Mass-radius bounds in massive gravity models

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Abstract. The mass-radius ratio bounds of spherically symmetric static compact objects are considered in the ghost-free dRGT massive gravity. In this type of modified gravity model, the graviton has a non-zero mass leading to the naturally generated cosmological constant term. Therefore, this may bring about to an explanation for the late-time accelerated expansion of the Universe without assuming any additional dark energy. The hydrostatic equilibrium (TOV) equation in this theory is derived to describe the structure of a spherical object such as a star. In this work, the generalized Buchdahl inequalities, providing the upper and lower limits of mass-radius ratio for high density compact objects, are obtained together with their crucial constraints. Finally, for theoretical testing these results may be proved in the context of astrophysical observations.

1. Introduction

Massive gravity is a modified general relativity by introducing the massive graviton. First started in 1939, Fierz and Pauli [1] proposed the linear model of massive gravity. Later, it was found that there exists a discrete difference between the theories of massless and massive in the limit m→0. To solve this discontinuity problem, the further studies in the nonlinear framework were required. Unfortunately, the nonlinearities usually generate a ghost instability. The ghost instability is the mode that the energy has no lower bound, so it is associated with instability of the system. After that, a number of ghost free massive gravity theories have been considered in the nonlinear framework, but one of the interesting theory is dRGT massive gravity. In 2010, de Rham, Gabadadze and Tolley [2, 3] succeeded to develop the first nonlinear fulfillment of the Fierz-Pauli theory free of the ghost instability. This theory naturally generates cosmological constant term. Therefore, it may be a possible explanation for late-time accelerated expansion of the Universe without any additional dark energy.

The study of the stability of compact objects in the general relativistic framework is of central importance for understanding the behavior of astrophysical systems. In order to understand the structure of a spherical object such as a star, we consider the TOV (Tolman, Oppenheimer, and Volkoff) equation which can describe the internal feature of a spherical object. Moreover, a simple but very powerful stability criterion was obtained by Buchdahl [4, 5], and it gives the condition for the stability of a compact object with total mass M and radius R as

$$\frac{2GM}{c^2 R} \leq \frac{8}{9}.$$

Furthermore, the mass-radius limits can be extended in massive gravity theories due to the possibility to give graviton a mass.

2. Formalism

2.1. Field equations of dRGT massive gravity

We start with the well-known Einstein-Hilbert gravitational action in the four-dimensional spacetime plus consistent nonlinear interaction terms interpreted as a graviton mass $m_{\rm g}$ which is given by [3]

$$S = \int d^4x \sqrt{-{\rm g}} \frac{1}{2\kappa} \left[R + 2\kappa \mathcal{L}_m + m_{\rm g}^2 \left(\mathcal{U}_2 + \alpha_3 \mathcal{U}_3 + \alpha_4 \mathcal{U}_4 \right) \right], \tag{1}$$

where $\kappa = 8\pi G/c^4$, the coefficients α_3 and α_4 are dimensionless free parameters, R is the scalar curvature, $\mathcal{L}_m$ is the matter Lagrangian, and $\mathcal{U}_i$ are dimensionless graviton potential terms defined as $\mathcal{U}_2 \equiv [\mathcal{K}]^2 - [\mathcal{K}^2]$, $\mathcal{U}_3 \equiv [\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3]$, and $\mathcal{U}_4 \equiv [\mathcal{K}]^4 - 6[\mathcal{K}]^2[\mathcal{K}^2] + 8[\mathcal{K}][\mathcal{K}^3] + 3[\mathcal{K}^2]^2 - 6[\mathcal{K}^4]$, respectively. The building block tensor is defined as

$$\mathcal{K}^\mu_\nu = \delta^\mu_\nu - \sqrt{{\rm g}^{\mu\sigma} f_{ab} \partial_\sigma \phi^a \partial_\nu \phi^b}, \tag{2}$$

where $[\mathcal{K}] = \mathcal{K}^\mu_\mu$, $[\mathcal{K}^n] = (\mathcal{K}^n)^\mu_\mu$, and we choose the unitary gauge $\phi^a = x^\mu \delta^a_\mu$ for the Stückelberg scalars. We follow the previous works by choosing a simple form of the fiducial metric to be [6, 7]

$$f_{\mu\nu} = {\rm diag}(0, 0, \lambda^2, \lambda^2 \sin^2 \theta), \tag{3}$$

where λ is a constant. This choice of interaction eliminates the BD ghost order by order. In four space-time dimensions, we consider a static and spherically symmetric metric of the following form

$$ds^2 = -n(r)d(ct)^2 + \frac{dr^2}{f(r)} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \tag{4}$$

After varying the action S, the modified Einstein field equations in the presence of the graviton potential are

$$G_{\mu\nu} - \kappa T_{\mu\nu} + m_{\rm g}^2 X_{\mu\nu} = 0. \tag{5}$$

We assume that the energy-momentum tensor of the matter is given by

$$T^\mu_\nu = (\rho c^2 + P) u^\mu u_\nu + P \delta^\mu_\nu, \tag{6}$$

i.e., by a perfect fluid, characterized by the matter density ρ, and the thermodynamic pressure P, respectively. Next, the effective energy-momentum tensor of massive graviton, obtained by varying the graviton potential terms in the action, takes the following form [6, 7]

$$X_{\mu\nu} = \mathcal{K}_{\mu\nu} - \mathcal{K} {\rm g}_{\mu\nu} - \alpha \left\{ \mathcal{K}^2_{\mu\nu} - \mathcal{K} \mathcal{K}_{\mu\nu} + \frac{[\mathcal{K}]^2 - [\mathcal{K}^2]}{2} {\rm g}_{\mu\nu} \right\}$$
$$+ 3\beta \left\{ \mathcal{K}^3_{\mu\nu} - \mathcal{K} \mathcal{K}^2_{\mu\nu} + \frac{1}{2} \mathcal{K}_{\mu\nu} \left\{ [\mathcal{K}]^2 - [\mathcal{K}^2] \right\} - \frac{1}{6} {\rm g}_{\mu\nu} \left\{ [\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3] \right\} \right\}, \tag{7}$$

where $\alpha_3 = (\alpha - 1)/3$, and $\alpha_4 = \beta/4 + (1 - \alpha)/12$.

Furthermore, the constraint from Bianchi identities gives separately the covariant derivatives of $T_{\mu\nu}$ and $X_{\mu\nu}$ equal to zero, according to the equations $\nabla^\mu X_{\mu\nu} = 0$, and $\nabla^\mu T_{\mu\nu} = 0$.

2.2. Hydrostatic equilibrium equation

The solution of the functional form f is obtained from equation (5) in the component $\mu = 0, \nu = 0$, and can be expressed as

$$f(r) = 1 - \frac{2G}{c^2} \frac{M(r)}{r} - \frac{\Lambda}{3} r^2 + \gamma r, \tag{8}$$

where $\Lambda = -3m_{\text{g}}^2(1 - 2\beta)$, and $\gamma = -\lambda m_{\text{g}}^2(1 - 3\beta)$.

From the continuity equation, $\nabla^\mu T_{\mu\nu} = 0$, it follows that

$$\frac{n'}{n} = -\frac{2P'}{\rho c^2 + P}. \tag{9}$$

By substituting equations (8) and (9) in equation (5) in the component $\mu = 1, \nu = 1$, the modified TOV equation can be obtained as

$$\frac{dP}{dr} = -\frac{(\rho c^2 + P)\left[\left(\frac{8\pi G}{c^4}P - \frac{2}{3}\Lambda\right)r^3 + \gamma r^2 + \frac{2G}{c^2}M(r)\right]}{2r^2\left[1 - \frac{2G}{c^2}\frac{M}{r} - \frac{\Lambda}{3}r^2 + \gamma r\right]}. \tag{10}$$

3. Buchdahl limits in dRGT massive gravity

We introduce the generalized Buchdahl variables (x, ω, ζ, y), defined as follows

$$x = r^2, \omega(r) = \frac{G}{c^2}\frac{M(r)}{r^3}, \zeta = n^{1/2}, y^2 = f(r) = 1 - 2\omega(r)r^2 - \frac{\Lambda}{3}r^2 + \gamma r.$$

By using equation (9) and equation (10), in terms of new variables, we obtain the generalized Buchdahl equation as follows

$$\frac{d}{dx}\left(y\frac{d\zeta}{dx}\right) - \frac{1}{2}\frac{\zeta}{y}\frac{d\omega}{dx} + \frac{\gamma}{8}\frac{\zeta}{y}x^{-3/2} = 0. \tag{11}$$

The density inside the spherically symmetric object is required to be decreasing functions of r. This requirement implies that

$$\frac{d}{dr}\left(\frac{M(r)}{r^3}\right) < 0, \tag{12}$$

leading to $d\omega/dx < 0$.

3.1. Mass-radius bounds for $\gamma > 0$

We introduce a new independent variable ℓ, obtained by changing the derivative $2y(d/dx) \to d/d\ell$, leading to the constraint,

$$\frac{d^2\zeta}{d\ell^2} < 0, \tag{13}$$

under the condition that $\gamma > 0$. By using the mean value theorem, we obtain the inequality

$$\frac{d\zeta}{d\ell} \leq \frac{\zeta(\ell) - \zeta(0)}{\ell - 0} \leq \frac{\zeta(\ell)}{\ell} \quad \to \quad \frac{1}{\zeta}\frac{d\zeta}{d\ell} \leq \frac{1}{\ell}. \tag{14}$$

We introduce now the new function $\alpha(r)$ defined by $y^2 = 1 - 2GM(r)\alpha(r)/c^2r$. We also assume the condition, $\alpha(r')M(r')/r' \geq (\alpha(r)M(r)/r)(r'/r)^2$, for all $r' < r$, by using the condition (12) together with the function $\alpha(r)$. Then, the right-hand side of inequality (14) can be bounded by above condition. By reorganizing the relation (14) and considering at the surface of the object, we eventually obtain the mass-radius ratio bounds in the presence of massive graviton for $\gamma > 0$,

$$\frac{4 + 3\gamma R}{9}\left[1 - \sqrt{1 - \frac{3(3\gamma^2 + 4\Lambda)R^2}{(4 + 3\gamma R)^2}}\right] \leq \frac{2GM}{c^2R} \leq \frac{4 + 3\gamma R}{9}\left[1 + \sqrt{1 - \frac{3(3\gamma^2 + 4\Lambda)R^2}{(4 + 3\gamma R)^2}}\right], \tag{15}$$

where R is the radius of the object, and M is the total mass. The validity of this inequality demands that the value in the square root must be greater than zero, a requirement which leads to the constraint $\Lambda < (4 + 6\gamma R)/3R^2$. Moreover, a nontrivial (positive) lower bound does exist only when the fraction in the square root is greater than zero, which gives another constraint for the negative Λ case, $\gamma > \sqrt{-4\Lambda/3}, \Lambda < 0$.

3.2. Mass-radius bounds for $\gamma < 0$

We introduce four new variables Γ, ψ, η and z, defined as $\Gamma(r) \equiv |\gamma|\zeta/8r^2, \psi = \zeta - \eta$, where $\eta = 4 \int_0^r \left(\int_0^{r_1} \Gamma(r_2)/\sqrt{1 - \Theta(r_2)/r_2} \, dr_2 \right) \cdot (r_1/\sqrt{1 - \Theta(r_1)/r_1}) \, dr_1$, while z is given by $dz = (1/y(x))dx$. The function $\Theta(r)$ is obviously defined by $y^2 = 1 - \Theta(r)/r$.

We assume two conditions that, for $r' < r$, $\Theta(r')/r' \geq (\Theta(r)/r) \cdot (r'/r)^2$, and $\Gamma(r') \geq \Gamma(r)$.

In terms of the new variables and two conditions defined above, the Buchdahl inequality (11) under the condition $\gamma < 0$ becomes

$$\frac{d^2}{dz^2}\psi(z) < 0. \tag{16}$$

Again, by using the mean value theorem, we find

$$\frac{d\psi}{dz} \leq \frac{\psi(z)}{z} \rightarrow \frac{d\zeta}{dz} - \frac{d\eta}{dz} \leq \frac{\zeta - \eta}{z}. \tag{17}$$

As the same way, we have obtained the following lower and upper bounds for the mass-radius ratio of compact objects in massive gravity for $\gamma < 0$,

$$\frac{4}{9}\left[1 - \sqrt{1 - \frac{3\left(\Lambda R + 3|\gamma|\right)R}{4}}\right] \leq \frac{2GM}{c^2 R} \leq \frac{4}{9}\left[1 + \sqrt{1 - \frac{3\left(\Lambda R + 3|\gamma|\right)R}{4}}\right]. \tag{18}$$

The inequality demands the value in the square root greater than zero which leads to a constraint $|\gamma| < 4/(9R) - \Lambda R/3$. Furthermore, a nontrivial (positive) lower bound in this case exists only when the fraction in the square root is greater than zero giving another constraint $|\gamma| > -\Lambda R/3$.

4. Conclusion

In this study, we have applied the massive gravity theory to the Buchdahl's technique, and investigated the mass-radius ratio bounds of spherically symmetric static compact objects, which also indicate their stability properties.

There are two possibilities for the important massive parameter γ, i.e. positive γ case, and negative γ case.

The results show that the upper mass-radius bound indeed exists in all case under different constraints of Λ, and if the lower mass-radius limit really exists, only the case of positive γ can be in Schwarzschild-Anti-de Sitter solution while the negative γ case can be in any space-time solution under its strictly constraint.

In the limit $\gamma = 0$, our upper and lower bounds in both positive and negative γ cases give the same result as the Buchdahl limits in the presence of a cosmological constant [8, 9].

To conclude, we have considered the mass-radius limits in dRGT massive gravity. The results obtained in this analysis may be proved in the context of astrophysical observations.

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