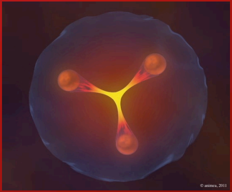
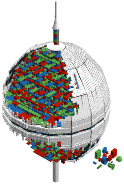
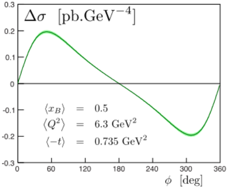
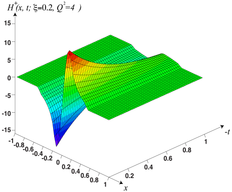
DE LA RECHERCHE À L'INDUSTRIE



Covariant extension of the GPD overlap representation







QCD Workshop | Hervé MOUTARDE

May 4$^{\text{th}}$, 2018

Context and key questions.

1. What do we need for high precision phenomenology?
2. How can we implement all theoretical constraints in flexible GPD parameterizations?
3. How do we relate all this to actual measurements?

What do we need for high precision phenomenology?

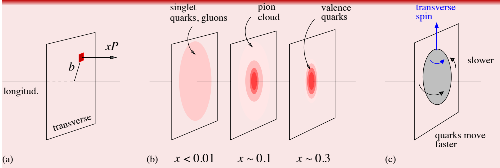
- **Probabilistic interpretation** of Fourier transform of GPD$(x, \xi = 0, t)$ in **transverse plane**.

$$\rho(x, b_\perp, \lambda, \lambda_N) = \frac{1}{2}\left[H(x, 0, b_\perp^2) + \frac{b_\perp^j \epsilon_{ji} S_\perp^i}{M} \frac{\partial E}{\partial b_\perp^2}(x, 0, b_\perp^2) + \lambda\lambda_N \tilde{H}(x, 0, b_\perp^2)\right] .$$

- Notations : quark helicity λ, nucleon longitudinal polarization λ_N and nucleon transverse spin $S_\perp$.

Burkardt, Phys. Rev. **D62**, 071503 (2000)

Can we obtain this picture from exclusive measurements?



Weiss, AIP Conf. Proc. **1149**, 150 (2009)

- Most general structure of matrix element of energy momentum tensor between nucleon states:

$$\left\langle N, P+\frac{\Delta}{2}\right| T^{\mu\nu} \left| N, P-\frac{\Delta}{2}\right\rangle = \bar{u}\left(P+\frac{\Delta}{2}\right)\left[A(t)\gamma^{(\mu}P^{\nu)} + B(t)P^{(\mu}i\sigma^{\nu)\lambda}\frac{\Delta_\lambda}{2M} + \frac{C(t)}{M}(\Delta^\mu\Delta^\nu - \Delta^2\eta^{\mu\nu})\right]u\left(P-\frac{\Delta}{2}\right),$$

with $t = \Delta^2$.

- Key observation: **link between GPDs and gravitational form factors**

$$\int \mathrm{d}x\, x H^q(x,\xi,t) = A^q(t) + 4\xi^2 C^q(t),$$

$$\int \mathrm{d}x\, x E^q(x,\xi,t) = B^q(t) - 4\xi^2 C^q(t).$$

Ji, Phys. Rev. Lett. **78**, 610 (1997)

- Spin sum rule:

$$\int \mathrm{d}x\, x\big(H^q(x,\xi,0) + E^q(x,\xi,0)\big) = A^q(0) + B^q(0) = 2J^q\ .$$

Ji, Phys. Rev. Lett. **78**, 610 (1997)

- **Shear** and **pressure** distributions:

$$\langle N \,|\, T^{ij}(\vec{r}) \,|\, N \rangle = s(r)\left(\frac{r^i r^j}{\vec{r}^2} - \frac{1}{3}\delta^{ij}\right) + p(r)\delta^{ij}\ .$$

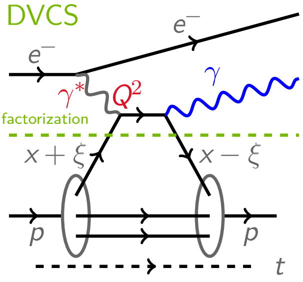
Polyakov and Shuvaev, hep-ph/0207153
Polyakov, Phys. Lett. **B555**, 57 (2003)

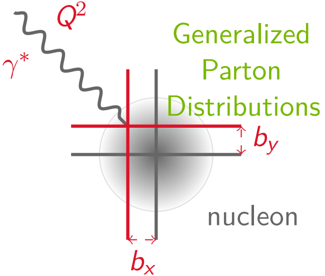
- **Energy**, **radial pressure** and **transverse pressure** distributions (u^μ the 4-velocity at spacetime location χ^ν):

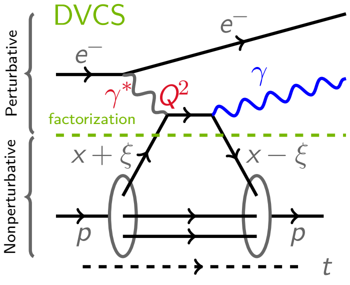
$$\langle N \,|\, T^{\mu\nu} \,|\, N \rangle = (\epsilon + p_t) u^\mu u^\nu - p_t \eta^{\mu\nu} + (p_r - p_t)\chi^\mu \chi^\nu\ .$$

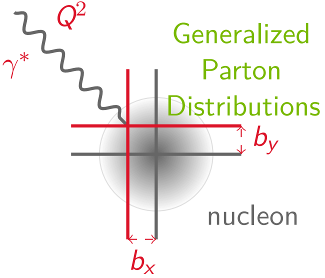
Trawinski *et al.*, *in preparation*

Exclusive processes of current interest (1/2).
Factorization and universality.



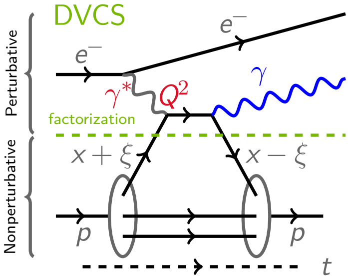


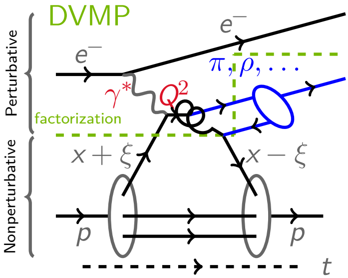

DVCS
Perturbative
Nonperturbative
e^-
e^-
γ^*
Q^2
γ
factorization
$x+\xi$
$x-\xi$
p
p
t

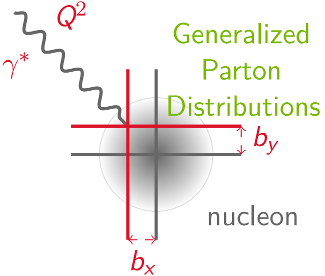

Q^2
γ^*
Generalized
Parton
Distributions
b_y
nucleon
b_x

Exclusive processes of current interest (1/2).

Factorization and universality.







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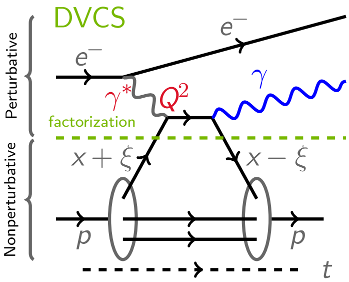
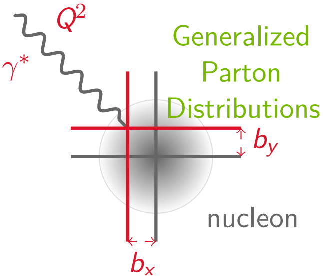
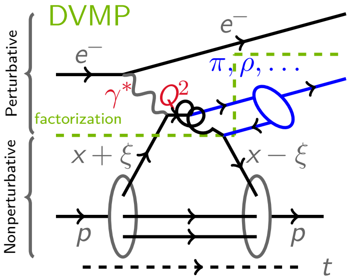
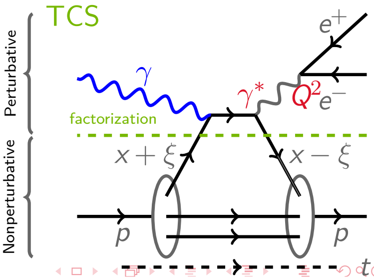
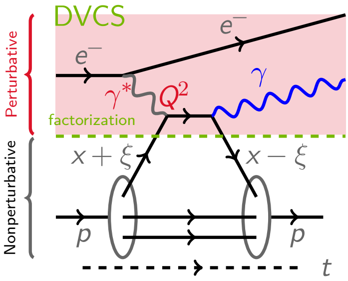
Factorization and universality.



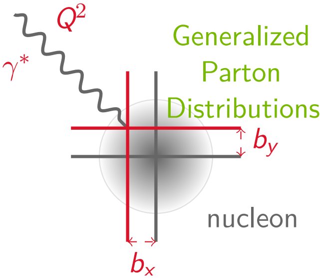




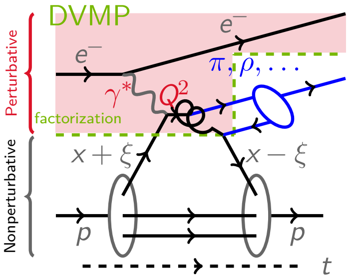




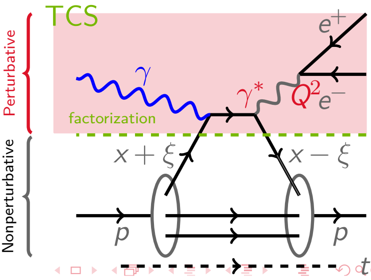
DVCS
Perturbative
Nonperturbative
e^-
e^-
γ^*
Q^2
γ
factorization
$x+\xi$
$x-\xi$
p
p
t



Q^2
γ^*
Generalized Parton Distributions
b_y
b_x
nucleon



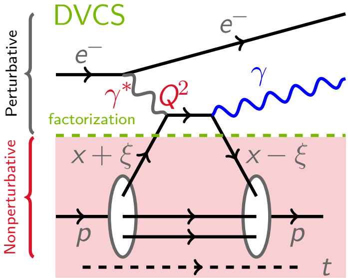
DVMP
Perturbative
Nonperturbative
e^-
e^-
$\pi, \rho, \ldots$
γ^*
Q^2
factorization
$x+\xi$
$x-\xi$
p
p
t

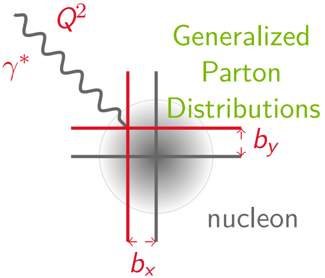


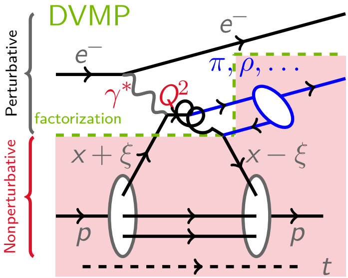
TCS
Perturbative
Nonperturbative
e^+
γ
γ^*
Q^2
e^-
factorization
$x+\xi$
$x-\xi$
p
p
t

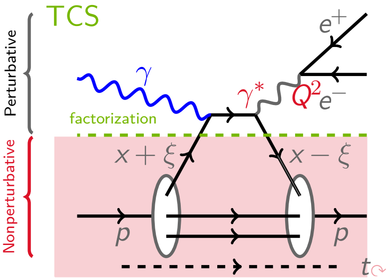
Exclusive processes of current interest (1/2).

Factorization and universality.









Bjorken regime : large Q^2 and fixed $xB \simeq 2\xi/(1+\xi)$

- Partonic interpretation relies on **factorization theorems**.
- All-order proofs for DVCS, TCS and some DVMP.
- GPDs depend on a (arbitrary) factorization scale μ_F.
- **Consistency** requires the study of **different channels**.

- GPDs enter DVCS through **Compton Form Factors** :

$$\mathcal{F}(\xi, t, Q^2) = \int_{-1}^{1} dx\, C\left(x, \xi, \alpha_S(\mu_F), \frac{Q}{\mu_F}\right) F(x, \xi, t, \mu_F) ,$$

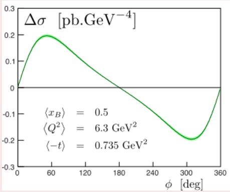
for a given GPD F.

- CFF $\mathcal{F}$ is a **complex function**.

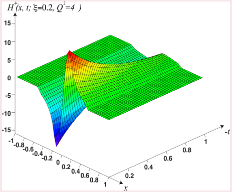
Imaging the nucleon. How?

Extracting GPDs is not enough...Need to extrapolate!

1. Experimental data fits



2. GPD extraction



3. Nucleon imaging

Images from Guidal et al.,
Rept. Prog. Phys. 76 (2013) 066202

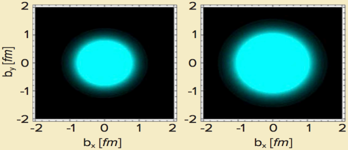
Reaching for the Horizon

The 2015 Long Range Plan for Nuclear Science

Sidebar 2.2: The First 3D Pictures of the Nucleon

A computed tomography (CT) scan can help physicians pinpoint minute cancer tumors, diagnose tiny broken bones, and spot the early signs of osteoporosis. Now physicists are using the principles behind the procedure to peer at the inner workings of the proton. This breakthrough is made possible by a relatively new concept in nuclear physics called generalized parton distributions.

An intense beam of high-energy electrons can be used



Imaging the nucleon. How?

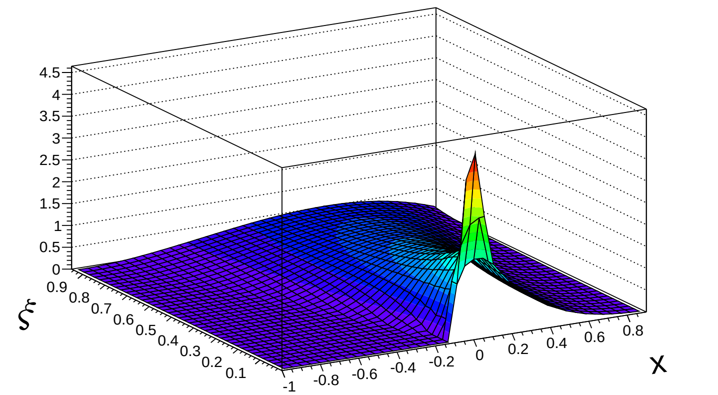
Extracting GPDs is not enough…Need to extrapolate!

1. **Extract** $H(x, \xi, t, \mu_F^{\mathrm{ref}})$ from experimental data.
2. **Extrapolate** to vanishing skewness $H(x, 0, t, \mu_F^{\mathrm{ref}})$.
3. **Extrapolate** $H(x, 0, t, \mu_F^{\mathrm{ref}})$ up to infinite t and down to vanishing t.
4. **Compute** 2D Fourier transform in transverse plane:

$$H(x, b_\perp) = \int_0^{+\infty} \frac{\mathrm{d}|\Delta_\perp|}{2\pi} |\Delta_\perp| \, J_0(|b_\perp||\Delta_\perp|) \, H(x, 0, -\Delta_\perp^2) \, .$$

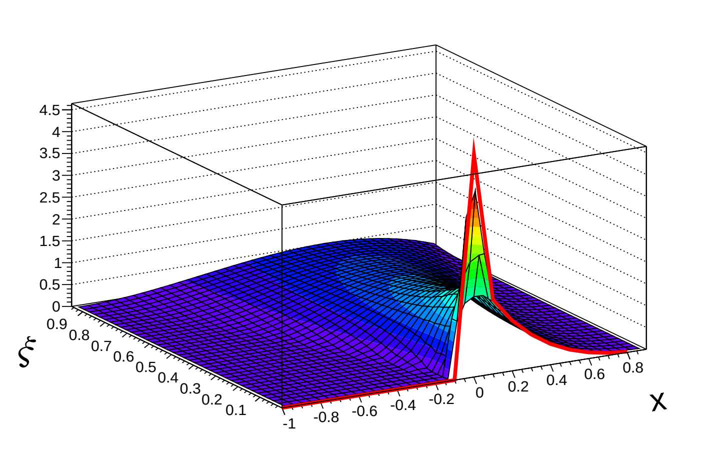
5. **Propagate** uncertainties.
6. **Control** extrapolations with an accuracy matching that of experimental data with **sound** GPD models.

GPD H at $t = -0.23$ GeV^2 and $Q^2 = 2.3$ GeV^2.



GPD model: see Kroll *et al.*, Eur. Phys. J. **C73**, 2278 (2013)

Need to know $H(x, \xi = 0, t)$ to do transverse plane imaging.

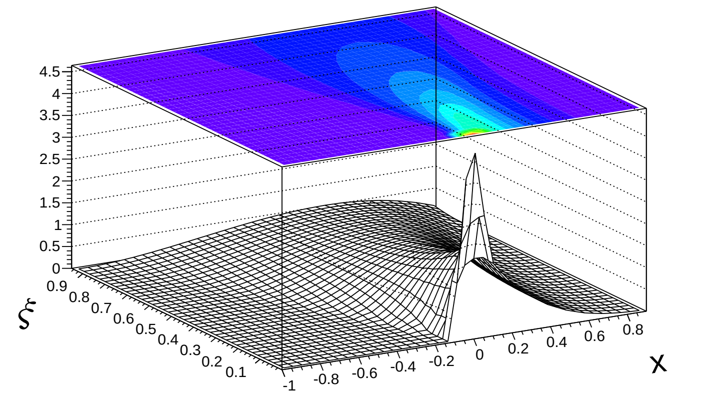


GPD model: see Kroll *et al.*, Eur. Phys. J. **C73**, 2278 (2013)

From principles to actual data.

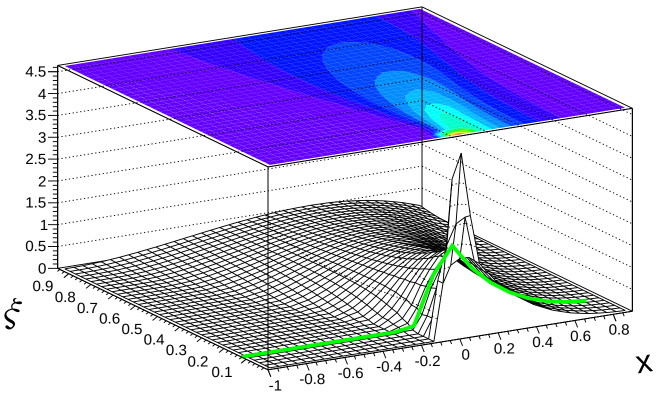
Direct experimental access to a restricted kinematic domain.

What is the physical region?



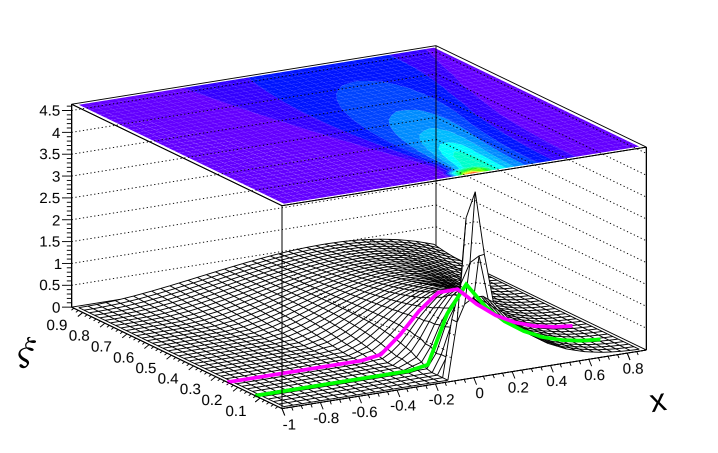
GPD model: see Kroll *et al.*, Eur. Phys. J. **C73**, 2278 (2013)

$\xi_{\min}$ from finite beam energy.



GPD model: see Kroll *et al.*, Eur. Phys. J. **C73**, 2278 (2013)

$\xi_{\max}$ from kinematic constraint on 4-momentum transfer.

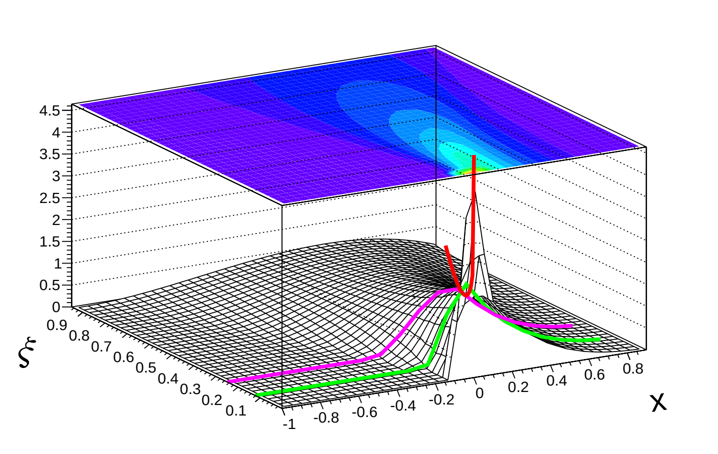


GPD model: see Kroll *et al.*, Eur. Phys. J. **C73**, 2278 (2013)

From principles to actual data.

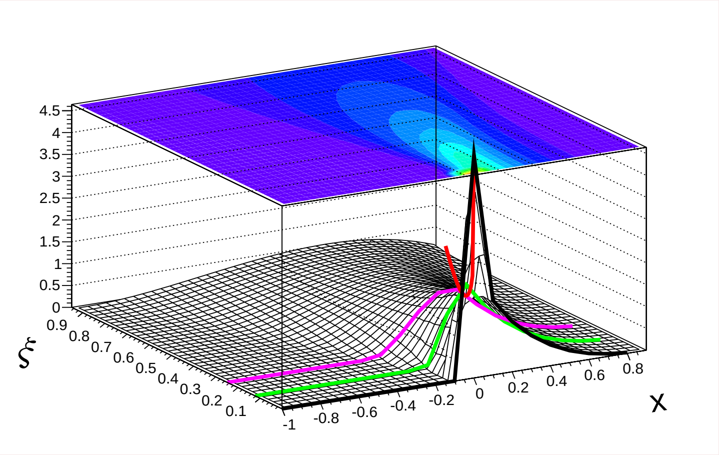
Direct experimental access to a restricted kinematic domain.

The cross-over line $x = \xi$.



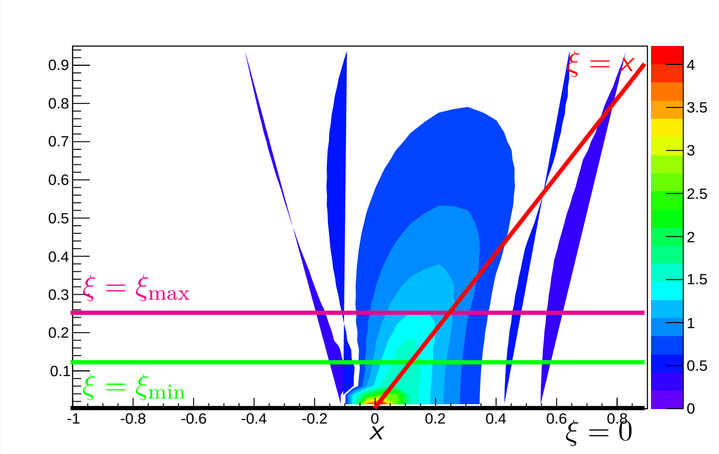
GPD model: see Kroll *et al.*, Eur. Phys. J. **C73**, 2278 (2013)

The black curve is what is needed for transverse plane imaging!



GPD model: see Kroll *et al.*, Eur. Phys. J. **C73**, 2278 (2013)

Density plot of H at $t = -0.23$ GeV^2 and $Q^2 = 2.3$ GeV^2



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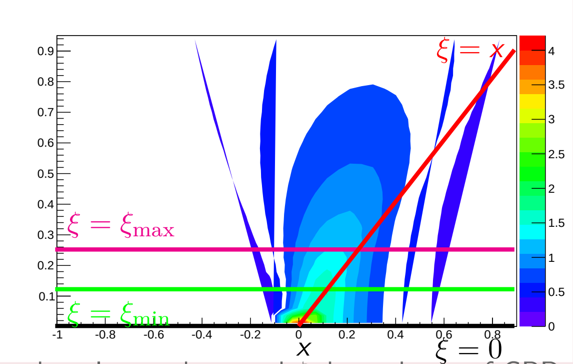


- Not a hopeless task: x and ξ dependence of GPDs are **strongly tied** (polynomiality)!
- Need for GPD modeling and flexible parameterizations.

- Write dispersion relation **at fixed t and Q^2**:

$$\mathrm{Re}\mathcal{H}(\xi) = \int_1^{\infty} \frac{\mathrm{d}\omega}{\pi} \mathrm{Im}\, C(\omega) \left\{ \int_{-1}^{+1} \mathrm{d}x \left[\frac{1}{\omega\xi - x} - \frac{1}{\omega\xi + x} \right] H\left(x, \frac{x}{\omega}\right) + \mathcal{I}(\omega) \right\} .$$

Diehl and Ivanov, Eur. Phys. J. **C52**, 919 (2007)

- At **leading order** in α_s (no kinematic corrections):

$$\mathrm{Im}\, C(\omega) \propto \pi \left[\delta(\omega - 1) - \delta(\omega + 1) \right] .$$

- Dispersion relation simplifies to:

$$\mathrm{Re}\mathcal{H}(\xi) \propto \int_{-1}^{+1} \mathrm{d}x \left[\frac{1}{\omega\xi - x} - \frac{1}{\omega\xi + x} \right] H(x, x) + \mathcal{I} ,$$
$$\mathrm{Im}\mathcal{H}(\xi) \propto H(\xi, \xi) - H(-\xi, \xi) .$$

- In principle tomography not possible **at leading order**...

DVCS analysis and fits.

No global GPD fit has been obtained so far.

- GPD fits **only in the small x_B region** with a **flexible** parameterization (kinematic simplifications).
- Global fits of CFFs in the sea and valence regions.
- Some GPD models with non-flexible parameterizations adjusted to experimental DVCS or DVMP data.

  Kumerički *et al.*, Eur. Phys. J. **A52**, 157 (2016)
- Unclear model-dependence on tomographic images obtained from CFF fits relying on **leading order** and **leading twist** analysis.

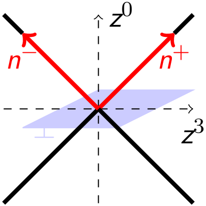
The situation can be improved!

- GPD parameterizations satisfying *a priori* all theoretical constraints on GPDs.
- Computing framework to go beyond leading order and leading twist analysis.

How can we implement all theoretical constraints in flexible GPD parameterizations?

$$H^q_\pi(x,\xi,t) = \frac{1}{2}\int \frac{\mathrm{d}z^-}{2\pi}\, e^{ixP^+z^-} \left\langle \pi, P+\frac{\Delta}{2}\right| \bar{q}\left(-\frac{z}{2}\right)\gamma^+ q\left(\frac{z}{2}\right)\left|\pi, P-\frac{\Delta}{2}\right\rangle_{\substack{z^+=0\\ z_\perp=0}}$$

with $t=\Delta^2$ and $\xi = -\Delta^+/(2P^+)$.



References

Müller *et al.*, Fortschr. Phys. **42**, 101 (1994)
Ji, Phys. Rev. Lett. **78**, 610 (1997)
Radyushkin, Phys. Lett. **B380**, 417 (1996)

- PDF **forward limit**

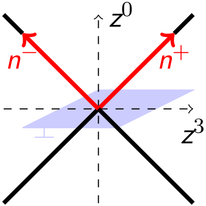
$$H^q(x,0,0) = q(x)$$

Spin-0 Generalized Parton Distribution.

Definition and simple properties.

$$H^q_\pi(x, \xi, t) = \frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \left\langle \pi, P + \frac{\Delta}{2} \right| \bar{q}\left(-\frac{z}{2}\right)\gamma^+ q\left(\frac{z}{2}\right) \left| \pi, P - \frac{\Delta}{2} \right\rangle_{\substack{z^+=0\\ z_\perp=0}}$$

with $t = \Delta^2$ and $\xi = -\Delta^+/(2P^+)$.



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Ji, Phys. Rev. Lett. **78**, 610 (1997)
Radyushkin, Phys. Lett. **B380**, 417 (1996)

- PDF **forward limit**
- Form factor **sum rule**

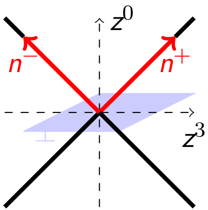
$$\int_{-1}^{+1} dx\, H^q(x, \xi, t) = F^q_1(t)$$

Spin-0 Generalized Parton Distribution.

Definition and simple properties.

$$H_\pi^q(x,\xi,t) = \frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \left\langle \pi, P+\frac{\Delta}{2} \middle| \bar{q}\left(-\frac{z}{2}\right)\gamma^+ q\left(\frac{z}{2}\right) \middle| \pi, P-\frac{\Delta}{2} \right\rangle_{\substack{z^+=0\\ z_\perp=0}}$$

with $t = \Delta^2$ and $\xi = -\Delta^+/(2P^+)$.



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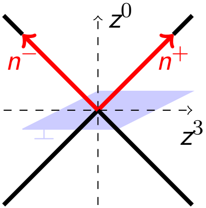
Ji, Phys. Rev. Lett. **78**, 610 (1997)
Radyushkin, Phys. Lett. **B380**, 417 (1996)

- PDF **forward limit**
- Form factor **sum rule**
- H^q is an **even function** of ξ from time-reversal invariance.

$$H^q_\pi(x, \xi, t) = \frac{1}{2}\int \frac{\mathrm{d}z^-}{2\pi} e^{ixP^+z^-} \left\langle \pi, P+\frac{\Delta}{2} \middle| \bar{q}\left(-\frac{z}{2}\right)\gamma^+ q\left(\frac{z}{2}\right) \middle| \pi, P-\frac{\Delta}{2} \right\rangle_{\substack{z^+=0\\ z_\perp=0}}$$

with $t = \Delta^2$ and $\xi = -\Delta^+/(2P^+)$.



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- PDF **forward limit**
- Form factor **sum rule**
- H^q is an **even function** of ξ from time-reversal invariance.
- H^q is **real** from hermiticity and time-reversal invariance.

Polynomiality

$$\int_{-1}^{+1} dx\, x^n H^q(x, \xi, t) = \text{polynomial in } \xi$$

Polynomiality

Lorentz covariance

- **Polynomiality**

Lorentz covariance

- **Positivity**

$$H^q(x, \xi, t) \leq \sqrt{q\left(\frac{x+\xi}{1+\xi}\right) q\left(\frac{x-\xi}{1-\xi}\right)}$$

- **Polynomiality**

  Lorentz covariance

- **Positivity**

  Positivity of Hilbert space norm

- **Polynomiality**

Lorentz covariance

- **Positivity**

Positivity of Hilbert space norm

- H^q has support $x \in [-1, +1]$.

- **Polynomiality**

Lorentz covariance

- **Positivity**

Positivity of Hilbert space norm

- H^q has support $x \in [-1, +1]$.

Relativistic quantum mechanics

- **Polynomiality**

Lorentz covariance

- **Positivity**

Positivity of Hilbert space norm

- H^q has support $x \in [-1, +1]$.

Relativistic quantum mechanics

- **Soft pion theorem** (pion target)

$$H^q(x, \xi = 1, t = 0) = \frac{1}{2}\phi^q_\pi\left(\frac{1+x}{2}\right)$$

- **Polynomiality**

  Lorentz covariance

- **Positivity**

  Positivity of Hilbert space norm

- H^q has support $x \in [-1, +1]$.

  Relativistic quantum mechanics

- **Soft pion theorem** (pion target)

  Dynamical chiral symmetry breaking

- **Polynomiality**

Lorentz covariance

- **Positivity**

Positivity of Hilbert space norm

- H^q has support $x \in [-1, +1]$.

Relativistic quantum mechanics

- **Soft pion theorem** (pion target)

Dynamical chiral symmetry breaking

How can we implement *a priori* these theoretical constraints?

- In the following, focus on **polynomiality** and **positivity**.
- Do not discuss the reduction to form factors or PDFs.

Polynomiality.

Mixed constraint from Lorentz invariance and discrete symmetries.

- Express Mellin moments of GPDs as **matrix elements**:

$$\int_{-1}^{+1} \mathrm{d}x \, x^m H^q(x, \xi, t) = \frac{1}{2(P^+)^{m+1}} \left\langle P + \frac{\Delta}{2} \middle| \bar{q}(0) \gamma^+ (i \overleftrightarrow{D}^+)^m q(0) \middle| P - \frac{\Delta}{2} \right\rangle$$

- Identify the **Lorentz structure** of the matrix element:

linear combination of $(P^+)^{m+1-k} (\Delta^+)^k$ for $0 \leq k \leq m+1$

- Remember definition of **skewness** $\Delta^+ = -2\xi P^+$.
- Select **even powers** to implement time reversal.
- Obtain **polynomiality condition**:

$$\int_{-1}^{1} \mathrm{d}x \, x^m H^q(x, \xi, t) = \sum_{\substack{i=0 \\ \text{even}}}^{m} (2\xi)^i C^q_{mi}(t) + (2\xi)^{m+1} C^q_{mm+1}(t) \; .$$

- Assume the existence of $D^q(z,t)$ such that:

$$\int_{-1}^{+1} \mathrm{d}z\, z^m D(z,t) = C^q_{mm+1}(t)\ .$$

- $H^q(x,\xi,t) - D(x/\xi,t)$ satisfies polynomiality at order m:

$$\int_{-1}^{1} \mathrm{d}x\, x^m \Big(H^q(x,\xi,t) - D(x/\xi,t) \Big) = \sum_{\substack{i=0 \\ \text{even}}}^{m} (2\xi)^i C^q_{mi}(t)\ .$$

- In the Radon transform framework, this is the **Ludwig-Helgason** consistency condition.
- Thus, there exists a function F_D such that:

$$H(x,\xi,t) = D(x/\xi,t) + \int_{\Omega_{\mathrm{DD}}} \mathrm{d}\beta \mathrm{d}\alpha\, F_D(\beta,\alpha,t)\, \delta(x-\beta-\alpha\xi)\ .$$

- The support $\Omega_{\mathrm{DD}} = \{|\alpha| + |\beta| \leq 1\}$ is related to the GPD physical domain $|x|, |\xi| \leq 1$.

- Most general representation of GPD:

$$H^q(x,\xi,t) = \int_{\Omega_{\mathrm{DD}}} \mathrm{d}\beta \mathrm{d}\alpha\, \delta(x-\beta-\alpha\xi)\big(F^q(\beta,\alpha,t)+\xi\, G^q(\beta,\alpha,t)\big)$$

- Support property: $x \in [-1,+1]$.
- Discrete symmetries: F^q is α-even and G^q is α-odd.
- **Gauge**: any representation (F^q, G^q) can be recast in one representation with a single DD f^q:

$$H^q(x,\xi,t) = x \int_{\Omega_{\mathrm{DD}}} \mathrm{d}\beta \mathrm{d}\alpha\, f^q_{\mathrm{BMKS}}(\beta,\alpha,t)\delta(x-\beta-\alpha\xi)$$

Belitsky *et al.*, Phys. Rev. **D64**, 116002 (2001)

$$H^q(x,\xi,t) = (1-x) \int_{\Omega_{\mathrm{DD}}} \mathrm{d}\beta \mathrm{d}\alpha\, f^q_{\mathrm{P}}(\beta,\alpha,t)\delta(x-\beta-\alpha\xi)$$

Pobylitsa, Phys. Rev. **D67**, 034009 (2003)

- Choose $F^q(\beta, \alpha) = 3\beta\theta(\beta)$ ad $G^q(\beta, \alpha) = 3\alpha\theta(\beta)$:

$$H^q(x, \xi) = 3x \int_\Omega \mathrm{d}\beta \mathrm{d}\alpha \, \delta(x - \beta - \alpha\xi)$$

- Simple analytic expressions for the GPD:

$$\begin{aligned} H(x, \xi) &= \frac{6x(1-x)}{1-\xi^2} \text{ if } 0 < |\xi| < x < 1, \\ H(x, \xi) &= \frac{3x(x+|\xi|)}{|\xi|(1+|\xi|)} \text{ if } -|\xi| < x < |\xi| < 1. \end{aligned}$$

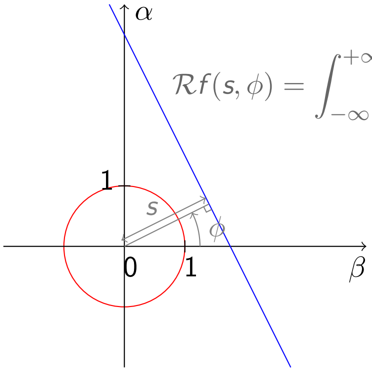
Double Distributions.
Lorentz covariance by example.

- Compute first Mellin moments.

n	$\int_{-\xi}^{+\xi} \mathrm{d}x\, x^n H(x,\xi)$	$\int_{+\xi}^{+1} \mathrm{d}x\, x^n H(x,\xi)$	$\int_{-\xi}^{+1} \mathrm{d}x\, x^n H(x,\xi)$
0	$\frac{1+\xi-2\xi^2}{1+\xi}$	$\frac{2\xi^2}{1+\xi}$	1
1	$\frac{1+\xi+\xi^2-3\xi^3}{2(1+\xi)}$	$\frac{2\xi^3}{1+\xi}$	$\frac{1+\xi^2}{2}$
2	$\frac{3(1-\xi)(1+2\xi+3\xi^2+4\xi^3)}{10(1+\xi)}$	$\frac{6\xi^4}{5(1+\xi)}$	$\frac{3(1+\xi^2)}{10}$
3	$\frac{1+\xi+\xi^2+\xi^3+\xi^4-5\xi^5}{5(1+\xi)}$	$\frac{6\xi^5}{5(1+\xi)}$	$\frac{1+\xi^2+\xi^4}{5}$
4	$\frac{1+\xi+\xi^2+\xi^3+\xi^4+\xi^5-6\xi^6}{7(1+\xi)}$	$\frac{6\xi^6}{7(1+\xi)}$	$\frac{1+\xi^2+\xi^4}{7}$

- Expressions get more complicated as n increases... But they always yield polynomials!

The Radon transform.
Definition and properties.



For $s > 0$ and $\phi \in [0, 2\pi]$:

$$\mathcal{R}f(s,\phi) = \int_{-\infty}^{+\infty} \mathrm{d}\beta \mathrm{d}\alpha\, f(\beta,\alpha)\delta(s - \beta\cos\phi - \alpha\sin\phi)$$

and:

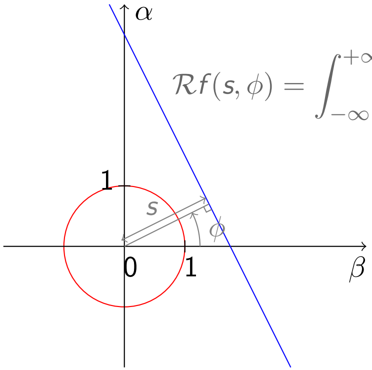
$$\mathcal{R}f(-s,\phi) = \mathcal{R}f(s,\phi \pm \pi)$$

Relation to GPDs:

$$x = \frac{s}{\cos\phi} \text{ and } \xi = \tan\phi$$

Relation between GPD and DD in Belistky *et al.* gauge

$$\frac{\sqrt{1+\xi^2}}{x} H(x,\xi) = \mathcal{R}f_{\mathrm{BMKS}}(s,\phi) \ ,$$



For $s > 0$ and $\phi \in [0, 2\pi]$:

$$\mathcal{R}f(s, \phi) = \int_{-\infty}^{+\infty} \mathrm{d}\beta \mathrm{d}\alpha \, f(\beta, \alpha) \delta(s - \beta \cos\phi - \alpha \sin\phi)$$

and:

$$\mathcal{R}f(-s, \phi) = \mathcal{R}f(s, \phi \pm \pi)$$

Relation to GPDs:

$$x = \frac{s}{\cos\phi} \text{ and } \xi = \tan\phi$$

Relation between GPD and DD in Pobylitsa gauge

$$\frac{\sqrt{1+\xi^2}}{1-x} H(x, \xi) = \mathcal{R}f_{\mathrm{P}}(s, \phi) \ ,$$

The range of the Radon transform.

The polynomiality property a.k.a. the Ludwig-Helgason condition.

- The Mellin moments of a Radon transform are **homogeneous polynomials** in $\omega = (\sin\phi, \cos\phi)$.
- The converse is also true:

Theorem (Hertle, 1983)

Let $g(s, \omega)$ an even compactly-supported distribution. Then g is itself the Radon transform of a compactly-supported distribution if and only if the ***Ludwig-Helgason consistency condition*** *hold:*

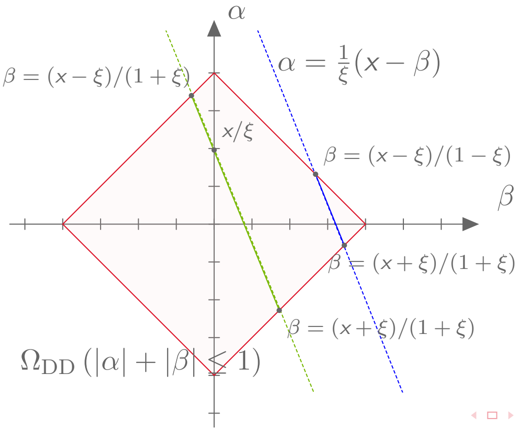
(i) *g is C^∞ in ω,*

(ii) *$\int \mathrm{d}s\, s^m g(s, \omega)$ is a homogeneous polynomial of degree m for all integer $m \geq 0$.*

- Double Distributions and the Radon transform are the **natural solution** of the polynomiality condition.

DGLAP and ERBL regions

$$(x, \xi) \in \text{DGLAP} \Leftrightarrow |s| \geq |\sin\phi| \,,$$
$$(x, \xi) \in \text{ERBL} \Leftrightarrow |s| \leq |\sin\phi| \,.$$



Each point (β, α) with $\beta \neq 0$ contributes to **both** DGLAP and ERBL regions.

Positivity.

A consequence of the positivity of the nom in a Hilbert space.

- Identify the matrix element defining a GPD as an **inner product** of two different states.
- Apply Cauchy-Schwartz inequality, and identify PDFs at specific kinematic points, *e.g.*:

$$|H^q(x,\xi,t)| \leq \sqrt{\frac{1}{1-\xi^2} q\left(\frac{x+\xi}{1+\xi}\right) q\left(\frac{x-\xi}{1-\xi}\right)}$$

- This procedures yields **infinitely many inequalities** stable under LO evolution.

Pobylitsa, Phys. Rev. **D66**, 094002 (2002)

- The **overlap representation** guarantees *a priori* the fulfillment of positivity constraints.

Overlap representation.

A first-principle connection with Light Front Wave Functions.

- Decompose an hadronic state $|H; P, \lambda\rangle$ in a Fock basis:

$$|H; P, \lambda\rangle = \sum_{N,\beta} \int [\mathrm{d}x \mathrm{d}\mathbf{k}_\perp]_N \psi_N^{(\beta,\lambda)}(x_1, \mathbf{k}_{\perp 1}, \ldots, x_N, \mathbf{k}_{\perp N}) \, |\beta, k_1, \ldots, k_N\rangle$$

- Derive an expression for the pion GPD in the DGLAP region $\xi \leq x \leq 1$:

$$H^q(x, \xi, t) \propto \sum_{\beta,j} \int [\mathrm{d}\bar{x} \mathrm{d}\bar{\mathbf{k}}_\perp]_N \delta_{j,q} \delta(x - \bar{x}_j) \big(\psi_N^{(\beta,\lambda)}\big)^* (\hat{x}', \hat{\mathbf{k}}'_\perp) \psi_N^{(\beta,\lambda)} (\tilde{x}, \tilde{\mathbf{k}}_\perp)$$

with $\tilde{x}, \tilde{\mathbf{k}}_\perp$ (resp. $\hat{x}', \hat{\mathbf{k}}'_\perp$) generically denoting incoming (resp. outgoing) parton kinematics.

Diehl *et al.*, Nucl. Phys. **B596**, 33 (2001)

- Similar expression in the ERBL region $-\xi \leq x \leq \xi$, but with overlap of N- and $(N+2)$-body LFWFs.

Overlap representation.

Advantages and drawbacks.

- Physical picture.
- Positivity relations are fulfilled **by construction**.
- Implementation of **symmetries of N-body problems**.

What is not obvious anymore

What is *not* obvious to see from the wave function representation is however the **continuity of GPDs at** $x = \pm\xi$ and the **polynomiality** condition. In these cases both the DGLAP and the ERBL regions must cooperate to lead to the required properties, and this implies **nontrivial relations between the wave functions** for the different Fock states relevant in the two regions. An *ad hoc* Ansatz for the wave functions would **almost certainly lead** to GPDs that **violate the above requirements**.

Diehl, Phys. Rept. **388**, 41 (2003)

For **any model of LFWF**, one has to address the following three questions:

1. Does the extension exist?
2. If it exists, is it unique?
3. How can we compute this extension?

Implementing Lorentz covariance.

Uniqueness of the extension.

Consider a GPD H vanishing on the DGLAP region and write it as a Radon transform:

$$H(x,\xi) = \int_{\Omega_{\mathrm{DD}}} \mathrm{d}\beta \mathrm{d}\alpha \left[F_D(\beta,\alpha) + \delta(\beta) D(\alpha) \right] \delta(x - \beta - \alpha\xi) \, .$$

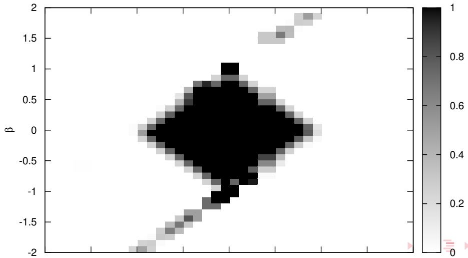
- $F_D(\beta,\alpha) = 0$ for all α and $\beta > 0$.
  Boman and Todd-Quinto, Duke Math. J. **55**, 943 (1987)
- Up to D-term-like contributions, the DGLAP region **completely characterizes** a GPD.
- Modeling strategy:
  1. Ensure positivity by modeling the DGLAP region as an overlap of LFWFs.
  2. Ensure polynomiality by inverting the Radon transform to identify an underlying DD.

Chouika *et al.*, Eur. Phys. J. **C77**, 906 (2017)

Ill-posedness in the sense of Hadamard.

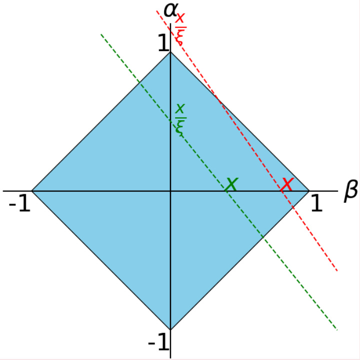
A first glimpse at the inverse Radon transform.

- Numerical evaluation ***almost unavoidable*** (polar vs cartesian coordinates).
- Ill-posedness by **lack of continuity**.
- The **unlimited** Radon inverse problem is **mildly** ill-posed while the **limited** one is **severely** ill-posed.
- Even if it existed, an analytic expression of the invert Radon transform would be of **limited practical use**.

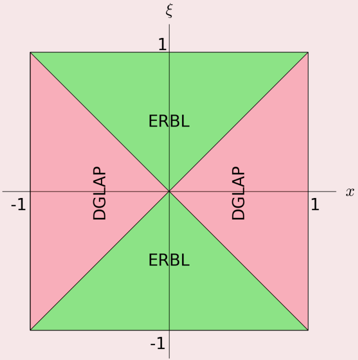


How can we get a DD from a GPD in the DGLAP region?

DD



GPD



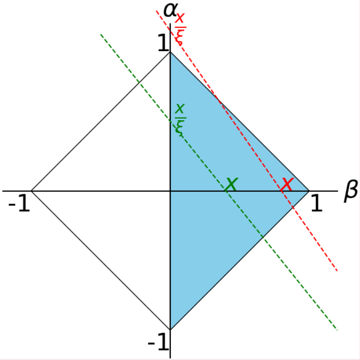
Computation of the extension.

Problem reduction.

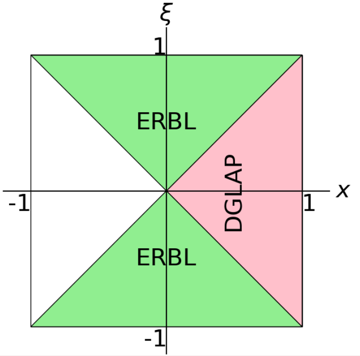
How can we get a DD from a GPD in the DGLAP region?

- Restrict to quark GPDs ($\beta > 0$).

DD



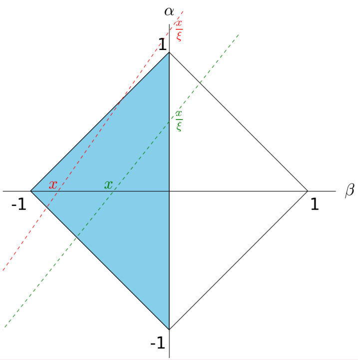
GPD



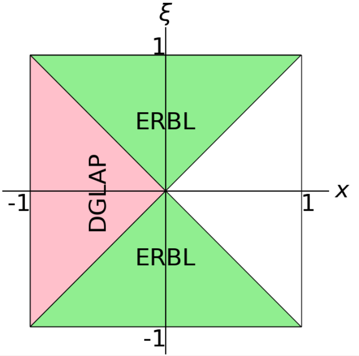
How can we get a DD from a GPD in the DGLAP region?

- Restrict to quark GPDs ($\beta > 0$).
- Only ERBL region "sees" both $\beta > 0$ and $\beta < 0$.

DD



GPD



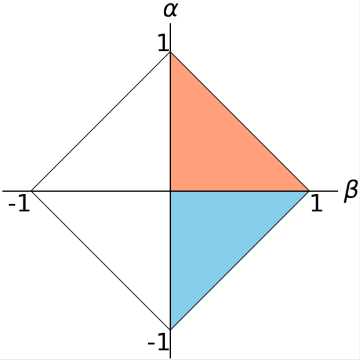
Computation of the extension.

Problem reduction.

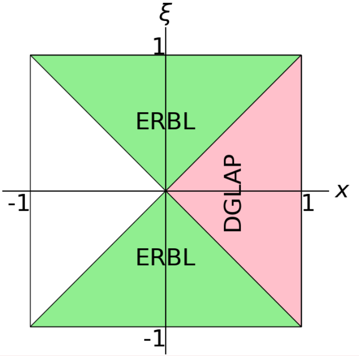
How can we get a DD from a GPD in the DGLAP region?

- Restrict to quark GPDs ($\beta > 0$).
- Only ERBL region "sees" both $\beta > 0$ and $\beta < 0$.
- Use α-parity of the DD.

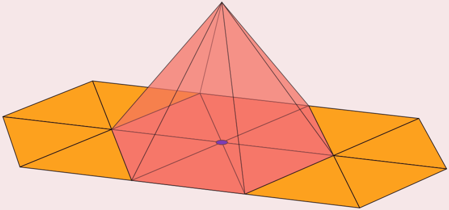
DD



GPD



Example of a P1 basis function



- Discretize the DD on a mesh with $n \simeq 800$ triangular cells.
- Compute the Radon transform of a P1 basis function.
- Sample $m \simeq 4n$ (x, ξ)-lines intersecting the DD support.
- Solve a linear system $AX = B$ with A a sparse $m \times n$ matrix.
- Adopt an iterative regularization method: LSMR.

Fong and Saunders, arXiv:1006.0758

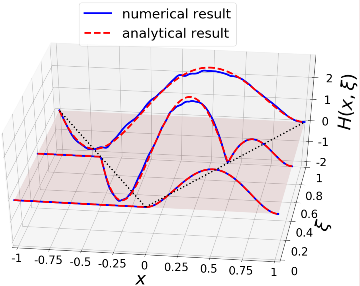
Examples - benchmarks (1/4).

Algebraic Bethe-Salpeter model.

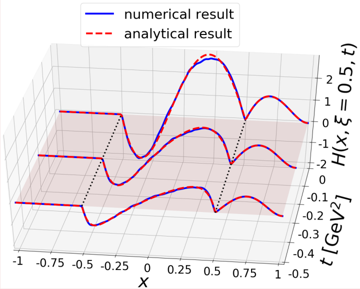
$$\Psi_{l=0}(x, \mathbf{k}_\perp) = 8\sqrt{15}\,\pi \frac{M^3}{\left(\mathbf{k}_\perp^2 + M^2\right)^2}(1-x)\,x\,,$$

$$i\,k_\perp^j\,\Psi_{l=1}(x, \mathbf{k}_\perp) = 8\sqrt{15}\,\pi \frac{k_\perp^j\,M^2}{\left(\mathbf{k}_\perp^2 + M^2\right)^2}(1-x)\,x\,, \qquad j = 1, 2$$

$H(x, \xi, t = 0)$



$H(x, \xi = 0.5, t)$



Chouika *et al.*, Eur. Phys. J. **C77**, 906 (2017)

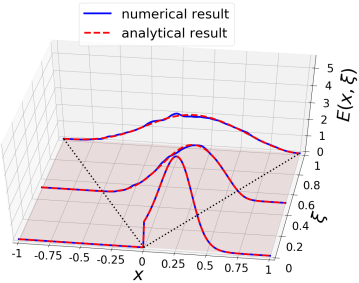
Examples - benchmarks (2/4).

Algebraic spectator model.

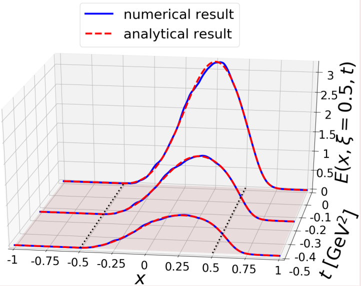
$$\varphi(x, \mathbf{k}_\perp) = \frac{gM^{2p}}{\sqrt{1-x}} x^{-p} \left(M^2 - \frac{\mathbf{k}_\perp^2 + m^2}{x} - \frac{\mathbf{k}_\perp^2 + \lambda^2}{1-x} \right)^{-p-1}$$

Hwang and Müller, Phys. Lett. **B660**, 350 (2008)

$H(x, \xi, t = 0)$



$H(x, \xi = 0.5, t)$



Chouika *et al.*, Eur. Phys. J. **C77**, 906 (2017)

Covariant extension

Phenomenology
Content of GPDs
Experimental access
Tomography
Dispersion relations

Modeling
Definition
Polynomiality
Radon transform
Positivity
Inverse Radon
Examples

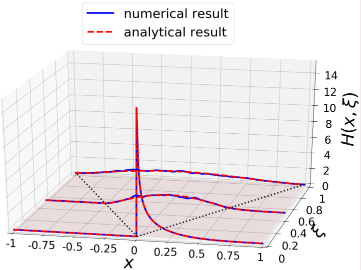
PARTONS
Design
Fits
Releases

Conclusion

Radyushkin DD Ansatz with phenomenological PDF:

$$q_{\mathrm{Regge}}(x) = \frac{35\,(1-x)^3}{32\,\sqrt{x}}\,.$$

$H(x, \xi, t = 0)$



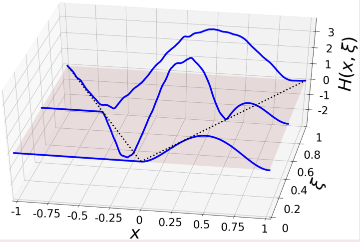
Chouika *et al.*, Eur. Phys. J. **C77**, 906 (2017)

Examples - benchmarks (4/4).

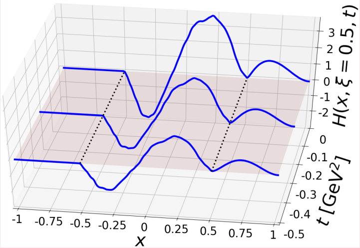
Gaussian wave function model.

$$\Psi\left(x, \mathbf{k}_\perp^2\right) = \frac{4\sqrt{15}\,\pi}{M}\sqrt{x(1-x)}\,e^{-\frac{\mathbf{k}_\perp^2}{4M^2(1-x)x}}\,.$$

$H(x, \xi, t = 0)$



$H(x, \xi = 0.5, t)$



Chouika *et al.*, Eur. Phys. J. **C77**, 906 (2017)

1. Modeling: GPDs (not CFFs) have to be extracted from measurements to learn about hadron structure.
2. Generic procedure to build models satisfying **all theoretical constraints**.
3. Remark: soft pion theorem can be fulfilled too!

   Chouika *et al.*, Phys. Lett. **B780**, 287 (2018)
4. Extension to spin-1/2 hadron in progress.
5. Integration in computing chain from GPDs to observables (PARTONS framework) in progress.
6. Still have to figure out how to input **phenomenological parameterizations** of PDFs and form factors for **global GPD fits**.

How do we relate all this to actual measurements?

PARtonic
Tomography
Of
Nucleon
Software

Computing chain design.

Differential studies: physical models and numerical methods.

Experimental data and phenomenology	Full processes
Computation of amplitudes	Small distance contributions
First principles and fundamental parameters	Large distance contributions

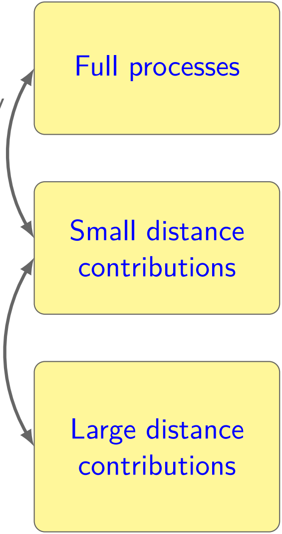
Computing chain design.

Differential studies: physical models and numerical methods.

Experimental data and phenomenology

Computation of amplitudes

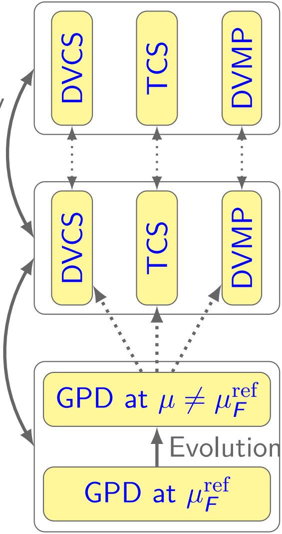
First principles and fundamental parameters



Experimental data and phenomenology

Computation of amplitudes

First principles and fundamental parameters



Computing chain design.

Differential studies: physical models and numerical methods.

Experimental data and phenomenology

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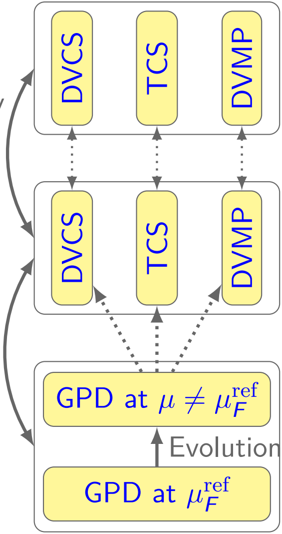
First principles and fundamental parameters



- Many observables.
- Kinematic reach.

Computing chain design.

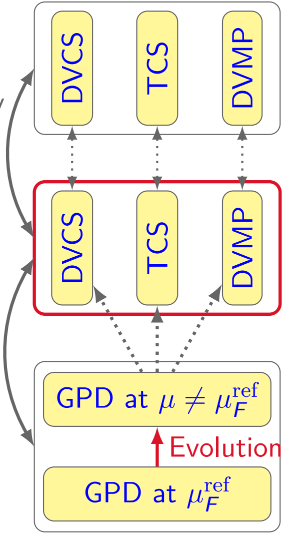
Differential studies: physical models and numerical methods.

Experimental data and phenomenology

Need for modularity

Computation of amplitudes

First principles and fundamental parameters



- Many observables.
- Kinematic reach.

- Perturbative approximations.
- Physical models.
- Fits.
- Numerical methods.
- Accuracy and speed.

Computing chain design.

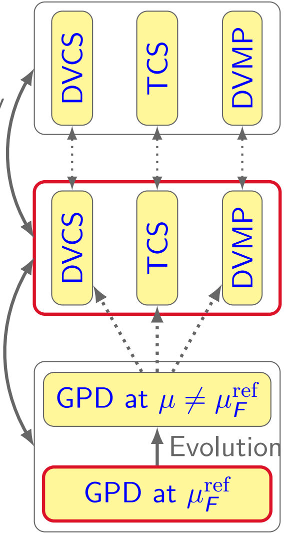
Differential studies: physical models and numerical methods.

Experimental data and phenomenology

Need for modularity

Computation of amplitudes

First principles and fundamental parameters



- Many observables.
- Kinematic reach.

- Perturbative approximations.
- Physical models.
- Fits.
- Numerical methods.
- Accuracy and speed.

Computing chain design.

Differential studies: physical models and numerical methods.

Experimental data and phenomenology

Need for modularity

Computation of amplitudes

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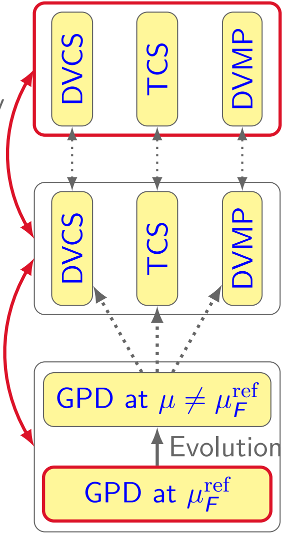

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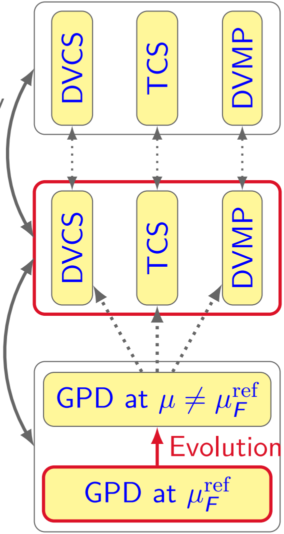

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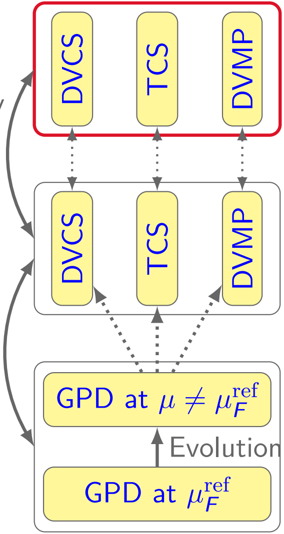

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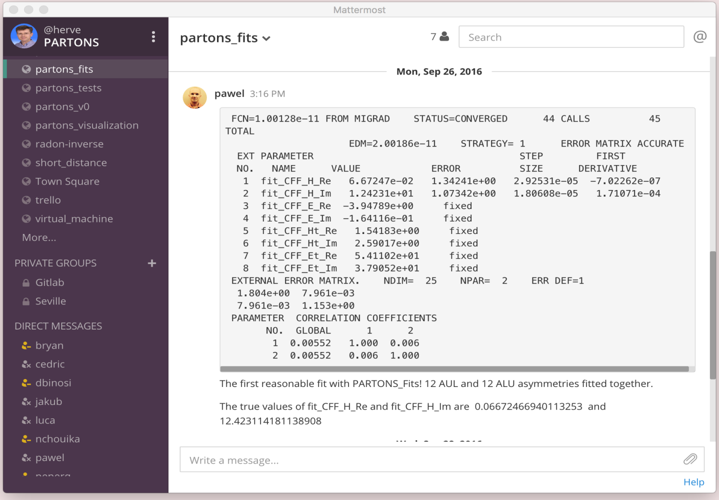
First local fit of pseudo DVCS data, Sep. 26$^{\text{th}}$, 2016



Parametric global fit of JLab DVCS data, Apr. 5^{th}, 2017

RESULTS

- Kinematic cuts — $Q^2 > 1.5$ GeV^2 (*where we can rely on LO approximation*)
  $-t / Q^2 < 0.25$ (*where we can rely on GPD factorization*)
- χ^2 / ndf — 3272.6 / (3433 - 7) ≈ 0.96
- Free parameters — a_{Hsea}, $a_{\tilde{H}val}$, $a_{\tilde{H}sea}$, C_{sub}, a_{sub}, N_E, $N_{\tilde{E}}$
- χ^2 / ndf per data set

[1] Phys. Rev. C 92, 055202 (2015)
[2] Phys. Rev. Lett. 115, 212003 (2015)
[3] Phys. Rev. D 91, 052014 (2015)

Experiment	Reference	Observables	N points all	N points selected	chi2	chi2 / ndf
Hall A	[1] KINX2	σUU	120	120	135.0	1.19
Hall A	[1] KINX2	ΔσLU	120	120	98.9	0.88
Hall A	[1] KINX3	σUU	108	108	274.8	2.72
Hall A	[1] KINX3	ΔσLU	108	108	107.3	1.06
CLAS	[2]	σUU	1933	1333	1089.2	0.82
CLAS	[2]	ΔσLU	1933	1333	1171.9	0.88
CLAS	[3]	AUL, ALU, ALL	498	305	338.1	1.13

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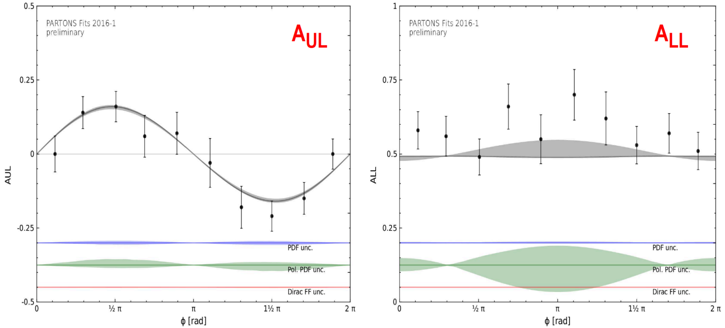
Comparison to CLAS data

RESULTS

CLAS: A_{UL} and A_{LL}
@ x_B = 0.26, t = -0.23 GeV², Q² = 2.0 GeV², E = 5.9 GeV

0.68 c.l.



Good description of experimental data, large systematics coming from Δq

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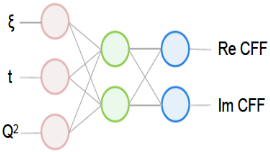
GPD or CFF fits (3/3).

Global fit of CFFs using neural networks.

Neural network global fit of CLAS asymmetries, May 31st, 2017

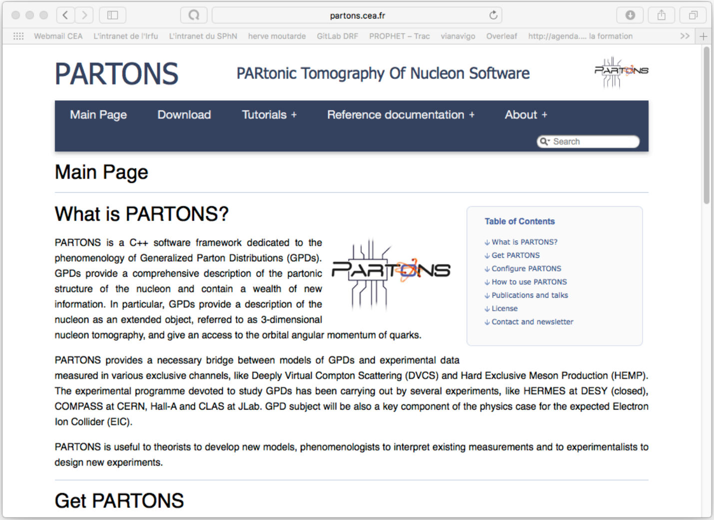
NEURAL NETWORK



- Our very first attempt to use NN technique → proof of feasibility
- Genetic algorithm (GA) to learn NN
- NN and GA libraries by PARTONS group
- Very simple design of NN
- CLAS asymmetry data only
- χ^2 / ndf = 273.9 / (305 - 68) ≈ 1.16

Open source release.

Publicly available on CEA GitLab server.



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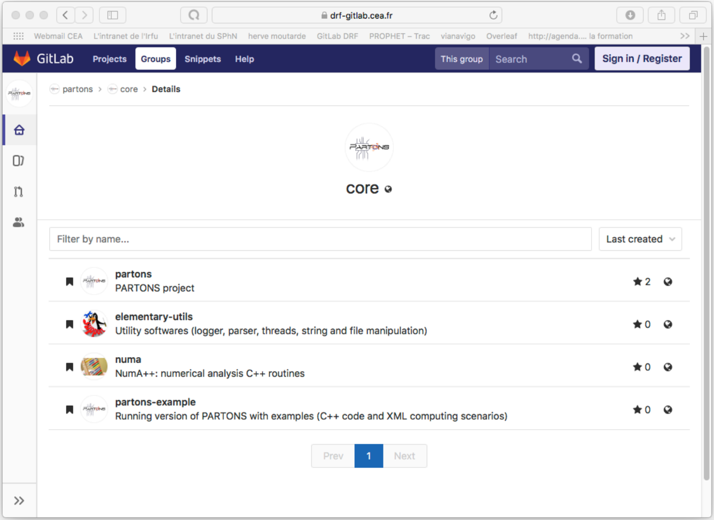

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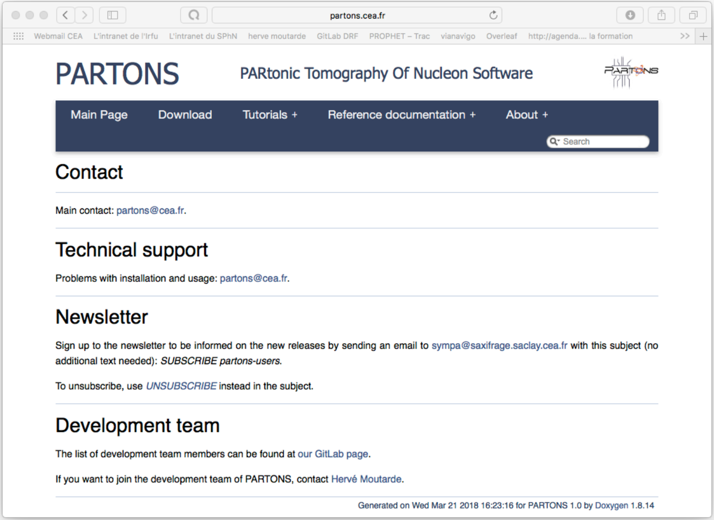

First release content.
DVCS channel only.

GPD modules
- GK
- VGG
- Vinnikov (evolution)
- MPSSW13 (NLO study)
- MMS13 (DD study)

DVCS modules
- VGG
- GV
- BMJ

CFF modules
- LO
- NLO
- NLO Noritzsch

Evolution modules
- Vinnikov (LO)

α_s modules
- 4-loop perturbation
- constant value

Future releases.

A lot remains to be integrated...Contributors welcome!

Channel modules

- DVMP
- TCS
- γM production
- ???

Other modules

- Mellin moments (EM tensor, lattice QCD)
- ???

Hadron structure modules

- DAs
- DDs
- Form factors
- PDFs
- LFWFs
- ???

Nonperturbative QCD modules

- Gap equation solver?
- α_s models?
- ???

Conclusion

- We can now build generic GPD model satisfying *a priori* **all theoretical constraints**.
- We now have tools to **systematically relate** these models to **experimental data**. Open source release under GPLv3.0. of the PARTONS framework.
- We have an **operating fitting engine** for global CFF fits.

New studies become possible!

- Global GPD fits.
- Energy-momentum structure of hadrons.
- Impact of nonperturbative QCD ingredients on 3D hadron structure studies.
- ???

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