

Double parton distributions

Evolution, initial conditions and transverse momentum dependence

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Outline

- Evolution equations for double Parton Distribution Functions (dPDFs)
- Sum rules
- Initial conditions
- Examples for the single channel: gluons
- Transverse momentum dependence in dPDFs

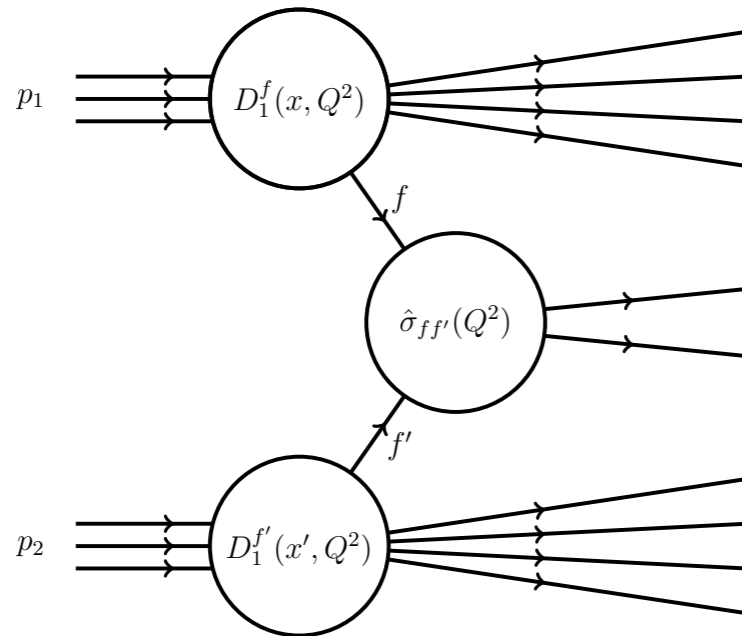
References:

Initial conditions: [Golec-Biernat, Lewandowska, Serino, Snyder, AS; 1507.08583](#)

Unintegrated dPDFs: [Golec-Biernat, AS; 1611.02033, 1801.00018](#)

Single scattering process

Single parton scattering: one hard process



Single collinear PDF:

$$D_1^f(x, Q^2)$$

Partonic cross section:

$$\hat{\sigma}^{ff'}(x_1, x_2, Q^2)$$

Collins, Soper, Sterman

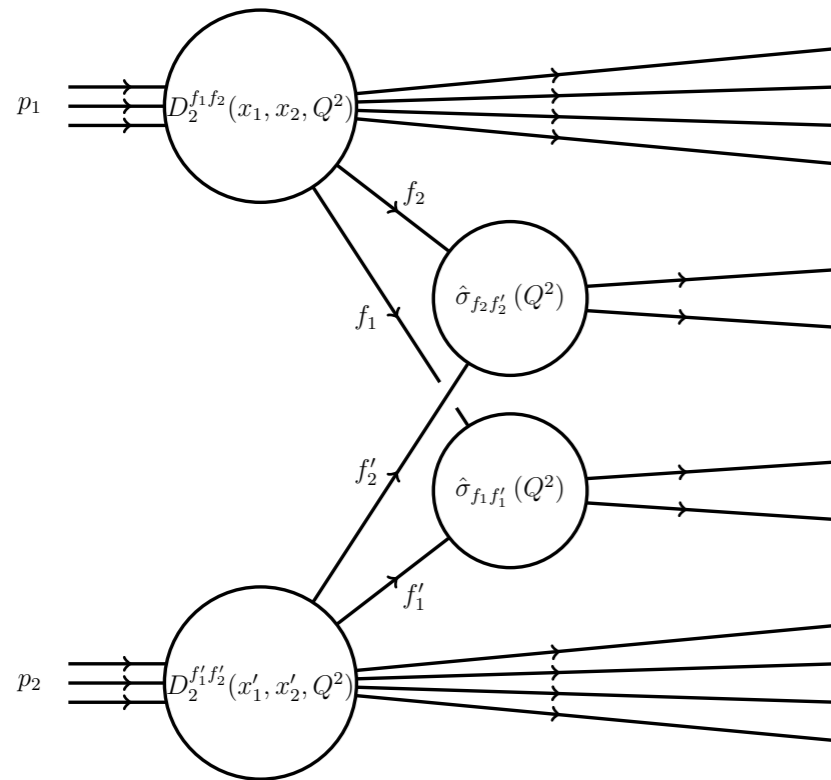
Collinear factorization:

Given the presence of the hard scale, the cross section (up to power corrections) can be factorized into perturbatively calculable partonic cross section and non-perturbative parton distribution functions.

$$\sigma = \sum_{ff'} \int dx_1 dx_2 D_1^f(x_1, Q^2) \hat{\sigma}^{ff'}(x_1, x_2, Q^2) D_1^{f'}(x_2, Q^2)$$

Double scattering process

Double parton scattering: two hard processes



Two types of partons:

$$f_1, f_2$$

Two momentum fractions:

$$x_1, x_2 \quad x_1 + x_2 \leq 1$$

Two hard scales:

$$Q_1, Q_2 \gg \Lambda_{QCD}$$

Relative transverse momentum:

$$\Delta$$

Double PDF (DPDF):

$$D_2^{f_1 f_2}(x_1, x_2; Q_1^2, Q_2^2; \Delta)$$

Factorization formula(?):

$$\sigma = \int dx_1 dx_2 dx'_1 dx'_2 d^2 \Delta D_2^{f_1 f_2}(x_1, x_2; Q_1^2, Q_2^2; \Delta) \hat{\sigma}^{f_1 f'_1}(\hat{s}_1, Q_1^2) \hat{\sigma}^{f_2 f'_2}(\hat{s}_2, Q_2^2) D_2^{f'_1 f'_2}(x'_1, x'_2, Q_1^2, Q_2^2; -\Delta)$$

Evolution equations for double PDFs

Evolution equations for dPDFs are known for the case of equal scales and momentum transfer zero: $D_2^{f_1 f_2}(x_1, x_2; Q^2, Q^2; \Delta = 0)$ (integrated over the transverse coordinate space)

In (leading logarithmic approximation) they correspond to the inclusive probability of finding two partons in a hadron with longitudinal momenta x_1 and x_2

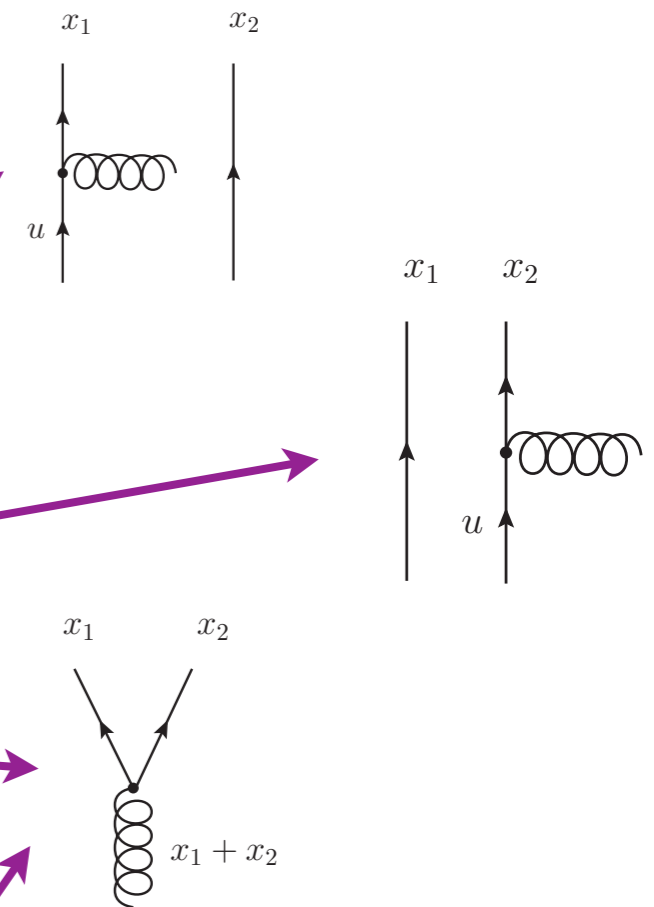
DGLAP evolution equation for double PDF:

$$\begin{aligned} \partial_t D_{f_1 f_2}(x_1, x_2, t) &= \sum_{f'} \int_0^{1-x_2} du \mathcal{K}_{f_1 f'}(x_1, u, t) D_{f' f_2}(u, x_2, t) \\ &+ \sum_{f'} \int_0^{1-x_1} du \mathcal{K}_{f_2 f'}(x_2, u, t) D_{f_1 f'}(x_1, u, t) \\ &+ \sum_{f'} \mathcal{K}_{f' \rightarrow f_1 f_2}^R(x_1, x_2, t) D_{f'}(x_1 + x_2, t) \end{aligned}$$

Konishi, Ukawa, Veneziano; Snigirev, Zinovev, Shelest

Non-homogeneous term
Splitting term of one parton into two:

$$\mathcal{K}_{f' \rightarrow f_1 f_2}^R(x_1, x_2, t) = \frac{\alpha_s(t)}{2\pi} \frac{1}{x_1 + x_2} P_{f' f_1}^{(0)}\left(\frac{x_1}{x_1 + x_2}\right)$$

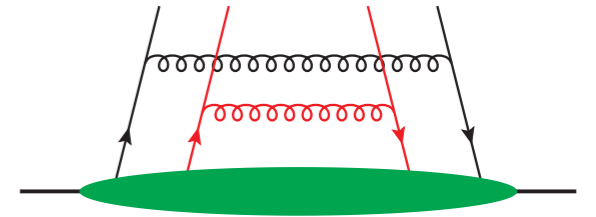


Evolution equations for double PDFs

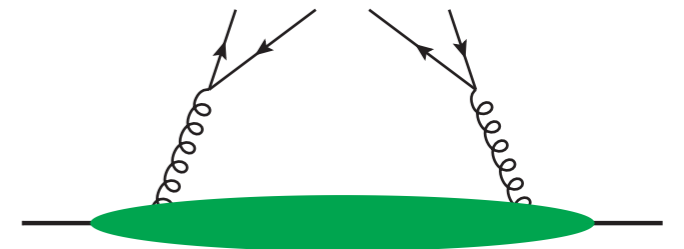
DGLAP evolution equation for double PDF:

$$\begin{aligned} \partial_t D_{f_1 f_2}(x_1, x_2, t) = & \sum_{f'} \int_0^{1-x_2} du \mathcal{K}_{f_1 f'}(x_1, u, t) D_{f' f_2}(u, x_2, t) \\ & + \sum_{f'} \int_0^{1-x_1} du \mathcal{K}_{f_2 f'}(x_2, u, t) D_{f_1 f'}(x_1, u, t) \\ & + \sum_{f'} \mathcal{K}_{f' \rightarrow f_1 f_2}^R(x_1, x_2, t) D_{f'}(x_1 + x_2, t), \end{aligned}$$

Homogeneous term



Non-homogeneous (splitting) term



DGLAP evolution equation for single PDF:

$$\partial_t D_f(x, t) = \sum_{f'} \int_0^1 du \mathcal{K}_{f f'}(x, u, t) D_{f'}(u, t)$$

Need to be solved together with suitable initial conditions.

Sum rules for single and double PDFs

Momentum sum rule for single PDFs

$$\sum_f \int_0^1 dx x D_f(x, t) = 1$$

Quark number sum rule for single PDFs

$$\int_0^1 dx \{D_{q_i}(x, t) - D_{\bar{q}_i}(x, t)\} = N_i$$

Momentum sum rule for double PDFs

$$\sum_{f_1} \int_0^{1-x_2} dx_1 x_1 \frac{D_{f_1 f_2}(x_1, x_2, t)}{D_{f_2}(x_2, t)} = 1 - x_2$$

Conditional probability to find the parton f_1 with the momentum fraction x_1 while keeping fixed the second parton f_2 with momentum x_2 .

Valence quark number sum rule for double PDFs

$$\int_0^{1-x_2} dx_1 \{D_{q_i f_2}(x_1, x_2, t) - D_{\bar{q}_i f_2}(x_1, x_2, t)\} = \begin{cases} N_i D_{f_2}(x_2, t) & \text{for } f_2 \neq q_i, \bar{q}_i \\ (N_i - 1) D_{f_2}(x_2, t) & \text{for } f_2 = q_i \\ (N_i + 1) D_{f_2}(x_2, t) & \text{for } f_2 = \bar{q}_i \end{cases}$$

If sum rules hold for initial conditions they will hold for higher scales after the evolution.

How to consistently impose the initial conditions for sPDF and dPDF with sum rules?

Problem of initial conditions in dPDFs

Usually simplifying assumption is taken:

$$D_{f_1 f_2}(x_1, x_2) = D_{f_1}(x_1) D_{f_2}(x_2)$$

Factorizable ansatz, could work well for rather small x but is inconsistent with sum rules.

Improvement with correlating factor:

Gaunt, Stirling

$$D_{f_1 f_2}(x_1, x_2) = D_{f_1}(x_1) D_{f_2}(x_2) \frac{(1 - x_1 - x_2)^2}{(1 - x_1)^{2+n_1} (1 - x_2)^{2+n_2}}$$

Takes into account some correlation but still does not obey sum rules exactly.

Initial conditions

If the single distribution is given by a Beta distribution

$$D(x) = N_1 x^{-\alpha} (1 - x)^{\beta}$$

There is a unique solution in terms of the Dirichlet distribution for the double parton density:

$$D(x_1, x_2) = N_2 x_1^{-\tilde{\alpha}} x_2^{-\tilde{\alpha}} (1 - x_1 - x_2)^{\tilde{\beta}}$$

With powers of the dPDF being related to the powers of sPDF

$$\tilde{\alpha} = \alpha, \quad \tilde{\beta} = \beta + \alpha - 1$$

Normalization for dPDF in this particular case is uniquely determined.

Small x powers for single and double PDFs are the same.

The large x power of the correlating factor in dPDF is related to the sum of large and small x powers of the single distribution.

Initial conditions: expansion

Realistic parametrizations are however more complicated than a single Beta distribution.

Example MSTW2008 gluon PDF: $x D_1^g(x, Q^2) = N_1 x^{-\delta_g} (1-x)^{\eta_g} (1 + \epsilon_g \sqrt{x} + \gamma_g x)$,

However, this parametrization is sum of Beta distributions of the form:

$$D(x) = N_1 \sum_{k=1}^K a_k x^{-\alpha_k} (1-x)^{\beta_k}$$

Assuming that the dPDF is the sum of Dirichlet distributions:

$$D(x_1, x_2) = N_2 \sum_{k=1}^K c_k x_1^{-\tilde{\alpha}_k} x_2^{-\tilde{\alpha}_k} (1-x_1-x_2)^{\tilde{\beta}_k}$$

Performing the same analysis as before (for single channel) one obtains the conditions for each k:

$$\tilde{\alpha}_k = \alpha_k$$

$$\tilde{\beta}_k = \beta_k - 1 + \alpha_k$$

The normalizations:

$$c_k = a_k \frac{B(\alpha_1 + \beta_1, 2 - \alpha_1)}{B(\beta_k + \alpha_k, 2 - \alpha_k)}$$

$$N_2 = N_1 \frac{1}{B(\alpha_1 + \beta_1, 2 - \alpha_1)}$$

Initial conditions for dPDFs: ratios

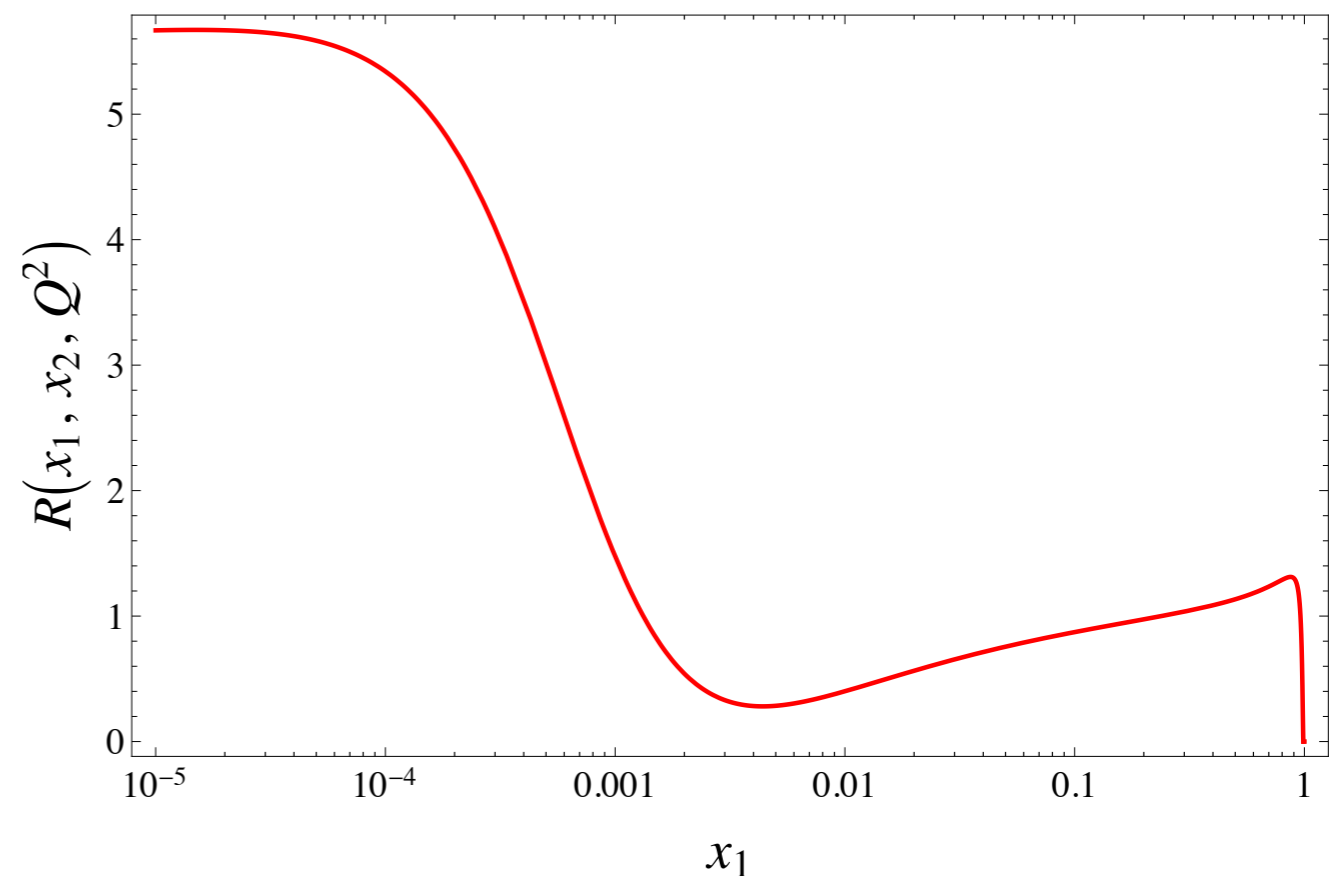
Ratio of double distribution to product of single distributions:

$$R^{gg}(x_1, x_2, Q^2) = \frac{D_2^{gg}(x_1, x_2, Q^2)}{D_1^g(x_1, Q^2) D_1^g(x_2, Q^2)}$$

- Measure of the correlations at the initial scale.
- For this parametrization the correlations are very significant.
- Ratio different from unity over wide range of x .
- Factorization of powers at small x but different normalization.
- In principle can extend to quarks, requires some constraints put onto the form of the single PDFs.

Ratio of Double Parton Distribution to Product of Single Parton Distributions

$$x_2 = 1. \times 10^{-2}, Q^2 = 1. \text{ GeV}^2$$



Evolution of single and double PDFs

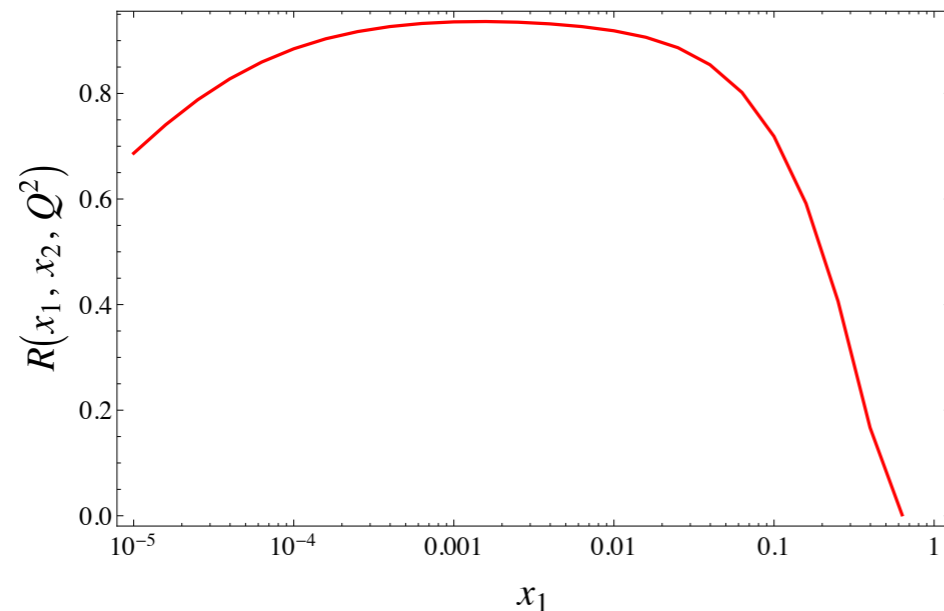
Evolve the dPDFs and sPDFs using DGLAP equations;

$$D_1^f(x, Q_0) \rightarrow D_1^f(x, Q)$$

$$D_2^{f_1 f_2}(x_1, x_2, Q_0) \rightarrow D_2^{f_1 f_2}(x_1, x_2, Q)$$

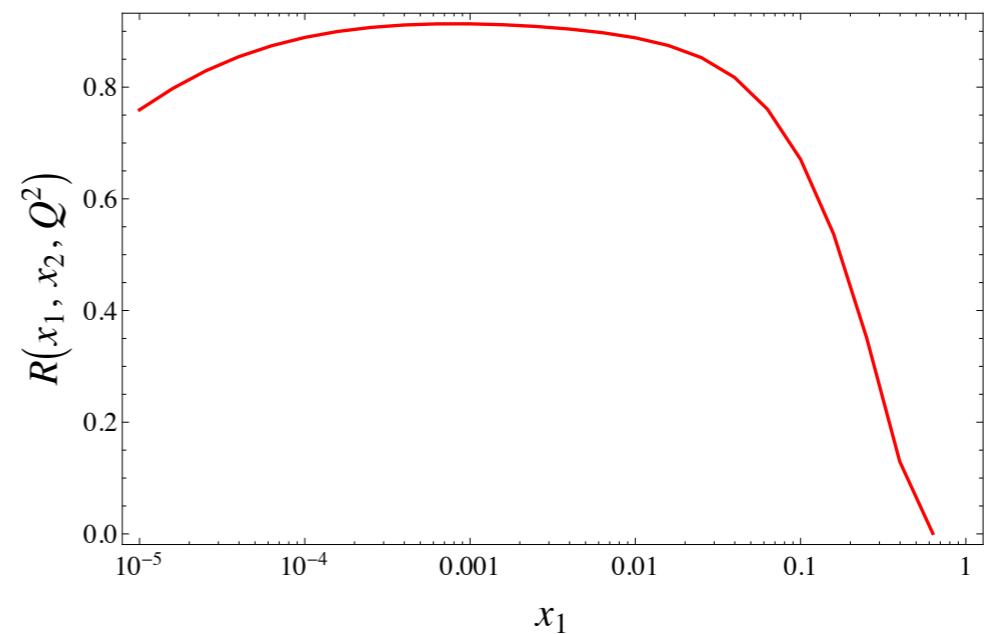
Ratio of Double Parton Distribution to Product
of Single Parton Distributions

$$x_2 = 3. \times 10^{-1}, Q^2 = 25. \text{ GeV}^2$$



Ratio of Double Parton Distribution to Product
of Single Parton Distributions

$$x_2 = 3. \times 10^{-1}, Q^2 = 100. \text{ GeV}^2$$



Correlation washed out by evolution except for large x.

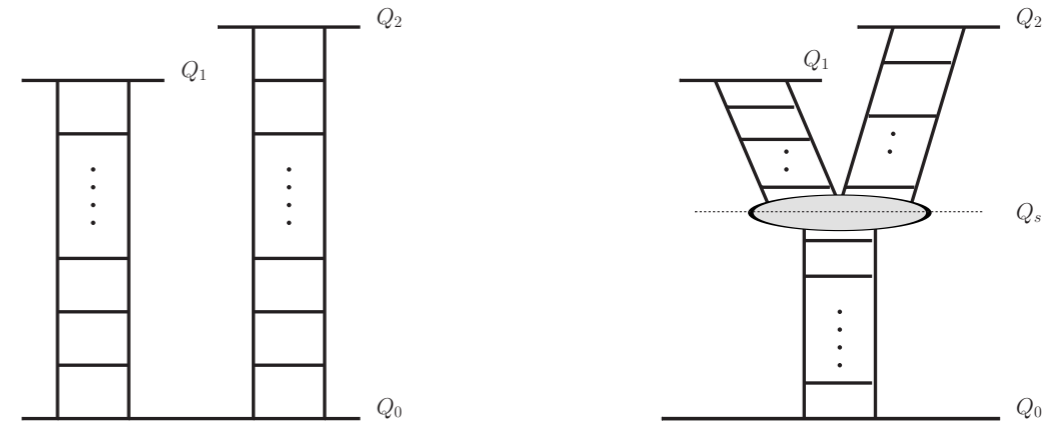
More on Factorization of dPDFs into sPDFs

How good is factorization of the dPDFs into product of sPDFs

$$D_{f_1 f_2}(x_1, x_2) = D_{f_1}(x_1) D_{f_2}(x_2) \quad ?$$

The solution to the evolution equations (generally for unequal scales) can be written as a sum of homogenous and inhomogenous terms

$$D_{f_1 f_2} = D_{f_1 f_2}^{(h)} + D_{f_1 f_2}^{(nh)}$$



$$D_{f_1 f_2} = \sum_{f', f''} \left\{ \int_{x_1}^{1-x_2} \frac{dz_1}{z_1} \int_{x_2}^{1-z_1} \frac{dz_2}{z_2} E_{f_1 f'} \left(\frac{x_1}{z_1}, Q_1, Q_0 \right) E_{f_2 f''} \left(\frac{x_2}{z_2}, Q_2, Q_0 \right) \times D_{f' f''}(z_1, z_2, Q_0, Q_0) \right. \\ \left. + \int_{Q_0^2}^{Q_{\min}^2} \frac{dQ_s^2}{Q_s^2} \int_{x_1}^{1-x_2} \frac{dz_1}{z_1} \int_{x_2}^{1-z_1} \frac{dz_2}{z_2} E_{f_1 f'} \left(\frac{x_1}{z_1}, Q_1, Q_s \right) E_{f_2 f''} \left(\frac{x_2}{z_2}, Q_2, Q_s \right) D_{f' f''}^{(sp)}(z_1, z_2, Q_s) \right\}$$

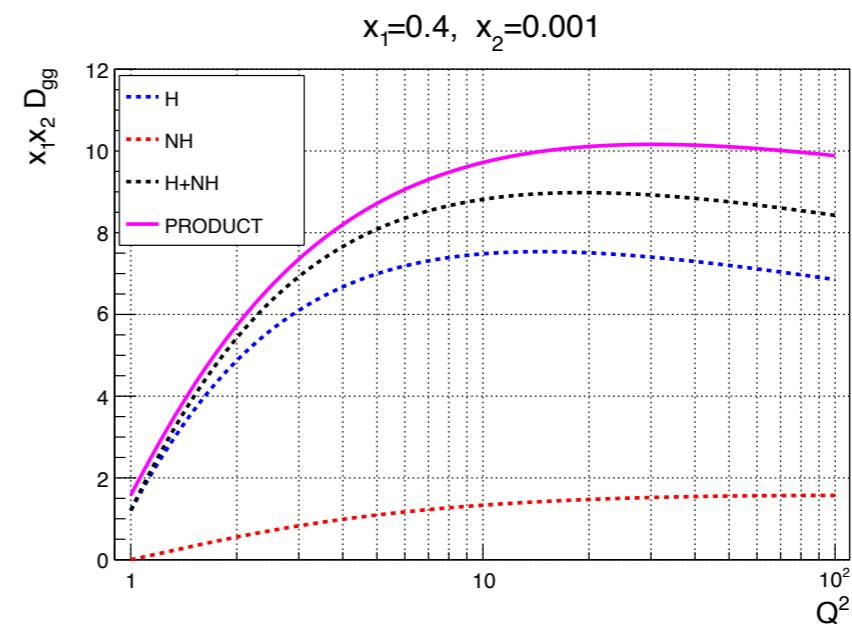
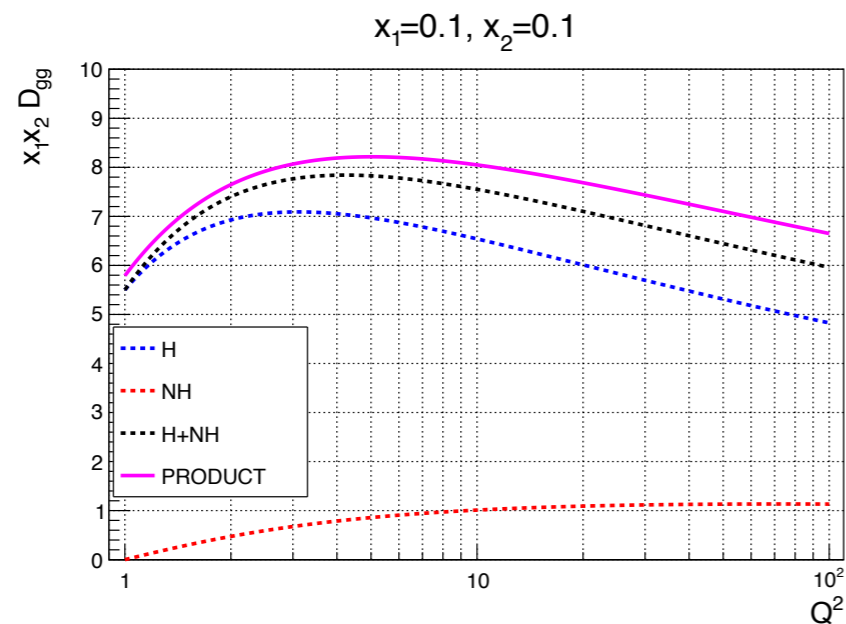
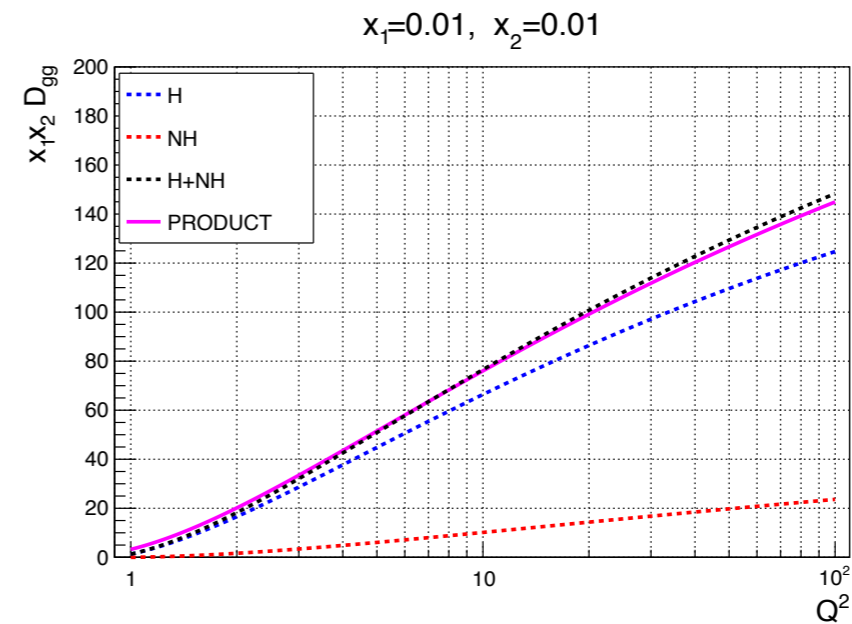
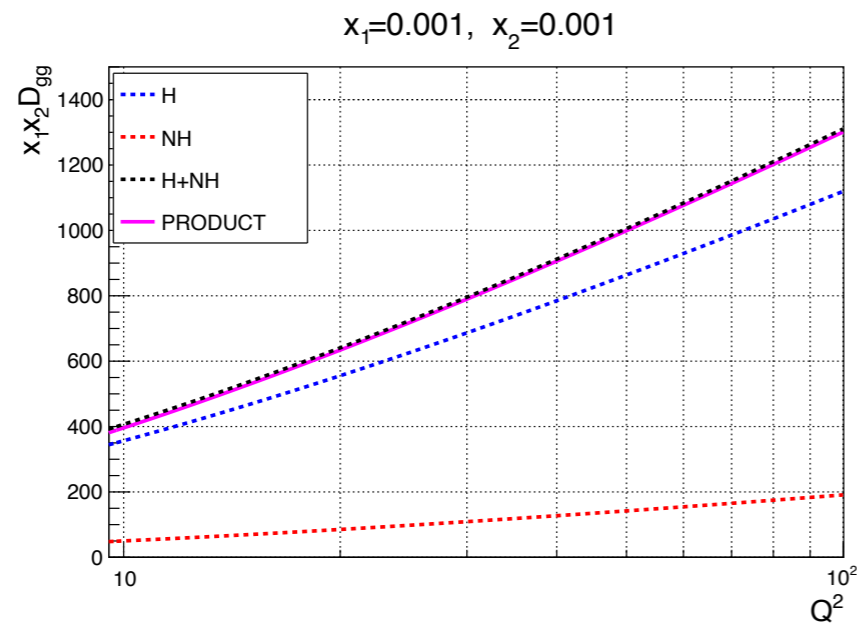
Green's functions for DGLAP evolution:

$$D_a(x, \mu) = \sum_b \int_x^1 \frac{dz}{z} E_{ab} \left(\frac{x}{z}, \mu, \mu_0 \right) D_b(z, \mu_0)$$

Splitting/non-homogenous term

$$D_{a'a''}^{(sp)}(x_1, x_2, Q_s) = \frac{\alpha_s(Q_s)}{2\pi} \sum_a \frac{D_a(x_1 + x_2, Q_s)}{x_1 + x_2} P_{a \rightarrow a'a''} \left(\frac{x_1}{x_1 + x_2} \right)$$

More on Factorization of dPDFs into sPDFs



Factorization works well for small x and large Q , provided momentum sum rule is implemented in the initial conditions.

Factorization holds for the sum of homogeneous and inhomogeneous terms.

Unintegrated DPDFs

What about the transverse momentum dependence of the DPDFs?

Possible formulation:

Small x Color Glass Condensate formalism: higher Wilson line correlators

TMD formulation *Buffing, Diehl, Kasemets*

Simple practical approach:

- *Kimber - Martin - Ryskin* approach to the unintegrated parton densities.
- Includes transverse momentum dependence in the parton densities.
- Practical approach for the phenomenology, using integrated densities, convoluted with the Sudakov form factors

Unintegrated PDFs

Martin, Kimber, Ryskin

DGLAP evolution for single PDF

$$\frac{\partial D_a(x, \mu)}{\partial \ln \mu^2} = \sum_{a'} \int_x^{1-\Delta} \frac{dz}{z} P_{aa'}(z, \mu) D_{a'}\left(\frac{x}{z}, \mu\right) - D_a(x, \mu) \sum_{a'} \int_0^{1-\Delta} dz z P_{a'a}(z, \mu)$$

real

virtual

after integrating out the virtual part

$$D_a(x, Q) = T_a(Q, Q_0) D_a(x, Q_0) + \int_{Q_0^2}^{Q^2} \frac{dk_{\perp}^2}{k_{\perp}^2} f_a(x, k_{\perp}, Q)$$

where the unintegrated density:

$$f_a(x, k_{\perp}, Q) \equiv T_a(Q, k_{\perp}) \sum_{a'} \int_x^{1-\Delta} \frac{dz}{z} P_{aa'}(z, k_{\perp}) D_{a'}\left(\frac{x}{z}, k_{\perp}\right)$$

or

$$f_a(x, k_{\perp}, Q) = \frac{\partial}{\partial \ln k_{\perp}^2} [T_a(Q, k_{\perp}) D_a(x, k_{\perp})]$$

with Sudakov
formfactor

$$T_a(Q, k_{\perp}) = \exp \left\{ - \int_{k_{\perp}^2}^{Q^2} \frac{dp_{\perp}^2}{p_{\perp}^2} \sum_{a'} \int_0^{1-\Delta} dz z P_{a'a}(z, p_{\perp}) \right\}$$

$$T_a(Q, k_{\perp}) \simeq 1, \quad Q \sim k_{\perp} \quad T_a(Q, k_{\perp}) \simeq 0, \quad Q \gg k_{\perp}$$

Unintegrated PDFs

$$f_a(x, k_\perp, Q) \equiv T_a(Q, k_\perp) \sum_{a'} \int_x^{1-\Delta} \frac{dz}{z} P_{aa'}(z, k_\perp) D_{a'}\left(\frac{x}{z}, k_\perp\right)$$

Dependence on two scales obtained in the last step of the evolution

Need to specify the cutoff :

DGLAP ordering:

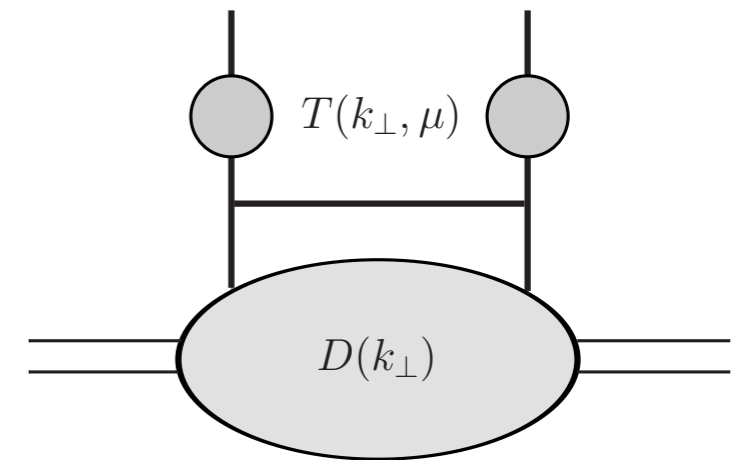
$$\Delta = \frac{k_\perp}{Q}$$

CCFM angular ordering:

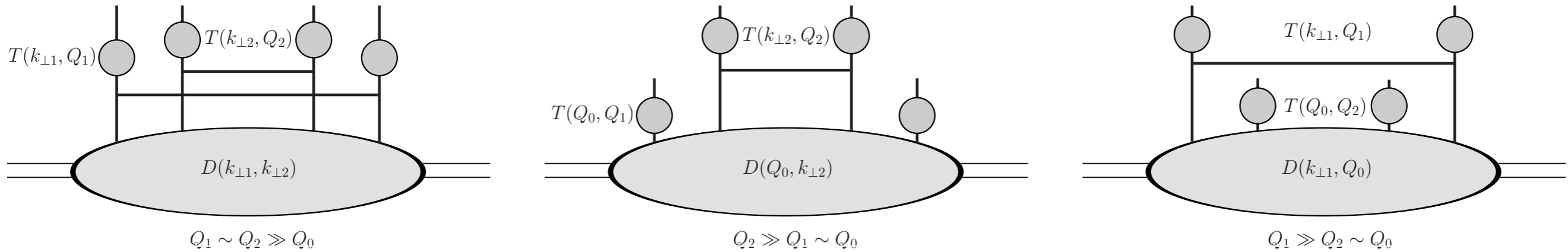
$$\Delta = \frac{k_\perp}{k_\perp + Q}$$

$$\Theta(\theta - \theta') \Rightarrow \mu > zk_t/(1-z) \quad z_{max} = \frac{\mu}{\mu + k_t}$$

Larger phase space for emissions, tail in transverse momentum extends to $k_\perp > Q$



Unintegrated DPDFs : homogeneous part



$$\underline{Q_1^2 \sim Q_2^2 \gg Q_0^2}$$

$k_{1\perp}$ unintegrated

$k_{2\perp}$ unintegrated

$$f_{a_1 a_2}^{(h)}(x_1, x_2, k_{1\perp}, k_{2\perp}, Q_1, Q_2) = T_{a_1}(Q_1, k_{1\perp}) T_{a_2}(Q_2, k_{2\perp}) \sum_{b,c} \int_{\frac{x_1}{1-x_2}}^{1-\Delta_1} \frac{dz_1}{z_1} \int_{\frac{x_2}{1-x_1/z_1}}^{1-\Delta_2} \frac{dz_2}{z_2} P_{a_1 b}(z_1, k_{1\perp}) P_{a_2 c}(z_2, k_{2\perp}) D_{bc}^{(h)}\left(\frac{x_1}{z_1}, \frac{x_2}{z_2}, k_{1\perp}, k_{2\perp}\right)$$

$$\underline{Q_1^2 \sim Q_0^2 \text{ and } Q_2^2 \gg Q_0^2}$$

$k_{1\perp}$ integrated

$k_{2\perp}$ unintegrated

$$f_{a_1, a_2}^{(h)}(x_1, x_2, k_{2\perp}, Q_1, Q_2) = T_{a_1}(Q_1, Q_0) T_{a_2}(Q_2, k_{2\perp}) \sum_b \int_{\frac{x_2}{1-x_1}}^{1-\Delta_2} \frac{dz_2}{z_2} P_{a_2 b}(z_2, k_{2\perp}) D_{a_1 b}^{(h)}\left(x_1, \frac{x_2}{z_2}, Q_0, k_{2\perp}\right)$$

$$\underline{Q_1^2 \gg Q_0^2 \text{ and } Q_2^2 \sim Q_0^2}$$

$k_{2\perp}$ integrated

$k_{1\perp}$ unintegrated

$$f_{a_1, a_2}^{(h)}(x_1, x_2, k_{1\perp}, Q_1, Q_2) = T_{a_1}(Q_1, k_{1\perp}) T_{a_2}(Q_2, Q_0) \sum_b \int_{\frac{x_1}{1-x_2}}^{1-\Delta_1} \frac{dz_1}{z_1} P_{a_1 b}(z_1, k_{1\perp}) D_{b a_2}^{(h)}\left(\frac{x_1}{z_1}, x_2, k_{1\perp}, Q_0\right)$$

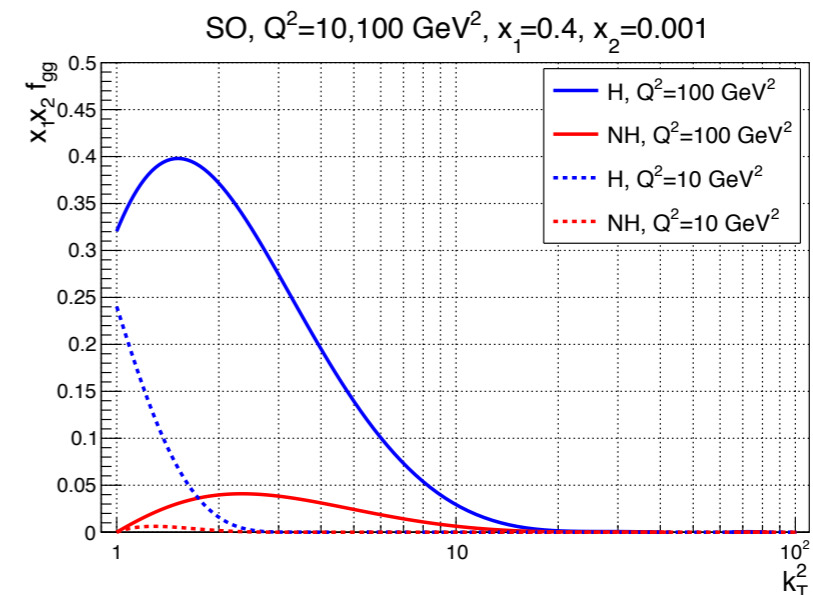
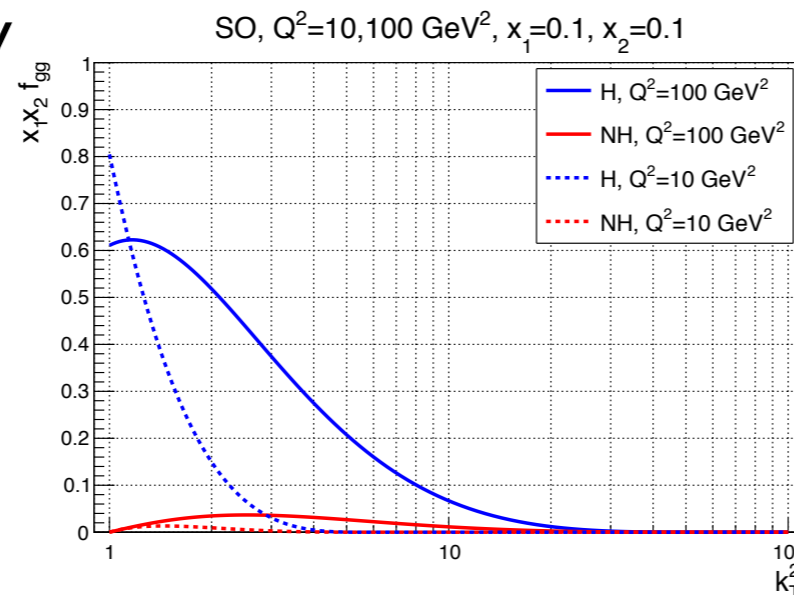
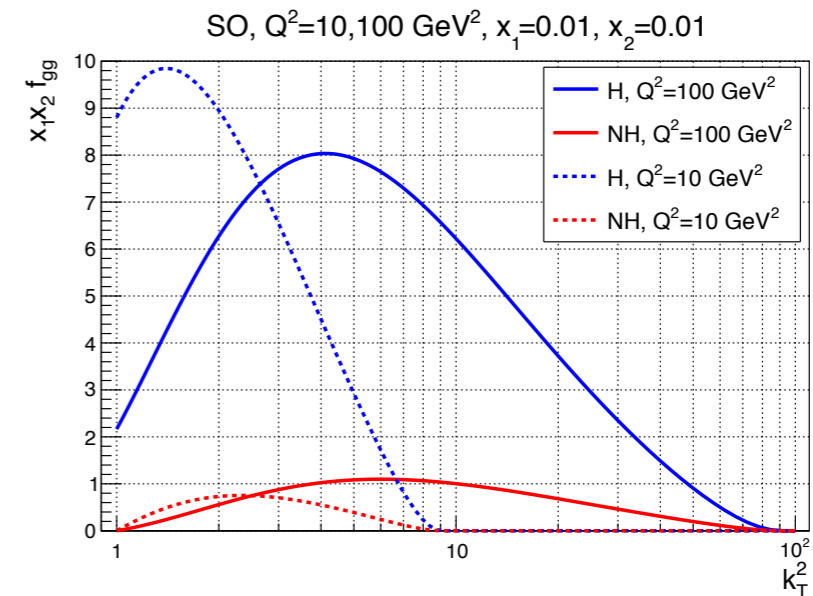
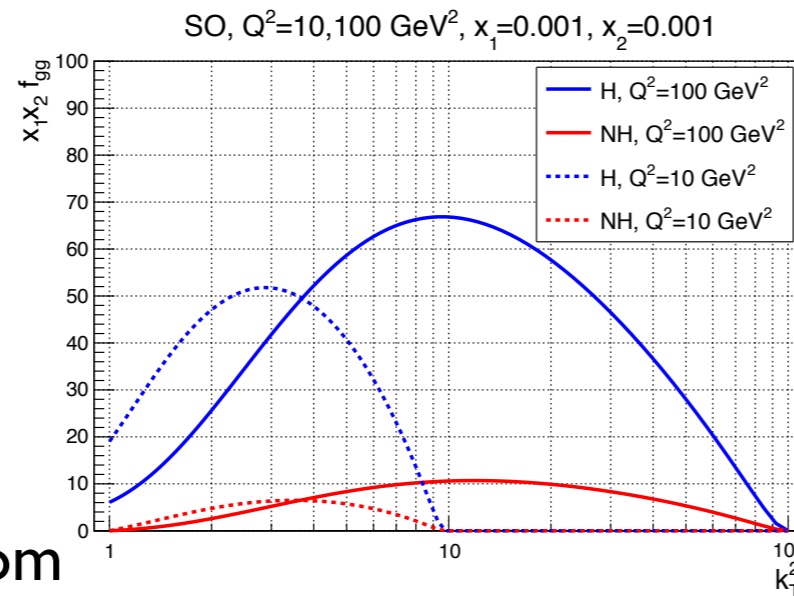
$$\underline{Q_1^2 \sim Q_2^2 \sim Q_0^2}$$

non-perturbative region - parametrized by integrated density

Results for unintegrated dPDFs

The same method can be applied to the non-homogeneous term.

$$f_{gg} = f_{gg}^{(h)}(x_1, x_2, k_{1\perp}, k_{2\perp}, Q_1, Q_2) + f_{gg}^{(nh)}(x_1, x_2, k_{1\perp}, k_{2\perp}, Q_1, Q_2)$$

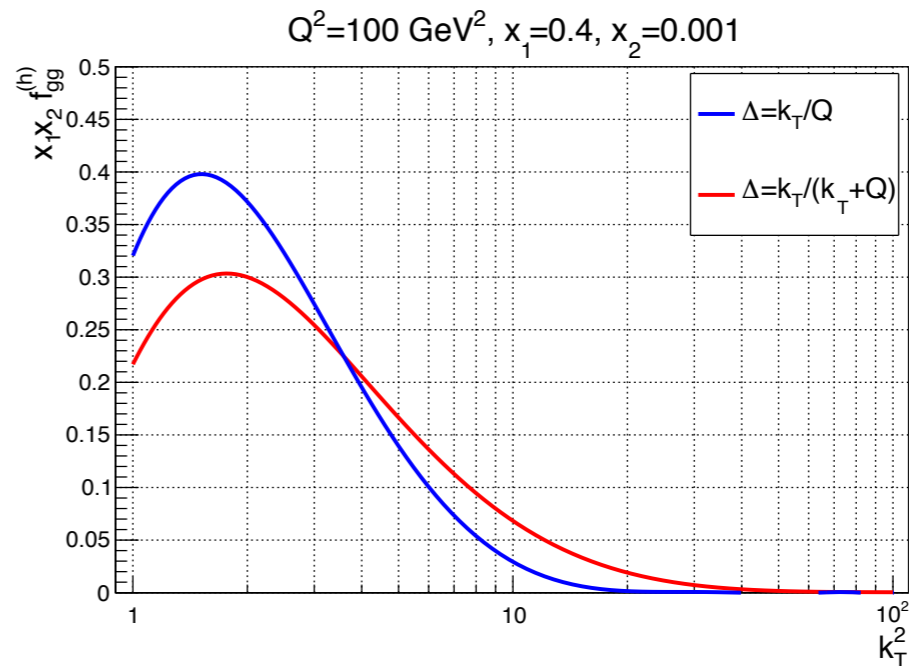
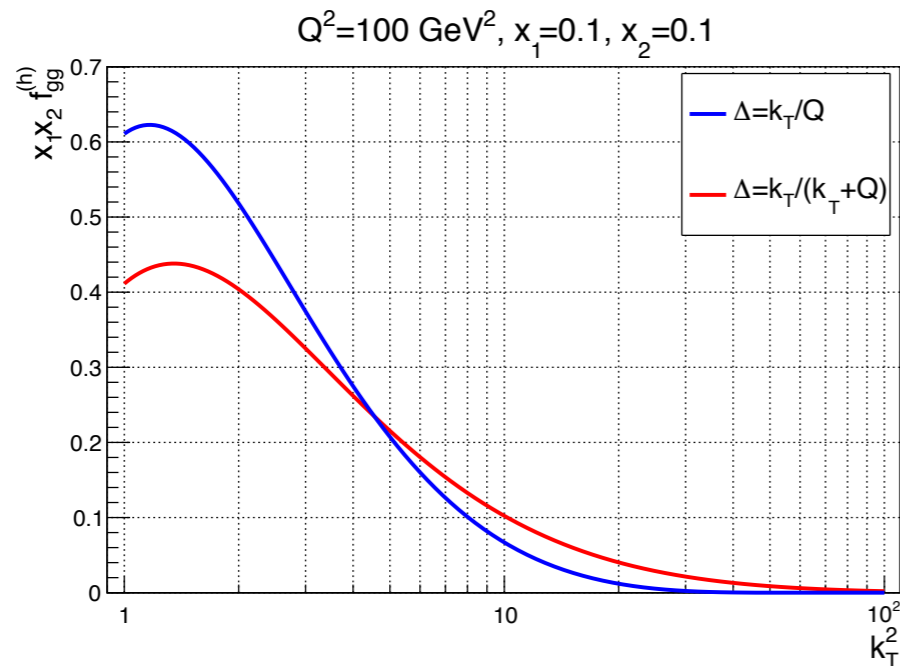
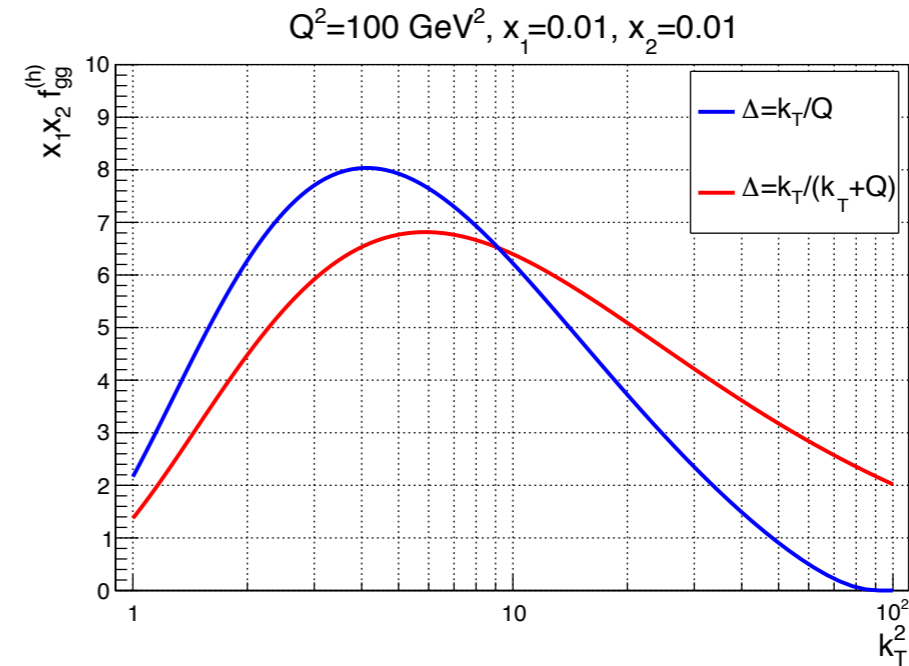
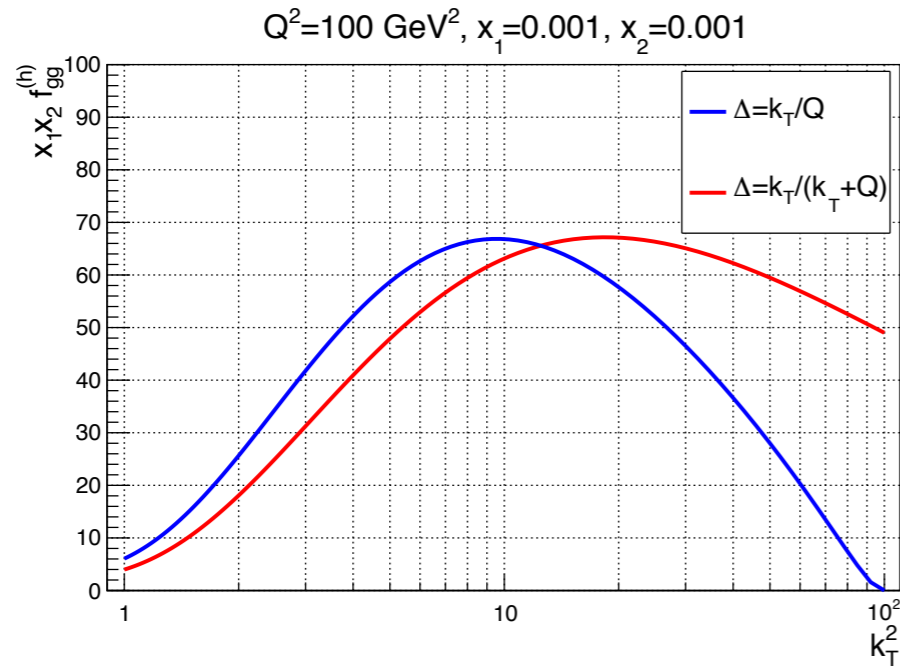


Showing contributions from perturbative region only

Non-homogenous term smaller than the homogenous one(that ratio could depend on the initial condition)

Higher Q , lower x , the maximum shifts to higher transverse momenta

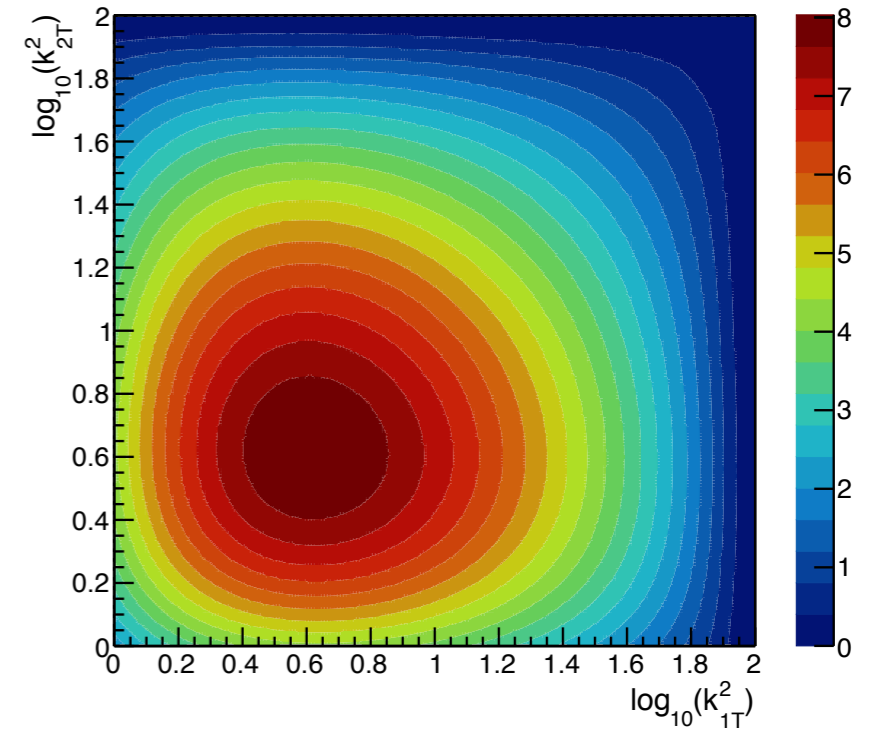
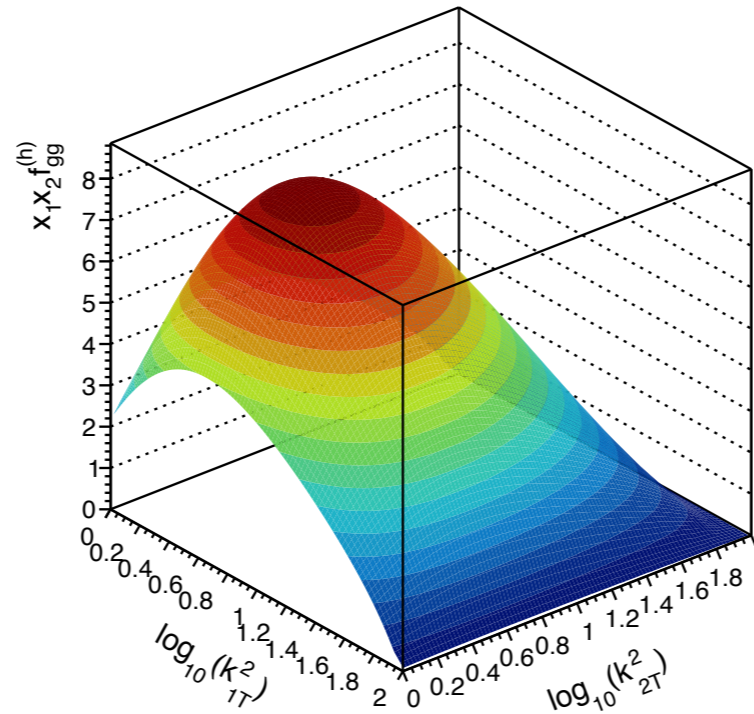
Results for unintegrated dPDFs



Effect of angular ordering cutoff: larger tails in k_T

Results for unintegrated dPDFs

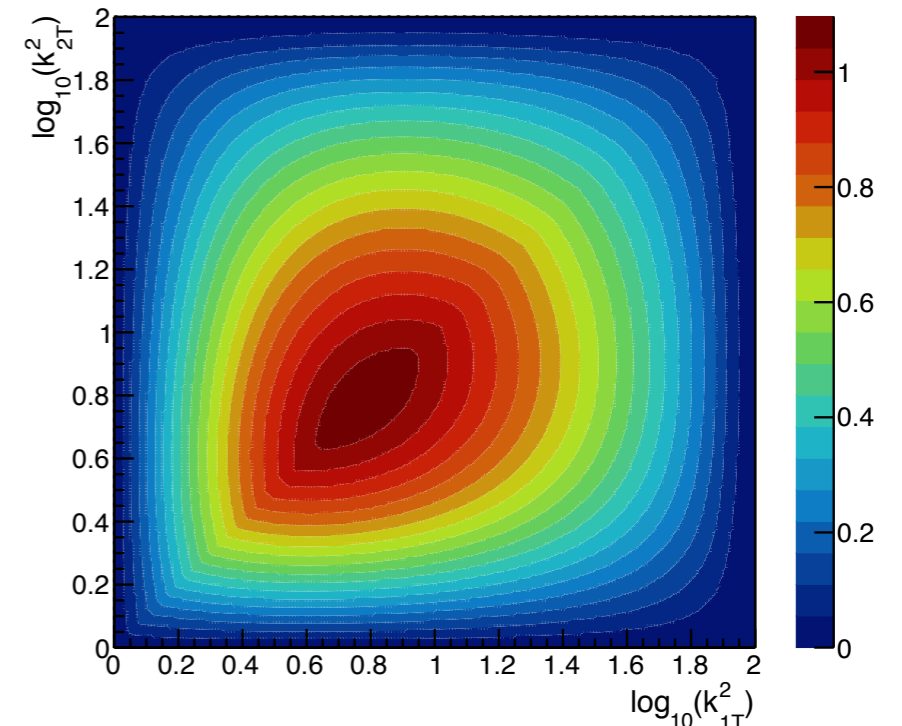
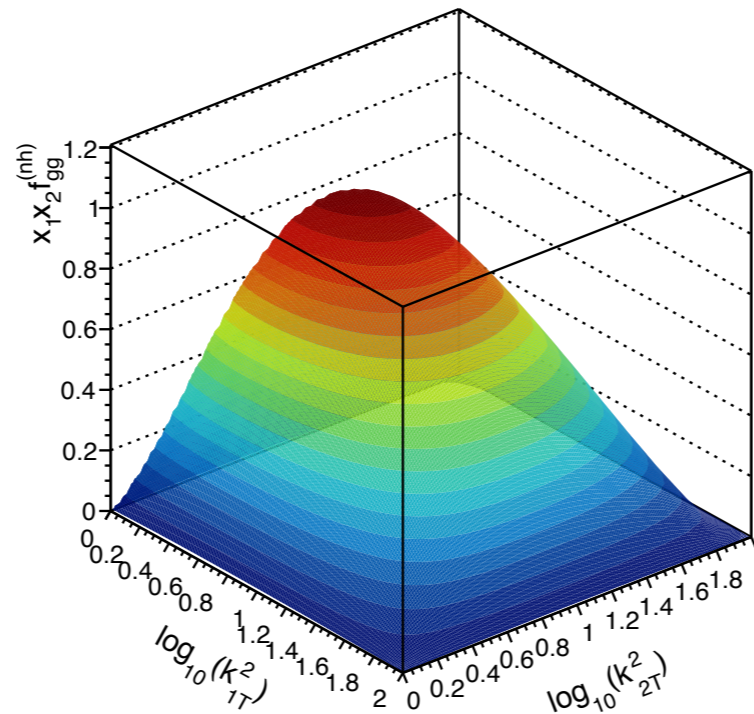
$Q^2=100 \text{ GeV}^2, x_1=0.01, x_2=0.01$ homogeneous



Fixed Q^2 , fixed x_1, x_2

Non-homogeneous term is more perturbative, higher transverse momenta

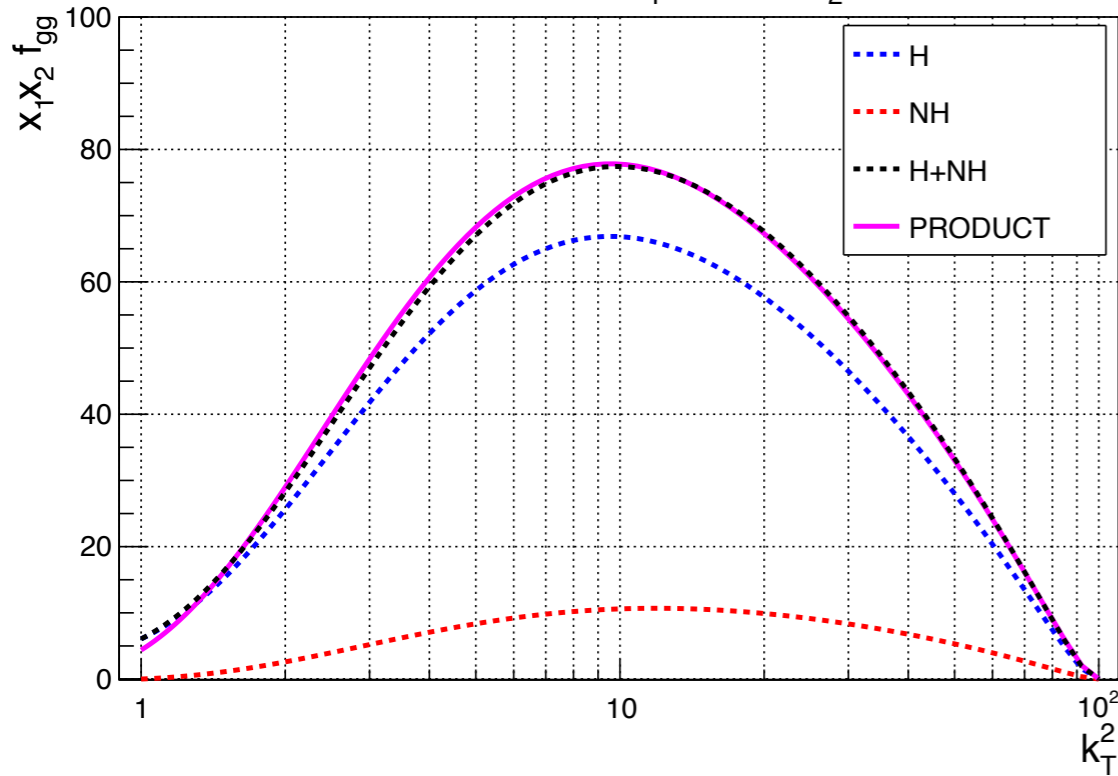
$Q^2=100 \text{ GeV}^2, x_1=0.01, x_2=0.01$ non-homogeneous



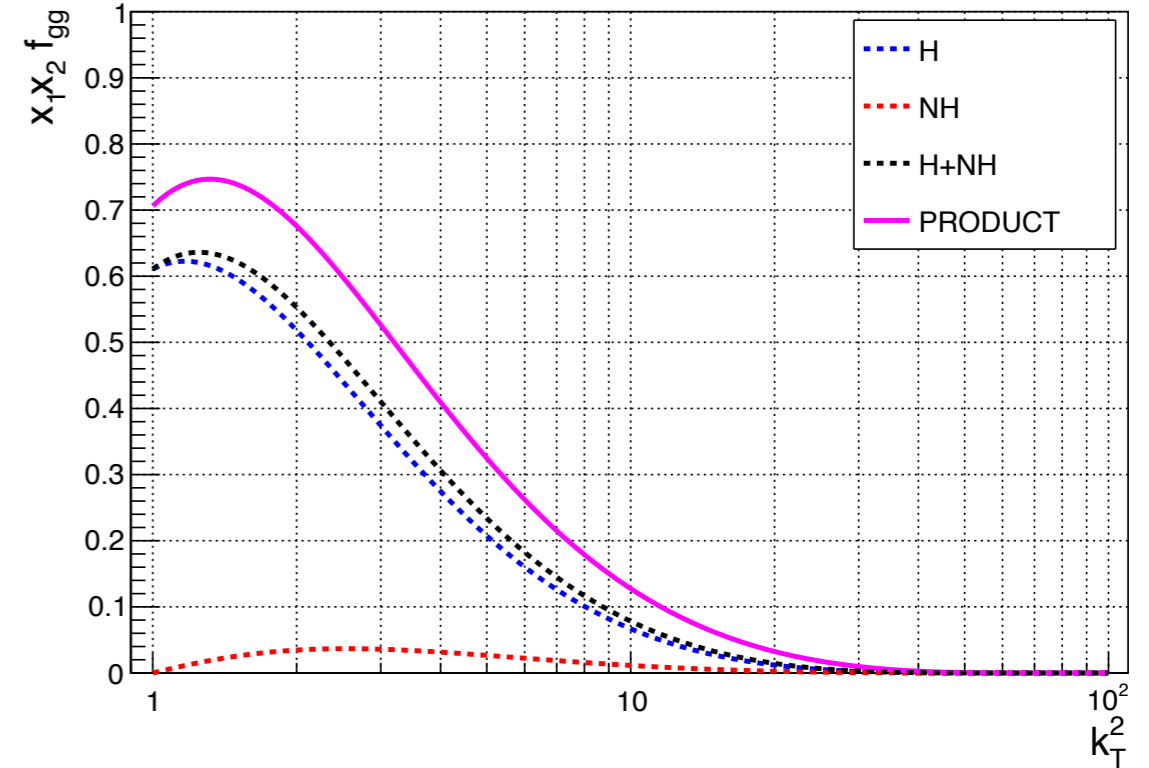
Smaller value for nonhomogeneous term.

Factorization of unintegrated dPDFs

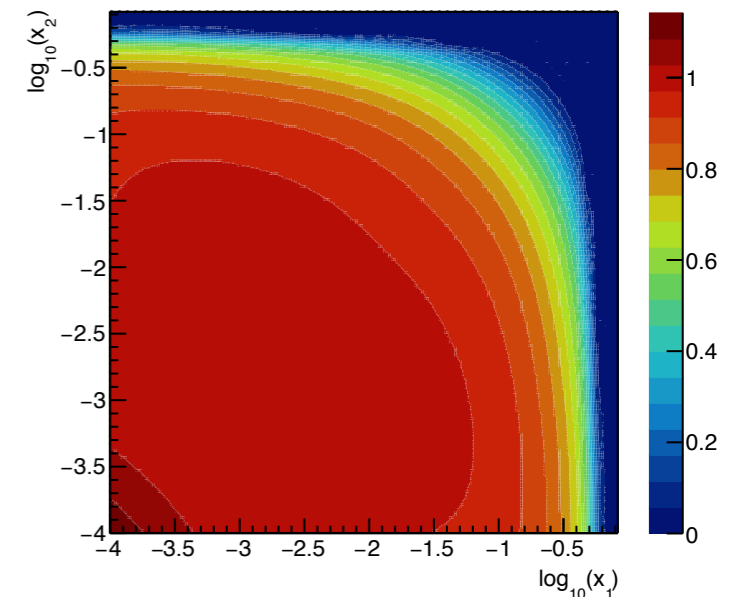
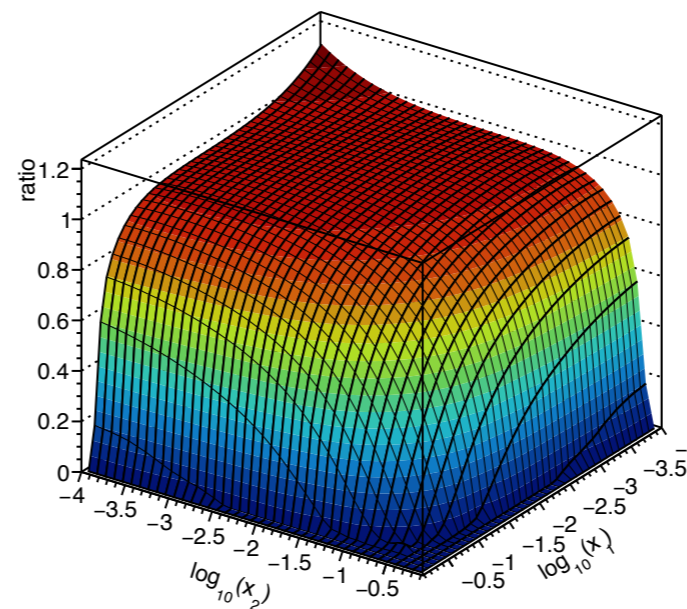
SO, $Q^2=100 \text{ GeV}^2$, $x_1=0.001$, $x_2=0.001$



SO, $Q^2=100 \text{ GeV}^2$, $x_1=0.1$, $x_2=0.1$



$Q^2=100 \text{ GeV}^2$, $k_T^2=10 \text{ GeV}^2$



$$r = \frac{f_{gg}(x_1, x_2, k_{1\perp}, k_{2\perp}, Q, Q)}{f_g(x_1, k_{1\perp}, Q) f_g(x_2, k_{2\perp}, Q)}$$

Summary and outlook

- Summary & outlook I - initial conditions:
 - Double integrated PDFs need consistent initial conditions for the evolution.
 - Beta functions for single PDF and Dirichlet distributions for double PDF with suitably matched powers and coefficients are good initial conditions. The momentum sum rule and quark number sum rule are satisfied simultaneously.
 - Extending the formalism: expansion in terms of Dirichlet distributions. First numerical tests with gluons. Sum rules provide relations between the powers at small and large x for single and double parton distributions.
 - In principle one can include quarks into the formalism; some additional constraints needed.
 - Is there any deeper physical meaning to the presented algorithm?

Summary and outlook

- Summary & outlook 2 - transverse momentum dependence:
 - Extended the KMR approach to dPDFs.
 - Expressions include correlations through the integrated dPDFs. Additional correlations enter through the regularization cutoffs.
 - Unintegrated dPDFs: small x , higher k_T values. Homogenous term dominates over the non-homogenous one.
 - Numerical simulations indicate very good factorization for large scales and small x for the sum of homogeneous and non-homogenous contributions. That property is contingent upon the momentum sum rule.
 - Include the momentum transfer.