Exploring one loop amplitude at four-points vertices by the OPP method

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Abstract. We construct the general formula of one-loop amplitude at four-point vertices using Ossola, Papadopoulos, and Pittau (OPP) method. The incoming and outgoing particles are defined as arbitrary massless particles, and the intermediate state contains arbitrary particles inside. In this works, the amplitude is reconstructed via finding four-type rational coefficients. First, box coefficient is extracted using the four-cut technique with linear algebra. We found that triangle and bubble coefficients can be extracted using three-cut and two-cut technique with Cauchy residue theorem instead of with discrete Fourier sum like the original version of the OPP. Tadpole coefficient can be dropped out, because its scalar integral, which contains only UV divergence, is completely absorbed by renormalization.

1. Introduction

Nowadays, searching for a sign of new physics in particle collision experiment is done through high experimental statistics. The highly efficient next to leading order (NLO) calculation is required in order to check theoretical and experimental matching. However, direct NLO calculation is not easy, when the number of vertices increases. the number of loop integral is increased extremely. In present, the Ossola, Papadopoulos, and Pittau (OPP) [1] method is found to be an effective tool to construct one-loop amplitude in form of a linear combination of 4-types of Master Integral (MI) [2] multiplied with rational coefficients.

In this research, we will explore details calculation of one-loop four-point vertices amplitude using the OPP method. All external particles are assigned to be massless particles while the internal intermediate particles are arbitrary. The amplitude will be written in form of a linear combination of four master integral multiplied with rational coefficients, ie., box, triangle, bubble, and tadpole. In the next section, we will introduce the basic construction of the OPP method based on the integrand reduction technique by the multiple-cut method. In section 3, the box coefficient is calculated by solving linear algebra. The triangle and bubble coefficients are calculable by the Cauchy residue theorem while in the original version of OPP calculation used discrete Fourier summation. The reason is that within triangle and bubble numerators, both can be rewritten as Laurent series [3] when the amplitude is cut by the Cutkorsky's rule [4]. Each coefficient will be on the different complex pole order. Therefore, they can be extracted using the Cauchy residue theorem. This is the different point to the original OPP calculation.

2. The OPP method

In general, the 4-point one-loop amplitude is generated by Feynman's rule. It contains many types of tensor integral, which depend on the numerator $N(p)$. It can be generally written as

$$i\mathcal{A} = \int \frac{d^4p}{(2\pi)^4}\frac{N(p)}{D_0D_1D_2D_3}, \tag{1}$$

where p is a loop momentum. Traditionally, the propagator is defined by momentum flow in the loop as $D_0 = p^2 - m_0^2, D_1 = (p+k_1)^2 - m_1^2, D_2 = (p+k_1+k_2)^2 - m_2^2, D_3 = (p-k_4)^2 - m_3^2$, when we take all external legs are incoming $k_1+k_2+k_3+k_4 = 0$. We would like to express the equation (1) in the form of the linear combination of MI multiplied by the rational coefficient as

$$i\mathcal{A} = c_0^{(0123)}I_{0123} + \sum_{i<j<k}^{3} c_{00}^{(ijk)}I_{ijk} + \sum_{i<j}^{3} c_{000}^{(ij)}I_{ij} + \sum_{i}^{3} c_{0000}^{(i)}I_i + \mathcal{R}, \tag{2}$$

where c_0, c_{00}, c_{000}, c_{0000} and $\mathcal{R}$ are box-, triangle-, bubble-, tadpole-coefficients and rational part respectively. The numerator of equation (1) is rewritable in term of rational coefficient plus irreducible scalar product (ISP) via Aguila and Pittau works [5] as

$$N(p) = \Delta_0^{(0123)}(p) + \sum_{i<j<k}\Delta_{00}^{(ijk)}(p)\prod_{l\neq ijk}D_l + \sum_{i<j}\Delta_{000}^{(ij)}(p)\prod_{k\neq ij}D_k + \sum_{i}\Delta_{0000}^{(i)}(p)\prod_{j\neq j}D_j, \tag{3}$$

when $\Delta^{i_0i_1..i_n}(p)$ contains a rational coefficient plus ISP, $\tilde{c}(p)$. It will not be reduced to $n-1$ point MI, however, it will vanish after integrating out of loop momentum.

$$\int \frac{d^4p}{(2\pi)^4}\frac{\Delta^{(i_0i_1..i_n)}(p)}{D_{i_0}D_{i_1}..D_{i_n}} = \int \frac{d^4p}{(2\pi)^4}\frac{c^{(i_0i_1..i_n)} + \tilde{c}^{(i_0i_1..i_n)}(p)}{D_{i_0}D_{i_1}..D_{i_n}} = c^{(i_0i_1..i_n)}I_{i_0i_1..i_n} \tag{4}$$

Their definitions are defined as following below

$$\Delta_0^{(0123)}(p) = c_0^{(0123)} + \tilde{c}_0^{(0123)}p.\omega, \tag{5}$$

$$\Delta_{00}^{(ijk)}(p) = c_{00}^{(ijk)} + \sum_{l=1}^{3}\left(c_{l0}((p+r_0).e_{ab})^l + c_{0l}((p+r_0).e_{ba})^l\right), \tag{6}$$

$$\Delta_{000}^{(ii+2)}(p) = c_{000}^{(ii+2)} + \tilde{c}_{000}^{(ii+2)}(2((p+r_0).(k_a-k_b))^2 - (p+r_0).e_{ab}(p+r_0).e_{ba}) + ..., \tag{7}$$

where $\omega^\mu = [2\gamma^\mu 1\rangle[13]\langle 32\rangle - \langle 2\gamma^\mu 1]\langle 13\rangle[32]$ and $e^\mu_{ab} = \langle k_a\gamma^\mu k_b]$. The choices of a, b have to correspond with the equation (4). Then, we apply the multiple cuts to the equation (1). The propagator D_i^{-1} which is cut will be substituted by the Dirac delta function $\delta(D_i)$, and the amplitude which is cut will be turned to the imaginary part. We cut 4-propagators (0123), 3-propagators (ijk), and 2-propagators $(i, i+2)$ respectively. We obtain

$$\Delta_0^{(0123)}(p) = N(p)|_{D_0=D_1=D_2=D_3=0}, \tag{8}$$

$$\Delta_{00}^{(ijk)}(p) = D_{i_0}^{-1}\left(N(p) - \Delta_0^{(0123)}(p)\right)|_{D_i=D_j=D_k=0}, \tag{9}$$

$$\Delta_{000}^{(ii+2)}(p) = D_{i_5}^{-1}D_{i_6}^{-1}(N(p) - \Delta_0^{(0123)}(p) - \sum_{i_0<i_1<i_2}\Delta_{00}^{(i_0i_1i_2)}(p)D_{i_4})|_{D_i=D_j=0}, \tag{10}$$

where $i_0 \neq i, j, k$, $i_4 \neq i_1, i_2, i_3$ and $i_5, i_6 \neq i, i+2$. We drop out the 1-cut propagator and the adjacent 2-cut propagators because their scalar integrals contain only an UV-divergence. They can be canceled out by cyclic permutation when the amplitude is sum overall possible channel.

3. Rational coefficients

4-cut propagators appear four-constrained equations, $D_0 = D_1 = D_2 = D_3 = 0$. It gives us the two solutions of loop momentum $p_\pm$, which is shown in the equation (A.1). The overall x_i are completely solved, thus the coefficient can be solved using a linear equation.

$$c_0^{(0123)} = \frac{N(p_+) + N(p_-)}{2} \tag{11}$$

$$\tilde{c}_0^{(0123)} = \frac{N(p_+) - N(p_-)}{2p_+.\omega} \tag{12}$$

3-cut propagators, the constrained equation is not enough to solve all variables of x_i on equation (A.2). However, the equation (6) can be expanded as Laurent series of x_3 or x_4. We found that each coefficient is attached on different complex pole order, thus it is solvable by Cauchy residue.

$$c_{00}^{(ijk)} = \mathrm{Res}\left(x_3^{-1}\Delta_{00}^{(ijk)}(p_1)\right)_{x_3=0}, \tag{13}$$

$$c_{l0}^{(ijk)} = (-1)^l (C_{ijk})^l\, \mathrm{Res}\left(x_3^{l-1}\Delta_{00}^{(ijk)}(p_1)\right)_{x_3=0}, \tag{14}$$

$$c_{0l}^{(ijk)} = (-1)^l (C_{ijk})^l\, \mathrm{Res}\left(x_4^{l-1}\Delta_{00}^{(ijk)}(p_2)\right)_{x_4=0}, \tag{15}$$

The definitions of $p_{1,2}$ and C_{ijk} are shown in (A.2) and Table (A1). Finally, 2-cut propagators use Cauchy residue theorem similar to 3-cut propagators.

$$c_{000}^{(i,i+2)} = \mathrm{Res}\left(\mathrm{Res}\left(\tilde{C}_{i,i+2}(x_2, x_4)\Delta_{000}^{(i,i+2)}(p_3)\right)_{x_2=0}\right)_{x_4=0}, \tag{16}$$

where $\tilde{C}_{i,i+2}(x_2, x_4) = x_2^{-1}x_4^{-1} + (4/3)x_2^{-3}x_4^{-1}x_3^{(i,i+2)}(0,1))$ and p_3, $x_3^{(i,i+2)}(x_2, x_4)$ are shown on equation (A.3) and Table (A2). However, rational part $\mathcal{R}$ cannot be detected by unitarity cut. Their origin comes from the small dimensional part of the amplitude. We can follow the reconstruction $\mathcal{R}$ in [6].

4. Calculational example

The example is QED photon-photon scattering, which leading order appears at four pointed one-loop level. All m_i in D_i are set equally. The numerator is writable from Feynman's rule

$$N_{++++}(p) = -e^4 \mathrm{Tr}[(\not{p} + m)\not{\epsilon}_1^+(\not{p} + \not{k}_1 + m)\not{\epsilon}_2^+(\not{p} + \not{k}_1 + \not{k}_2 + m)\not{\epsilon}_3^+(\not{p} - \not{k}_4 + m)\not{\epsilon}_4^+]. \tag{17}$$

The other two channels of amplitude are obtained by interchanging the momentum index $3 \leftrightarrow 4$ and $2 \leftrightarrow 3$. Putting equation (17) into (8)-(16) and using the method in [6]

$$i\mathcal{A}_{++++} = -8ie^4m^4(I_{box}(s_{12}, s_{14}, m) + I_{box}(s_{12}, s_{13}, m) + I_{box}(s_{13}, s_{14}, m)) + \frac{e^4}{4\pi^2}, \tag{18}$$

where $s_{ij} = (k_i + k_j)^2$ and $I_{box}(s_{12}, s_{14}, m) = I_{0123}$. The scalar MIs existing are symbolically calculated in the condition $s_{ij} \ll m$, where ω is incoming photon's energy.

$$i\mathcal{A}_{++++} \approx -\frac{e^4(s_{12}^2 + s_{14}s_{12} + s_{14}^2)}{240\pi^2 m^4} = -\frac{\alpha^2(3 + \cos^2\theta)\omega^4}{15m^4}, \tag{19}$$

where θ is the scattering angle. This result is in agreement with the result of Karplus [7].

5. Conclusions

We found that box-coefficient is solvable using linear algebra while triangle and bubble coefficients are solvable using the Cauchy residue theorem. However, the adjacent bubble and tadpole coefficients are ignored in our processes, the amplitude is obtained correctly.

Appendix A.

This section shows the parametrized loop momentum in various cut channel and its solution.

$$p^\mu_\pm = k^\mu_1 x_1 + k^\mu_2 x_2 + e^\mu_{12} x^\pm_3 + e^\mu_{21} x^\pm_4, \tag{A.1}$$

where $x_1 = s^{-1}_{12} M_{21} - 1$, $x_2 = s^{-1}_{12} M_{10}$, $x_3 = c(4e_{12}.k_4)^{-1}\mathcal{F}_\mp$, $x_4 = c(4e_{21}.k_4)^{-1}\mathcal{F}_\pm$, which $M_{ij} = m^2_i - m^2_j$, $c = M_{03} - s_{14}x_1 - s_{13}x_2$, $\mathcal{F}_\pm = 1 \pm \sqrt{1 + 4s_{13}s_{14}c^{-2}\left(s^{-1}_{12}m^2_0 - x_1x_2\right)}$.

$$\begin{aligned} p^\mu_1 &= -r^\mu_0 + k^\mu_a d_1 + k^\mu_b d_2 + e^\mu_{ab} x_3 + \left(2s_{ab}C_{ijk}x_3\right)^{-1} e^\mu_{ba}, \\ p^\mu_2 &= -r^\mu_0 + k^\mu_a d_1 + k^\mu_b d_2 + \left(2s_{ab}C_{ijk}x_4\right)^{-1} e^\mu_{ab} + e^\mu_{ba} x_4, \end{aligned} \tag{A.2}$$

Table A1. a and b are the choices of parametrized momentum, and d_1, d_2, C_{ijk} are solutions of (A.2) in several cut channels of 3-cut propagators.

Cut (ijk)	r_0	a	b	d_1	d_2	C_{ijk}
012	0	1	2	$s^{-1}_{12}M_{21} - 1$	$s^{-1}_{12}M_{10}$	$2s_{12}(M_{01}M_{12} - m^2_1 s_{12})^{-1}$
013	0	1	4	$s^{-1}_{14}M_{03}$	$s^{-1}_{14}M_{10}$	$2s_{14}(M_{10}M_{03} - m^2_0 s_{14})^{-1}$
023	0	3	4	$s^{-1}_{12}M_{03}$	$s^{-1}_{12}M_{32} + 1$	$2s_{12}(M_{03}M_{32} - m^2_3 s_{12})^{-1}$
123	k_1	2	3	$s^{-1}_{14}M_{32} - 1$	$s^{-1}_{14}M_{21}$	$2s_{14}(M_{12}M_{23} - m^2_2 s_{14})^{-1}$

$$\begin{aligned} p^\mu_3 &= -r^\mu_0 + (k_a + k_b)^\mu n + (k_a - k_b)^\mu x_2 + e^\mu_{ab} x^{(i,i+2)}_3(x_2, x_4) + e^\mu_{ba} x_4, \\ x^{(i,i+2)}_3(x_2, x_4) &= \frac{1}{16s^2_{ab}x_4}\left(s^2_{ab}(1 - 4x^2_2) + M^2_{i,i+2} - 2s_{ab}(m^2_i + m^2_{i+2})\right), \end{aligned} \tag{A.3}$$

Table A2. a and b are the choices of parametrized momentum, and n are solutions of (A.3) in several cut channels of the nonadjacent 2-cut propagators.

Cut (ij)	r_0	a	b	n
02	0	1	2	$(2s_{12})^{-1}M_{20} - 2^{-1}$
13	k_1	1	4	$(2s_{14})^{-1}M_{13} + 2^{-1}$

References

[1] Ossola G, Papadopoulos C G and Pittau R 2007 *Nucl. Phys.* B **763** 147–69
[2] Passarino G and Veltman M J G 1979 *Nucl. Phys.* B **160** 151–207
[3] Mastrolia P, Mirabella E and Peraro T 2012 *JHEP* **06** 095
[4] Cutkosky R E 1960 *J. Math. Phys.* **1** 429
[5] Aguila F and Pittau R 2004 *JHEP* **07** 017
[6] Ossola G, Papadopoulos C G and Pittau R 2008 *JHEP* **05** 004
[7] Karplus R and Neuman M 1950 *Phy. Rev.* **83** 776–84