Birdtracks for SU(N)

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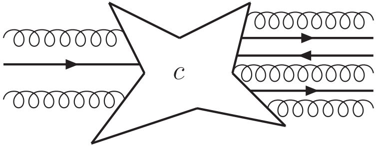
PSR 18 Lund

Plan

- construct bases for colour space
- using projectors onto SU(N) irreps
- written as birdtracks

- QCD process



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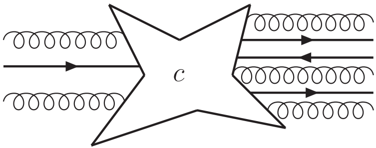

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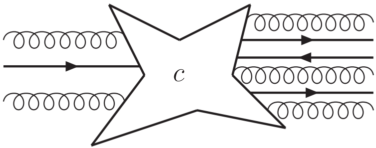

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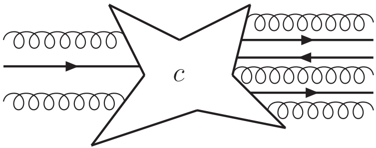

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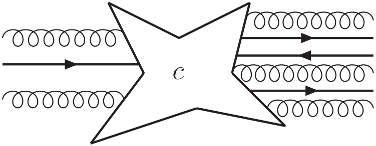

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- total amplitude as linear combination of colour structures

  requires basis of $\underbrace{\{\text{singlets} \subset (\overline{V} \otimes V)^{\otimes n_q} \otimes A^{\otimes n_g}\}}_{\text{colour space}}$

- birdtrack notation

$$T \in V \otimes A \otimes \overline{V} \otimes A$$

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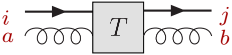
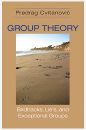
assign indices



$$= T^{ia}{}_{j}{}^{b}$$



Predrag Cvitanović
Group Theory – Birdtracks, Lie's and Exceptional Groups
Princeton University Press 2008

[Nordita lecture notes 1984 & PRD **14** (1976) 1536]

- birdtrack notation

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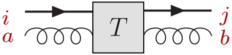

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- btw: Cvitanović uses thin instead of curly gluon lines

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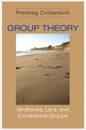

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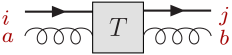

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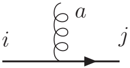
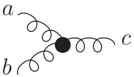
- examples



$= (t^a)^i{}_j$,



$= \mathrm{i} f^{abc}$

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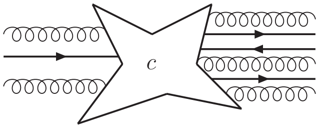
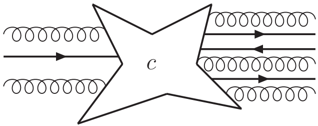
- frequently used: trace basis



- frequently used: trace basis
- consistes of products of (traces of products of) generators
  - ☺ easy to construct



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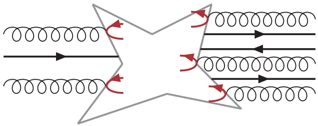

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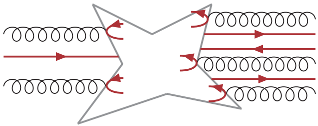

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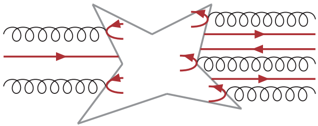

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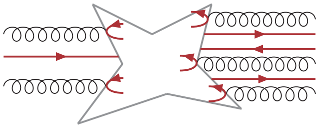

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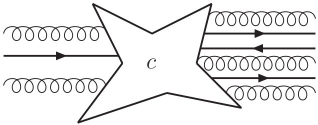
- compare to: multiplet basis
- constructed from projectors



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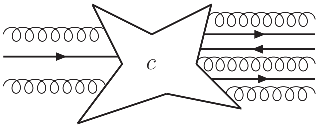
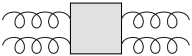
- compare to: multiplet basis
- constructed from projectors
  - ☺ orthogonal & minimal



	number of multiplets		dimension of colour space	
	$N=3$	$N=\infty$	$N=3$	$N=\infty$
$A^{\otimes 2} \to A^{\otimes 2}$	6	7	8	9
$A^{\otimes 3} \to A^{\otimes 3}$	29	51	145	265
$A^{\otimes 4} \to A^{\otimes 4}$	166	513	3 598	14 833
$A^{\otimes 5} \to A^{\otimes 5}$	1 002	6 345	107 160	1 334 961



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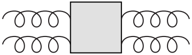

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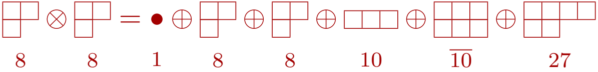

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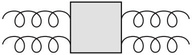

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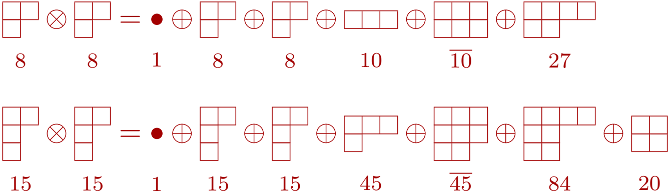

$N=3$

$N=4$

constructing multiplet bases

- quarks only ⇝ Hermitian Young operators
  SK & M. Sjödahl, J. Math. Phys. **55** (2014) 021702, arXiv:1307.6147
  J. Alcock-Zeilinger & H. Weigert, J. Math. Phys. **58** (2017) 051702, arXiv:1610.10088
  J. Alcock-Zeilinger & H. Weigert, J. Math. Phys. **58** (2017) 051703, arXiv:1610.08802
- gluons only
  SK & M. Sjödahl, JHEP **09** (2012) 124, arXiv:1207.0609
- quarks & anti-quarks
  work in progress (with J. Alcock-Zeilinger)
- quarks, anti-quarks & gluons ⇝ (at least) two strategies
  SK & M. Sjödahl, JHEP **09** (2012) 124, arXiv:1207.0609
  and work in progress (Alcock-Zeilinger & SK / Sjödahl & Thorén)

working with multiplet bases

- download bases for 6 external partons
  SK & M. Sjödahl, JHEP **09** (2012) 124, arXiv:1207.0609
- software
  M. Sjödahl: ColorMath/ColorFull (Mathematica/C++ packages)
- expanding into multiplet bases ⇝ Wigner 3j & 6j coefficients
  M. Sjödahl & J. Thorén JHEP **09** (2015) 055, arXiv:1507.03814

backup: constructing multiplet bases

- (conventional) Young operators are not Hermitian
  $\rightsquigarrow$ do not yield orthogonal bases

$Y_{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}} = \frac{4}{3}$  , $Y_{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}} = \frac{4}{3}$ 

- Hermitian Young operators ☺

$P_{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}} = \frac{4}{3}$  , $P_{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}} = \frac{4}{3}$ 

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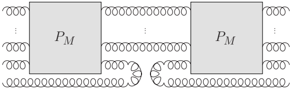
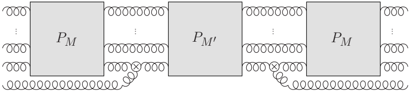
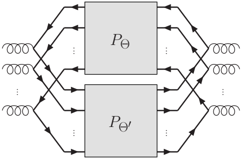
- first occurrence n_f

  $M' \subset M \otimes A \qquad \Rightarrow \qquad |n_f(M) - n_f(M')| \leq 1$

- rules for different cases

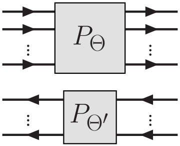
  - $P_{M'} = \frac{\dim M'}{\dim M}$ 

  - $P_{M'} = \gamma$ 

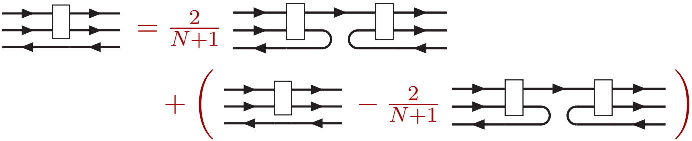
  - $T =$ 

SK & M. Sjödahl, JHEP **09** (2012) 124, `arXiv:1207.0609`

- apply two Hermitian Young operators



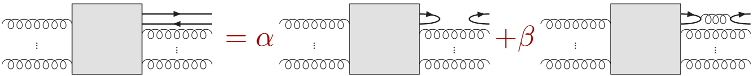
- further decompose by subtracting contractions, e.g.



work in progress (with J. Alcock-Zeilinger)

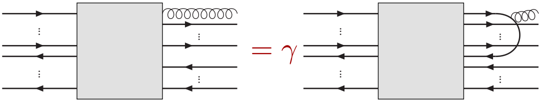
two strategies

- based on gluon projectors



SK & M. Sjödahl, JHEP **09** (2012) 124, `arXiv:1207.0609`

- based on $q\bar{q}$ projectors



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