Numerical resummation in SCET

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Resummation in SCET uses factorization theorems and RG equations

In SCET, collinear and soft degrees of freedom factorize at level of Lagrangian.

$$\mathcal{L} = \mathcal{L}_{n_1} + \ldots \mathcal{L}_{n_n} + \mathcal{L}_s$$

If collinear and soft factor in definition of observable, one obtains factorization theorem

$$\Sigma(v, Q) = H(\mu, \mu_H) J(\mu, \mu_J) \otimes J(\mu, \mu_J) \otimes S(\mu, \mu_S)$$

$$\mu_F \equiv \mu_F[v, Q]$$

Most important part of above: Each term in factorization theorem depends only on single scale

Resummation in SCET uses factorization theorems and RG equations

$$\Sigma(v,Q) = H(\mu,\mu_H)J(\mu,\mu_J) \otimes J(\mu,\mu_J) \otimes S(\mu,\mu_S)$$

One can now derive RG equations for each piece of factorization formula, and one finds in general (F=H,J,S)

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} F(\mu,\mu_F) = \gamma_F(\mu,\mu_F) \otimes F(\mu,\mu_F)$$

From this one can write

$$F(\mu,\mu_F) = U(\mu,\mu_F) \otimes F(\mu_F,\mu_F)$$

Solution to RGE resums logarithms.

Precision determined by loops in anom. dimensions (and matching)

Resummation in SCET uses factorization theorems and RG equations

1. SCET requires factorization theorem
2. For each ingredient of factorization theorem compute and solve RG equations

Downsides of approach:

1. Only works for observables where factorization formula can be derived
2. Need a different calculation for each observable
3. Clearly does not work if factorization formula does not exist

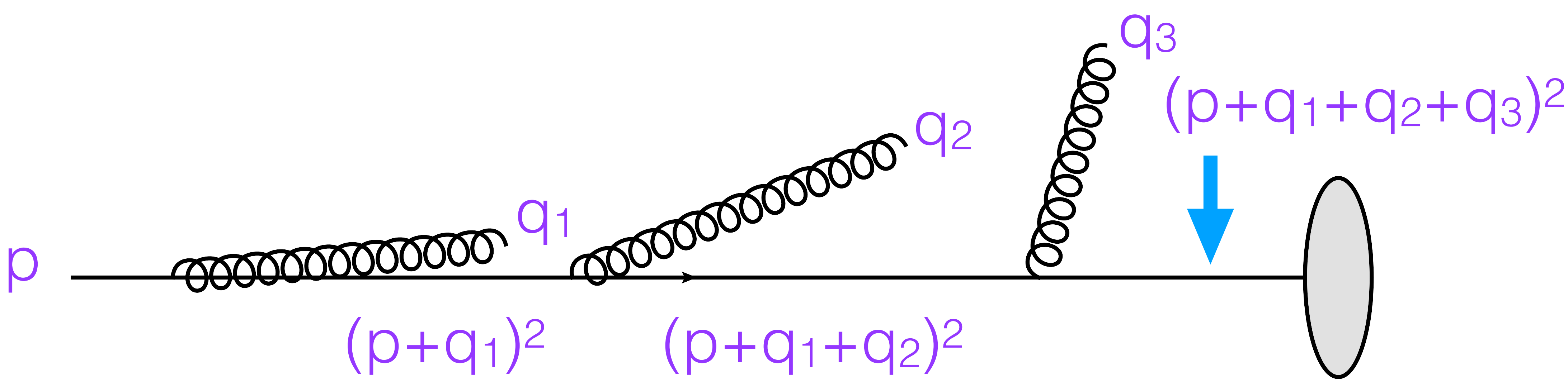
A numerical resummation approach was developed based on the coherent branching formalism

Catani, Webber, Marchesini ('91)
Catani, Turnock, Webber, Trentadue ('91)
Banfi, Salam, Zanderighi ('04)

One starts from the generic all order expression

$$\Sigma(v) = \mathcal{V}(\Phi_B) \sum_{n=0}^{\infty} \frac{1}{n!} \int [\mathrm{d}k_1] \ldots [\mathrm{d}k_n] |M(k_1, \ldots, k_n)|^2 \, \Theta\left[V(k_1, \ldots, k_n) < v\right]$$

To proceed, need to simplify the generic $|M(k_1,\ldots,k_n)|^2$



Leading divergence only if $q^T{}_1 \ll q^T{}_2 \ll q^T{}_3$ (strongly ordered)

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Leading divergence only if $q^T{}_1 \ll q^T{}_2 \ll q^T{}_3$ (strongly ordered)

In strongly ordered limit

$$|M(k_1, \ldots, k_n)|^2 = |M(k_1)|^2 \, |M(k_2)|^2 \ldots |M(k_n)|^2$$

This gives

$$\Sigma^{\mathrm{LL}}(v) = \mathcal{V}(\Phi_B) \sum_{n=0}^{\infty} \frac{1}{n!} \left[\prod_i \int [\mathrm{d}k_i] |M(k_i)|^2 \right] \Theta\left[V(k_1, \ldots, k_n) < v\right]$$

For simple enough observable can be solved very easily

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Consider an observable that satisfies

$$V_{\max}(k_1, \ldots, k_n) = \max \left[V(k_1), \ldots, V(k_n) \right]$$

Then one can easily exponentiate the result

$$\Sigma^{\mathrm{LL}}_{\max}(v) = \mathcal{V}(\Phi_B) \frac{1}{n!} \prod_{i=1}^{n} \left\{ \int [\mathrm{d}k_i] |M(k_i)|^2 \Theta \left[V(k_i) < v \right] \right\}$$

$$= e^{-R_{\mathrm{LL}}(v)}$$

Can this be applied for more complicated observables?

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$$\Sigma^{\mathrm{LL}}(v) = \mathcal{V}(\Phi_B) \sum_{n=0}^{\infty} \frac{1}{n!} \left[\prod_i \int [\mathrm{d}k_i] |M(k_i)|^2 \right] \Theta \left[V(k_1, \ldots, k_n) < v \right]$$

$$\Sigma^{\mathrm{LL}}_{\mathrm{max}}(v) = \mathcal{V}(\Phi_B) \frac{1}{n!} \prod_{i=1}^{n} \left\{ \int [\mathrm{d}k_i] |M(k_i)|^2 \Theta \left[V(k_i) < v \right] \right\}$$

Can one relate the two expressions to one another?

Most observables satisfy the following property

$$V(k_1, k_2, \ldots, k_n)\, \Theta[V(k_1) < \delta v] = V(k_2, \ldots, k_n)$$

For these, one can make additional simplifications

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$$\Sigma^{\mathrm{LL}}(v) = \mathcal{V}(\Phi_B) \sum_{n=0}^{\infty} \frac{1}{n!} \left[\prod_i \int [\mathrm{d}k_i] |M(k_i)|^2 \right] \Theta \left[V(k_1, \ldots, k_n) < v \right]$$

$$V(k_1, k_2, \ldots, k_n)\, \Theta[V(k_1) < \delta v] = V(k_2, \ldots, k_n)$$

Splitting each integration into a part <δv and one >δv one finds

$$\Sigma^{\mathrm{LL}}(v) = \mathcal{V}(\Phi_B) \left\{ \sum_{n=0}^{\infty} \frac{1}{n!} \left[\prod_i \int^{\delta v} [\mathrm{d}k_i] |M(k_i)|^2 \right] \right\} \left\{ \sum_{n=0}^{\infty} \frac{1}{n!} \left[\prod_i \int_{\delta v} [\mathrm{d}k_i] |M(k_i)|^2 \right] \right\} \Theta \left[V(k_1, \ldots, k_n) < v \right]$$

$$= \Sigma^{\mathrm{LL}}_{\max}(\delta v) \sum_{n=0}^{\infty} \frac{1}{n!} \left[\prod_i \int_{\delta v} [\mathrm{d}k_i] |M(k_i)|^2 \right] \Theta \left[V(k_1, \ldots, k_n) < v \right]$$

Rewrite as

$$\frac{\Sigma^{\mathrm{LL}}(v)}{\Sigma^{\mathrm{LL}}_{\max}(\delta v)} = \sum_{n=0}^{\infty} \frac{1}{n!} \left[\prod_i \int_{\delta v} [\mathrm{d}k_i] |M(k_i)|^2 \right] \Theta \left[V(k_1, \ldots, k_n) < v \right]$$

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Allows to write

$$\Sigma(v) = \Sigma_{\max}(v)\mathcal{F}(v)$$

with

$$\mathcal{F}(v) = \frac{\Sigma^{\mathrm{LL}}(\delta v)}{\Sigma^{\mathrm{LL}}_{\max}(v)} \sum_{n=0}^{\infty} \frac{1}{n!} \left[\prod_i \int_{\delta v} [\mathrm{d}k_i] |M(k_i)|^2\right] \Theta\left[V(k_1, \ldots, k_n) < v\right]$$

For simple enough observables, can perform integral analytically

IR divergences regulated, can be calculated in 4 dimension

Can always be calculated numerically

A numerical resummation approach was developed based on the coherent branching formalism

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In summary, for any observable one can write

$$\Sigma(v) = \Sigma_{\mathrm{max}}(v)\mathcal{F}(v)$$

where Σ_{max} can be computed easily analytically and

$$\mathcal{F}(v) = \frac{\Sigma^{\mathrm{LL}}(\delta v)}{\Sigma^{\mathrm{LL}}_{\mathrm{max}}(v)} \sum_{n=0}^{\infty} \frac{1}{n!} \left[\prod_i \int_{\delta v} [\mathrm{d}k_i] |M(k_i)|^2\right] \Theta\left[V(k_1, \ldots, k_n) < v\right]$$

can be calculated numerically

Downsides of approach:

1. Computation of the max observable not trivial at higher orders
2. Computation of the ratio again complicated at higher orders

Pros and Cons for both: Coherent branching formalism being more generic and SCET more systematic

SCET	Coherent branching
Only works for observables for which factorization theorem exists	Works for any observable (that is rIRC safe)
Purely analytical calculations (although numerical techniques for parts exist)	Only need analytical resummation for simple observable. Everything else done numerically
Very systematic way to go to higher orders (more loops in anomalous dimensions)	Somewhat of an art to go to higher orders (need to know exactly what was and was not included before)

Combining two approaches allows for generic resummation in SCET, without need for factorization theorem

	Combination
1	Derive general factorization formula that only uses separation of modes in SCET Lagrangian
2	Use SCET to obtain resummed result for a simple observable
3	Obtain relation between general and simplified observable through numerical algorithm within SCET

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1	Derive general factorization formula that only uses separation of modes in SCET Lagrangian
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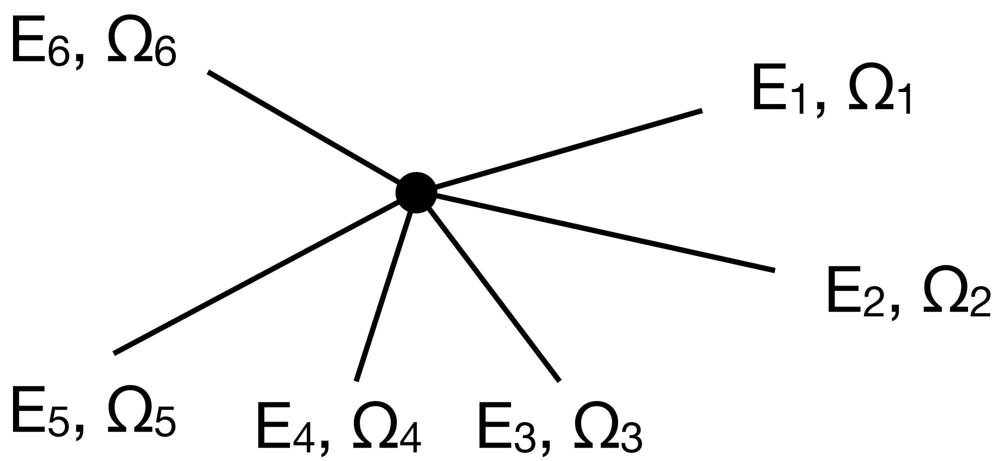
1) Write a "factorized" expression using only the separation of modes in SCET Lagrangian

CWB, Fleming, Lee, Sterman ('08)
CWB, Hornig Tackmann ('08)

Write the energy distribution of a generic event as



$$\omega_X(\Omega) = \sum_{i=1}^{n} E_i \, \delta(\Omega - \Omega_i)$$

Write a general observable in terms of this energy distribution

$$\Sigma(\Phi_B, v) \equiv \int \mathcal{D}\omega \, \frac{\delta\sigma}{\mathrm{d}\Phi_B \delta\omega} \, \theta(V[\Phi_B, \omega] < v)$$

with

$$\omega_k = \int_{\Omega_k} \mathrm{d}\Omega \, \omega(\Omega) \qquad \mathcal{D}\omega(\Omega) \equiv \mathcal{D}\omega(\Omega) \, \theta[\omega(\Omega)] = \prod_k \mathrm{d}\omega_k \, \theta(\omega_k)$$

1) Write a "factorized" expression using only the separation of modes in SCET Lagrangian

CWB, Fleming, Lee, Sterman ('08)
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Almost any observable can be written in terms of fully differential energy distribution

$$\Sigma(\Phi_B, v) \equiv \int \mathcal{D}\omega \, \frac{\delta\sigma}{\mathrm{d}\Phi_B \delta\omega} \, \theta(V[\Phi_B, \omega] < v)$$

Using the fact that SCET Lagrangian completely factors into collinear and soft pieces

$$\Sigma(v) = H \int \mathcal{D}\omega_n \, \frac{\delta\sigma_n}{\mathrm{d}\Phi_B \delta\omega_n} \int \mathcal{D}\omega_{\bar{n}} \, \frac{\delta\sigma_{\bar{n}}}{\mathrm{d}\Phi_B \delta\omega_{\bar{n}}} \int \mathcal{D}\omega_s \, \frac{\delta\sigma_S}{\mathrm{d}\Phi_B \delta\omega_s} \theta(V[\omega_n + \omega_{\bar{n}} + \omega_s] < v)$$

This applies to any observable

Observable can be factorizable or not

Combining two approaches allows for generic resummation in SCET, without need for factorization theorem

	Combination
1	Derive general factorization formula that only uses separation of modes in SCET Lagrangian
2	Use SCET to obtain resummed result for a simple observable
3	Obtain relation between general and simplified observable through numerical algorithm within SCET

2) For the simplified observable, can obtain resummation using the standard SCET methods

$$\Sigma(v) = H \int \mathcal{D}\omega_n \frac{\delta\sigma_n}{\mathrm{d}\Phi_B \delta\omega_n} \int \mathcal{D}\omega_{\bar{n}} \frac{\delta\sigma_{\bar{n}}}{\mathrm{d}\Phi_B \delta\omega_{\bar{n}}} \int \mathcal{D}\omega_s \frac{\delta\sigma_S}{\mathrm{d}\Phi_B \delta\omega_s} \theta(V[\omega_n + \omega_{\bar{n}} + \omega_s] < v)$$

Define max observable satisfying

$$V_{\max}[\Phi_B, \omega_n + \omega_{\bar{n}} + \omega_s] = \max\left[V_{\max}[\Phi_B, \omega_n] + V_{\max}[\Phi_B, \omega_{\bar{n}}] + V_{\max}[\Phi_B, \omega_s]\right]$$

Gives trivial factorization

$$\Sigma_{\max}(v) = H(\mu, \mu_H)\, \Sigma_n^{\max}(\mu, \mu_J)\, \Sigma_{\bar{n}}^{\max}(\mu, \mu_J)\, \Sigma_S^{\max}(\mu, \mu_S)$$

Resum using RG equations for each ingredient

Combining two approaches allows for generic resummation in SCET, without need for factorization theorem

	Combination
1	Derive general factorization formula that only uses separation of modes in SCET Lagrangian
2	Use SCET to obtain resummed result for a simple observable
3	Obtain relation between general and simplified observable through numerical algorithm within SCET

3) Difference between desired and simplified observable can now be computed numerically

Relate the "max" observable to the observable we want

$$\Sigma(v) = \Sigma_{\max}(v)\,\mathcal{F}(v)$$

Using the factorization results from above

$$\Sigma(v) = H \int \mathcal{D}\omega_n \frac{\delta\sigma_n}{\mathrm{d}\Phi_B \delta\omega_n} \int \mathcal{D}\omega_{\bar{n}} \frac{\delta\sigma_{\bar{n}}}{\mathrm{d}\Phi_B \delta\omega_{\bar{n}}} \int \mathcal{D}\omega_s \frac{\delta\sigma_S}{\mathrm{d}\Phi_B \delta\omega_s} \theta(V[\omega_n + \omega_{\bar{n}} + \omega_s] < v)$$

$$\Sigma_{\max}(v) = H(\mu, \mu_H)\,\Sigma_n^{\max}(\mu, \mu_J)\,\Sigma_{\bar{n}}^{\max}(\mu, \mu_J)\,\Sigma_S^{\max}(\mu, \mu_S)$$

one finds

$$\mathcal{F}(v) = \int \mathcal{D}\omega_n\,\mathcal{F}'(\omega_n, v) \int \mathcal{D}\omega_{\bar{n}}\,\mathcal{F}'(\omega_{\bar{n}}, v) \int \mathcal{D}\omega_s\,\mathcal{F}'(\omega_s, v)\theta(V[\omega_n + \omega_{\bar{n}} + \omega_s] < v)$$

with

$$\mathcal{F}'(\omega_F, v) = \frac{\frac{\delta\sigma_F}{\mathrm{d}\Phi_B \delta\omega_F}}{\Sigma_{\max}(\mu_H, \mu_F)}$$

3) Difference between desired and simplified observable can now be computed numerically

$$\mathcal{F}'(\omega_F, v) = \frac{\frac{\delta\sigma_F}{\mathrm{d}\Phi_B \delta\omega_F}}{\Sigma_{\max}(\mu_H, \mu_F)}$$

For the jet function one can show that with appropriate definitions

$$\frac{\delta\sigma_{n\bar{n}}^{\mathrm{LL}}}{\mathrm{d}\Phi_B \delta\omega_{n,\bar{n}}} = \delta(\omega_{n,\bar{n}})\,, \qquad \Sigma_{n,\bar{n}}^{\max,\mathrm{LL}}(\mu_H, \mu_J) = 1$$

$$\mathcal{F}'^{\mathrm{NLL}}(\omega_{n,\bar{n}}, v) = \delta(\omega_{n,\bar{n}})$$

3) Difference between desired and simplified observable can now be computed numerically

$$\mathcal{F}'(\omega_F, v) = \frac{\frac{\delta\sigma_F}{\mathrm{d}\Phi_B \delta\omega_F}}{\Sigma_{\max}(\mu_H, \mu_F)}$$

For the soft functions, do manipulations similar to coherent branching

$$\frac{\delta\sigma_S}{\mathrm{d}\Phi_B \delta\omega_s} = \mathcal{V}_\mathcal{S}(\Phi_B) \sum_{n=0}^{\infty} |M_S(\Phi_B; k_1, \ldots, k_n)|^2$$

1. Replace $|M_S(k_1, k_2, \ldots, k_n)|^2$ by strongly ordered limit (product of tree-level, 1-particle emissions)
2. Introduce resolution scale δ to regulate IR (cancels between numerator and denominator
3. Perform various algebraic simplifications on result

3) Difference between desired and simplified observable can now be computed numerically

$$\mathcal{F}'_F(\Phi_B, \omega_F, v) = \frac{\frac{\delta\sigma_F}{\mathrm{d}\Phi_B \delta\omega_F}}{\Sigma_n^{\max}(\Phi_B, v)}$$

After all manipulations, one finds

$$\mathcal{F}^{\mathrm{NLL}}(\Phi_B; v) = \delta^{R'_{\mathrm{LL}}(\Phi_B; v)} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^{n} \int_{\delta v}^{\infty} \frac{\mathrm{d}v_i}{v_i} \int [\mathrm{d}k_i]\, R'_{\mathrm{LL}}(\Phi_B; v, k_i)\, \Theta\left[V(\Phi_B; k_1, \ldots, k_n) < v\right]$$

with

$$R'_{\mathrm{LL}}(\Phi_B; v, k) \equiv \left| M_S^{(0)}(\Phi_B; k) \right|^2 \delta\left[V(\Phi_B; k) - v\right]$$

3) Difference between desired and simplified observable can now be computed numerically

$$\mathcal{F}^{\mathrm{NLL}}(\Phi_B; v) = \delta^{R'_{\mathrm{LL}}(\Phi_B;v)} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^{n} \int_{\delta v}^{\infty} \frac{\mathrm{d}v_i}{v_i} \int [\mathrm{d}k_i]\, R'_{\mathrm{LL}}(\Phi_B; v, k_i)\, \Theta\left[V(\Phi_B; k_1, \ldots, k_n) < v\right]$$

This can be written as

$$\begin{aligned}\mathcal{F}_{\mathrm{NLL}}(\Phi_B; v) = \Bigg[& \left(\frac{v}{\delta v}\right)^{-R'_{\mathrm{LL}}(\Phi_B;v)} \\ & + \int_{\delta v}^{v} \frac{\mathrm{d}v_1}{v_1} \left(\frac{v}{v_1}\right)^{-R'_{\mathrm{LL}}(\Phi_B;v)} \int [\mathrm{d}k_1] R'_{\mathrm{LL}}(\Phi_B; v, k_1) \left(\frac{v_1}{\delta v}\right)^{-R'_{\mathrm{LL}}(\Phi_B;v)} \\ & + \int_{\delta v}^{v} \frac{\mathrm{d}v_1}{v_1} \left(\frac{v}{v_1}\right)^{-R'_{\mathrm{LL}}(\Phi_B;v)} \int [\mathrm{d}k_1] R'_{\mathrm{LL}}(\Phi_B; v, k_1) \\ & \quad \times \int_{\delta v}^{v_1} \frac{\mathrm{d}v_2}{v_2} \left(\frac{v_1}{v_2}\right)^{-R'_{\mathrm{LL}}(\Phi_B;v)} \int [\mathrm{d}k_2] R'_{\mathrm{LL}}(\Phi_B; v, k_2) \left(\frac{v_2}{\delta v}\right)^{-R'_{\mathrm{LL}}(\Phi_B;v)} \\ & + \ldots \Bigg] \Theta\left[V(\Phi_B; k_1, \ldots, k_n) < v\right] .\end{aligned}$$

3) Difference between desired and simplified observable can now be computed numerically

$$\mathcal{F}^{\rm NLL}(\Phi_B; v) = \delta^{R'_{\rm LL}(\Phi_B;v)} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^{n} \int_{\delta v}^{\infty} \frac{dv_i}{v_i} \int [dk_i]\, R'_{\rm LL}(\Phi_B; v, k_i)\, \Theta\left[V(\Phi_B; k_1, \dots, k_n) < v\right]$$

Or this

Algorithm 1: Generating NLL transfer function

Start with $i = 0$ and $v_0 = v$;
while true **do**
- i=i+1;
- Generate v_i randomly according to $(v_{i-1}/v_i)^{-R'_{\rm LL}(\Phi_B;v)} = r$, with $r \in [0,1]$;
- **if** $v_i < \delta v$ **then**
  - break;
- **end**
- Generate a momentum k_i randomly according to $R'_{\rm LL}(\Phi_B; v, k_i)$ subject to the constraint $V(\Phi_B; k_i) = v$;

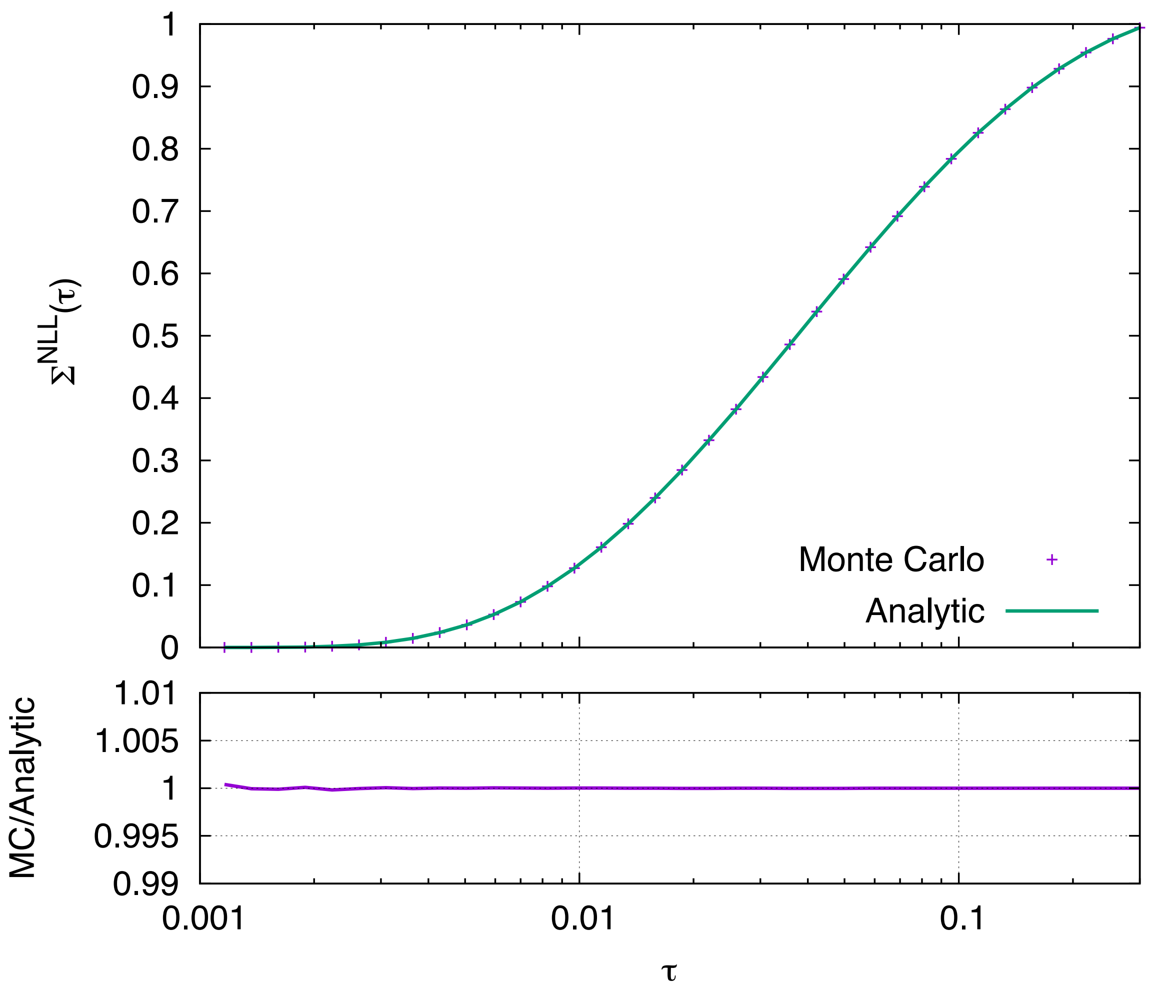
end

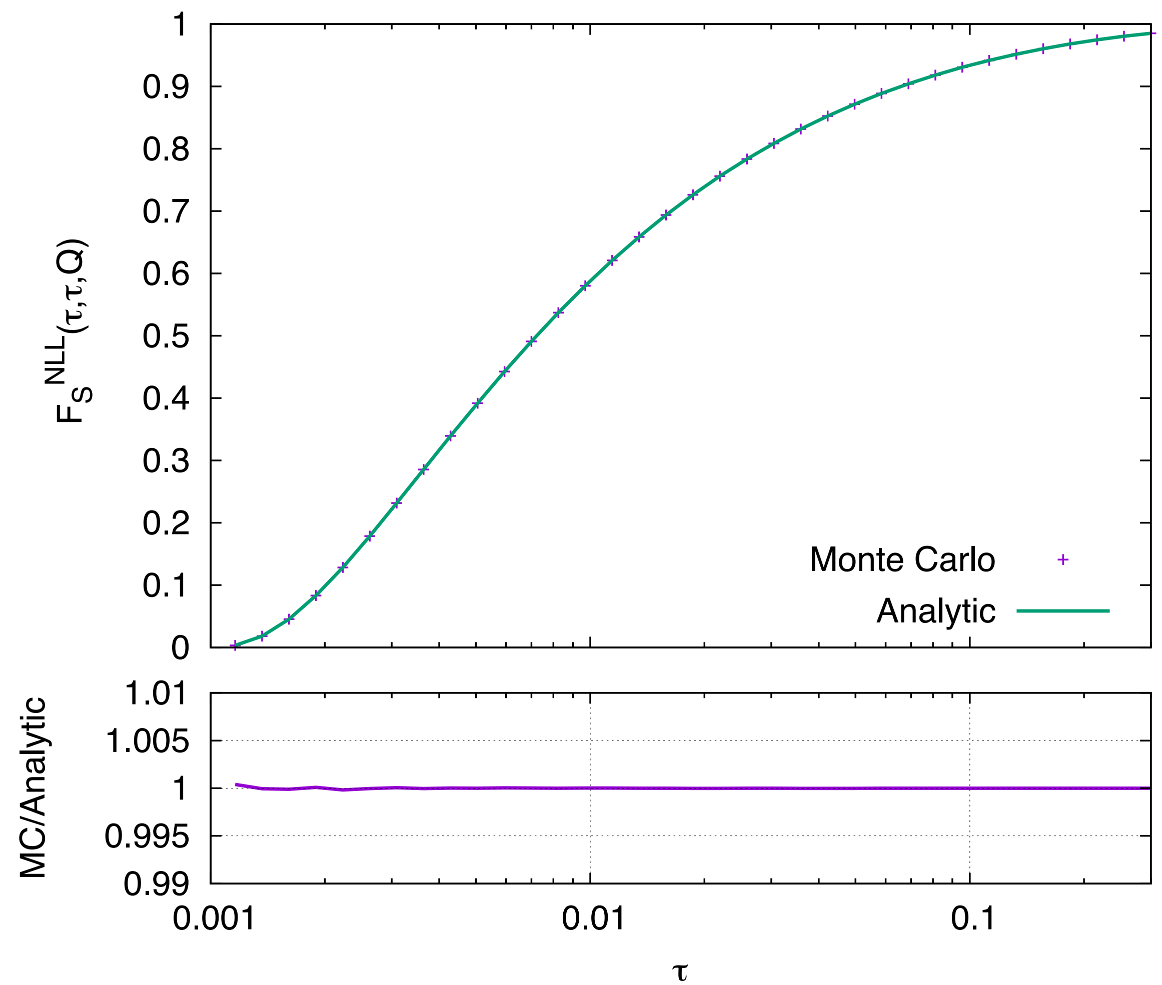
One now has a set of momenta k_i. If $V(\Phi_B; k_1, \dots k_n)$ is less than v, accept the event, otherwise reject it. The value of $\mathcal{F}_{\rm NLL}(\Phi_B; v)$ is equal to the fraction of the accepted events

3) Difference between desired and simplified observable can now be computed numerically

One can check that this works by comparing analytical to numerical calculation

$$\Sigma^{\mathrm{NLL}}(\tau) = \Sigma_{\max}(\tau)\mathcal{F}_S^{\mathrm{NLL}}(\tau,\tau,Q)$$





This approach opens door for resummation for a large class of observables

I have shown that one can combine two different methods of performing resummation to obtain a numerical method that is both systematic and applicable to wide variety of observables

Questions?

1. Find simplified observable for class of observables
2. Compute analytical resummation to given order
3. Run generic numerical algorithm to compute resummation for any observable in given class