Inhomogeneous initial conditions and the start of inflation

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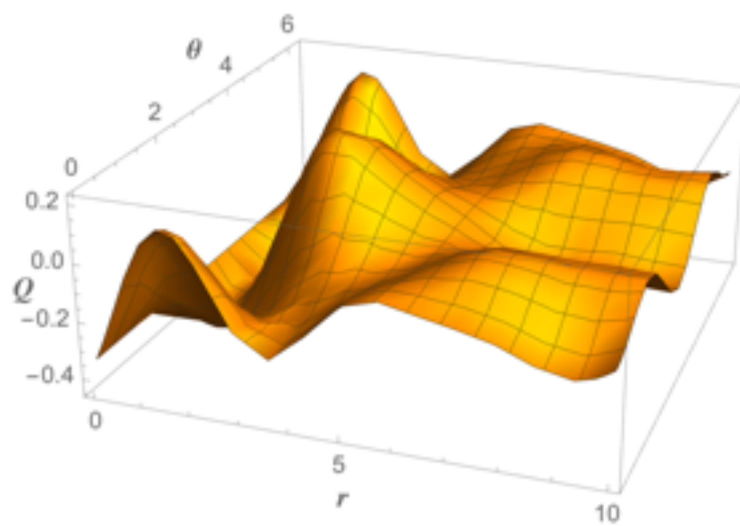

Can a really lumpy spacetime with inhomogeneities on length scales around and well within the Hubble radius, when we include the effects of nonlinear back-reaction, nonetheless flow into inflation?

with David Kaiser
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Initial conditions problem

- Inflation explains high degree of flatness and homogeneity observed today in our universe at horizon scales
- criticism: in order to begin inflation may require homogeneity over many Hubble volumes
- if inflationary expansion fails to begin under sufficiently inhomogeneous initial conditions, such that inflation requires fine-tuning of its initial state to occur, then its naturalness is challenged.
- we study this problem using a well defined set of nonlinear interactions in the Hartree approximation
- our results are consistent with recent simulations in full (3+1) numerical relativity (e.g. East et al. 2016, Clough et al. 2017)…
- however, by using the Hartree approximation to study certain nonlinear interactions, our numerical approach can be applied more efficiently to a wide range of models tracking the evolution of perturbations across a wide range of scales.

Linearized perturbations around FRW spacetime

- single-field models with minimal couplings to gravity and canonical kinetic term

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{\rm pl}^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V\left(\phi\right)\right]$$

- expand both scalar field and spacetime metric to first order around their background values

$$\phi\left(x^\mu\right) = \phi\left(t\right) + \delta\phi\left(x^\mu\right) \qquad g_{\mu\nu}\left(x^\lambda\right) + h_{\mu\nu}\left(x^\lambda\right)$$

$$ds^2 = -\left(1+2A\right)dt^2 + 2a\left(\partial_i B\right)dtdx^i + a^2\left[\left(1-2\psi\right)h_{ij} + 2\partial_i\partial_j E\right]dx^i dx^j$$

- parameterize background FRW spatial sections as

$$h_{ij}dx^i dx^j = d\chi^2 + S_K^2\left(\chi\right)d\Omega_{(2)}^2$$

$$d\Omega_{(2)}^2 = d\theta^2 + \sin^2\theta d\phi^2$$

$$\chi \equiv \int_0^r \frac{dr'}{\sqrt{1-Kr'^2}} = \begin{cases} \arcsin r & K=1 \\ r & K=0 \\ \mathrm{arcsinh} r & K=-1 \end{cases}$$

$$S_K^2\left(\chi\right) \equiv \begin{cases} \sin\chi & K=1 \\ \chi & K=0 \\ \sinh\chi & K=-1 \end{cases}$$

- to first order in metric perturbations we may form the gauge-invariant Bardeen potentials:

$$\Phi \equiv A - \partial_t\left[a^2\left(\dot{E} - \frac{B}{a}\right)\right] \qquad \Psi \equiv \psi + a^2 H\left(\dot{E} - \frac{B}{a}\right)$$

Linearized perturbations around FRW spacetime

- Longitudinal gauge: $E = B = 0$

  $\Phi = A \quad \Psi = \psi$

- single-field models with minimal couplings, to linear order in perturbations no anisotropic pressure in stress-energy tensor $\rightarrow \Phi = \Psi$

- we consider linearized perturbations to Einstein's field equations

$$\bar{G}_{\mu\nu} + \delta G_{\mu\nu} = \frac{1}{M_{\rm pl}^2}\left[\bar{T}_{\mu\nu} + \delta T_{\mu\nu}\right]$$

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V\left(\phi\right)\right]$$

- together with the Euler-Lagrange equation of motion for the field, obtain the linearized equations of motion to first order in $\delta\phi \;\; \Psi$

- expand spatially varying quantities in comoving Fourier modes k …

Mode expansion

- typically we study behavior of field and metric perturbation modes after inflation has persisted for several e-folds so the universe has become spatially homogeneous and isotropic to a high degree of accuracy

$$\mathcal{R}_{\mathbf{k}} \to q_k\left(t\right) Z_{\mathbf{k}}\left(\mathbf{x}\right)$$

- We are interested in the behavior of perturbations before inflation has begun, we do not assume spatial homogeneity and isotropy to begin with

- we expand our perturbations in eigenfunctions of the comoving spatial Laplacian

$$\mathcal{R}\left(x^\mu\right) = \int_0^\infty dk \sum_{lm} \mathcal{R}_{klm}\left(x^\mu\right)$$

$$\nabla^2 \mathcal{R}_{\mathbf{k}} \equiv \frac{1}{\sqrt{h}}\partial_i\left[\sqrt{h}h^{ij}\partial_j\mathcal{R}\right] = -k^2\mathcal{R}_{\mathbf{k}}$$

- solutions:

$$\mathcal{R}_{\mathbf{k}}\left(x^\mu\right) = q_{klm}\left(t\right) J_{kl}\left(r\right) Y_{lm}\left(\theta,\phi\right)$$

$$J_{lk}\left(\chi\right) = \begin{cases} N_{klm}\frac{1}{\sqrt{\sin\chi}}P^{-l-1/2}_{\sqrt{k^2+1}-1/2}\left(\cos\chi\right) & K=1 \\ \sqrt{\frac{2}{\pi}}kj_l\left(kr\right) & K=0 \\ M_{klm}\frac{1}{\sqrt{\sinh\chi}}P^{-l-1/2}_{-i\sqrt{k^2-1}-1/2}\left(\cosh\chi\right) & K=-1 \end{cases}$$

$$\int_0^\infty \sqrt{h}d\chi \int_0^\pi d\theta \int_0^{2\pi} d\phi Z^*_{klm}\left(\chi,\theta,\phi\right) Z_{k',l',m'}\left(\chi,\theta,\phi\right) = \delta\left(k-k'\right)\delta_{ll'}\delta_{mm'}$$

Linearized equations of motion

- Linearized equations of motion for $\phi(t) \quad a(t) \quad \delta\phi_{ki} \quad \Psi_{ki}$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V(\phi_{\mathrm{cl}})}{\partial \phi} = 0$$

$$H^2 = \frac{1}{3M_{\mathrm{pl}}^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) - \frac{K}{a^2}$$

$$\delta\ddot{\phi}_{\mathbf{k}} + 3H\delta\dot{\phi}_{\mathbf{k}} + \left(\frac{k^2}{a^2} + \frac{\partial^2 V(\phi_{\mathrm{cl}})}{\partial \phi^2}\right)\delta\phi_{\mathbf{k}} = -2\frac{\partial V(\phi_{\mathrm{cl}})}{\partial \phi}\Psi_{\mathbf{k}} + 4\dot{\phi}\dot{\Psi}_{\mathbf{k}}$$

$$\dot{\Psi}_{\mathbf{k}} + H\Psi_{\mathbf{k}} = \frac{1}{2M_{pl}^2}\dot{\phi}\delta\phi_{\mathbf{k}}$$

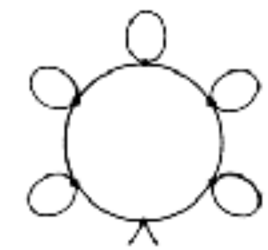
- along with the constraint (coming from 00 and 0i components of perturbed Einstein field equations

$$\left(\dot{H} + \frac{k^2}{a^2}\right)\Psi_{\mathbf{k}} = \frac{1}{2M_{\mathrm{pl}}^2}\left(\ddot{\phi}\delta\phi_{\mathbf{k}} - \dot{\phi}\delta\dot{\phi}_{\mathbf{k}}\right)$$

- Does not capture nonlinear effects of back-reaction

Hartree corrections

- Hartree approximation incorporates certain nonlinear structure and gravitational back-reaction for the self-coupled system.
- nonperturbative approximation obtained by resumming an infinite set of Feynman diagrams of a particular class ("daisy" diagrams) to all orders (Dolan and Jackiw 1974)



- incorporate $\mathcal{O}(\hbar)$ corrections to EOM for ϕ …

$$V_{,\phi}(\phi) \to V_{,\phi}(\phi) + \frac{1}{2}V^{(3)}(\phi)(\delta\phi)^2 + \mathcal{O}\left(\hbar^{3/2}\right)$$

- …and for $\delta\phi_k$

$$V_{,\phi\phi}\delta\phi \to V_{,\phi\phi}(\phi)\,\delta\phi + \frac{1}{6}V^{(4)}(\phi)(\delta\phi)^3 + \mathcal{O}\left(\hbar^2\right)$$

- implemented by substitutions among nonlinear terms involving $\delta\phi$ in the equations of motion

$$\left(\delta\hat{\phi}\right)^2 \to \left\langle \left(\delta\hat{\phi}\right)^2 \right\rangle \to \sum_k |\delta\phi_{k00}|^2$$

$$\left(\delta\hat{\phi}\right)^3 \to 3\left\langle \left(\delta\hat{\phi}\right)^2 \right\rangle \delta\phi$$

$$\left\langle \left(\delta\hat{\phi}\right)^2 \right\rangle = \langle 0|\,\delta\hat{\phi}\,(x^\mu)\,\delta\,(y^\mu)\,|0\rangle\,|_{x^\mu \to y^\mu}$$

$$\left(\delta\hat{\phi}\right)^4 \to 6\left\langle \left(\delta\hat{\phi}\right)^2 \right\rangle (\delta\phi)^2 - 3\left\langle \left(\delta\hat{\phi}\right)^2 \right\rangle^2$$

- operates like a mean-field approximation: spherically symmetric in k-space

Equations of motion with Hartree corrections

- incorporate $\mathcal{O}\left(\hbar\right)$ corrections to EOM for ϕ $\delta\phi$ and make Hartree approximation

$$\ddot{\phi}+3H\dot{\phi}+\frac{\partial V\left(\phi_{\rm cl}\right)}{\partial\phi}+\frac{1}{2}\frac{\partial^3 V\left(\phi_{cl}\right)}{\partial\phi^3}\left\langle\left(\delta\hat{\phi}\right)^2\right\rangle=0$$

$$H^2=\frac{1}{3M_{\rm pl}^2}\left(\frac{1}{2}\dot{\phi}^2+V\left(\phi\right)+\frac{1}{2}\left\langle\left(\delta\dot{\hat{\phi}}\right)^2\right\rangle+\frac{1}{2a^2}h^{ij}\left\langle\partial_i\delta\hat{\phi}\partial_j\delta\hat{\phi}\right\rangle+\frac{1}{2}\frac{\partial^2 V\left(\phi_{cl}\right)}{\partial\phi^2}\left\langle\left(\delta\hat{\phi}\right)^2\right\rangle\right)-\frac{K}{a^2}$$

$$\delta\ddot{\phi}_{\mathbf{k}}+3H\delta\dot{\phi}_{\mathbf{k}}+\left(\frac{k^2}{a^2}+\frac{\partial^2 V\left(\phi_{\rm cl}\right)}{\partial\phi^2}+\frac{1}{2}\frac{\partial^4 V\left(\phi_{cl}\right)}{\partial\phi^4}\left\langle\left(\delta\hat{\phi}\right)^2\right\rangle\right)\delta\phi_{\mathbf{k}}=-2\left(\frac{\partial V\left(\phi_{\rm cl}\right)}{\partial\phi}+\frac{1}{2}\frac{\partial^3 V}{\partial\phi^3}\left\langle\left(\delta\hat{\phi}\right)^2\right\rangle\right)\Psi_{\mathbf{k}}+4\dot{\phi}$$

$$\dot{\Psi}_{\mathbf{k}}+H\Psi_{\mathbf{k}}=\frac{1}{2M_{pl}^2}\dot{\phi}\delta\phi_{\mathbf{k}}$$

- along with the constraint

$$\left(\dot{H}+\frac{k^2}{a^2}\right)\Psi_{\mathbf{k}}=\frac{1}{2M_{\rm pl}^2}\left(\ddot{\phi}\delta\phi_{\mathbf{k}}-\dot{\phi}\delta\dot{\phi}_{\mathbf{k}}\right)$$

- expanded:

$$\left\langle\left(\delta\hat{\phi}\right)^2\right\rangle\to\sum_k\left|\delta\phi_{k00}\right|^2$$

$$\left\langle\left(\delta\dot{\hat{\phi}}\right)^2\right\rangle\to\sum_k\left|\delta\dot{\phi}_{k00}\right|^2$$

$$\frac{1}{a^2}h^{ij}\left\langle\partial_i\delta\hat{\phi}\partial_j\delta\hat{\phi}\right\rangle\to\frac{k^2}{a^2}\left|\delta\phi_{k00}\right|^2$$

Numerical calculation for $V(\phi) = \lambda\phi^4$

- scenarios in which $\rho(t_0) + \delta\rho(t_0) \sim M_{\rm pl}^4 \qquad \delta\rho(t_0) \leq \rho(t_0)$

- significant inhomogeneities on length-scales around and within the (initial) Hubble radius:

$$k_1 = 10^{-1} H_0$$
$$k_2 = 10^{-1/2} H_0$$
$$k_3 = 10^{0} H_0$$
$$k_4 = 10^{1/2} H_0$$
$$k_5 = 10^{1} H_0$$
$$k_6 = 10^{3/2} H_0$$

- parameterize initial mode amplitudes (away from Bunch-Davies):

$$\delta\phi_{klm} = R_{klm} \exp(-k/\kappa) M_{\rm pl} \qquad \delta\dot{\phi}_{klm}(t_0) = D_{klm} M_{\rm pl}^2$$

$$R_{klm} \in \{-1, 1\} \qquad \kappa \to \max\left[\delta\rho(t_0)\right]$$

- track each mode $\delta\phi_k$ Ψ_k coupled to background

- track evolution into onset of inflation $\epsilon(t_{\rm start}) < 1$ through end of inflation $\epsilon(t_{\rm end}) = 1$

$$\epsilon \equiv -\frac{\dot{H}}{H^2}$$

- record onset of inflation and $N \equiv \int_{t_0}^{t_{\rm end}} H(t)\, dt$ over range of initial conditions

Results

- structure within the Hubble radius is rapidly being damped out, producing smooth patch on Hubble-radius scales

k1=0.0816496580928
k2=0.258198889747
k3=0.816496580928
k4=2.58198889747
k5=8.16496580928
k6=25.8198889747

$\phi(t_0) = 23.7$

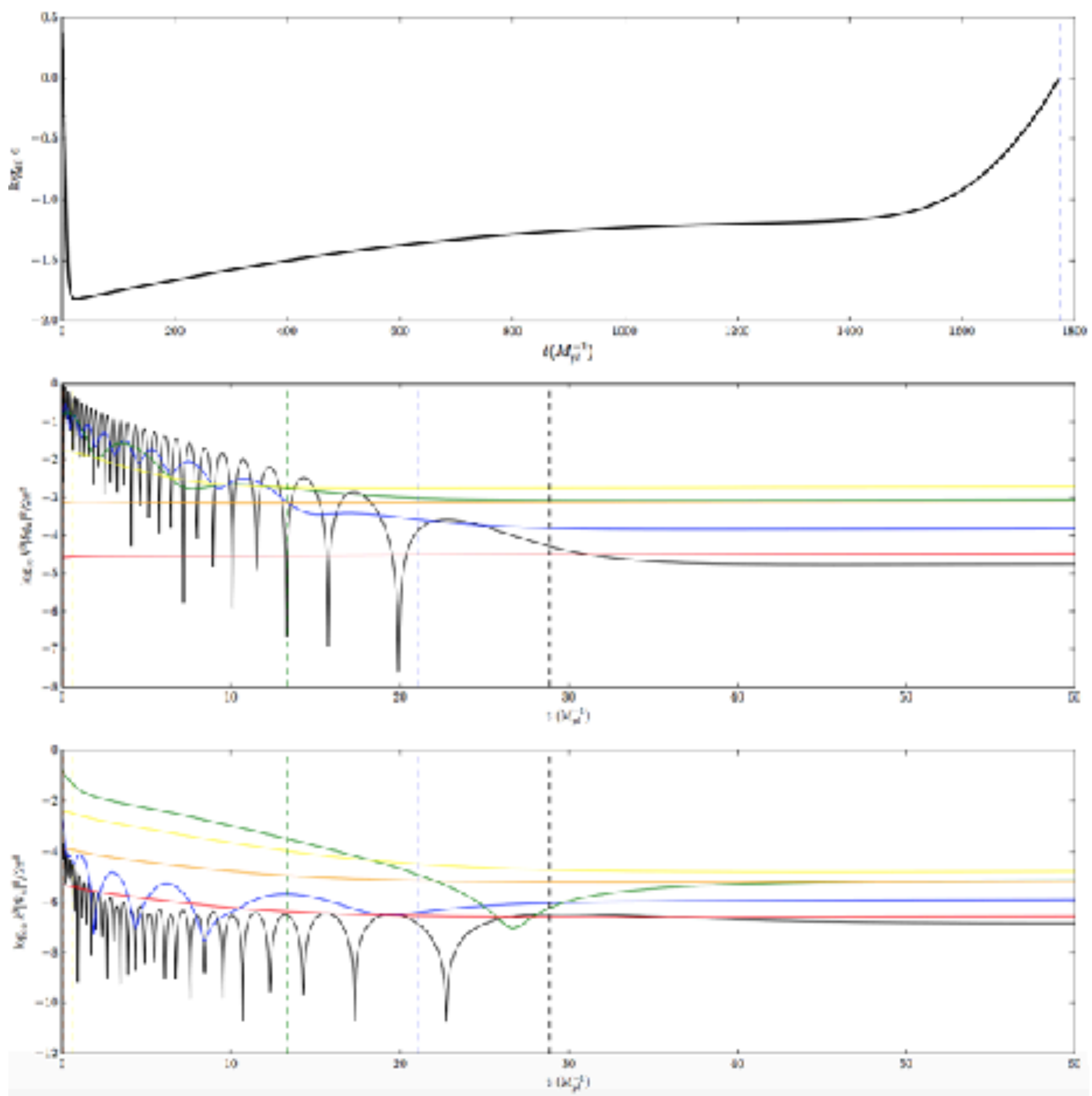
$\dot{\phi}(t_0) = -1.0$

$K = 0$

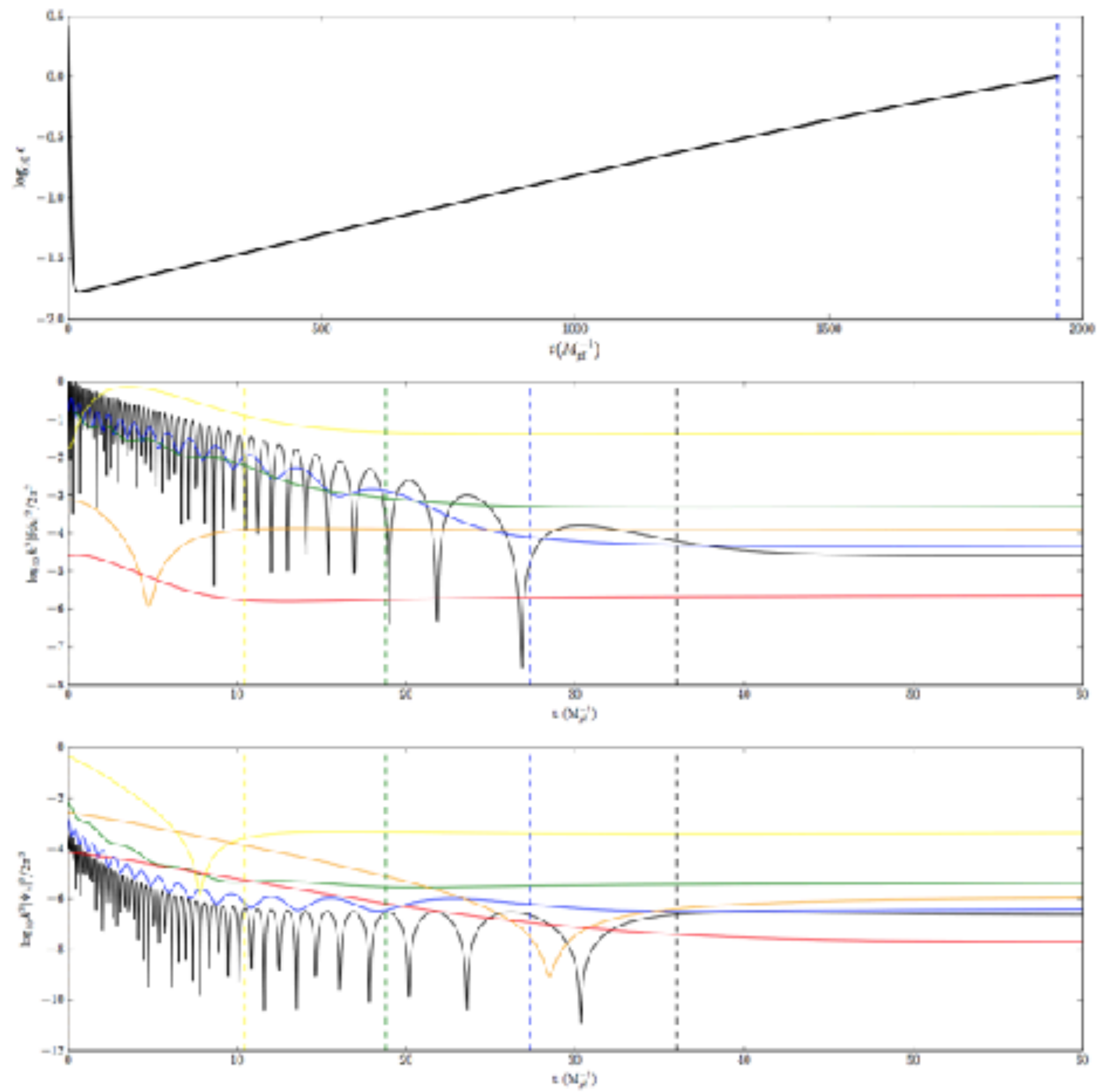
$\lambda = 10^{-6}$

$\kappa = 3.1 \rightarrow \delta\rho \leq M_{\rm pl}^4$

Hartree corrections: Nef=66



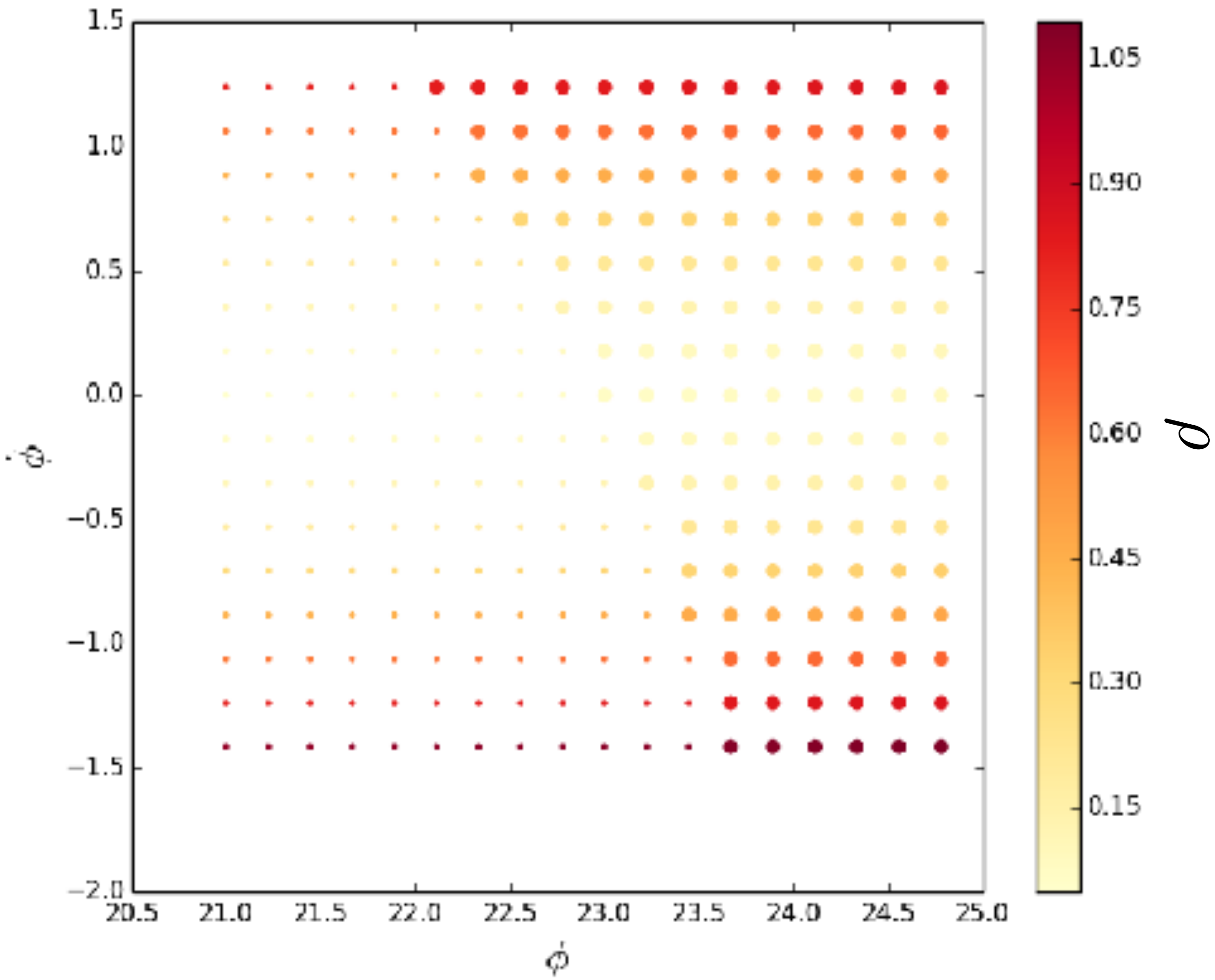
no Hartree corrections: Nef=63



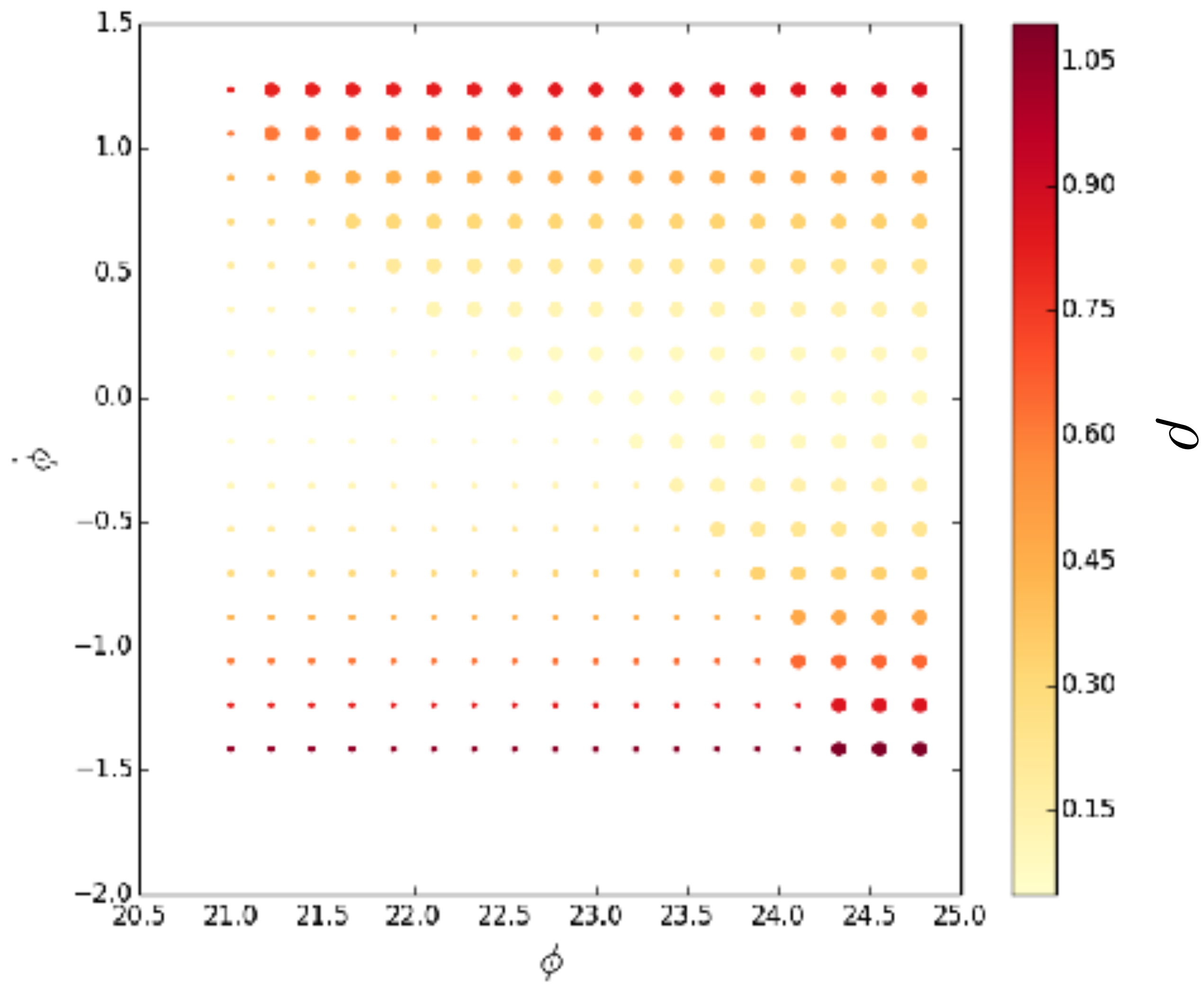
Results: phase space

- nontrivial adjustments to expected number of efolds at large positive and negative initial field velocity
- large-field inflation in a simple potential like lambda-phi-4 robust in the face of significant inhomogeneities

Hartree corrections



no Hartree corrections



Conclusions

- including Hartree corrections, metric perturbations that begin inside Hubble radius fall rapidly in amplitude (structure within the Hubble radius is rapidly being damped out, producing smooth patch on Hubble-radius scales)
- system still finds inflationary attractor, even in the face of significant initial lumpiness
- in phase space plots, there are nontrivial adjustments to expected number of efolds at large positive and negative initial field velocity,
- but in general, large-field inflation in a simple potential like lambda-phi-4 appears robust in the face of significant inhomogeneities
- our computationally simpler and more efficient approach (compared to numerical relativity studies) allows us to explore the initial conditions problem for more models over a wide range of scales