Radiation reaction of a charged particle with the external rectangular force

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Abstract. We considered the equation of motion of an accelerating point charged particle which produced the radiation reaction force that described by the Abraham-Lorentz equation. Then we considered another equation of motion from a different approach so-called Landau-Lifshitz equation which came from a limit in the case of a finite spherical shell charged particle. This latter equation shows that an acceleration of charged particle depends on an external force and its derivative. We showed the solution of the acceleration from the external rectangular force of these equations. So we solved the velocity and position of the charged particle. As a result, we compared the solutions from the Abraham-Lorentz and Landau-Lifshitz equations of motion.

1. Introduction

When a charged particle is moving with the acceleration, it emits the electromagnetic radiation. The radiation electromagnetic fields of a point charged particle q with the mass m is moving with an acceleration in the low velocity limit are [1–3]

$$\vec{E} = \frac{\mu_0 q}{4\pi\rho^3}(\vec{\rho} \times (\vec{\rho} \times \vec{a})), \qquad \vec{B} = \frac{\vec{\rho}}{\rho} \times \vec{E}, \tag{1}$$

where $\vec{a}$ is an acceleration of the particle and $\vec{\rho} = \vec{r} - \vec{r}'$ is the displacement vector of the distance between the field point and particle point.

So it can be seen that the energy rate of electromagnetic radiation is

$$P = \frac{\mu_0 q^2 a^2}{6\pi c}. \tag{2}$$

From the conservation law of energy, we considered the energy loss of an accelerating point charged particle. There is a recoil force from the radiation fields shown in the formula [1–3]

$$\vec{F}_{rad} = \left(\frac{\mu_0 q^2}{6\pi c}\right)\dot{\vec{a}}. \tag{3}$$

It is called the Abraham-Lorentz force which describes the radiation reaction from an accelerated moving charged particle. When an additional external force is acting on the accelerated charged particle, the equation of motion will be

$$\vec{a} - \tau\dot{\vec{a}} = \frac{\vec{F}_{ext}}{m}. \tag{4}$$

This is the Abraham-Lorentz equation, where $\tau = \dfrac{\mu_0 q^2}{6\pi mc}$ is the characteristic time and $\vec{F}_{ext}$ is the external force acting to the particle [4]. For an electron, $\tau = 6.26 \times 10^{-24}s$. It means that the characteristic time implies that the recoil force from radiation fields effect the motion of a charged particle under this short-time scale [2,4].

When we considered an extended point charged particle to be a spherical shell charged particle with radius R and, later, took the limit $R \to 0$ back to the point charged particle in consideration. The equation of motion of this charged spherical shell which accounts its radiation fields can be written in the formula [4–6]

$$\left(1 - \frac{c\tau}{R}\right)\frac{d\vec{u}}{dt} - \frac{c^2\tau}{2R^2}\left[\vec{u}(t - 2R/c) - \vec{u}(t)\right] = \frac{\vec{F}_{ext}}{m}. \tag{5}$$

We solved this equation by the Fourier transform method. Let $\tilde{\vec{u}}(\omega)$ is the transformed velocity which is expressed as

$$\tilde{\vec{u}}(\omega) = \int_{-\infty}^{\infty} \vec{u}(t)e^{-i\omega t}dt. \tag{6}$$

We put this equation to the equation of motion, then the particle velocity can be expressed in the formula

$$\vec{u}(t) = \frac{1}{2\pi m}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\vec{F}_{ext}(t')e^{-i\omega t'}dt'}{i\omega(1 - c\tau/R) + (c^2\tau/2R^2)[1 - e^{-2i\omega R/c}]}e^{i\omega t}d\omega. \tag{7}$$

From this velocity, it can be found out that the acceleration of finite spherical shell charged particle with the limit $R \to 0$ brings to the Landau-Lifshitz equation [3–6]

$$\vec{a} = \frac{\vec{F}_{ext}}{m} + \frac{\tau\dot{\vec{F}}_{ext}}{m}. \tag{8}$$

This equation shows that the acceleration of a particle depends on an external force and its time derivative.

2. The solution of Abraham-Lorentz and Landau-Lifshitz equations under the external rectangular force

Now we are interested in finding and comparing the solutions of the Abraham-Lorentz and Landau-Lifshitz equations in the case of the external rectangular force in one dimension because this is simple case for consideration

$$F_{ext}(t) = \begin{cases} 0, & (t \leq 0) \\ F, & (0 < t < T) \\ 0, & (t \geq T). \end{cases} \tag{9}$$

In the case of the Abraham-Lorentz equation the acceleration of a particle has taken the form [4]

$$a(t) = \begin{cases} \frac{F}{m}(1 - e^{T/\tau})e^{t/\tau}, & (t \leq 0) \\ \frac{F}{m}(1 - e^{T/\tau}), & (0 < t < T) \\ 0, & (t \geq T). \end{cases} \tag{10}$$

Next, we solved the velocity and distance from this acceleration which has the form,

$$u(t) = \begin{cases} \frac{F\tau}{m}(1 - e^{T/\tau})e^{t/\tau}, & (t \leq 0) \\ \frac{F}{m}(t + \tau - \tau e^{(t-T)/\tau})), & (0 < t < T) \\ \frac{FT}{m}, & (t \geq T), \end{cases} \tag{11}$$

$$x(t) = \begin{cases} \frac{F\tau^2}{m}(1 - e^{T/\tau})e^{t/\tau} + x_0, & (t \leq 0) \\ \frac{F}{m}(\frac{t^2}{2} + \tau t - \tau^2 e^{(t-T)/\tau}), & (0 < t < T) \\ x_{02}(t - T), & (t \geq T). \end{cases} \tag{12}$$

The acceleration from the Landau-Lifshitz equation for the finite spherical shell charged particle has the form [4]

$$a(t) = \begin{cases} 0, & (t \leq 0) \\ F/m, & (0 < t < T) \\ 0, & (t \geq T). \end{cases} \tag{13}$$

Now we considered to solve this equation for each time interval. In the interval $t \leq 0$ we defined the initial condition $u_1(t = 0) = u_{01}$, so the solution for this interval is

$$u_1(t) = u_{01}.$$

Accordingly, in the intervals $0 < t < T$ and $t \geq T$ we chose the initial conditions $u_2(0) = u_{01}$ and $u_3(0) = u_{02}$, therefore the velocity is written as

$$u(t) = \begin{cases} u_{01}, & (t \leq 0) \\ u_{01} + \dfrac{Ft}{m}, & (0 < t < T) \\ u_{02}, & (t \geq T). \end{cases} \tag{14}$$

Finally, the solution of coordinate in all time interval is expressed in the form,

$$x(t) = \begin{cases} u_{01}t, & (t \leq 0) \\ u_{01}t + \dfrac{F}{2m}t^2, & (0 < t < T) \\ u_{02}(t - T), & (t \geq T). \end{cases} \tag{15}$$

Exactly, in the case of the rectangular force when $t \leq 0$, the acceleration of a point charged particle is equal to $F/m(1 - e^{T/\tau})e^{t/\tau}$. We saw that the acceleration from the Abraham-Lorentz equation causes one of the unphysical problems called the preacceleration. It implies that the acceleration occurs before the force acts at a point charged particle. Because there is no force acting on the charged particle in the time interval $t \leq 0$ but the acceleration appears. This problem is the contradiction of the principle of causality in the theory of relativity.

Consequently, we extended from the point charged to the finite spherical shell charged particle and its acceleration which was solved from the Landau-Lifshitz equation can exclude the preacceleration problem. Obviously, the solution of the Landau-Lifshitz equation had shown that in the time interval $t \leq 0$ there was no force acting on the charged particle and the acceleration did not appear. The acceleration appeared in the interval $0 < t < T$ because the constant force F was taking to the charged particle. Then the acceleration was causally connected with the force acting on the finite spherical shell limit. Obviously, the velocity and the distance was solved from the Landau-Lifshitz equation also accorded with the condition of causality because they satisfied the acceleration solution in all time intervals. But the velocity and distance did not causally match to the acceleration in the case of the Abraham-Lorentz equation.

3. Conclusion

In this work, we had shown that the unphysical problem of the solution of Abraham-Lorentz equation can be fixed by changing the model of the point charged particle to the model of the finite spherical shell charged particle. This is the reason why we use the Landau-Lifshitz equation to describe the motion of the charged particle without the preacceleration problem.

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