FLASY 2018: 7th Workshop on Flavour Symmetries and
Consequences in Accelerators and Cosmology
University of Basel, 2-5 July, 2018

Low-Scale seesaw and the CP violation in neutrino oscillations

Lepton masses and mixing from **modular S**4 symmetry

João Penedo
(SISSA/INFN, Trieste, Italy)

Leitmotiv: Symmetry predictions for CPV phases

1[st] part

Froggatt-Nielsen

2[nd] part

Modular Symmetry

Low-Scale seesaw and the CP violation in neutrino oscillations

Nucl. Phys. B **929** (2018) 377

arXiv: 1712.09922

in collaboration with S. T. Petcov and T. T. Yanagida

SM + 2 right-handed neutrinos

$$\mathcal{L} \supset -\overline{\nu_R}\,\mathbf{Y_D^T}\,L\,H^{c\dagger} - \frac{1}{2}\overline{\nu_R}\,\mathbf{M_N}\,\nu_L^C + \text{h.c.}$$

$$\mathbf{Y_D} = \begin{pmatrix} 0 & g_{e2} \\ 0 & g_{\mu 2} \\ 0 & g_{\tau 2} \end{pmatrix} \qquad \mathbf{M_N} = \begin{pmatrix} 0 & M \\ M & 0 \end{pmatrix}$$

Limit of a conserved **lepton number**, with $\begin{cases} L(\nu_{1R}) = -1 \\ L(\nu_{2R}) = +1 \end{cases}$

→ **Symmetry-protected scenario**

recall: O. Fischer talk on Monday; See also Shaposhnikov, '07; Kersten, Smirnov, '07

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$$\mathbf{Y_D} = \begin{pmatrix} g_{e1} & g_{e2} \\ g_{\mu 1} & g_{\mu 2} \\ g_{\tau 1} & g_{\tau 2} \end{pmatrix} \qquad \mathbf{M_N} = \begin{pmatrix} 0 & M \\ M & \mu \end{pmatrix}$$

"linear" component

similar to texture in
Malinsky, Romão, Valle, '05

subleading "inverse" component

Wyler, Wolfenstein, '83
Mohapatra, '86
Mohapatra, Valle, '86

~~Symm~~etry-protected scenario

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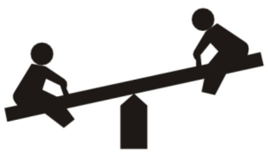
$$\mathbf{Y}_\mathbf{D} = \begin{pmatrix} g_{e1} & g_{e2} \\ g_{\mu 1} & g_{\mu 2} \\ g_{\tau 1} & g_{\tau 2} \end{pmatrix} \qquad \mathbf{M}_\mathbf{N} = \begin{pmatrix} 0 & M \\ M & \mu \end{pmatrix}$$

$$\mathbf{m}_\nu \simeq -\,v^2\,\mathbf{Y}_\mathbf{D}\,\mathbf{M}_\mathbf{N}^{-1}\,\mathbf{Y}_\mathbf{D}^T$$



$M \sim$ GeV – TeV

Pseudo-Dirac Neutrino

Ibarra, Molinaro, Petcov
1007.2378, 1103.6217

$$(\mathbf{m}_\nu)_{\ell\ell'} \simeq -\frac{v^2}{M}\left(g_{\ell 1} g_{\ell' 2} + g_{\ell 2} g_{\ell' 1}\right) \sim \frac{v^2}{M}\,g_1\,g_2$$

$v = 174$ GeV

$$|g_{\ell 1}| \sim 10^{-12} - 10^{-8}$$
$$|g_{\ell 2}| \sim 10^{-4} - 10^{-2}$$

A Froggatt-Nielsen realisation

SUSY setup with a broken U(1)FN, flavour scale Λ and charges:

	$\hat{S}$	$\hat{N}_1$	$\hat{N}_2$	$\hat{H}_u$	$\hat{L}_e$	$\hat{L}_\mu$	$\hat{L}_\tau$	$\hat{e}^c$	$\hat{\mu}^c$	$\hat{\tau}^c$
$Q_{\rm FN}$	-1	$n>0$	-1	0	2	1	1	4	2	0

"Lopsided" (for $\hat{L}_e$, $\hat{L}_\mu$, $\hat{L}_\tau$)
Sato, Yanagida, '00

$\epsilon \simeq \lambda_C \simeq 0.2$ (for $\hat{e}^c$, $\hat{\mu}^c$, $\hat{\tau}^c$)

$\epsilon \simeq 0.06$ also possible
Buchmuller, Yanagida, '99

$$\langle S \rangle = \epsilon\, \Lambda$$

$$W_\nu \sim \Lambda\,(\epsilon^{2n}\,\hat{N}_1\,\hat{N}_1 + \epsilon^{n-1}\,\hat{N}_1\,\hat{N}_2) + (\epsilon\,\hat{L}_e + \hat{L}_\mu + \hat{L}_\tau)\,(\epsilon^{n+1}\,\hat{N}_1 + g_2\,\hat{N}_2)\,\hat{H}_u$$

$\mu \sim \epsilon^{2n}\,\Lambda \qquad M \sim \epsilon^{n-1}\,\Lambda \qquad g_{\ell 1} \sim \epsilon^{n+1}$

A Froggatt-Nielsen realisation

SUSY setup with a broken $U(1)_{FN}$, flavour scale Λ and charges:

	$\hat{S}$	$\hat{N}_1$	$\hat{N}_2$	$\hat{H}_u$	$\hat{L}_e$	$\hat{L}_\mu$	$\hat{L}_\tau$	$\hat{e}^c$	$\hat{\mu}^c$	$\hat{\tau}^c$
Q_{FN}	-1	$n > 0$	-1	0	2	1	1	4	2	0

A large charge n mimicks L-conservation

$$\mathbf{Y_D} \sim \begin{pmatrix} g_{e1} & \epsilon\, g_2 \\ g_{\mu 1} & g_2 \\ g_{\tau 1} & g_2 \end{pmatrix} \sin\beta \qquad \begin{cases} |g_{e2}| : |g_{\mu 2}| : |g_{\tau 2}| \simeq \epsilon : 1 : 1 \quad \bigstar \\ |g_{e1}| : |g_{\mu 1}| : |g_{\tau 1}| \simeq ? : ? : ? \end{cases}$$

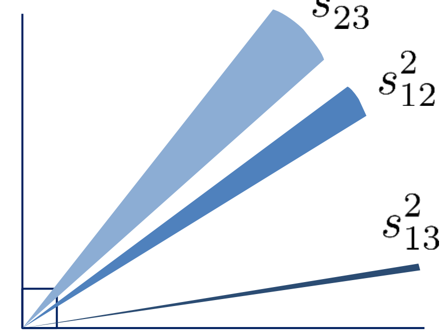
$$|g_{\ell 1}| \ll |g_{\ell' 2}|$$

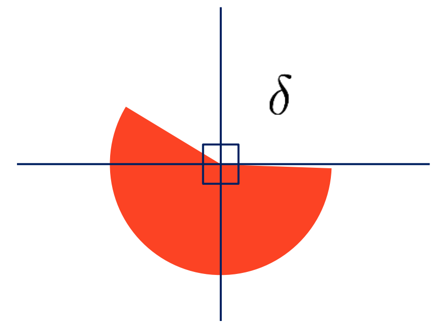
Interlude: neutrino mixing

For a **NO spectrum**, favoured at the 3.1σ CL:

Parameter	Best fit	1σ range	3σ range
$\Delta m^2_{\odot}/10^{-5}\ \mathrm{eV}^2$	7.34	7.20 – 7.51	6.92 – 7.91
$\lvert\Delta m^2_{\mathrm{A}}\rvert/10^{-3}\ \mathrm{eV}^2$	2.49	2.46 – 2.53	2.39 – 2.59
$\sin^2\theta_{12}/10^{-1}$	3.04	2.91 – 3.18	2.65 – 3.46
$\sin^2\theta_{13}/10^{-2}$	2.14	2.07 – 2.23	1.90 – 2.39
$\sin^2\theta_{23}/10^{-1}$	5.51	4.81 – 5.70	4.30 – 6.02
δ/π	1.32	1.14 – 1.55	0.83 – 1.99

Capozzi, Lisi, Marrone, Palazzo, 1804.09678





Parameterisation:

$$\mathbf{U}_{\mathrm{PMNS}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \mathrm{diag}(1, e^{i\alpha/2}, 1)$$

a single Majorana phase, $\alpha_{21} - \alpha_{31} \to \alpha$

Predictions for CPV phases

Casas-Ibarra parameterisation (NO):

$$\mathbf{Y_D} = \frac{i}{v}\,\mathbf{U}^*_{\rm PMNS}\sqrt{\hat{m}}\,\mathbf{O}\,\sqrt{\hat{M}}\,\mathbf{V}^\dagger$$

$$\mathbf{O} \equiv \begin{pmatrix} 0 & 0 \\ \cos z & \pm\sin z \\ -\sin z & \pm\cos z \end{pmatrix},\; z \in \mathbb{C}$$

$$\mathbf{Y_D} = \begin{pmatrix} g_{e1} & g_{e2} \\ g_{\mu 1} & g_{\mu 2} \\ g_{\tau 1} & g_{\tau 2} \end{pmatrix}$$

small $|\mathrm{Im}\, z|$

$$\mathbf{m}_\nu = \mathbf{U}^*\,\mathrm{diag}(m_1, m_2, m_3)\,\mathbf{U}^\dagger$$

$$\mathbf{U}_{\rm PMNS} = \mathbf{U}_l^\dagger(1+\eta)\mathbf{U} \simeq \mathbf{U}$$

→ (very small) deviations from unitarity

→ possible corrections, but power counting in $\mathbf{m}_\nu$ unaffected

Predictions for CPV phases

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$$\mathbf{Y_D} = \begin{pmatrix} g_{e1} & g_{e2} \\ g_{\mu 1} & g_{\mu 2} \\ g_{\tau 1} & g_{\tau 2} \end{pmatrix}$$

large $|\mathrm{Im}\, z|$ ✓

see also Drewes et al, 1609.09069

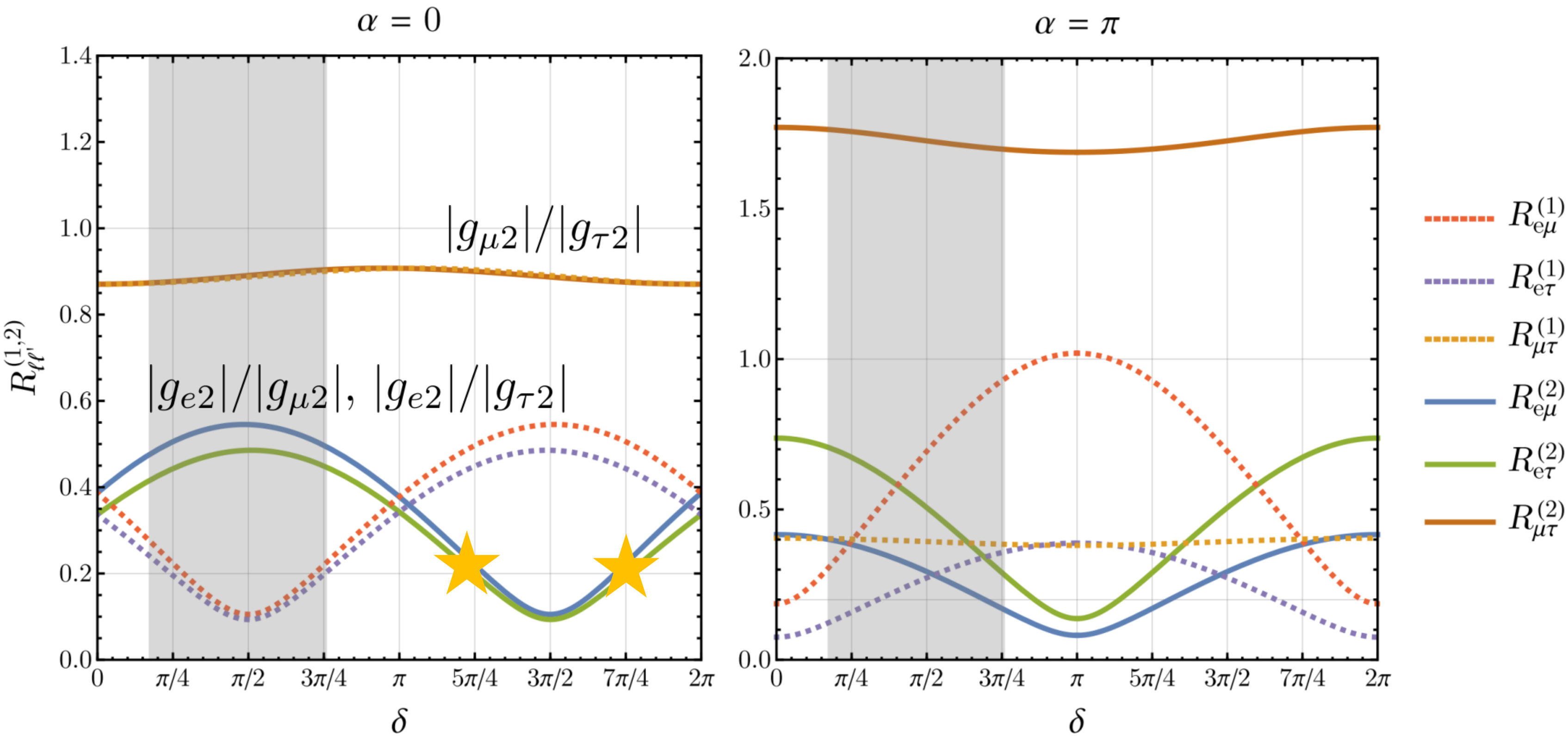
Ratios of couplings independent of the Casas-Ibarra parameter

$$R^{(1)}_{\ell\ell'} \equiv \frac{|g_{\ell 1}|}{|g_{\ell' 1}|} \simeq \frac{\left|\sqrt{m_2}\,U^*_{\ell 2} \pm i\,\sqrt{m_3}\,U^*_{\ell 3}\right|}{\left|\sqrt{m_2}\,U^*_{\ell' 2} \pm i\,\sqrt{m_3}\,U^*_{\ell' 3}\right|}$$

$$R^{(2)}_{\ell\ell'} \equiv \frac{|g_{\ell 2}|}{|g_{\ell' 2}|} \simeq \frac{\left|\sqrt{m_2}\,U^*_{\ell 2} \mp i\,\sqrt{m_3}\,U^*_{\ell 3}\right|}{\left|\sqrt{m_2}\,U^*_{\ell' 2} \mp i\,\sqrt{m_3}\,U^*_{\ell' 3}\right|}$$

Predictions for CPV phases

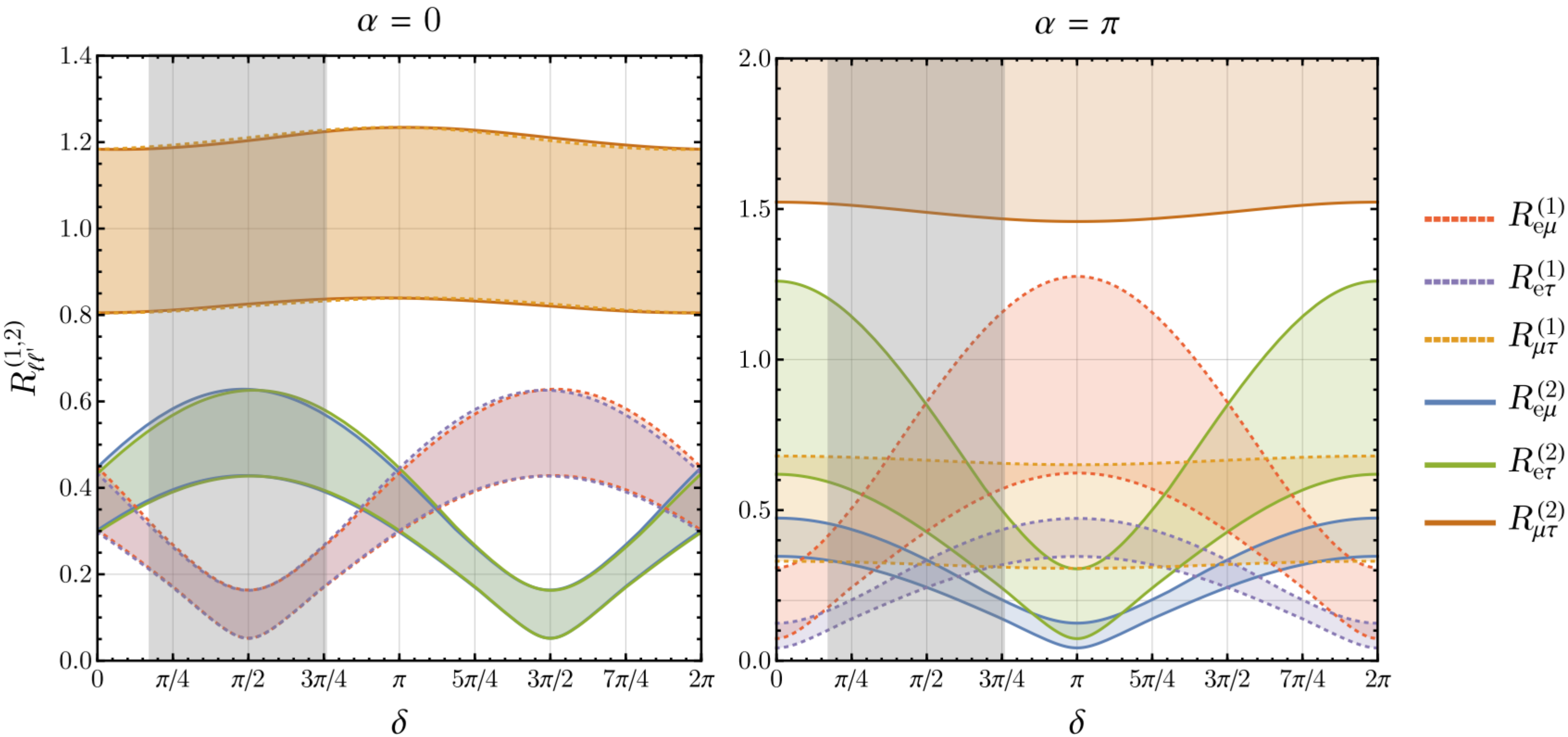
Using best-fit values of mass-squared differences and mixing angles:



$$\epsilon = 0.2 \;\rightarrow\; \delta \sim \pm\pi/4,\, \pm 3\pi/4$$

Predictions for CPV phases

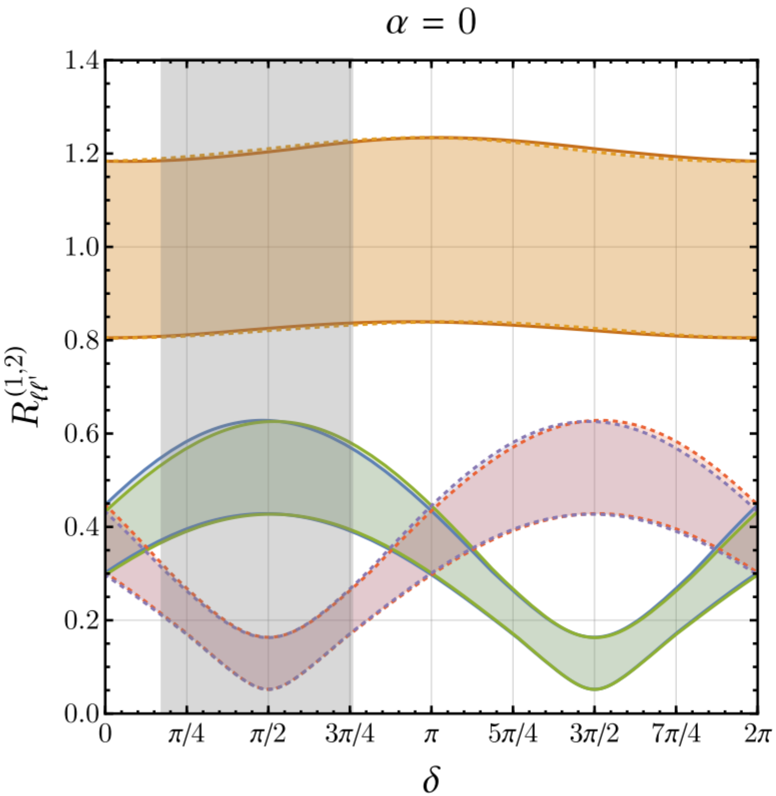
Using 3σ ranges for mass-squared differences and mixing angles:



$$\epsilon = 0.2 \;\rightarrow\; \delta \sim \pm\pi/4,\, \pm 3\pi/4$$

Predictions for CPV phases

Using 3σ ranges for mass-squared differences and mixing angles:



Ratio	Allowed range
$R_{e\mu}^{(1)}$	$0.05 - 1.28$
$R_{e\tau}^{(1)}$	$0.04 - 0.63$
$R_{\mu\tau}^{(1)}$	$0.31 - 1.23$
$R_{e\mu}^{(2)}$	$0.04 - 0.63$
$R_{e\tau}^{(2)}$	$0.05 - 1.26$
$R_{\mu\tau}^{(2)}$	$0.80 - 3.21$

There are patterns which cannot be realised, e.g. **1 : 1 : 1**

$$\epsilon = 0.2 \rightarrow \delta \sim \pm\pi/4, \pm 3\pi/4$$

Bounds on the couplings $g_1,\ g_2$

Mass matrix entries: $\left|(\mathbf{m}_\nu)_{\ell\ell'}\right| = \left|\mathbf{U}^*_{\ell j}\, m_j\, \mathbf{U}^*_{\ell' j}\right|$

constrains $\left|g_{\ell 1} g_{\ell' 2} + g_{\ell' 1} g_{\ell 2}\right|$ $\qquad g_1\, g_2$

Non-unitarity of PMNS:

$\eta = -(\mathbf{RV})\,(\mathbf{RV})^\dagger/2$ constrains $\left|g_{\ell 2} g_{\ell' 2}\right|$

$$|\eta| < \begin{pmatrix} 1.3\times 10^{-3} & 1.2\times 10^{-5} & 1.4\times 10^{-3} \\ 1.2\times 10^{-5} & 2.2\times 10^{-4} & 6.0\times 10^{-4} \\ 1.4\times 10^{-3} & 6.0\times 10^{-4} & 2.8\times 10^{-3} \end{pmatrix} \qquad g_2$$

for $M > v$ Fernandez-Martinez et al, 1605.08774, Blennow et al, 1609.08637

$\mathbf{R}^* \simeq v\,\mathbf{Y_D}\,\mathbf{M_N}^{-1}$

$\mathbf{M_N} \simeq \mathbf{V}^*\,\mathrm{diag}(M_1, M_2)\,\mathbf{V}^\dagger$

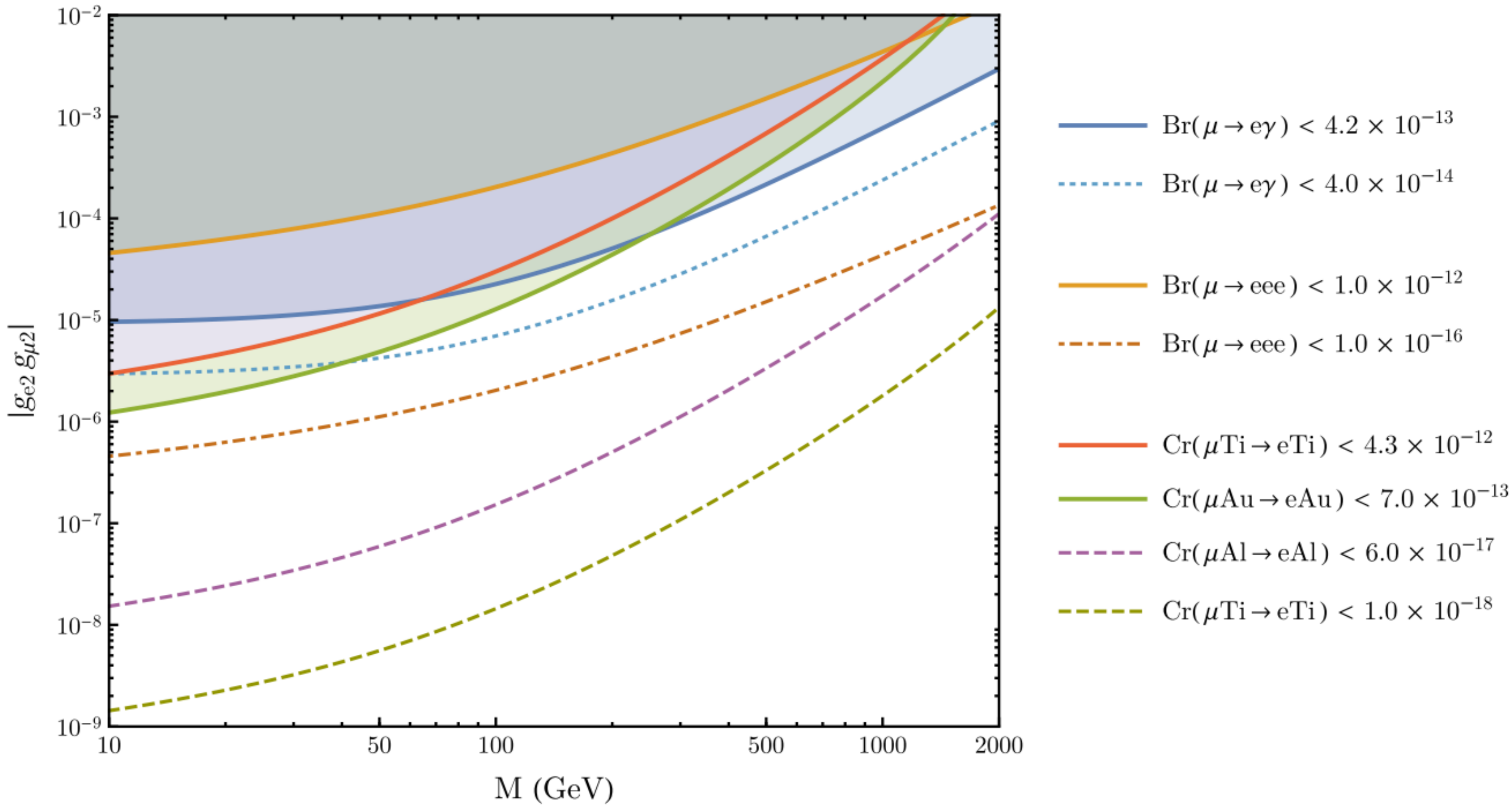
$$\mathbf{RV} \simeq \frac{1}{\sqrt{2}}\,\frac{v}{M}\begin{pmatrix} g^*_{e2} & -i\,g^*_{e2} \\ g^*_{\mu 2} & -i\,g^*_{\mu 2} \\ g^*_{\tau 2} & -i\,g^*_{\tau 2} \end{pmatrix}$$

controls pheno,

in particular...

Bounds on the couplings – LFV

see also Ibarra, Molinaro, Petcov, 1103.6217; Dinh, Ibarra, Molinaro, Petcov, 1205.4671



For the $\epsilon : 1 : 1$ structure, LFV processes like $\tau \to e\gamma$, $\tau \to \mu\gamma$, $\tau \to \mu\mu\mu$ do not provide stronger constraints on the magnitude of the couplings

Additional phenomenology

Collider physics: Antusch, Fischer, 1502.05915; Deppisch, Dev, Pilaftsis, 1502.06541; Das, Okada, 1702.04668; Das, Dev, Kim, 1704.00880; and others...

Higgs decays: if new pseudo-Dirac neutrino is light enough, it can affect Higgs **branching ratios**

see also Cely, Ibarra, Molinaro, Petcov, 1208.3654; Antusch, Fischer, 1502.05915

$0\nu\beta\beta$-decay: contribution from new states **suppressed** with respect to standard 3 neutrino case

(Anti-)leptognesis: the resonant condition is not satisfied

$$\mu \sim g_1 M \quad \rightarrow \quad \mu \gg \Gamma/2$$

the addition of a third RH neutrino and the suppression of μ (e.g. using an extra dim.) may allow to generate BAU

Fukugita, Yanagida, '02

Summary (1st part)

- Froggatt-Nielsen (FN) can be responsible for **approximate *L*-conservation**.
- RH neutrinos can have GeV–TeV (low-scale) masses.
- Preferences for CPV phases can be derived, namely $\delta \sim \pm\pi/4, \pm 3\pi/4, \alpha \sim 0$ for $\epsilon \simeq \lambda_C$.
- Sizeable LFV signals and collider phenomenology are possible.

Lepton masses and mixing from **modular S4** symmetry

arXiv: 1806.11040

in collaboration with S. T. Petcov

STAY TUNED...

Thu 05/07 (tomorrow)

9:00

F. Feruglio: "Are neutrino masses modular forms?"

12:00

T. Tatsuishi: "Neutrino mixing from finite modular groups"

Modular invariance shaping the leptonic sector

Feruglio, 1706.08749

Principal congruence subgroups, for $N = 1, 2, \ldots$

$$\Gamma(N) \equiv \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{Z} \wedge \det\gamma = 1 \wedge \gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \; (\mathrm{mod}\; N) \right\}$$

Modular group: $\Gamma(1) \simeq \mathrm{SL}(2, \mathbb{Z})$

$$\tau \to \gamma\tau = \frac{a\tau + b}{c\tau + d},$$

with $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \overline{\Gamma}(N)$

$$\{\tau \in \mathbb{C} \,|\, \mathrm{Im}(\tau) > 0\}$$

Groups of linear fractional transformations:

$\overline{\Gamma}(N) \equiv \Gamma(N)/\{1, -1\}$, for $N = 1, 2$

$\overline{\Gamma}(N > 2) \equiv \Gamma(N)$

$\overline{\Gamma}(1) : \; S^2 = (ST)^3 = 1$

Finite modular groups:

$$\Gamma_N \equiv \overline{\Gamma}(1)/\overline{\Gamma}(N)$$

$$S^2 = (ST)^3 = T^N = 1$$

$$S : \; \tau \to -1/\tau, \qquad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$T : \; \tau \to \tau + 1, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Modular-invariant SUSY actions

Ferrara et al, '89

$$\mathcal{S} = \int d^4x\, d^2\theta\, d^2\overline{\theta}\; K(\chi_i, \overline{\chi}_i; \tau, \overline{\tau}) + \int d^4x\, d^2\theta\; W(\chi_i; \tau) + \text{h.c.}$$

$$W(\chi_i;\tau) = \sum_n \sum_{\{i_1,\ldots,i_n\}} \left(Y_{\{i_1,\ldots,i_n\}}(\tau)\; \chi_{i_1} \cdots \chi_{i_n}\right)_{\mathbf{1}}$$

$$\text{with } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_N \quad \begin{cases} \tau \;\to\; \gamma\tau = \dfrac{a\tau + b}{c\tau + d} \\[2ex] \chi_i \;\to\; (c\tau + d)^{-k_i}\, \rho_i(\gamma)\, \chi_i \\[2ex] Y(\tau) \;\to\; Y(\gamma\tau) = (c\tau + d)^{2k_Y}\, \rho_Y(\gamma)\, Y(\tau) \end{cases}$$

Bottom-up approach

For top-down, see e.g. Kobayashi et al, 1804.06644

Modular-invariant SUSY actions

$$\text{with } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_N \quad \begin{cases} \tau \;\to\; \gamma\tau = \dfrac{a\tau + b}{c\tau + d} \\[2ex] \chi_i \;\to\; (c\tau + d)^{-k_i}\, \rho_i(\gamma)\, \chi_i \\[2ex] Y(\tau) \;\to\; Y(\gamma\tau) = (c\tau + d)^{2k_Y}\, \rho_Y(\gamma)\, Y(\tau) \end{cases}$$

Exponents are “weights”

τ is a dimensionless spurion, $\langle\tau\rangle$ only source of modular sym. breaking

$Y(\tau)$ are **modular forms** obeying $\begin{cases} 2k_Y - k_{i_1} - \ldots - k_{i_n} = 0 \\ \rho_Y \otimes \rho_{i_1} \otimes \ldots \otimes \rho_{i_1} \supset \mathbf{1} \end{cases}$

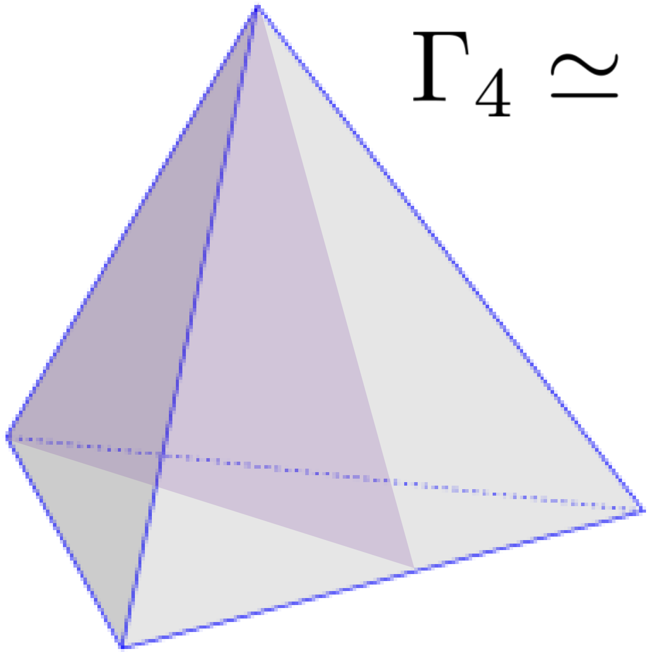
Leading-order predictions potentially modified by corrections from Kähler, SUSY breaking

Modular S4

S3 and A4 explored in Feruglio, 1706.08749 and more recently in Kobayashi, Tanaka and Tatsuishi, 1803.10391

$$\Gamma_4 \simeq S_4$$

$$\Gamma_2 \simeq S_3,\ \Gamma_3 \simeq A_4$$



rotations + **reflection**
$\Leftrightarrow$ **permutation of vertices**

see e.g. Ishhimori et al, 1003.3552

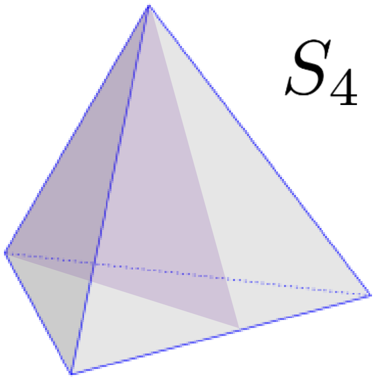
Usual presentation:

$$S^2 = T^3 = U^2 = (ST)^3$$
$$= (SU)^2 = (TU)^2 = (STU)^4 = 1$$

not convenient

Modular S4

$$S^2 = (ST)^3 = T^4 = 1$$



$$\mathbf{1}: \quad \rho(S) = 1, \quad \rho(T) = 1$$

$$\mathbf{1}': \quad \rho(S) = -1, \quad \rho(T) = -1$$

$$\mathbf{2}: \quad \rho(S) = \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix}, \quad \rho(T) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\mathbf{3}: \quad \rho(S) = \frac{1}{3}\begin{pmatrix} -1 & 2\omega^2 & 2\omega \\ 2\omega & 2 & -\omega^2 \\ 2\omega^2 & -\omega & 2 \end{pmatrix}, \quad \rho(T) = \frac{1}{3}\begin{pmatrix} -1 & 2\omega & 2\omega^2 \\ 2\omega & 2\omega^2 & -1 \\ 2\omega^2 & -1 & 2\omega \end{pmatrix}$$

$$\mathbf{3}': \quad \rho(S) = -\frac{1}{3}\begin{pmatrix} -1 & 2\omega^2 & 2\omega \\ 2\omega & 2 & -\omega^2 \\ 2\omega^2 & -\omega & 2 \end{pmatrix}, \quad \rho(T) = -\frac{1}{3}\begin{pmatrix} -1 & 2\omega & 2\omega^2 \\ 2\omega & 2\omega^2 & -1 \\ 2\omega^2 & -1 & 2\omega \end{pmatrix}$$

we have adapted group theoretical results from
Bazzocchi, Merlo, Morisi, 0901.2086

$$\omega = e^{2\pi i/3}$$

Generators of modular forms – Dedekind eta

$$\eta(z) \equiv q^{1/24} \prod_{k=1}^{\infty} (1 - q^k) \qquad q \equiv e^{2\pi i\, z}$$

$$\eta(z+1) = e^{i\pi/12}\,\eta(z)$$

$$\eta(-1/z) = \sqrt{-iz}\,\eta(z)$$

At level 4, corresponding to S4

S :

$$\{\eta_i\} = \left\{\eta\left(\tau + \frac{1}{2}\right), \eta\left(4\tau\right), \eta\left(\frac{\tau}{4}\right), \eta\left(\frac{\tau+1}{4}\right), \eta\left(\frac{\tau+2}{4}\right), \eta\left(\frac{\tau+3}{4}\right)\right\}$$

T :

up to multiplicative factors

Generators of modular forms – basis

$$Y(a_1,\dots,a_6|\tau) \equiv a_1\frac{\eta'(\tau+1/2)}{\eta(\tau+1/2)} + 4\,a_2\frac{\eta'(4\tau)}{\eta(4\tau)} + \frac{1}{4}\Big[a_3\frac{\eta'(\tau/4)}{\eta(\tau/4)} + a_4\frac{\eta'((\tau+1)/4)}{\eta((\tau+1)/4)} + a_5\frac{\eta'((\tau+2)/4)}{\eta((\tau+2)/4)} + a_6\frac{\eta'((\tau+3)/4)}{\eta((\tau+3)/4)}\Big]$$

$$\sum_i a_i = 0$$

$$S:\ Y(a_1,\dots,a_6|\tau) \to Y(a_1,a_2,a_3,a_4,a_5,a_6|-1/\tau) = \tau^2\,Y(a_5,a_3,a_2,a_6,a_1,a_4|\tau)$$

$$T:\ Y(a_1,\dots,a_6|\tau) \to Y(a_1,a_2,a_3,a_4,a_5,a_6|\tau+1) = Y(a_1,a_2,a_6,a_3,a_4,a_5|\tau)$$

Lowest weight forms arrange into:

$$Y_{\mathbf{2}}(\tau) \equiv \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \end{pmatrix}$$ **doublet**

$$Y_{\mathbf{3'}}(\tau) \equiv \begin{pmatrix} Y_3(\tau) \\ Y_4(\tau) \\ Y_5(\tau) \end{pmatrix}$$ **triplet'**

with

$$Y_1(\tau) \equiv Y(1,1,\omega,\omega^2,\omega,\omega^2|\tau)$$
$$Y_2(\tau) \equiv Y(1,1,\omega^2,\omega,\omega^2,\omega|\tau)$$
$$Y_3(\tau) \equiv Y(1,-1,-1,-1,1,1|\tau)$$
$$Y_4(\tau) \equiv Y(1,-1,-\omega^2,-\omega,\omega^2,\omega|\tau)$$
$$Y_5(\tau) \equiv Y(1,-1,-\omega,-\omega^2,\omega,\omega^2|\tau)$$

Constraints

$$\frac{1}{3}\left(Y_3^2 + 2Y_4Y_5\right) = Y_1Y_2\,, \qquad -\frac{1}{\sqrt{3}}\left(Y_3^2 - Y_4Y_5\right) = Y_1Y_4 - Y_2Y_5\,,$$
$$\frac{1}{3}\left(Y_4^2 + 2Y_3Y_5\right) = Y_2^2\,, \qquad -\frac{1}{\sqrt{3}}\left(Y_5^2 - Y_3Y_4\right) = Y_1Y_5 - Y_2Y_3\,,$$
$$\frac{1}{3}\left(Y_5^2 + 2Y_3Y_4\right) = Y_1^2\,, \qquad -\frac{1}{\sqrt{3}}\left(Y_4^2 - Y_3Y_5\right) = Y_1Y_3 - Y_2Y_4\,.$$

Related to non-linear realisation of the symmetry as suggested in 1706.08749?

Modular forms of higher weight

From tensor products of $Y_{\mathbf{2}}(\tau)$ and $Y_{\mathbf{3}'}(\tau)$:

Non-zero $\mathbf{1}'$ arises only at weight 6

$$Y_{\mathbf{1}}^{(4)} = Y_1Y_2 \sim \mathbf{1}$$
$$Y_{\mathbf{2}}^{(4)} = (Y_2^2,\, Y_1^2)^T \sim \mathbf{2}$$
$$Y_{\mathbf{3}}^{(4)} = (Y_1Y_4 - Y_2Y_5,\, Y_1Y_5 - Y_2Y_3,\, Y_1Y_3 - Y_2Y_4)^T \sim \mathbf{3}$$
$$Y_{\mathbf{3}'}^{(4)} = (Y_1Y_4 + Y_2Y_5,\, Y_1Y_5 + Y_2Y_3,\, Y_1Y_3 + Y_2Y_4)^T \sim \mathbf{3}'$$

Building minimal models

$$W = \sum_i \alpha_i \left(E_i^c\, L\, H_d\, Y_{\mathbf{2}}^{a_i}\, Y_{\mathbf{3}'}^{b_i}\right)_{\mathbf{1}} + \frac{g}{\Lambda}\left(L\, H_u\, L\, H_u\, Y_{\mathbf{2}}^{c}\, Y_{\mathbf{3}'}^{d}\right)_{\mathbf{1}}$$

Building minimal models

Guidelines

- **No flavons** are introduced,
- Neutrino masses arise from the **Weinberg operator**,

- Higgs multiplets transform trivially,
- Lepton doublets transform as an S4 triplet,
- Lepton singlets transform as S4 singlets, and

- Lowest possible weights are chosen such that a **rank 3 charged-lepton mass matrix** is possible without additional shaping symmetries.

$$W = \sum_i \alpha_i \left(E_i^c \, L \, H_d \, Y_{\mathbf{2}}^{a_i} \, Y_{\mathbf{3'}}^{b_i}\right)_{\mathbf{1}} + \frac{g}{\Lambda}\left(L \, H_u \, L \, H_u \, Y_{\mathbf{2}}^{c} \, Y_{\mathbf{3'}}^{d}\right)_{\mathbf{1}}$$

Model I ($k_L = 1$)

$\alpha = 1,\ \beta = 1.7 \times 10^{-4},$
$\gamma = 0.025,\ \tau = 0.19 + 0.99i$

Minimality guidelines imply:

	H_u	H_d	L	E_1^c	E_2^c	E_3^c
ρ_i	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}$ / $\mathbf{3'}$	$\mathbf{1'}$ / $\mathbf{1}$	$\mathbf{1}$ / $\mathbf{1'}$	$\mathbf{1'}$ / $\mathbf{1}$
k_i	0	0	1	1	3	3

IO spectrum

$$\frac{m_e}{m_\mu} \simeq 0.0049\,,\ \sin^2\theta_{12} \simeq 0.146\,,\quad \delta \simeq 1.23\pi\,,$$

$$\frac{m_\mu}{m_\tau} \simeq 0.0533\,,\ \sin^2\theta_{13} \simeq 0.116\,,\ \alpha_{21} \simeq 1.92\pi\,,$$

$$r \simeq 0.0287\,,\ \sin^2\theta_{23} \simeq 0.548\,,\ \alpha_{31} \simeq 0.50\pi\,.$$

$$M_\nu^{\mathrm{I}} = \frac{2g_1 v_u^2}{\Lambda}\begin{pmatrix} 0 & Y_2 & Y_1 \\ Y_2 & Y_1 & 0 \\ Y_1 & 0 & Y_2 \end{pmatrix}$$

$$M_e^{\mathrm{I}} = v_d \begin{pmatrix} \alpha\, Y_3 & \alpha\, Y_5 & \alpha\, Y_4 \\ \beta\,(Y_1Y_4 - Y_2Y_5) & \beta\,(Y_1Y_3 - Y_2Y_4) & \beta\,(Y_1Y_5 - Y_2Y_3) \\ \gamma\,(Y_1Y_4 + Y_2Y_5) & \gamma\,(Y_1Y_3 + Y_2Y_4) & \gamma\,(Y_1Y_5 + Y_2Y_3) \end{pmatrix}^\dagger$$

Model II ($k_L = 2$)

$\alpha = 0.12,\ \beta = 4.2 \times 10^{-4},$
$\gamma = 1,\ g/g' = 0.34 - 0.94i,$
$g''/g' = 1.93 - 0.53i,\ \tau = -0.1 + 1.2i$

Minimality guidelines imply:

	H_u	H_d	L	E_1^c	E_2^c	E_3^c
ρ_i	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}$ / $\mathbf{3'}$	$\mathbf{1'}$ / $\mathbf{1}$	$\mathbf{1}$ / $\mathbf{1'}$	$\mathbf{1'}$ / $\mathbf{1}$
k_i	0	0	2	0	2	2

NO spectrum

$$\frac{m_e}{m_\mu} \simeq 0.0052\,,\ \sin^2\theta_{12} \simeq 0.294\,,\quad \delta \simeq 0.32\pi\,,$$
$$\frac{m_\mu}{m_\tau} \simeq 0.0542\,,\ \sin^2\theta_{13} \simeq 0.021\,,\ \alpha_{21} \simeq 0.96\pi\,,$$
$$r \simeq 0.0300\,,\ \sin^2\theta_{23} \simeq 0.574\,,\ \alpha_{31} \simeq 1.59\pi\,.$$

$$M_e^{\mathrm{II}} = M_e^{\mathrm{I}}$$
$$M_\nu^{\mathrm{II}} = \frac{2g'v_u^2}{\Lambda}\left[\begin{pmatrix} (g/g')Y_1Y_2 & Y_2^2 & Y_1^2 \\ Y_2^2 & Y_1^2 & (g/g')Y_1Y_2 \\ Y_1^2 & (g/g')Y_1Y_2 & Y_2^2 \end{pmatrix} + \frac{1}{2}\frac{g''}{g'}\begin{pmatrix} 2(Y_1Y_4 - Y_2Y_5) & -(Y_1Y_3 - Y_2Y_4) & -(Y_1Y_5 - Y_2Y_3) \\ -(Y_1Y_3 - Y_2Y_4) & 2(Y_1Y_5 - Y_2Y_3) & -(Y_1Y_4 - Y_2Y_5) \\ -(Y_1Y_5 - Y_2Y_3) & -(Y_1Y_4 - Y_2Y_5) & 2(Y_1Y_3 - Y_2Y_4) \end{pmatrix}\right]$$

Model II ($k_L = 2$)

$\alpha = 0.12,\ \beta = 1,$
$\gamma = 4.5 \times 10^{-4},\ g/g' = 15.1 + 7.7i,$
$g''/g' = -0.40 - 1.08i,\ \tau = 0.35 + 0.85i$

Minimality guidelines imply:

	H_u	H_d	L	E_1^c	E_2^c	E_3^c
ρ_i	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}$ / $\mathbf{3'}$	$\mathbf{1'}$ / $\mathbf{1}$	$\mathbf{1}$ / $\mathbf{1'}$	$\mathbf{1'}$ / $\mathbf{1}$
k_i	0	0	2	0	2	2

NO spectrum ✓

$$\frac{m_e}{m_\mu} \simeq 0.0045\,,\ \sin^2\theta_{12} \simeq 0.278\,,\quad \delta \simeq 1.37\pi\,,$$

$$\frac{m_\mu}{m_\tau} \simeq 0.0557\,,\ \sin^2\theta_{13} \simeq 0.021\,,\ \alpha_{21} \simeq 0.25\pi\,,$$

$$r \simeq 0.0296\,,\ \sin^2\theta_{23} \simeq 0.480\,,\ \alpha_{31} \simeq 1.22\pi\,.$$

$$|\langle m \rangle| \simeq 0.043 \text{ eV}$$

$$M_e^{\text{II}} = M_e^{\text{I}}$$

$$M_\nu^{\text{II}} = \frac{2g' v_u^2}{\Lambda}\left[\begin{pmatrix} (g/g')Y_1Y_2 & Y_2^2 & Y_1^2 \\ Y_2^2 & Y_1^2 & (g/g')Y_1Y_2 \\ Y_1^2 & (g/g')Y_1Y_2 & Y_2^2 \end{pmatrix} + \frac{1}{2}\frac{g''}{g'}\begin{pmatrix} 2(Y_1Y_4 - Y_2Y_5) & -(Y_1Y_3 - Y_2Y_4) & -(Y_1Y_5 - Y_2Y_3) \\ -(Y_1Y_3 - Y_2Y_4) & 2(Y_1Y_5 - Y_2Y_3) & -(Y_1Y_4 - Y_2Y_5) \\ -(Y_1Y_5 - Y_2Y_3) & -(Y_1Y_4 - Y_2Y_5) & 2(Y_1Y_3 - Y_2Y_4) \end{pmatrix}\right]$$

Summary (2^{nd} part)

- We have explored the consequences of broken **modular invariance** in the lepton sector. We considered the finite modular subgroup $\Gamma_4 \simeq S_4$.
- A minimal model with **no flavons** can accommodate lepton masses and mixing while predicting Dirac and Majorana CPV phases.
- Model-building avenue worthy of **future study**.

Thank you / Merci vilmal !