Are Neutrino Masses Modular Forms?

Bale, July 5, 2018

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Based on:
- F.F. 1706.08749
- Juan Carlos Criado, F.F., 1807.01125

Precision Era for Neutrino Physics

	IO	NO	
$r \equiv \|\Delta m^2_{sol}/\Delta m^2_{atm}\|$	0.0301(8)	0.0299(8)	2.7%
$\sin^2\theta_{12}$	0.303(13)	0.304(13)	4.3%
$\sin^2\theta_{13}$	0.0218(8)	0.0214(8)	3.7%
$\sin^2\theta_{23}$	0.56(3)	0.55(3)	5.4%
δ/π	1.52(14)	1.32(19)	≈ 10%

[Capozzi et al. 1804.09678]

independent global fits:
de Salas, Gariazzo, Mena, Ternes , Tortola, 1806.11051,
Gariazzo, Archidiacono, de Salas, Mena, Ternes, Tortola, 1801.04946
de Salas, Forero, Ternes, Tortola, J. W. F. Valle, 1708.01186
Esteban, Gonzalez-Garcia, Maltoni, Martinez-Soler,Schwetz, 1611.01514

NO preferred over the IO

stimulating time for
for models of neutrino masses
and mixing angles.

$y_e(m_Z)$	$2.794745(16) \times 10^{-6}$
$y_\mu(m_Z)$	$5.899863(19) \times 10^{-4}$
$y_\tau(m_Z)$	$1.002950(91) \times 10^{-2}$

[Antusch and Maurer 1306.6879]

Symmetry approach

One of the few tools we have, but with several obstacles

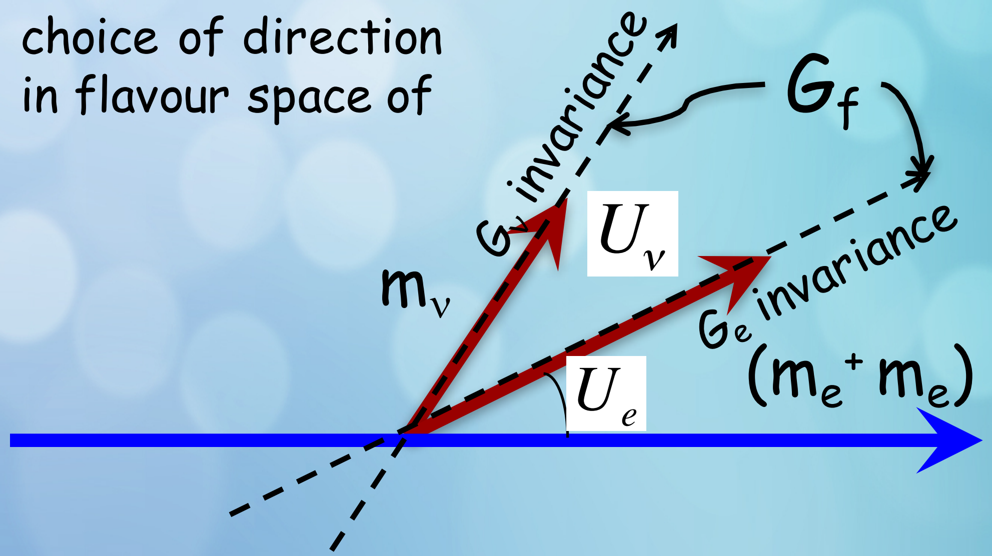
reviews:
Ishimori, Kobayashi, Ohki, Shimizu, Okada, Tanimoto , 1003.3552;
King, Luhn, 1301.1340;
King, Merle, Morisi, Shimizu, Tanimoto, 1402.4271;
King, 1701.04413
Hagedorn, 1705.00684;
.
.
.

1. predictability

high number of free parameters
- Lowest order Lagrangian parameters
- complicated SB sector
- higher dimensional operators
- SUSY breaking effects
- RGE corrections (Λ_{UV}, m_{SUSY} , $\tan\beta$)

2. vacuum alignment

choice of direction in flavour space of



dynamically selected ?
(minimum of energy density)
…
by hand ?
…
anthropic selection ?

This proposal [1706.08749]

a) neutrino masses and mixings depend on a small number of fields [here (τ,φ)] ➡ $m_{ij}(\tau,\varphi)$

b) dependence of m_{ij} on (τ,φ) is holomorphic ➡ supersymmetric model

c) flavour symmetry acts non-linearly [to determine all higher dimensional operators] ➡ $\begin{cases} \tau \to F(\tau) \\ \varphi \to G(\tau,\varphi) \end{cases}$

a) + b) + c) ➡ the functional form of $m_{ij}(\tau,\varphi)$ is completely determined

d) the VEVs (τ,φ) are selected by some unknown mechanism

Here: a) + b) + c) from modular invariance as flavour symmetry

some results [Juan Carlos Criado, F.F.]

$$w_\nu = -\frac{1}{\Lambda}(H_u L)^T \mathcal{W} (H_u L) + \dots$$

$$\mathcal{W} = \begin{cases} \mathcal{C} & \texttt{Model 1} \\ \frac{1}{2}\left(\mathcal{Y}_\nu^T \mathcal{C}^{-1} \mathcal{Y}_\nu\right) & \texttt{Model 2} \end{cases}$$

$$\mathcal{Y}_\nu = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathcal{C} = \begin{pmatrix} 2Y_1(\tau) & -Y_3(\tau) & -Y_2(\tau) \\ -Y_3(\tau) & 2Y_2(\tau) & -Y_1(\tau) \\ -Y_2(\tau) & -Y_1(\tau) & 2Y_3(\tau) \end{pmatrix}$$

$(Y_1(\tau), Y_2(\tau), Y_3(\tau))$
completely determined up to a common overall constant, to any order in τ

contribution to lepton mixing from charged lepton sector

$$U_e = \begin{pmatrix} 1 & 0 & \varphi_3 \\ 0 & 1 & -\varphi_3 \\ -\varphi_3 & \varphi_3 & 1 \end{pmatrix} P_{23} + \dots$$ terms of $O\left(\varphi_3^2, \frac{m_e^2}{m_\mu^2}\varphi_3, \frac{m_\mu^2}{m_\tau^2}\varphi_3\right)$

$\varphi_3 \approx 0.1$

⇒ all dimensionless neutrino data are determined in terms of 3 vacuum parameters
no corrections to superpotential in the exact SUSY limit

Fit to Model 1

	best value	pull
$r \equiv \lvert\Delta m^2_{sol}/\Delta m^2_{atm}\rvert$	$0.0302(11)$	$+0.13$
m_3/m_2	$0.0150(5)$	$-$
$\sin^2\theta_{12}$	$0.304(17)$	$+0.08$
$\sin^2\theta_{13}$	$0.0217(8)$	-0.13
$\sin^2\theta_{23}$	$0.577(4)$	$+0.67$
δ/π	$1.529(3)$	$+0.07$
α_{21}/π	$0.135(6)$	$-$
α_{31}/π	$1.728(18)$	$-$

Inverted mass Ordering

- no SUSY breaking effects
- no RGE corrections

best fit parameters

τ	$0.0117 + i\ 0.9948$
φ_3	-0.086

close to the self-dual critical point

$\chi^2_{min} = 0.4$

8 dimensionless physical quantities independent on any coupling constant!

$$m_1 = 4.90(3) \times 10^{-2}\text{eV} \quad , \quad m_2 = 4.98(2) \times 10^{-2}\text{eV} \quad , \quad m_3 = 7.5(3) \times 10^{-4}\text{eV}$$

$$|m_{ee}| = 4.73(4) \times 10^{-2}\text{eV}$$

by reproducing individually Δm^2_{sol} and Δm^2_{atm}

Fit to Model 2

	best value	pull
$r \equiv \|\Delta m^2_{sol}/\Delta m^2_{atm}\|$	$0.0299(12)$	0.0
m_3/m_2	$3.68(5)$	$-$
$\sin^2\theta_{12}$	$0.306(11)$	$+0.15$
$\sin^2\theta_{13}$	$0.0211(12)$	-0.42
$\sin^2\theta_{23}$	$0.459(5)$	-3.04
δ/π	$1.438(8)$	$+0.62$
α_{21}/π	$1.704(5)$	$-$
α_{31}/π	$1.201(16)$	$-$

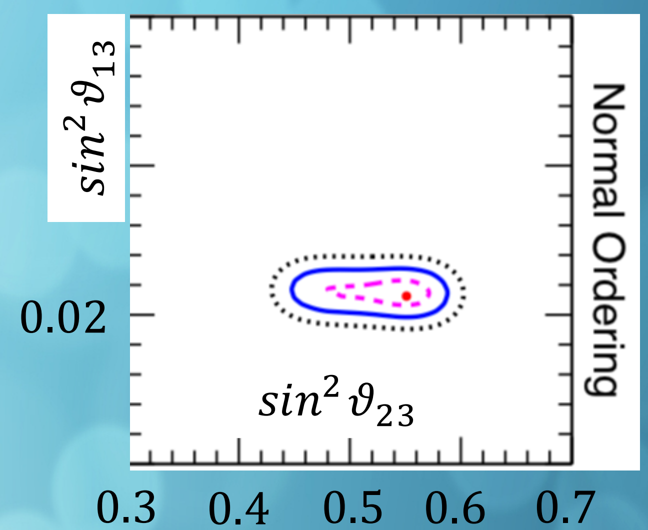
- no SUSY breaking effects
- no RGE corrections

best fit parameters

τ	$-0.2005 + i\ 1.0578$
φ_3	0.117

$\chi^2_{min} = 9.9$

Normal mass Ordering



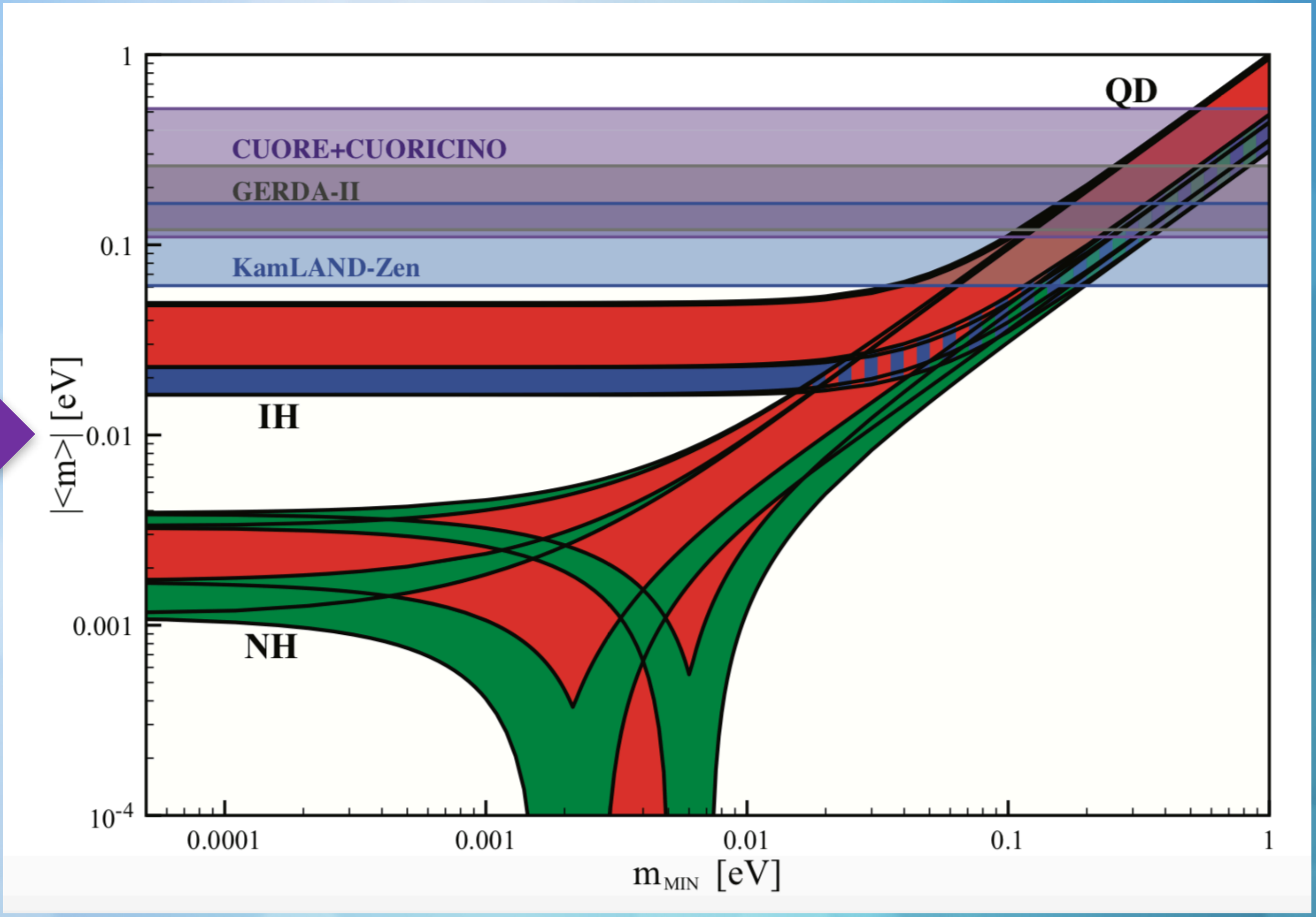
[Capozzi et al. 1804.09678]

$$m_1 = 1.09(3)\times 10^{-2}\text{eV} \quad , \quad m_2 = 1.39(2)\times 10^{-2}\text{eV} \quad , \quad m_3 = 5.11(4)\times 10^{-2}\text{eV}$$

$$|m_{ee}| = 1.04(2)\times 10^{-2}\text{eV}$$

by reproducing individually Δm^2_{sol} and Δm^2_{atm}

[from pdg 2017]



Modular Invariance as Flavour Symmetry

modular transformations

$$\tau \to \gamma\tau \equiv \frac{a\tau + b}{c\tau + d}$$

a,b,c,d integers
ad-bc=1

τ is a complex field,
Im(τ) > 0

they form the (discrete, infinite) modular group $\overline{\Gamma}$ generated by

$$S : \tau \to -\frac{1}{\tau} \quad , \qquad T : \tau \to \tau + 1$$

$$S^2 = \mathbb{1} \quad , \quad (ST)^3 = \mathbb{1}$$

duality (S) — discrete shift symmetry (T)

most general transformation on a set of $\mathcal{N}$=1 SUSY chiral multiplets $\varphi^{(I)}$

$$\begin{cases} \tau \to \gamma\tau \equiv \dfrac{a\tau + b}{c\tau + d} \\ \varphi^{(I)} \to (c\tau + d)^{k_I} \rho^{(I)}(\gamma) \varphi^{(I)} \end{cases}$$

e.g.

$$\varphi^{(I)} = \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix}$$

k_I: the weight, a real number

$\rho^{(I)}$: unitary representation of the finite modular group

$$\Gamma_N \equiv \overline{\Gamma}/\overline{\Gamma}(N)$$

$\mathcal{N}$=1 SUSY modular invariant theories

focus on Yukawa interactions and $\mathcal{N}$=1 global SUSY
[extension to $\mathcal{N}$=1 SUGRA straightforward]

$$\mathcal{S} = \int d^4x d^2\theta d^2\bar\theta\ K(\Phi,\bar\Phi) + \int d^4x d^2\theta\ w(\Phi) + h.c.$$

$$\Phi = (\tau, \varphi)$$

S invariant if

$$\begin{cases} w(\Phi) \to w(\Phi) \\ K(\Phi,\bar\Phi) \to K(\Phi,\bar\Phi) + f(\Phi) + f(\bar\Phi) \end{cases}$$

- $$K(\Phi,\bar\Phi) = -h \log(-i\tau + i\bar\tau) + \sum_I (-i\tau + i\bar\tau)^{+k_I} |\varphi^{(I)}|^2$$

"minimal"
Kahler potential

- $$w(\Phi) = \sum_n Y_{I_1 \ldots I_n}(\tau) \varphi^{(I_1)} \ldots \varphi^{(I_n)}$$

field-dependent
Yukawa couplings

invariance of w(Φ) guaranteed by an holomorphic Y such that

$$Y_{I_1 \ldots I_n}(\gamma\tau) = (c\tau + d)^{k_Y(n)} \rho(\gamma)\ Y_{I_1 \ldots I_n}(\tau)$$

modular forms
of level N and weight k_Y

1. the weights sum to zero: $k_Y(n) + k_{I_1} + \cdots + k_{I_n} = 0$
2. the product $\rho \times \rho^{I_1} \times \ldots \times \rho^{I_n}$ contains an invariant singlet

Models 1 and 2 are based on Γ_3

Why Γ_3 ? Γ_3 is isomorphic to A_4, smallest group of the Γ_N series possessing a 3-dimensional irreducible representation

[Ma, Rajasekaran, 0106291
Babu, Ma, Valle 0206292]

[recent extensions to Γ_2 and Γ_4 in Kobayashi, Tanaka, Tatsuishi, 1803.10391;
Penedo, Petcov 1806.11040]

	(E_1^c, E_2^c, E_3^c)	N^c	L	H_d	H_u	φ
$SU(2)_L \times U(1)_Y$	$(1, +1)$	$(1, 0)$	$(2, -1/2)$	$(2, -1/2)$	$(2, +1/2)$	$(1, 0)$
$\Gamma_3 \equiv A_4$	$(1, 1'', 1')$	3	3	1	1	3
k_I	$(k_{E_1}, k_{E_2}, k_{E_3})$	k_N	k_L	k_d	k_u	k_φ

Table 1: Chiral supermultiplets, transformation properties and weights. Model 1 has no gauge singlets N^c.

	k_{E_i}	k_N	k_L	k_d	k_u	k_φ
Model 1	-2	$-$	-1	0	0	$+3$
Model 2	-4	-1	$+1$	0	0	$+3$

Table 2: Weights of chiral multiplets. Model 1 has no gauge singlets N^c.

modular invariance broken by

τ

$\varphi = (1, 0, \varphi_3)$

real

Modular forms of level 3 [1706.08749]

dimension of linear space $\mathcal{M}_k(\Gamma(3))$ is (k+1) , k > 0 even integer

- 3 linearly independent modular forms of level 3 and minimal weight $k_I = 2$

$$
\begin{aligned}
Y_1(\tau) &= \frac{i}{2\pi}\left[\frac{\eta'\left(\frac{\tau}{3}\right)}{\eta\left(\frac{\tau}{3}\right)} + \frac{\eta'\left(\frac{\tau+1}{3}\right)}{\eta\left(\frac{\tau+1}{3}\right)} + \frac{\eta'\left(\frac{\tau+2}{3}\right)}{\eta\left(\frac{\tau+2}{3}\right)} - \frac{27\eta'(3\tau)}{\eta(3\tau)}\right] \\
Y_2(\tau) &= \frac{-i}{\pi}\left[\frac{\eta'\left(\frac{\tau}{3}\right)}{\eta\left(\frac{\tau}{3}\right)} + \omega^2\frac{\eta'\left(\frac{\tau+1}{3}\right)}{\eta\left(\frac{\tau+1}{3}\right)} + \omega\frac{\eta'\left(\frac{\tau+2}{3}\right)}{\eta\left(\frac{\tau+2}{3}\right)}\right] \\
Y_2(\tau) &= \frac{-i}{\pi}\left[\frac{\eta'\left(\frac{\tau}{3}\right)}{\eta\left(\frac{\tau}{3}\right)} + \omega\frac{\eta'\left(\frac{\tau+1}{3}\right)}{\eta\left(\frac{\tau+1}{3}\right)} + \omega^2\frac{\eta'\left(\frac{\tau+2}{3}\right)}{\eta\left(\frac{\tau+2}{3}\right)}\right] \quad .
\end{aligned}
$$

can be expressed in terms of the Dedekind eta function

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \qquad q \equiv e^{i2\pi\tau}$$

- they transform in a triplet 3 of Γ_3

$$Y(-1/\tau) = \tau^2\,\rho(S)Y(\tau) \qquad Y(\tau+1) = \rho(T)Y(\tau)$$

$$\rho(S) = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \qquad \rho(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \qquad \omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

- they generate the whole ring $\mathcal{M}(\Gamma(3))$

any modular form of level 3 and weight 2k can be written as an homogeneous polynomial in Y_i of degree k

Corrections from SUSY breaking

- unknown breaking mechanism. Here:

F-component of a chiral supermultiplet, gauge and modular invariant

$$X = \vartheta^2 F$$

messenger scale M

SUSY-breaking scale

$$m_{SUSY} = \frac{F}{M}$$

- most general correction term to lepton masses and mixing angles

$$\delta \mathcal{S} = \frac{1}{M^2} \int d^4x d^2\theta d^2\bar{\theta}\ X^\dagger\ f(\Phi, \bar{\Phi}) + h.c.$$

$f(\Phi, \bar{\Phi})$ has dimension 3, determined by gauge invariance and lepton number conservation (treating Λ as spurion with L=+2)

⇨ $$\delta \mathcal{W}/\mathcal{W} \approx \delta \mathcal{Y}/\mathcal{Y} \approx \frac{m_{SUSY}}{M}$$

tiny, if sufficient gap between m_{SUSY} and M

10^{-10} for $m_{SUSY} = 10^8$ GeV, $M = 10^{18}$ GeV

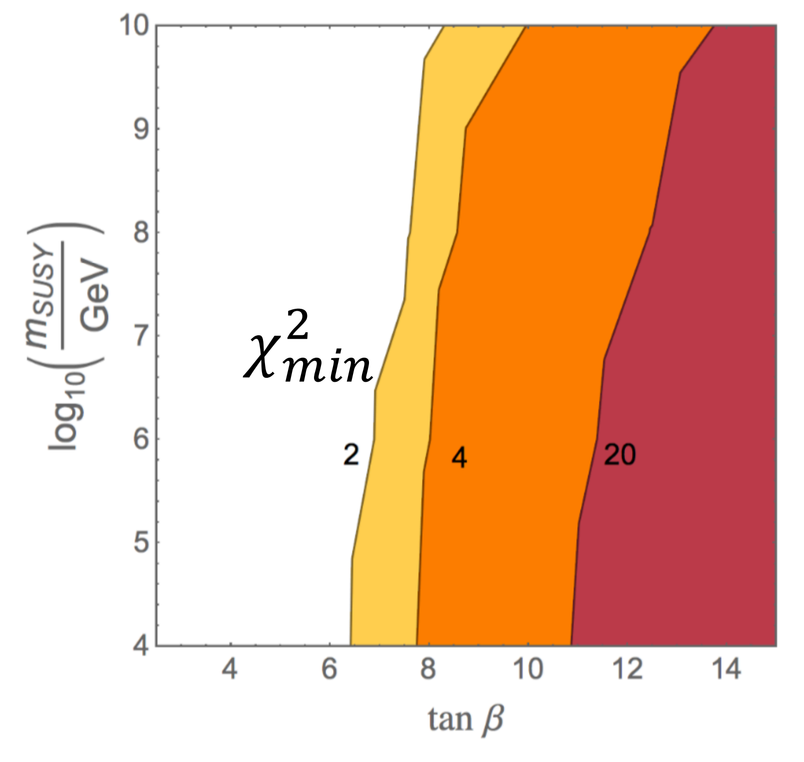
Corrections from RGE

Model 1 (IO)

- r and $sin^2\vartheta_{12}$ mostly affected, at large $tan\beta$

$\Lambda = 10^{15}$ GeV

m_{SUSY}	Quantity	$\tan\beta = 2.5$	$\tan\beta = 10$	$\tan\beta = 15$
10^4 GeV	r	0.0302	0.0292	0.0288
	$\sin^2\theta_{12}$	0.304	0.345	0.418
	χ^2_{min}	0.4	12.2	82.0
10^8 GeV	r	0.0302	0.0294	0.0286
	$\sin^2\theta_{12}$	0.303	0.335	0.389
	χ^2_{min}	0.4	7.0	47.7



Model 2 (NO)

negligible corrections for $tan\beta$ up to 25 and m_{SUSY} as low as 10^4 GeV

Conclusions

- Modular invariance provides strong constraints in SUSY model building: couplings of τ to matter multiplets are completely fixed to any order in the τ power expansion

- in a minimal model, level N=3, all dimensionless neutrino physical quantities independent on any coupling constant!
Once the vacuum is fixed they are all determined.

- Model 1 (IO) has an excellent χ^2_{min} ,
Model 2 (NO) has a reasonable χ^2_{min}

- accounting for SUSY breaking and RGE corrections, a finite region of the parameter space exists ($m_{SUSY} \ll M$ and $\tan\beta < 10$ for Model 1) where data and theory agree

- Open questions:
  - role of modular symmetry in charged lepton/quark Yukawa couplings
  - additional experimental tests to distinguish models (sum rules,…)
  - vacuum selection
  - …

Backup Slides

if we go minimal

	L	H_u	Y
$SU(2)\times U(1)$	$(2,-1/2)$	$(2,+1/2)$	$(1,0)$
$\Gamma_3 \equiv A_4$	3	1	3
k_I	+1	0	+2

the operator

$$w_\nu = \frac{1}{\Lambda}(H_u H_u \, LL \, Y)_1$$

is completely specified up to an overall constant

we get

$$m_\nu = \begin{pmatrix} 2Y_1 & -Y_3 & -Y_2 \\ -Y_3 & 2Y_2 & -Y_1 \\ -Y_2 & -Y_1 & 2Y_3 \end{pmatrix} \frac{v_u^2}{\Lambda}$$

a familiar matrix but now Y_i are determined by the choice of τ

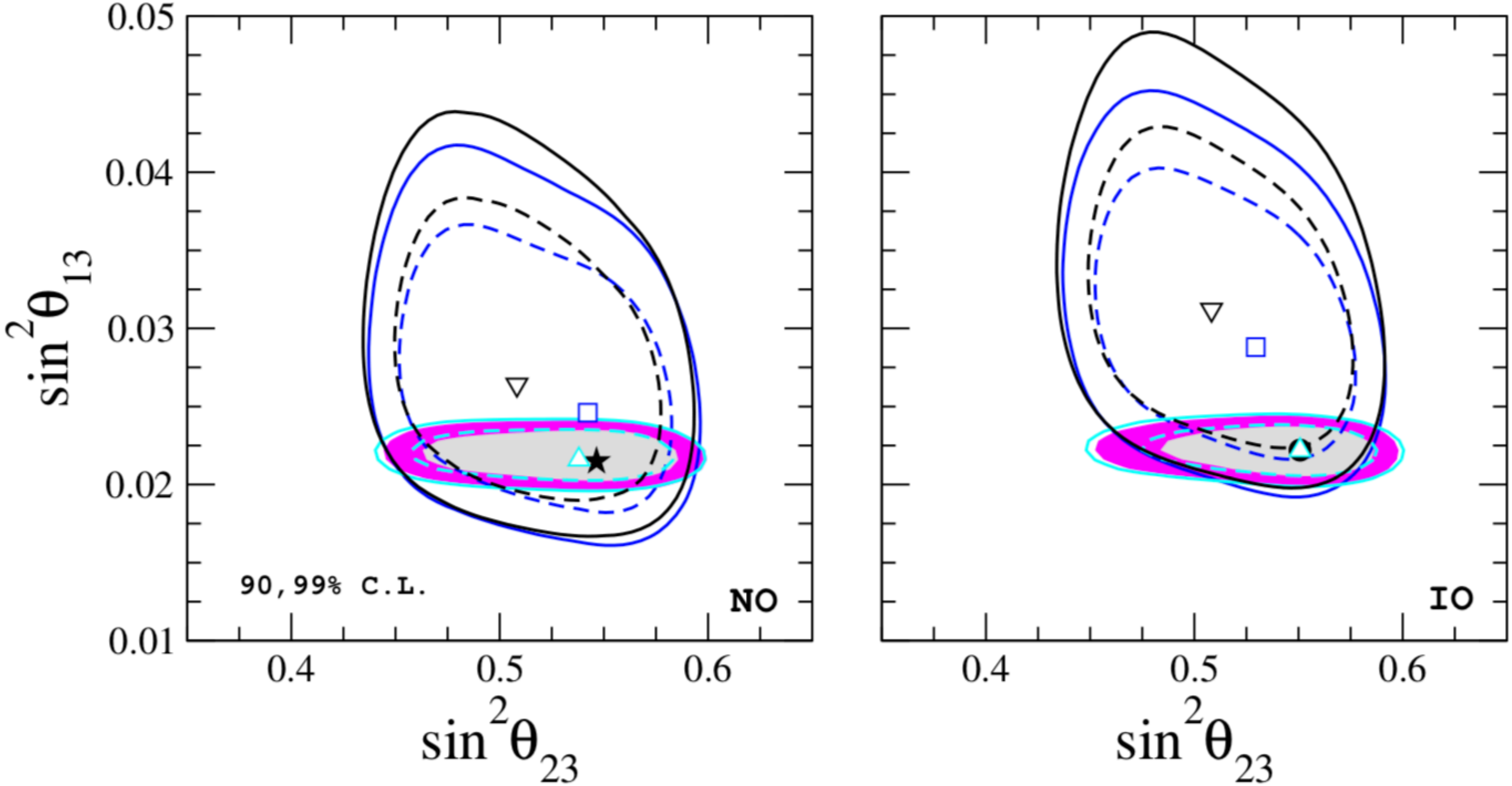
by scanning τ VEVs the best agreement is obtained for $\tau = 0.0111 + 0.9946i$

	$\frac{\Delta m^2_{sol}}{\Delta m^2_{atm}}$	$\sin^2\vartheta_{12}$	$\sin^2\vartheta_{13}$	$\sin^2\vartheta_{23}$	$\frac{\delta_{CP}}{\pi}$	$\frac{\alpha_{21}}{\pi}$	$\frac{\alpha_{31}}{\pi}$
Exp	0.0292	0.297	0.0215	0.5	1.4	–	–
1σ	0.0008	0.017	0.0007	0.1	0.2	–	–
prediction	0.0292	0.295	0.0447	0.651	1.55	0.22	1.80

many σ away

2-parameter fit to 5 physical quantities

8 dimensionless physical quantities independent on any coupling constant!



Status of neutrino oscillations 2018: 3 σ hint for normal mass ordering and improved CP sensitivity

P.F. de Salas (Valencia U., IFIC), D.V. Forero (Campinas State U. & Virginia Tech.), C.A. Ternes, M. Tortola, J.W.F. Valle (Valencia U., IFIC). Aug 3, 2017. 8 pp.

Published in **Phys.Lett. B782 (2018) 633-640**

1σ parameter space

Intervals where $\chi^2 \leq \chi^2_{\rm min} + 1$:

	IO	NO
$\mathrm{Re}(\tau)$	$[0.0113, 0.0120]$	$[-0.2023, -0.1987]$
$\mathrm{Im}(\tau)$	$[0.9944, 0.9951]$	$[1.0522, 1.0633]$
$\mathrm{Re}(\varphi_3)$	$[-0.090, -0.082]$	$[0.113, 0.121]$

Charged Lepton Sector

$$\mathcal{Y}_e = \begin{pmatrix} a\ \varphi_1 & a\ \varphi_3 & a\ \varphi_2 \\ b\ \varphi_2 & b\ \varphi_1 & b\ \varphi_3 \\ c\ \varphi_3 & c\ \varphi_2 & c\ \varphi_1 \end{pmatrix}$$

$$U_e = \begin{pmatrix} 1 & \varphi_3 & 0 \\ 0 & -\varphi_3 & 1 \\ -\varphi_3 & 1 & \varphi_3 \end{pmatrix} + \ldots$$

where dots stand for terms of order φ_3^2, $(m_e^2/m_\mu^2)\varphi_3$ and $(m_\mu^2/m_\tau^2)\varphi_3$.

Fit to Yukawa couplings

Model 1

$a\cos\beta$	2.806923×10^{-6}
$b\cos\beta$	9.992488×10^{-3}
$c\cos\beta$	5.899778×10^{-4}

$y_e(m_Z)$	2.794745×10^{-6}	0.0
$y_\mu(m_Z)$	5.899864×10^{-4}	+0.05
$y_\tau(m_Z)$	1.002950×10^{-2}	0.0

Model 2

$a\cos\beta$	2.809569×10^{-6}
$b\cos\beta$	9.961316×10^{-3}
$c\cos\beta$	5.899455×10^{-4}

$y_e(m_Z)$	2.794745×10^{-6}	0.0
$y_\mu(m_Z)$	5.899863×10^{-4}	0.0
$y_\tau(m_Z)$	1.002950×10^{-2}	0.0

Level N modular forms

N	g	$d_{2k}(\Gamma(N))$	μ_N	Γ_N
2	0	$k+1$	6	S_3
3	0	$2k+1$	12	A_4
4	0	$4k+1$	24	S_4
5	0	$10k+1$	60	A_5
6	1	$12k$	72	
7	3	$28k-2$	168	

Table 1: Some properties of modular forms: g is the genus of the space $\mathcal{H}/\Gamma(N)$ after compactification, $d_{2k}(\Gamma(N))$ the dimension of the linear space $\mathcal{M}_{2k}(\Gamma(N))$, μ_N is the dimension of the quotient group $\Gamma_N \equiv \overline{\Gamma}/\overline{\Gamma}(N)$, which, for $N \leq 5$, is isomorphic to a permutation group.

Fundamental domain of Γ(3)

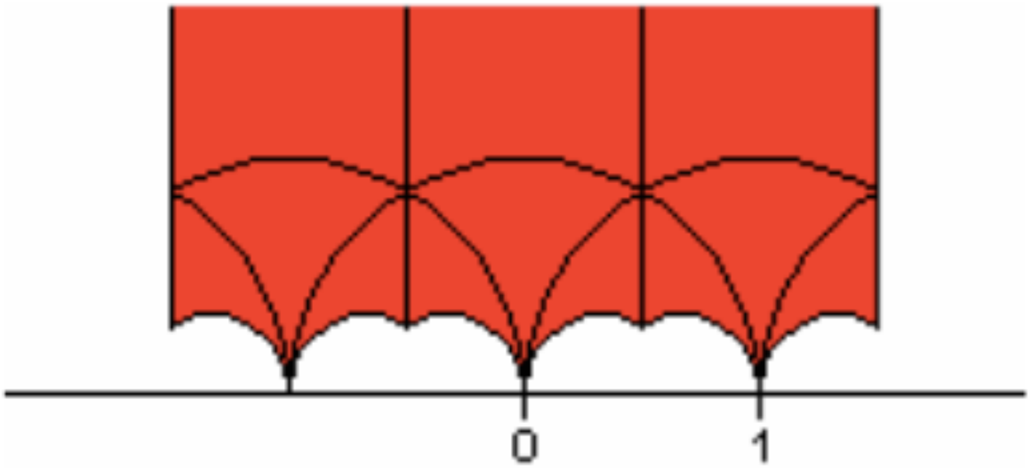


Figure 1: Fundamental domain for $\Gamma(3)$.

Q-expansion

$$\begin{aligned} Y_1(\tau) &= 1 + 12q + 36q^2 + 12q^3 + \ldots \\ Y_2(\tau) &= -6q^{1/3}(1 + 7q + 8q^2 + \ldots) \\ Y_3(\tau) &= -18q^{2/3}(1 + 2q + 5q^2 + \ldots) \quad . \end{aligned}$$

$$Y_2^2 + 2Y_1Y_3 = 0$$

some VEVs

$$(Y_1, Y_2, Y_3)|_{\tau=i\infty} = (1, 0, 0)$$

$$(Y_1, Y_2, Y_3)|_{\tau=i} = Y_1(i)(1, 1-\sqrt{3}, -2+\sqrt{3})$$

$$Y(-1/\tau)|_{\tau=i} = -\rho(S)Y(\tau)|_{\tau=i}$$

Ring of level-3 modular forms

$$Y_2^2 + 2Y_1Y_3 = 0$$

As discussed explicitly in Appendix D, the constraint (30) is essential to recover the correct dimension of the linear space $\mathcal{M}_{2k}(\Gamma(3))$. On the one side from table 1 we see that this space has dimension $2k+1$. On the other hand the number of independent homogeneous polynomial $Y_{i_1}Y_{i_2}\cdots Y_{i_k}$ of degree k that we can form with Y_i is $(k+1)(k+2)/2$. These polynomials are modular forms of weight $2k$ and, to match the correct dimension, $k(k-1)/2$ among them should vanish. Indeed this happens as a consequence of eq. (30). Therefore the ring $\mathcal{M}(\Gamma(3))$ is generated by the modular forms $Y_i(\tau)$ $(i = 1, 2, 3)$.

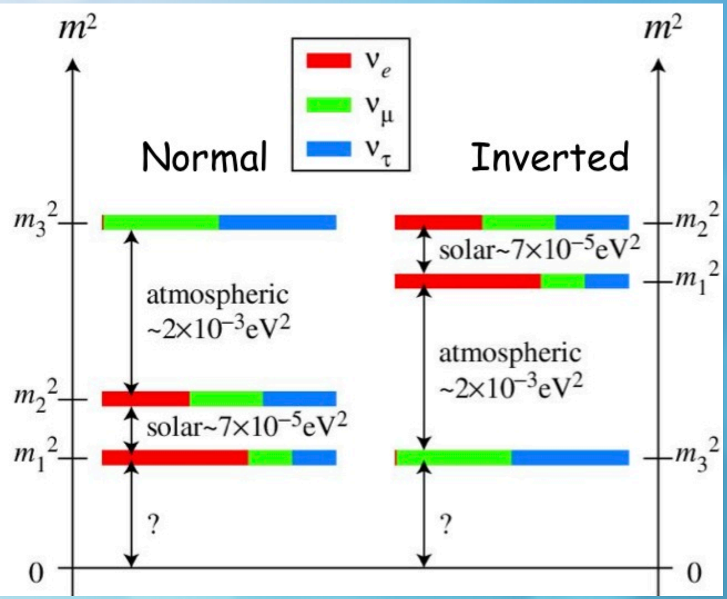
relevant parameters

$m_1 < m_2 \; [\Delta m^2_{ij} \equiv m^2_i - m^2_j]$

$\Delta m^2_{21} < |\Delta m^2_{32}|, |\Delta m^2_{31}|$

i.e. 1 and 2 are, by definition, the closest levels

two possibilities: NO and IO



Mixing matrix U_{PMNS} (Pontecorvo,Maki,Nakagawa,Sakata)

$$L_{CC} = -\frac{g}{\sqrt{2}} W^-_\mu \, \overline{e}_L \gamma^\mu U_{PMNS} \nu_L$$

$0 \le \vartheta_{ij} \le \pi/2$

$0 \le \delta < 2\pi$

Majorana phases

standard parametrization

$$U_{PMNS} = \begin{matrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{matrix} \overset{\begin{matrix}\nu_1 & \nu_2 & \nu_3\end{matrix}}{\begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\ -c_{12}s_{13}c_{23}e^{i\delta} + s_{12}s_{23} & -s_{12}s_{13}c_{23}e^{i\delta} - c_{12}s_{23} & c_{13}c_{23} \end{pmatrix}} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\frac{\alpha_{21}}{2}} & 0 \\ 0 & 0 & e^{i\frac{\alpha_{31}}{2}} \end{pmatrix}$$

Action of modular invariance on flavor space

most general transformation on a set of $\mathcal{N}$=1 SUSY chiral multiplets $\varphi^{(I)}$

$$\begin{cases} \tau \to \dfrac{a\tau + b}{c\tau + d} \\ \varphi^{(I)} \to (c\tau + d)^{-k_I} \rho^{(I)}(\gamma) \varphi^{(I)} \end{cases}$$

e.g. $\varphi^{(I)} = \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix}$

the weight, a real number (k_I)

unitary representation of the finite modular group ($\rho^{(I)}$) $\Gamma_N \equiv \overline{\Gamma}/\overline{\Gamma}(N)$

- Γ_N are finite groups

Γ_2	Γ_3	Γ_4	Γ_5
S_3	A_4	S_4	A_5

- if all k_I=0, the construction collapses to the well-known models based on linear, unitary flavor symmetries.

Few facts about (level-N) Modular Forms

- transformation property under the modular group

$$f_i(\gamma\tau) = (c\tau + d)^k \rho(\gamma)_{ij} f_j(\tau)$$

unitary representation of the finite modular group $\Gamma_N \equiv \overline{\Gamma}/\overline{\Gamma}(N)$

- q-expansion

$$f(\tau + N) = f(\tau) \quad \Rightarrow \quad f(\tau) = \sum_{i=0}^{\infty} a_n q_N^n \qquad q_N = e^{\frac{i2\pi\tau}{N}}$$

- $k < 0 \Rightarrow f(\tau) = 0$

  $k = 0 \Rightarrow f(\tau) =$ constant

  $k > 0$ (even integer) $\Rightarrow f(\tau) \in \mathcal{M}_k(\Gamma(N))$ finite-dimensional linear space

- ring of modular forms generated by few elements

$$\mathcal{M}(\Gamma(N)) = \bigoplus_{k=0}^{\infty} \mathcal{M}_{2k}(\Gamma(N))$$

an explicit example in a moment

$\mathcal{N}$=1 SUSY modular invariant theories

- known since late 1980s

S. Ferrara, D. Lust, A. D. Shapere and S. Theisen, Phys. Lett. B **225** (1989) 363.

S. Ferrara, .D. Lust and S. Theisen, Phys. Lett. B **233** (1989) 147.

focus on Yukawa interactions and $\mathcal{N}$=1 global SUSY

$$\mathcal{S} = \int d^4x d^2\theta d^2\bar{\theta}\ K(\Phi,\bar{\Phi}) + \int d^4x d^2\theta\ w(\Phi) + h.c.$$

$$\Phi = (\tau, \varphi)$$

Kahler potential, kinetic terms

superpotential, holomorphic function of Φ Yukawa interactions

S invariant if

$$\begin{cases} w(\Phi) \to w(\Phi) \\ K(\Phi,\bar{\Phi}) \to K(\Phi,\bar{\Phi}) + f(\Phi) + f(\bar{\Phi}) \end{cases}$$

- invariance of the Kahler potential easy to achieve. For example:

$$K(\Phi,\bar{\Phi}) = -h\log(-i\tau + i\bar{\tau}) + \sum_I (-i\tau + i\bar{\tau})^{-k_I} |\varphi^{(I)}|^2$$

minimal K

- extension to $\mathcal{N}$=1 SUGRA straightforward: ask invariance of $G=K+\log|w|^2$

- invariance of the superpotential much less trivial. Expand w(Φ) in powers of the matter supermultiplets

$$w(\Phi) = \sum_n Y_{I_1 \dots I_n}(\tau)\, \varphi^{(I_1)} \dots \varphi^{(I_n)}$$

field-dependent Yukawa couplings

- invariance of w(Φ) guaranteed by an holomorphic Y such that

$$Y_{I_1 \dots I_n}(\gamma\tau) = (c\tau + d)^{k_Y(n)} \rho(\gamma)\; Y_{I_1 \dots I_n}(\tau)$$

modular forms of level N and weight k_Y

$$k_Y(n) = k_{I_1} + \dots + k_{I_n}$$

The product $\rho \times \rho^{I_1} \times \dots \times \rho^{I_n}$ contains an invariant singlet

Variants

neutrino masses from see-saw mechanism

$$w_\nu = g\,(N^c H_u L)_1 + \Lambda (N^c N^c Y)_1$$

assignement

	L	N^c	H_u	Y
$SU(2)\times U(1)$	$(2,-1/2)$	$(1,0)$	$(2,+1/2)$	$(1,0)$
$\Gamma_3 \equiv A_4$	3	3	1	3
k_I	k_L	$+1$	k_u	$+2$

$1+k_L+k_u=0$

we get the best agreement at $\tau = -0.195 + 1.0636i$

	$\frac{\Delta m^2_{sol}}{\Delta m^2_{atm}}$	$\sin^2\vartheta_{12}$	$\sin^2\vartheta_{13}$	$\sin^2\vartheta_{23}$	$\frac{\delta_{CP}}{\pi}$	$\frac{\alpha_{21}}{\pi}$	$\frac{\alpha_{31}}{\pi}$
Exp	0.0292	0.297	0.0215	0.5	1.4	–	–
1σ	0.0008	0.017	0.0007	0.1	0.2	–	–
prediction	0.0280	0.291	0.0486	0.331	1.47	1.83	1.26

Normal mass ordering is predicted

$m_1 = 1.096 \times 10^{-2}\ eV$ $\quad m_2 = 1.387 \times 10^{-2}\ eV$ $\quad m_3 = 5.231 \times 10^{-2}\ eV$