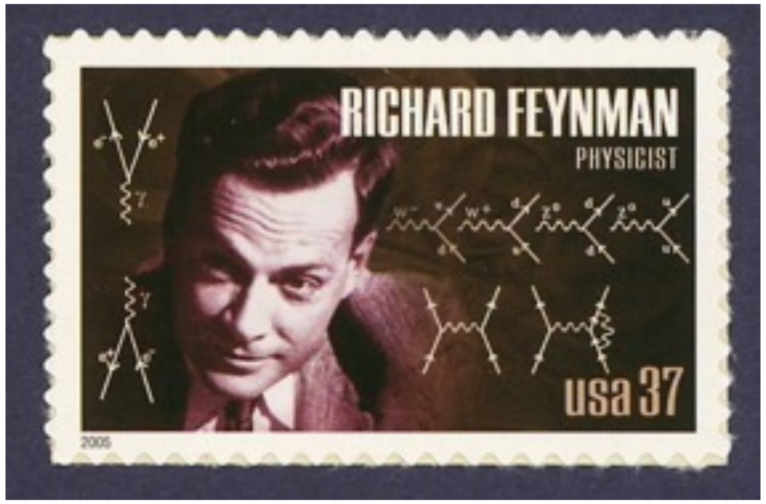
- in high energy scattering, sometimes masses may be neglected; symmetry enhanced from Poincaré to conformal symmetry

- broad applications: gauge theories, Yukawa vertices, ϕ^4 ; ϕ^3 in D=6 dimensions

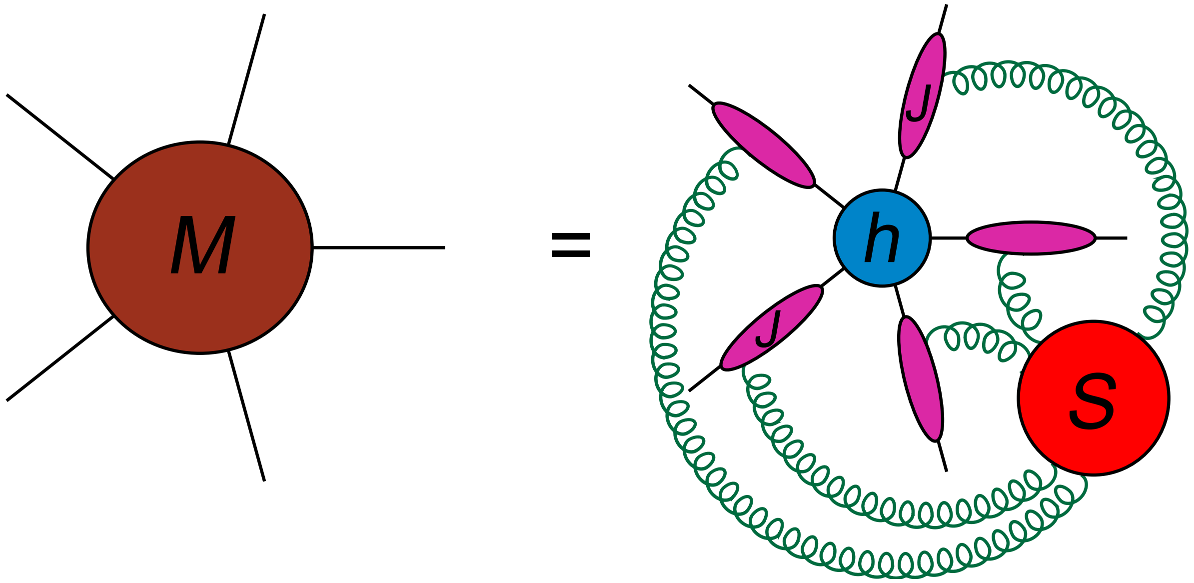
- most studies so far deal with correlation functions in position space; what are the consequences for on-shell scattering processes?

Symmetry for finite `hard functions`

- application: complicated amplitudes from symmetry?



- two quantum sources of symmetry breaking: soft/collinear and ultraviolet effects

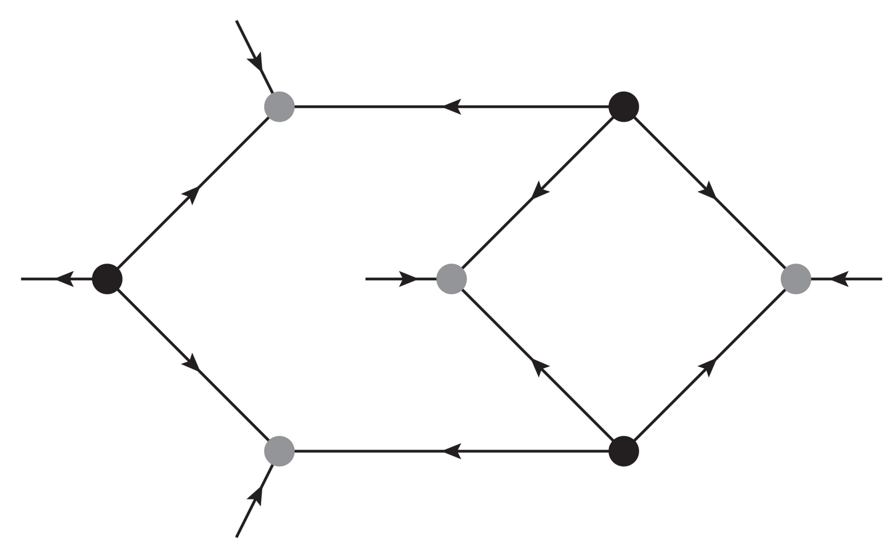


[Figure: L. Dixon, J.Phys A44 (2011) 454001]

- this talk: study effect of symmetry on finite `remainder functions`, i.e. hard processes

Plan of the talk

- (Loop-level) conformal Ward identities
- Application: \`bootstrapping\` 5-particle integrals
- Superconformal symmetry: from 2nd order PDE to 1st order PDE
- First result for a non-trivial hexa-box integral



Conformal symmetry

- important in many areas: string theory, AdS/CFT, conformal bootstrap, solid state physics, mathematics

- all local (re)scalings of the measure
  - Poincaré group,
  - dilations, $x^\mu \to \lambda x^\mu$
  - special conformal boosts $x^\mu \to \dfrac{x^\mu - b^\mu x^2}{1 - 2(b \cdot x) + b^2 x^2}$

- powerful symmetry!

Conformal symmetry: momentum space

- off-shell special conformal generator K_μ
  2nd order in momentum space

$$K^\mu_\Delta = -q^\mu \Box_q + 2q^\nu \partial_{q^\nu}\partial_{q_\mu} + 2(D-\Delta)\partial_{q_\mu}$$

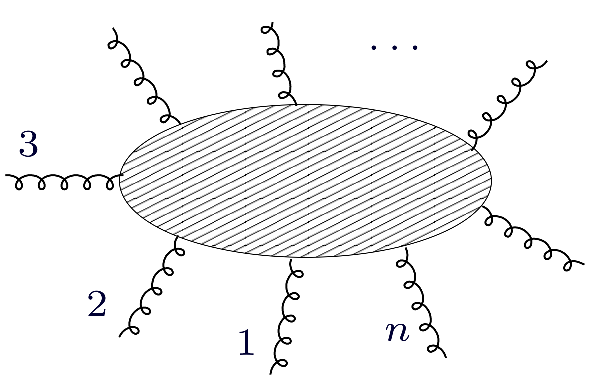
  Conformal dimension Δ

- amputate external legs; on-shell generator $\mathbb{K}_\mu$
- in D=4, simple spinor-helicity form [Witten 2003]

$$\sigma^\mu_{\alpha\dot\alpha} p_\mu = \lambda_\alpha \tilde\lambda_{\dot\alpha} \quad , \quad \mathbb{K}_\mu = 2\,\tilde\sigma^{\dot\alpha\alpha}_\mu \frac{\partial^2}{\partial\lambda^\alpha \partial\tilde\lambda^{\dot\alpha}}$$

- conformal invariance:

$$\left(\sum_{i=1}^{n} \mathbb{K}^\mu_i\right) \mathcal{I}(p_1, \ldots p_n) = 0$$



Examples of conformal interactions

- at classical level ϕ^4 , e.g. six-particle scattering

$$\mathcal{I}_6 = \frac{\delta^{(6)}(\sum_i p_i)}{(p_1 + p_2 + p_3)^2}$$

$$\mathbb{K}^\mu \, \mathcal{I}_6 = \delta^{(6)}(\sum_i p_i) \, \mathbb{K}^\mu \, \frac{1}{(p_1 + p_2 + p_3)^2} = 0$$

- all tree-level gluon amplitudes

$$\mathbb{K}^\mu \mathcal{I}(p_1 \, , \ldots \, , p_n) = 0$$

- Questions:

— what modifications are needed at loop level?

— how powerful are these symmetries?

Holomorphic anomaly

- tree-level MHV amplitude of n gluons

$$\mathcal{A}_{n;\text{tree}}^{\text{MHV}} = \frac{\langle 12\rangle^3\,\delta^{(4)}\left(\sum_{i=1}^{n}\lambda_i\,\tilde{\lambda}_i\right)}{\langle 23\rangle\langle 34\rangle\ldots\langle n1\rangle}\,, \qquad \langle ij\rangle = \lambda_i^{\alpha}\epsilon_{\alpha\beta}\lambda_j^{\beta}$$

- holomorphic anomaly [Cachazo, Svrcek, Witten 2004]

$$\frac{\partial}{\partial\tilde{\lambda}^{\dot{\alpha}}}\frac{1}{\langle\lambda\chi\rangle} = 2\pi\tilde{\chi}_{\dot{\alpha}}\delta(\langle\lambda\chi\rangle)\delta([\tilde{\lambda}\tilde{\chi}]) \qquad \Longleftarrow \qquad \frac{\partial}{\partial\bar{z}}\frac{1}{z} = \pi\delta^2(z)$$

- anomaly of tree amplitudes is localized on collinear configurations of particles (contact terms)

[Beisert et al. 2009]

- studied at level of cuts (discontinuities) of loop amplitudes

[Korchemsky and Sokatchev, 2009]

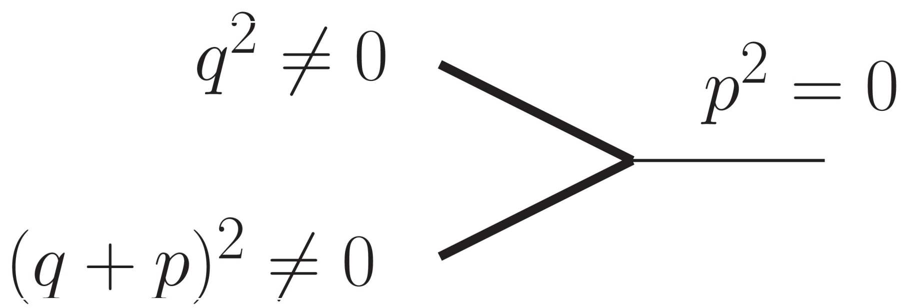
[Beisert et al. 2010]

- here: study directly for loop corrections

6D vertex function ϕ^3

[Chicherin and Sokatchev, 2018]

- mixed off-shell/on-shell object



$$= \langle\, \phi(q)\, \phi(-q-p) \mid \phi(p)\, \rangle_g$$

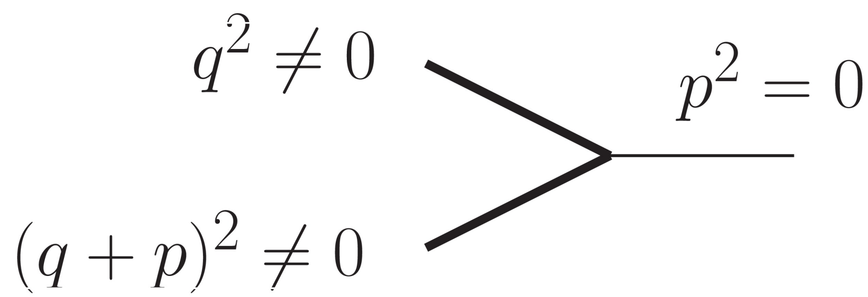
$$(K^\mu_{\Delta=2} + \mathbb{K}^\mu)\ \frac{1}{(q^2+i0)((q+p)^2+i0)}$$

$$= \quad ???$$

6D vertex function ϕ^3

[Chicherın and Sokatchev, 2018]

- mixed off-shell/on-shell object

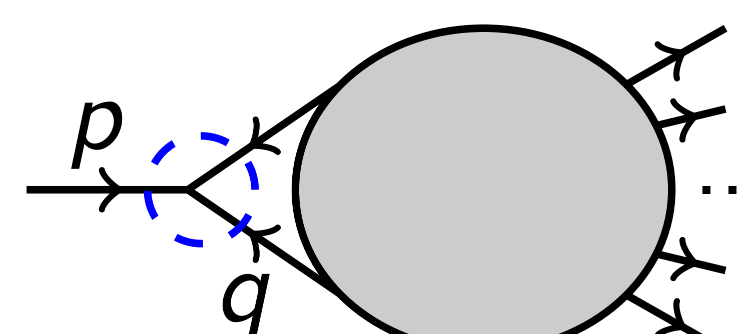


$$= \langle\, \phi(q)\, \phi(-q-p) \mid \phi(p)\,\rangle_g$$

$$(K^{\mu}_{\Delta=2} + \mathbb{K}^{\mu}) \; \frac{1}{(q^2 + i0)((q+p)^2 + i0)}$$

$$= 4i\pi^3\, p^{\mu} \int_0^1 d\xi\, \xi(1-\xi)\, \delta^{(6)}(q + \xi p)$$

- anomaly is contact type and lives on collinear configurations $q \sim p$

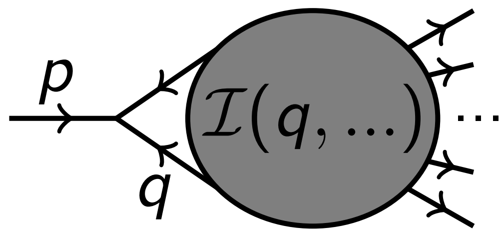


Conformal Ward identities

[Chicherin and Sokatchev, 2018]

- contact anomaly localizes loop integration

$$\int d^6 k$$



$$\longrightarrow \int_0^1 d\xi \xi(1-\xi)\mathcal{I}(q=-\xi p, \ldots)$$

- system of inhomogeneous 2nd order PDE

Example

[Chicherin and Sokatchev, 2018]

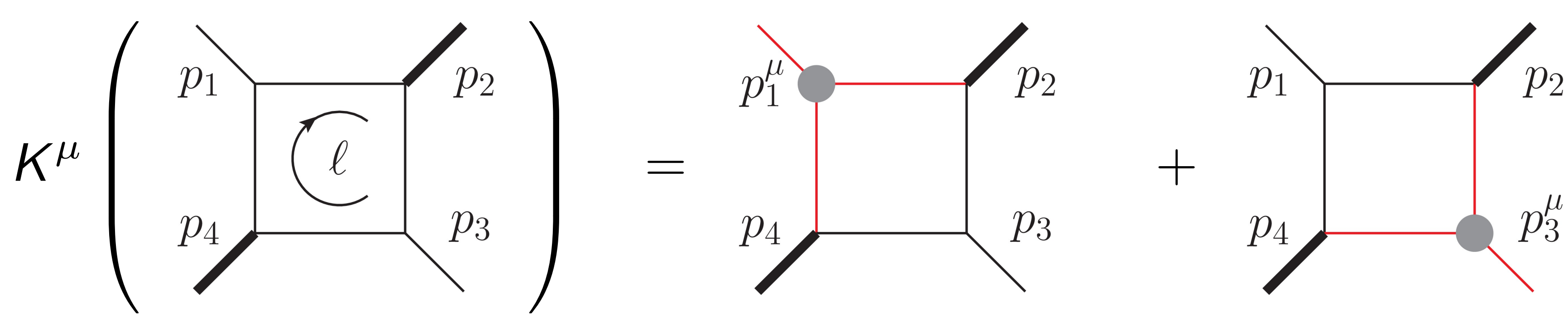
- consider 6-D two-mass box

  (corresponds to finite part of 4-D box)

  built from conformal ϕ^3 vertices

- conformal anomaly (2nd-order inhom. DE)

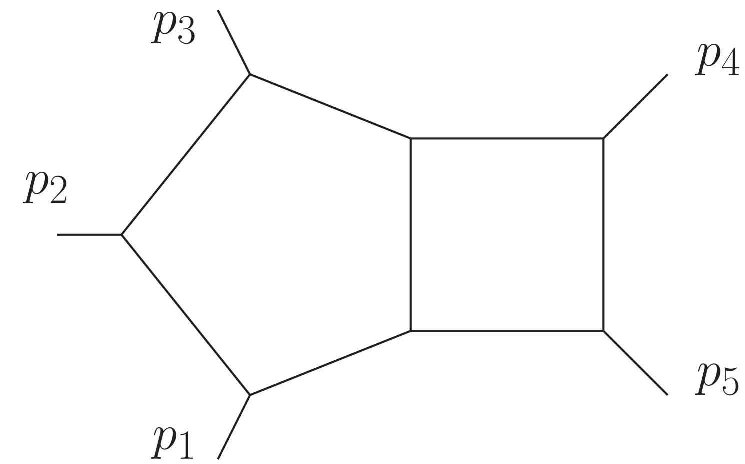
  $K^\mu \equiv \mathbb{K}_1^\mu + K_2^\mu + \mathbb{K}_3^\mu + K_4^\mu$



$$K^\mu\,\mathcal{I}_{(\ell)} = \int_0^1 d\xi\; A^\mu_{(\ell-1)}(\xi)$$

Bootstrap of multi-loop integrals

- 2nd order DE are difficult to solve, but they are efficient for the bootstrap!



- example: 6-D scalar penta-box

— 5-particle scattering: 31-letter alphabet [Gehrmann, JMH, Lo Presti, 2015] [Chicherin, JMH, Mitev, 2018]

— ansatz of weight-5 integrable symbols

$$\mathcal{S}(\mathcal{I}_5) = \frac{1}{\sqrt{\Delta}} \sum_{i_1,\ldots,i_5} c_{i_1\ldots i_5}(W_{i_1} \otimes \ldots \otimes W_{i_5}) \,, \qquad \Delta = \det(p_i \cdot p_j)$$

— 161 free coefficients; uniquely fixed by just one projection

$$(n\cdot K)\,\mathcal{S}(\mathcal{I}_5) = (n\cdot p_1)A_1 + (n\cdot p_3)A_3 \,, \qquad (n\cdot p_i) = 0 \;\; \text{at} \;\; i = 2,4,5$$

Summary of this part

- Conformal symmetry: anomalous Ward identities for K_μ are 2nd order DE that are hard to solve

- knowing the function alphabet (and leading singularities) we can bootstrap the answer

Next:

- Superconformal symmetry yields 1st order PDE
- They can be integrated directly! No assumptions about alphabet!

N=1 matter supergraphs with on-shell states

- WZ model in 4D; off-shell super fields

$$\Phi(x,\theta) = \phi(x) + \theta^\alpha \psi_\alpha(x) + (\theta)^2 F(x)\,, \quad \bar\Phi(x,\bar\theta) = \phi(x) + \bar\theta_{\dot\alpha}\bar\psi^{\dot\alpha}(x) + (\bar\theta)^2 \bar F(x)$$

$$S_{WZ} = \int d^4x\, d^2\theta d^2\bar\theta\, \bar\Phi\Phi + \frac{g}{3!}\int d^4x\, d^2\theta\, \Phi^3 + \frac{g}{3!}\int d^4x\, d^2\bar\theta\, \bar\Phi^3$$

- Classical superconformal symmetry su(2,2|1)

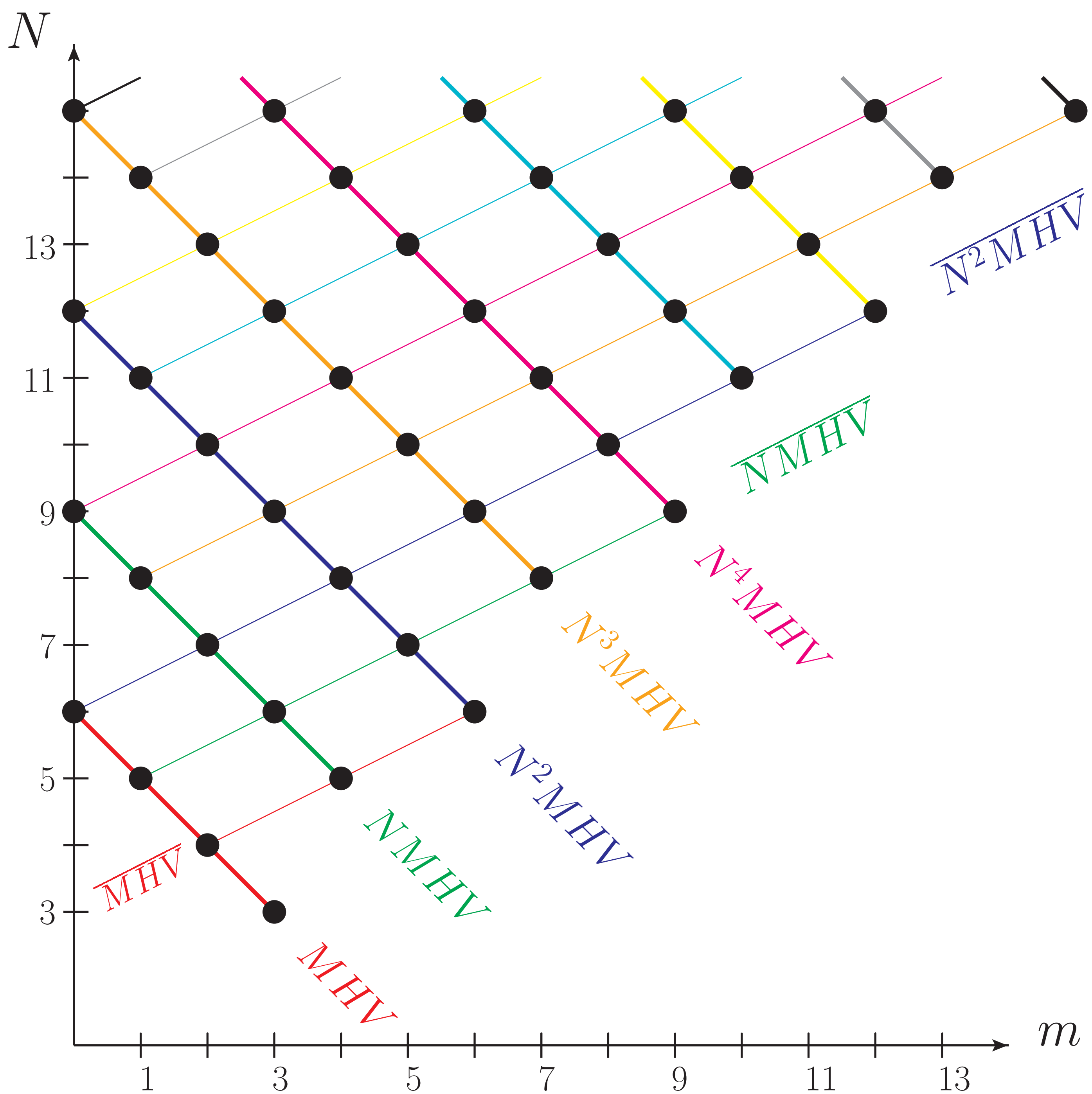
- Two superstates with $\eta \equiv \tilde\lambda_{\dot\alpha}\bar\theta^{\dot\alpha}$

state	$\lvert\bar\psi\rangle$	$\lvert\bar\phi\rangle$	$\lvert\phi\rangle$	$\lvert\psi\rangle$
helicity	$-\frac{1}{2}$	0	0	$\frac{1}{2}$

$$\Psi(p,\eta) = |\psi\rangle + \eta|\phi\rangle$$
$$\bar\Phi(p,\eta) = |\bar\phi\rangle + \eta|\bar\psi\rangle$$

helicity classification

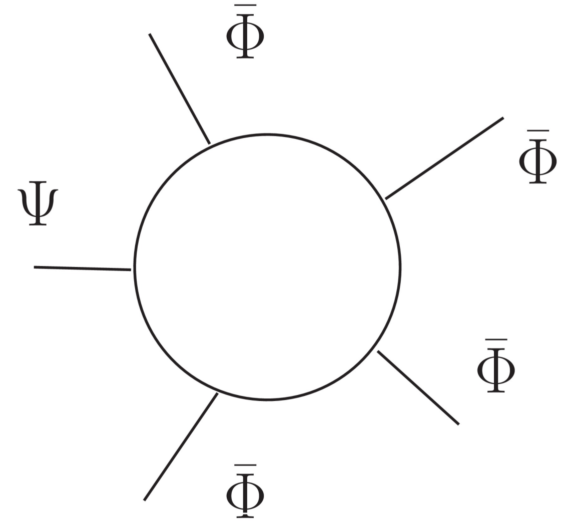
- superamplitudes N=m+n, m $\bar{\Phi}(p,\eta)$, n $\Psi(p,\eta)$



five-particle $\overline{MHV}$ superamplitudes

- we consider finite amplitude supergraphs

$$\mathcal{A}_5^{\mathrm{NMHV}} =$$



$$= \delta^{(4)}(P)\, \underbrace{\delta^{(2)}(Q) \cdot \Xi}_{\text{R-charge} = 3} \cdot \mathcal{I}(\{\lambda, \tilde\lambda\})$$

- supercharges $\quad Q_\alpha = \sum_i \eta_i \lambda_{i,\alpha}\,, \qquad \bar Q_{\dot\alpha} = \sum_i \tilde\lambda_{i,\dot\alpha} \frac{\partial}{\partial \eta_i}$

- unique superinvariant at five points

$$\bar Q\, \Xi = 0 \;\;\Rightarrow \qquad \Xi_{ijk} = \eta_i[jk] + \eta_j[ki] + \eta_k[ij]\,, \qquad [ij] := \tilde\lambda_{\dot\alpha} \epsilon^{\dot\alpha\dot\beta} \tilde\lambda_{\dot\beta}$$

$\longrightarrow$ single bosonic function (Feynman integral) $\mathcal{I}$!

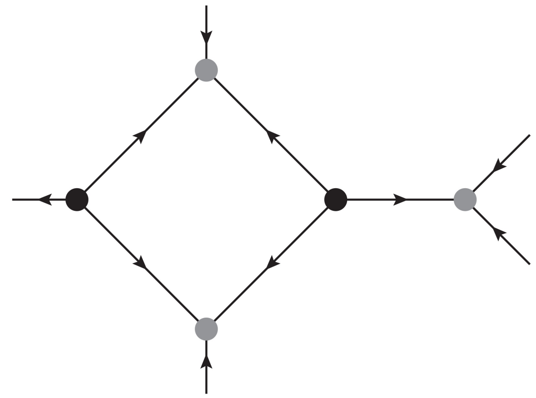
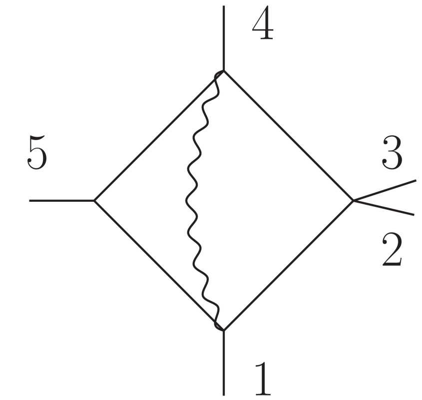
- S-susy gives rise to twistor collinearity operator

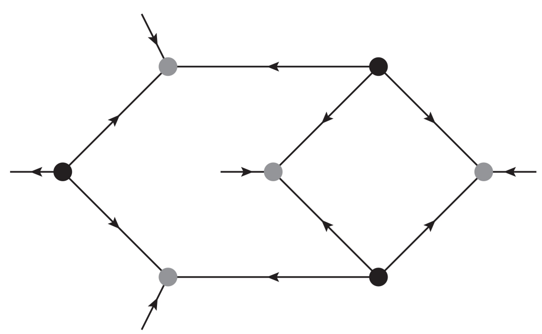
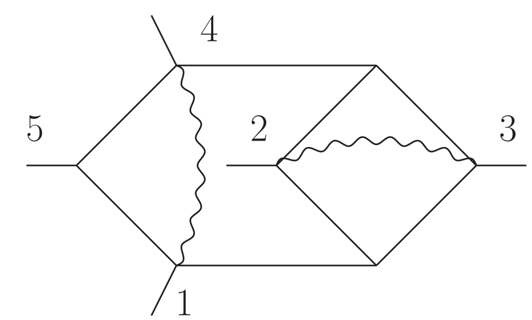
$$\{\mathbb{S}_\alpha\,, \Xi_{ijk}\} = (F_{ijk})_\alpha \equiv [jk]\frac{\partial}{\partial \lambda_i^\alpha} + [ki]\frac{\partial}{\partial \lambda_j^\alpha} + [ij]\frac{\partial}{\partial \lambda_k^\alpha}$$

[Witten 2003]

Ward identities for 5-point integrals

- integrals with `magic numerators` [Arkani-Hamed, Bourjaily, Cachazo, Trnka, 2010]

$\mathcal{A}_5^{\mathrm{NMHV}} =$  $\Longrightarrow \quad \mathcal{I}_5^{(1)}(\{\lambda, \tilde{\lambda}\}) =$ 

$\mathcal{A}_5^{\mathrm{NMHV}} =$  $\Longrightarrow \quad \mathcal{I}_5^{(2)}(\{\lambda, \tilde{\lambda}\}) =$ 

- S-variation of $\mathcal{A}_5$ anomalous
- PDE for Feynman integral $\mathcal{I}_5^{(\ell)}(\{\lambda, \tilde{\lambda}\})$ with collinearity operator

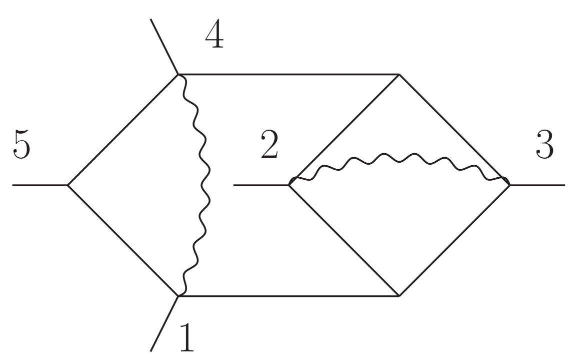
$$F^{\alpha}_{ijk}\, \mathcal{I}_5^{(\ell)}(\{\lambda, \tilde{\lambda}\}) = \sum_{r=1,2,3,4} \lambda^{\alpha}_r \int_0^1 d\xi\, A_r^{(\ell-1)}(\xi, \{\lambda, \tilde{\lambda}\})$$

Solving the DE for the non-planar hexa-box

- five-particle kinematics $\mathcal{I} = \mathcal{I}(x_1, x_2, x_3, x_4)$

$$x_1 = -1 - \frac{s_{14}}{s_{15}}\,, \quad x_2 = -1 - \frac{s_{14}}{s_{45}}\,, \quad x_3 = \frac{[12][34]}{[23][41]}\,, \quad x_4 = \frac{[23][45]}{[34][52]}$$

- Ward identity

$\mathcal{I} =$ 

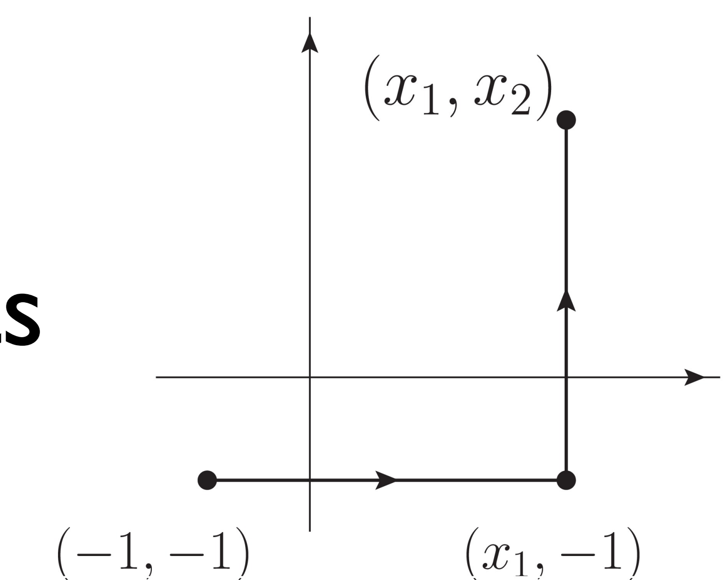
$$\tilde{d}\mathcal{I}(x_1, x_2, x_3, x_4) = a_1\, \tilde{d} \log x_1 + a_4\, \tilde{d} \log x_2$$

$$+ a_2\, \tilde{d} \log \frac{1 - x_1 x_2}{(1+x_2)(x_3-1)x_4 + (1+x_1)(x_3 x_4 - 1)}$$

$$+ a_3\, \tilde{d} \log \frac{1 - x_1 x_2}{(1+x_2)x_3 x_4 + (1+x_1)(x_3 x_4 - 1)}$$

where $\tilde{d} = dx_1\, \partial_{x_1} + dx_2\, \partial_{x_2}$; a_k – anomaly of k-th leg, weight-3 pure functions

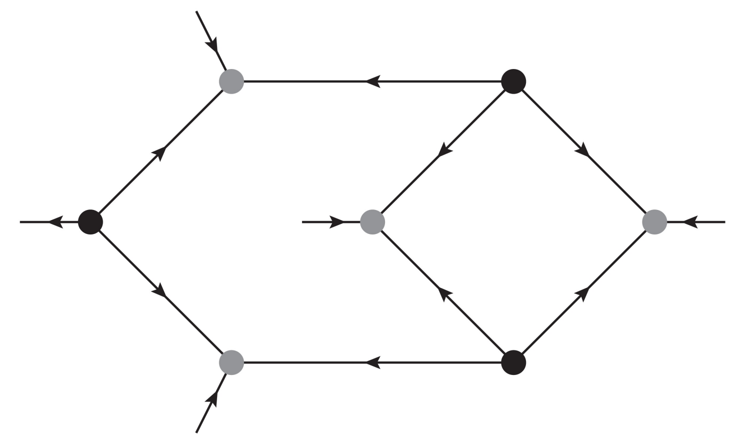
- boundary conditions
  - — $\mathcal{I}(x_1 = -1, x_2 = -1) = 0$, i.e. at $s_{14} = 0$
  - — OR: from absence of unphysical cuts



Current status hexa-box integrals

- first result for a non-trivial hexa-box integral

  [Chicherin, JMH, Sokatchev, 2018]



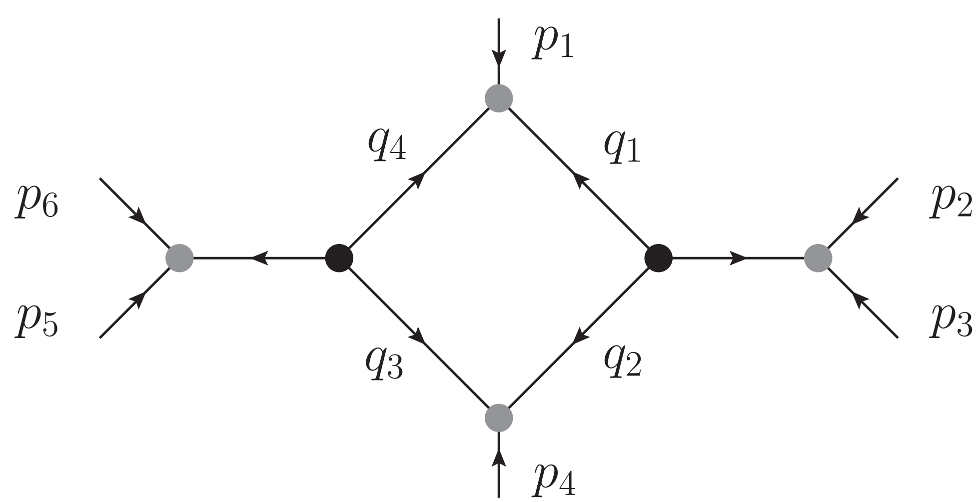
in agreement with conjectured non-planar pentagon function alphabet [Chicherin, JMH, Mitev, 2018]

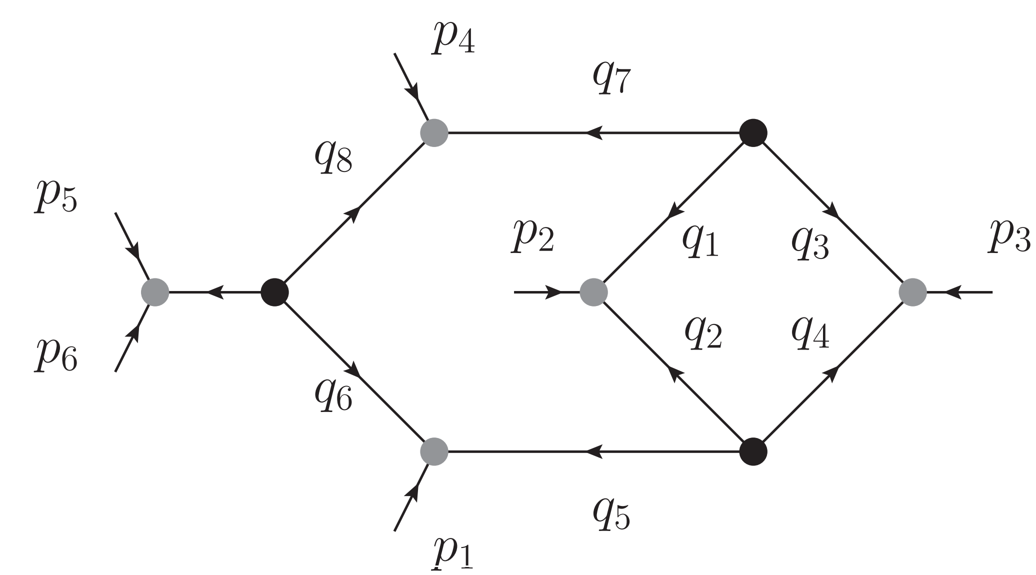
- IBP reductions [Böhm, Georgoudis, Larsen, Schönemann, Zhang, 2018]
- differential equations for all hexa-box integrals [Abreu, Page, Zeng, 2018]
- differential equations and solution [Chicherin, Gehrmann, Lo Presti, JMH, Mitev, Wasser, 2018]

  agrees with result for superconformal integral

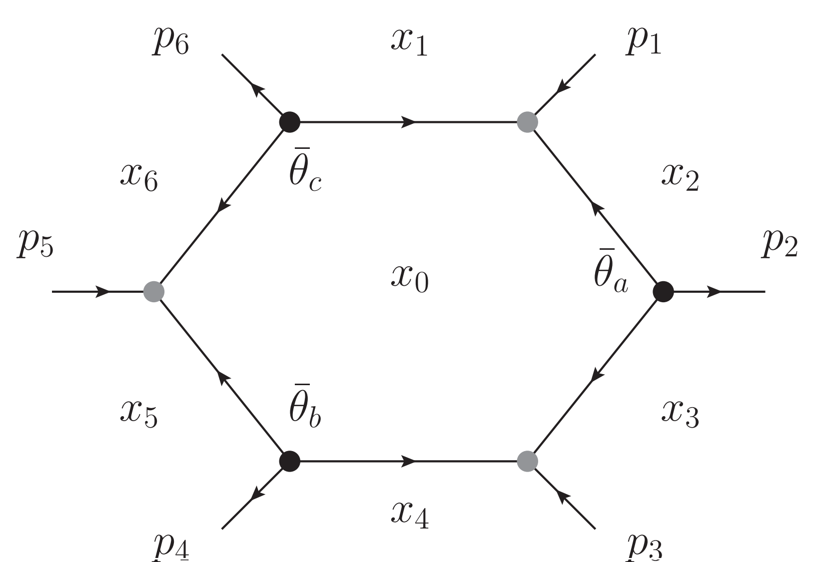
Further examples

- six-particle $\overline{MHV}$ supergraphs (single bosonic function)





- six-particle NMHV supergraph (two bosonic functions)



Summary

- Conformal symmetry (2nd order PDE)

— anomalous Ward identity of Feynman diagrams

— efficiently solved using bootstrap assumptions

⟶ [see talk at Loops & Legs 2018 by S. Zoia]

- Superconformal symmetry (1st order PDE)

— 4-D Wess-Zumino model of N=1 matter

— Ward identities easy to solve, no assumptions needed

- Future directions:

— include N=1 gauge sector

— gauge invariance requires sum of diagrams; IR divergences?

— study interplay with beta function

Thank you!