Statistical Properties of Helical Magnetic Fields

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Small seed (primordial) fields can be amplified by various mechanisms.

What is the origin of these PMFs?

Generation mechanism affects the statistical properties.
Generation Mechanisms

Inflationary Magnetogenesis

- PMFs arise from vacuum fluctuations\(^a\) - very large correlation lengths.
- Involves the breaking of conformal symmetry.
- Scale invariant (or nearly) power spectrum.
- Typically involves couplings like \(R^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}\) or \(f(\phi)F_{\mu\nu} F^{\mu\nu}\).

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Phase Transition Magnetogenesis

- An out of equilibrium, first-order transition is typically needed.
- Violent bubble nucleation generates significant turbulence\(^a\).
- *Causal* processes\(^b\) - limited correlation lengths (\(H_\chi^{-1}\)).
- Two main phase transitions are:
  1. Electroweak Phase Transition (\(T \sim 100\) GeV)
  2. QCD Phase Transition (\(T \sim 150\) MeV)


There is Helicity!

- Generation mechanisms can involve significant parity ($P$) violation.
- This can lead to *helical* PMFs - the evolution is affected\(^2\).
- Helicity can grow with evolution\(^3\).
- Can help us understand phenomena like *Baryogenesis*.


Magnetic Helicity

This is given by

\[ H = \int_V \mathbf{A} \cdot \mathbf{B} \, d^3r = \int_{\tilde{V}} \tilde{\mathbf{A}} \cdot \tilde{\mathbf{B}} \, d^3x \quad (1) \]

\( V \) denotes a closed volume with fully contained fields lines.

\( H \) is invariant under \( \mathbf{A} \rightarrow \mathbf{A} + \nabla \Lambda \) if \( \mathbf{B} \) vanishes at the boundaries.

The evolution of \( H \) is,

\[ \frac{dH}{d\tau} = -2\tilde{\eta} \int_{\tilde{V}} \tilde{\mathbf{B}} \cdot (\nabla \times \tilde{\mathbf{B}}) \, d^3x \quad (2) \]

It is seen\(^4\) that:

- Partial magnetic helicity evolves to full helicity.
- Kinetic helicity is converted to magnetic helicity.

Motivation

- We study the evolution of correlation lengths ($\lambda$) of fields generated at inflation.
- Investigating how the *smoothed* fields are related to $\lambda$ and the *realizability condition*.
- Making the *realizability condition* hold consistently for scale invariant fields.
- Based on arXiv:1804.01177.
Modeling Magnetic Fields

We work with the correlation function in $k$-space,

$$B_{ij}(r) \equiv \langle B_i(x)B_j(x + r) \rangle \quad \Rightarrow \quad \mathcal{F}_{ij}^{(B)}(k) = \int d^3 r \: e^{i k \cdot r} \: B_{ij}(r)$$ (3)

This gives the symmetric and helical parts,

$$\frac{\mathcal{F}_{ij}^{(B)}(k)}{(2\pi)^3} = P_{ij}(\hat{k}) \frac{E_M(k)}{4\pi k^2} + i\epsilon_{ijl}k_l \frac{H_M(k)}{8\pi k^2}$$ (4)

- Mean energy density: $\mathcal{E}_M = \int dk \: E_M(k)$
- Integral length scale: $\xi_M = (\int dk \: k^{-1} E_M(k)) / \mathcal{E}_M$
- Mean helicity density: $\mathcal{H}_M = \int dk \: H_M(k)$
The Realizability Condition

This relates the symmetric and helical components.

$$|\mathcal{H}_M| \leq 2\xi_M \mathcal{E}_M \Rightarrow |H_M(k)| \leq 2k^{-1}E_M(k) \quad (5)$$

For scale-invariant spectrum, at large length scales, $E_M \sim k^{-1}$. $\xi_M$ is unbounded.

Then Helicity is divergent. One must have $E_M \sim k^4$ at superhorizon scales.

![Scale Invariant Spectrum](image)

**Figure**: Scale Invariant Spectrum.
We use the **Pencil Code** to study the evolution of $\mathcal{E}_M(t)$.

**Figure:** Magnetic (red) and kinetic (blue) energy spectra at early times. The green symbols denote the position of $k_\ast(t)$. Black symbols denote the location of the horizon wavenumber $k_{\text{hor}}(t)$.

$$k_\ast(t) \approx \xi_M(t)(\eta_{\text{turb}}t)^{-1/2}, \quad k_{\text{hor}}(t) = (ct)^{-1}$$

(6)
Figure: The late time evolution. We have the usual inverse cascade, with an increase of $\xi_M(t) \sim t^q$. 

*Graph showing the evolution of $E(k,t)$ and $E_K(k,t)$.*
Cosmological Applications

- The Planck Collaboration\(^5\) derived upper limits on PMFs.

- They use a fixed spectral shape, and neglect the presence of a *turbulent regime*.

- Fields generated during EW and QCD Phase Transitions cannot leave any imprint on the CMB.

- Only scale-invariant fields can leave observable traces on the CMB.

Thank you
Supplementary Slides
Some Wave Numbers

The magnetic dissipation wavenumber is,

\[ k_{\text{MD}}^4 = \frac{2}{\eta^2} \int_0^\infty dk \ k^2 \ \tilde{E}_M(k) \]  

(7)

The weak turbulence dissipation wavenumber is,

\[ \text{Lu}(k_{\text{WT}}) = \frac{v_A(k_{\text{WT}})}{\eta k_{\text{WT}}} = 1 \]  

(8)

with \( v_A^2 = 2kE_M(k) \).
In the early universe, $H$ is conserved. This leads to (for divergenceless $\mathbf{B}$),

$$L_- |\mathbf{B}(x)|^2 \leq (\text{curl}^{-1} \mathbf{B}) \cdot \mathbf{B} \leq L_+ |\mathbf{B}(x)|^2$$

where $L_- < 0 < L_+$ are the eigenvalues of curl$^{-1}$. This implies,

$$\left| \frac{(\text{curl}^{-1} \mathbf{B}) \cdot \mathbf{B}}{L_+} \right| \leq |\mathbf{B}(x)|^2$$

Taking the ensemble average,

$$\left| \frac{\mathcal{H}_M}{L_+} \right| \leq 2\mathcal{E}_M$$

$L_+ \sim \xi_M$. 
Turbulence and MHD

Loitsiansky Integral: \( L = \int r^2 \langle u(x) \cdot u(x + r) \rangle \, dr \propto \ell^5 u_\ell^2 \)

Saffman Integral: \( S = \int \langle u(x) \cdot u(x + r) \rangle \, dr \propto \ell^3 u_\ell^2 \)

\[ \text{Re} = \frac{u_{rms} \xi_M}{\nu} \]

Kolmogorov spectrum: \( E(k) \sim k^{-5/3} \) - comes from requirement of scale invariance. Inertial forces causes KE transfer.
Decay Laws

We take the maximum *comoving* correlation length at the epoch of EW Phase transition,

\[ \xi_* \equiv \xi_{\text{max}} = H_*^{-1} \left( \frac{a_0}{a_*} \right) \sim 6 \times 10^{-11} \text{ Mpc} \]

and the maximum *mean* energy density as,

\[ \mathcal{E}_* = 0.1 \times \frac{\pi^2}{30} g_* T_*^4 \sim 4 \times 10^{58} \text{ eV cm}^{-3} \]

We use \( \eta(a) = \frac{2}{\sqrt{\Omega_{m,0} H_0}} \left[ \sqrt{a_{\text{eq}}} + a - \sqrt{a_{\text{eq}}} \right] \).

Non-helical case: \( \frac{\xi}{\xi_*} = \left( \frac{\eta}{\eta_*} \right)^{\frac{1}{2}} \), \( \frac{\mathcal{E}}{\mathcal{E}_*} = \left( \frac{\eta}{\eta_*} \right)^{-1} \).

Helical case: \( \frac{\xi}{\xi_*} = \left( \frac{\eta}{\eta_*} \right)^{\frac{2}{3}} \), \( \frac{\mathcal{E}}{\mathcal{E}_*} = \left( \frac{\eta}{\eta_*} \right)^{-\frac{2}{3}} \).

Partial: Turnover when \( \left( \frac{\eta^{\frac{1}{2}}}{\eta_*} \right) = \exp \left( \frac{1}{2} \sigma \right) \).
We solve the hydromagnetic equations for an isothermal relativistic gas with pressure \( p = \rho/3 \)

\[
\frac{\partial \ln \rho}{\partial t} = -\frac{4}{3} (\nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \ln \rho) + \frac{1}{\rho} \left[ \mathbf{u} \cdot (\mathbf{J} \times \mathbf{B}) + \eta J^2 \right],
\]

\[
\frac{\partial \mathbf{u}}{\partial t} = -\mathbf{u} \cdot \nabla \mathbf{u} + \frac{\mathbf{u}}{3} (\nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \ln \rho) - \frac{\mathbf{u}}{\rho} \left[ \mathbf{u} \cdot (\mathbf{J} \times \mathbf{B}) + \eta J^2 \right]
\]

\[
-\frac{1}{4} \nabla \ln \rho + \frac{3}{4\rho} \mathbf{J} \times \mathbf{B} + \frac{2}{\rho} \nabla \cdot (\rho \nu \mathbf{S}),
\]

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B} - \eta \mathbf{J}),
\]

where \( S_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) - \frac{1}{3} \delta_{ij} \nabla \cdot \mathbf{u} \) is the rate-of-strain tensor, \( \nu \) is the viscosity, and \( \eta \) is the magnetic diffusivity.