Neutrino mixing from finite modular groups

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Based on

[1] Tatsuo Kobayashi, Kentaro Tanaka, T.H.T [arXiv:1803.10391]

[2] T.K, T.H.T, N.Omoto (Hokkaido U.),
Y.Shimizu, K.Takagi (Hiroshima U.),
M.Tanimoto (Niigata U.)

Work in progress

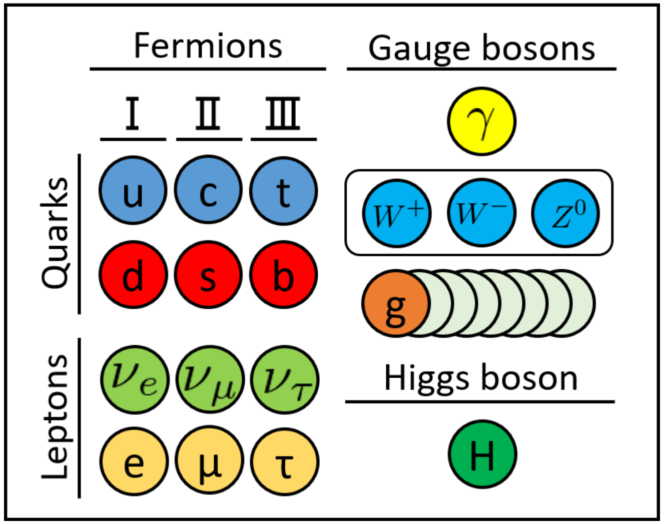
Introduction

The Standard Model is successfull.
However, there are unsolved mysteries:

- gravity
- dark matter, inflation
- Flavor anomalies
- etc.

⇒ Theories beyond the SM are studied

The Standard Model



candidate:

Superstring theory $(10D)$

- ■ Include gravity
- ■ $(4D$ effective theory) include various gauge groups
- ■ include various matters

Introduction

String Phenomenology: derive the SM from String theory

Superstring theory: $10D$

Compactification $10D \to 4D$

The Standard Model (-like structure)

Qualitative aspects:

- Gauge symmetry: $SU(3)_C \times SU(2)_L \times U(1)_Y$
- Chiral structure: $Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, u_R, d_R$
- 3 generations

Quantitative aspects:

- Gauge couplings
- Quark mass, mixing
- lepton mass, mixing ← Today's Topic
- Etc.

Introduction

Superstring theory

$10D \to 4D$
(assumption)

First half

Neutrino mixing from finite modular groups

$\Gamma(N)$: finite discrete groups

Weinberg operator for effective neutrino mass

$$\mathcal{L}_{\mathrm{eff}} \supset \frac{1}{\Lambda} HHLLY$$

Neutrino L, coupling constant Y
→ Non-trivial rep. of $\Gamma(N)$

Model search for realistic U_{PMNS}

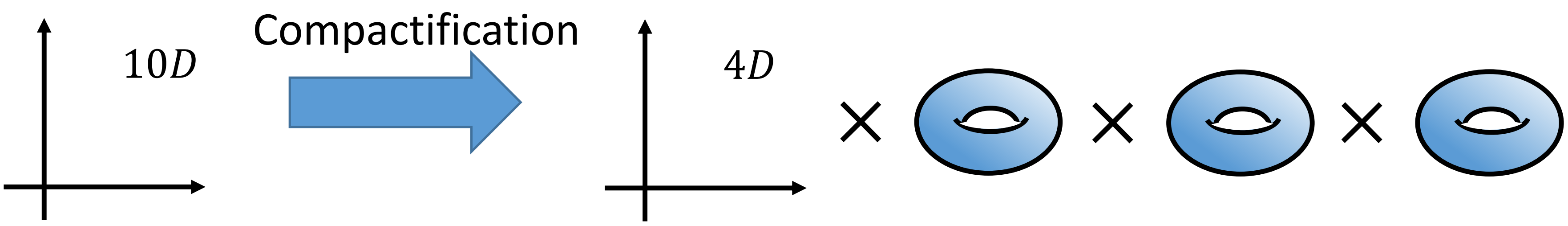
latter half

Effective Theory and Moduli

- Superstring theory is $10D$
- Our universe is $4D$

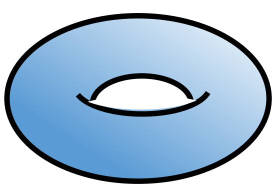
→ The extra $6D$ should be compact.

Torus compactification



We get 4D effective Lagrangian by integrating out over 6D.

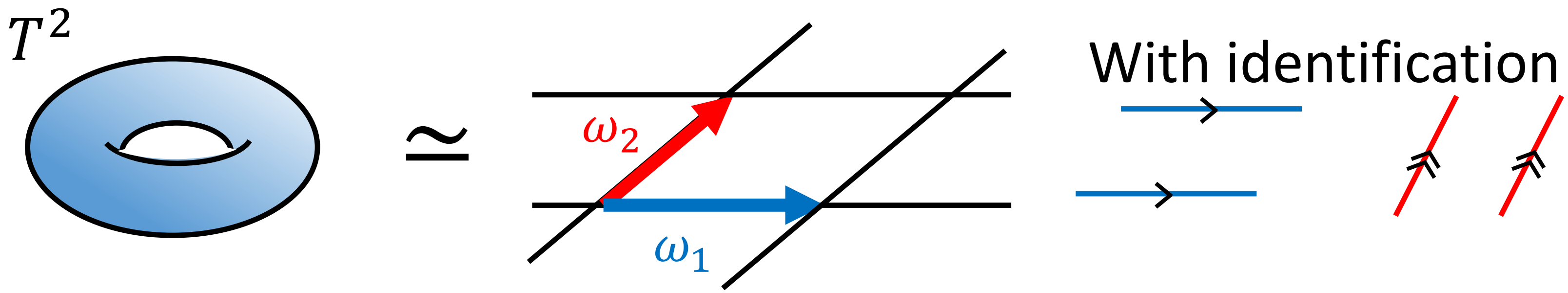
$$S = \int d^4x d^6y \, \mathcal{L}_{10D} \to \int d^4x \, \mathcal{L}_{\mathrm{eff}}$$

→ $\mathcal{L}_{\mathrm{eff}}$ depends on the structure of 

➢ $4D$ effective theory depends on internal space

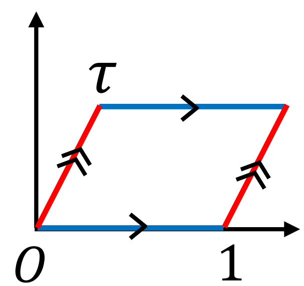
Effective Theory and Moduli

The structure of a torus $T^2 \simeq$ The structure of a lattice on $\mathbb{C}$-plane



Without loss of generality,

$$(\omega_1, \omega_2) \to \left(1, \frac{\omega_2}{\omega_1}\right) \equiv (1, \tau)$$

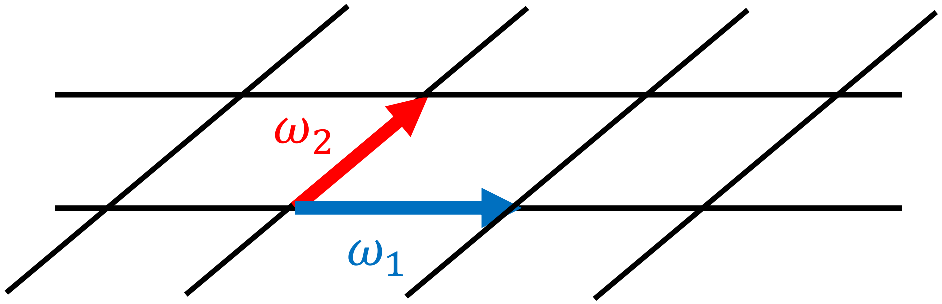


$\mathcal{L}_{\text{eff}}$ depends on τ. e.g.) $\mathcal{L}_{\text{eff}} \supset Y(\tau)_{ij}\phi\overline{\psi_i}\psi_j + \cdots$

- $4D$ effective theory depends on a modulus τ

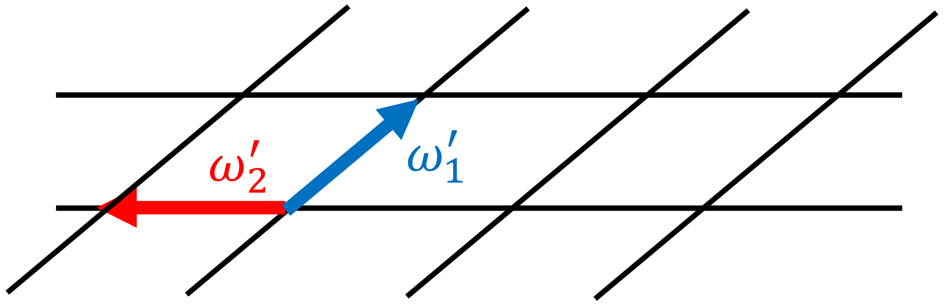
Modular Transformation

There are two independent lattice invariant transformations.



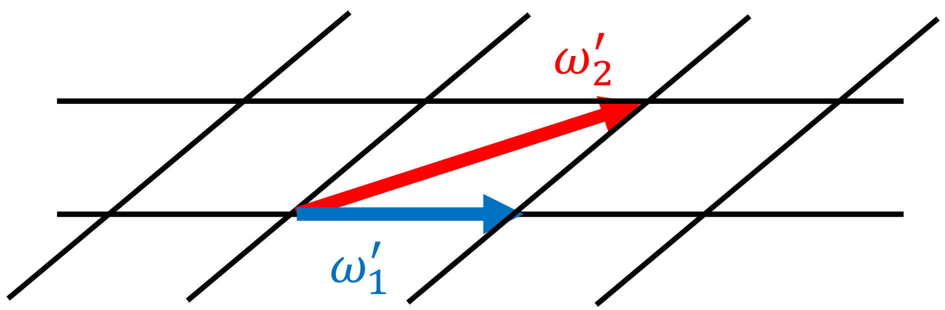
S-transformation

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \omega_2 \\ -\omega_1 \end{pmatrix}$$



T-transformation

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 + \omega_1 \end{pmatrix}$$



Modular Transformation

General form of lattice invariant transformation

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \qquad \begin{matrix} ad - bc = 1 \\ a, b, c, d \in \mathbb{Z} \end{matrix}$$

It is equivalent to

$$\tau' = \frac{a\tau + b}{c\tau + d}, \qquad \begin{matrix} ad - bc = 1 \\ a, b, c, d \in \mathbb{Z} \end{matrix}$$

Modular transformation

Modular group Γ

Modular transf. does not change the lattice (torus)

$4D$ effective theory (depends on τ) must be invariant under modular transf.

Subgroups of Modular Group

When we use more complicated compactification than torus, modular group Γ can be (partially) broken.

Torus T^2 → Modular group Γ

↓ More complicated geometry ↓ break

e.g.) toroidal orbifold T^2/Z_N → Subgroup of Γ

Modular group

$\Gamma \simeq \{S, T | S^2 = \mathbb{I}, (ST)^3 = \mathbb{I}\}$

Principle congruence subgroup of Γ

$\Gamma(N) \simeq \{S, T | S^2 = \mathbb{I}, (ST)^3 = \mathbb{I}, T^N = \mathbb{I}\}$

$\Gamma(2) \simeq S_3$

$\Gamma(3) \simeq A_4$

$\Gamma(4) \simeq S_4$

$\Gamma(5) \simeq A_5$

Subgroups of Modular Group

Modular group

$$\Gamma \simeq \{S, T | S^2 = \mathbb{I}, (ST)^3 = \mathbb{I}\}$$

Principle congruence subgroup

$$\Gamma(N) \simeq \{S, T | S^2 = \mathbb{I}, (ST)^3 = \mathbb{I}, T^N = \mathbb{I}\}$$

Examples: [T.K, S.N, S.T, S.T, T.H.T, arXiv:1804.06644]

- Magnetized T^2 with $M = 2$

⇒ $S^2 = \mathbb{I}, (ST)^3 = e^{\pi i/4}\mathbb{I}, T^4 = \mathbb{I}$

⇒ $(Z_8 \times Z_4) \rtimes S_3$

- Magnetized T^2 with $M = 4$
- T^2/Z_2 orbifold

⇒ $S^2 = \mathbb{I}, (ST)^3 = e^{\pi i/4}\mathbb{I}, T^8 = \mathbb{I}$

⇒ $(Z_8 \times Z_8) \rtimes A_4$

Modular Form

An example for modular invariant $\mathcal{L}$

$$\mathcal{L}_{\text{ex}} = f(\tau)\phi^{(1)} \cdots \phi^{(n)}$$

- $f(\tau)$: coupling constant
- $\phi^{(I)}$: fields

- $f(\tau), \phi^{(I)}$ are non-trivial representations of modular group Γ

Modular transformation:

$$\gamma \in \Gamma, \qquad \tau \to \frac{a\tau + b}{c\tau + d}, \qquad ad - bc = 1$$

$$f(\tau) \to (c\tau + d)^k \rho(\gamma) f(\tau)$$

← Modular form with weight k

$$\phi^{(I)} \to (c\tau + d)^{-k_I} \rho^{(I)}(\gamma) \phi^{(I)}$$

↓

When $k = \sum_I k_I$, $\mathcal{L}_{\text{ex}}$ is modular invariant.

Modular Form

We can expect effective theories with $\Gamma(N)$ symmetry.

$$\mathcal{L}_{\rm eff} \in f(\tau)\phi^{(1)} \cdots \phi^{(n)}$$

$f(\tau), \phi^{(I)}$: non-trivial rep. of $\Gamma(N)$

In some cases, concrete form of function $f(\tau)$ is found.

The modular form with weight 2 for

- $\Gamma(3) \simeq A_4$ ⇒ [F.Feruglio, arXiv:1706.08749]
- $\Gamma(2) \simeq S_3$ ⇒ Our work
- $\Gamma(4) \simeq S_4$ ⇒ [J.T.Penedo, S.T.Petcov, arXiv1806.11040]

Weight “even” = the product of weight 2

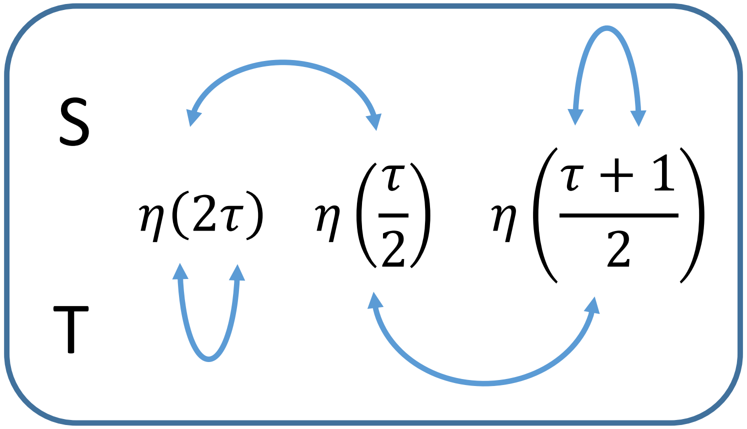
Modular Form

S_3 doublet (weight 2)   Representations: $1, 1', 2$

$$Y_1(\tau) = \frac{i}{4\pi}\left(\frac{\eta'(\tau/2)}{\eta(\tau/2)} + \frac{\eta'((\tau+1)/2)}{\eta((\tau+1)/2)} - \frac{8\eta'(2\tau)}{\eta(2\tau)}\right)$$

$$Y_2(\tau) = \frac{\sqrt{3}i}{4\pi}\left(\frac{\eta'(\tau/2)}{\eta(\tau/2)} - \frac{\eta'((\tau+1)/2)}{\eta((\tau+1)/2)}\right)$$

Multiplied by overall coefficient



Dedekind Eta function

Definition: $\eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)$, $q = e^{2\pi i\tau}$

Modular transformation: $\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau)$, $\eta(\tau+1) = e^{\frac{i\pi}{12}}\eta(\tau)$

- $\eta^{24}(\tau)$ is a modular form with weight 12

Model Search

- We don't consider the concrete setup in stringtheory
- We assume the form of $\mathcal{L}_{\mathrm{eff}}$

We have studied two models

- $\Gamma(3) \simeq A_4$
  - The same model as [F.Feruglio,arXiv:1706.08749]
  - Extended models

- $\Gamma(2) \simeq S_3$

Observables

PMNS Matrix

$$\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix} = U_{PMNS}\begin{pmatrix}\nu_1\\ \nu_2\\ \nu_3\end{pmatrix} \qquad \begin{pmatrix}1 & 0 & 0\\ 0 & c_{23} & s_{23}\\ 0 & -s_{23} & c_{23}\end{pmatrix}\begin{pmatrix}c_{13} & 0 & s_{13}e^{-i\delta_{CP}}\\ 0 & 1 & 0\\ -s_{13}e^{i\delta_{CP}} & 0 & c_{13}\end{pmatrix}\begin{pmatrix}c_{12} & s_{12} & 0\\ -s_{12} & c_{12} & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & e^{i\alpha_2/2} & 0\\ 0 & 0 & e^{i\alpha_3/2}\end{pmatrix}$$

Mass squared difference

$$\delta m^2 = m_2^2 - m_1^2 \qquad r = \frac{\delta m^2}{|\Delta m^2|}$$

$$\Delta m^2 = m_3^2 - \frac{m_1^2+m_2^2}{2}$$

Parameter	Normal Ordering	Inverted Ordering
$\delta m^2/10^{-5}\mathrm{eV}^2$	$7.37^{+0.17}_{-0.16}$	$7.37^{+0.17}_{-0.16}$
$\vert\Delta m^2\vert/10^{-3}\mathrm{eV}^2$	$2.525^{+0.042}_{-0.030}$	$2.505^{+0.034}_{-0.032}$
$\sin^2\theta_{12}/10^{-1}$	$2.97^{+0.17}_{-0.16}$	$2.97^{+0.17}_{-0.16}$
$\sin^2\theta_{13}/10^{-2}$	$2.15^{+0.07}_{-0.07}$	$2.16^{+0.08}_{-0.07}$
$\sin^2\theta_{23}/10^{-1}$	$4.25^{+0.21}_{-0.15}$	$5.89^{+0.16}_{-0.22} \oplus 4.33^{+0.15}_{-0.16}$
δ_{CP}/π	$1.38^{+0.23}_{-0.20}$	$1.31^{+0.31}_{-0.19}$
r	$2.92^{+0.10}_{-0.11}\times 10^{-2}$	$2.94^{+0.11}_{-0.10}\times 10^{-2}$

[Global constraints, arXiv:1703.04471] (other than r)

A_4 model

- Rep. of A_4 : $1, 1', 1'', 3$
- Flavon ϕ takes VEV $\langle\phi\rangle = (u, 0, 0)$: assumption
- No right-handed neutrino

The same model as [F.Feruglio,arXiv:1706.08749]

	$SU(2)_L, U(1)_Y$	A_4	k_I
$e^c_{R_1}$	$(\mathbf{1}, +1)$	$\mathbf{1}$	-4
$e^c_{R_2}$	$(\mathbf{1}, +1)$	$\mathbf{1}''$	-4
$e^c_{R_3}$	$(\mathbf{1}, +1)$	$\mathbf{1}'$	-4
L	$(\mathbf{2}, -1/2)$	$\mathbf{3}$	1
H_u	$(\mathbf{2}, +1/2)$	$\mathbf{1}$	0
H_d	$(\mathbf{2}, -1/2)$	$\mathbf{1}$	0
ϕ	$(\mathbf{1}, 0)$	$\mathbf{3}$	3

Superpotential

◆ Charged lepton mass term : diagonalized

$$w_e = \beta_1 e_1^c H_d (L\phi)_1 + \beta_2 e_2^c H_d (L\phi)_{1'} + \beta_3 e_3^c H_d (L\phi)_{1''}$$

⇒ $$m_e = u v_d \,\mathrm{diag}(\beta_1, \beta_2, \beta_3)$$

A_4 model

- Neutrino mass term:

$$w_\nu = \frac{1}{\Lambda}(H_u H_u L L Y)_1$$

$Y(\tau) = (Y_1, Y_2, Y_3)$:
Triplet, modular form with weight 2

$$\Rightarrow \quad m_\nu = \frac{v_u^2}{\Lambda}\begin{pmatrix} 2Y_1 & -Y_3 & -Y_2 \\ & 2Y_2 & -Y_1 \\ & & 2Y_3 \end{pmatrix}$$

- Mass matrix is strongly restricted
- No realistic values of Y_i for $(\pm 3\sigma)$

Extended models in A_4 (Preliminary)

Collaborate with Y. Shimizu, K. Takagi, M. Tanimoto

Model A

- Charged lepton mass: diagonalized by flavon
- Dirac neutrino, no see-saw

Model B

- Charged lepton mass: No flavon
    - B-1: Dirac neutrino, no see-saw
    - B-2: Majorana neutrino, no right-handed neutrino

$$w_e = \sum_{i=1,2,3} \alpha_i (Y L H_d e_i)_1$$

There are no realistic solution in Model A, B.

Extended models in A_4 (Preliminary)

Model C

- Charged lepton mass: No flavon
- Dirac neutrino
- See-saw

	$SU(2)_L, U(1)_Y$	A_4	k_I
e_1^c	$(\mathbf{1}, +1)$	$\mathbf{1}$	1
e_2^c	$(\mathbf{1}, +1)$	$\mathbf{1}''$	1
e_3^c	$(\mathbf{1}, +1)$	$\mathbf{1}'$	1
L	$(\mathbf{2}, -1/2)$	$\mathbf{3}$	1
N^c	$(\mathbf{1}, 0)$	$\mathbf{3}$	1
H_u	$(\mathbf{2}, +1/2)$	$\mathbf{1}$	0
H_d	$(\mathbf{2}, -1/2)$	$\mathbf{1}$	0

$$w_e = \sum_{i=1,2,3} \alpha_i (Y L H_d e_i)_1$$

$$w_\nu = g_1\left((Y)_3 (L H_u N^c)_{3s}\right)_1 + g_1\left((Y)_3 (L H_u N^c)_{3A}\right)_1 + \Lambda (Y N^c N^c)_1$$

$Y(\tau)$: A_4 triplet, modular form with weight 2

α_i: real

g_i: complex

Extended models in A_4 (Preliminary)

$$w_e = \sum_{i=1,2,3} \alpha_i (YLH_d e_i)_1$$

$$w_\nu = g_1\left((Y)_3 (LH_u N^c)_{3s}\right)_1 + g_1\left((Y)_3 (LH_u N^c)_{3A}\right)_1 + \Lambda(YN^c N^c)_1$$

- α_i are determined by charged lepton mass

Parameters for neutrino sector: 4 reals

$$\left(\frac{g_1}{g_2}\right), \tau$$

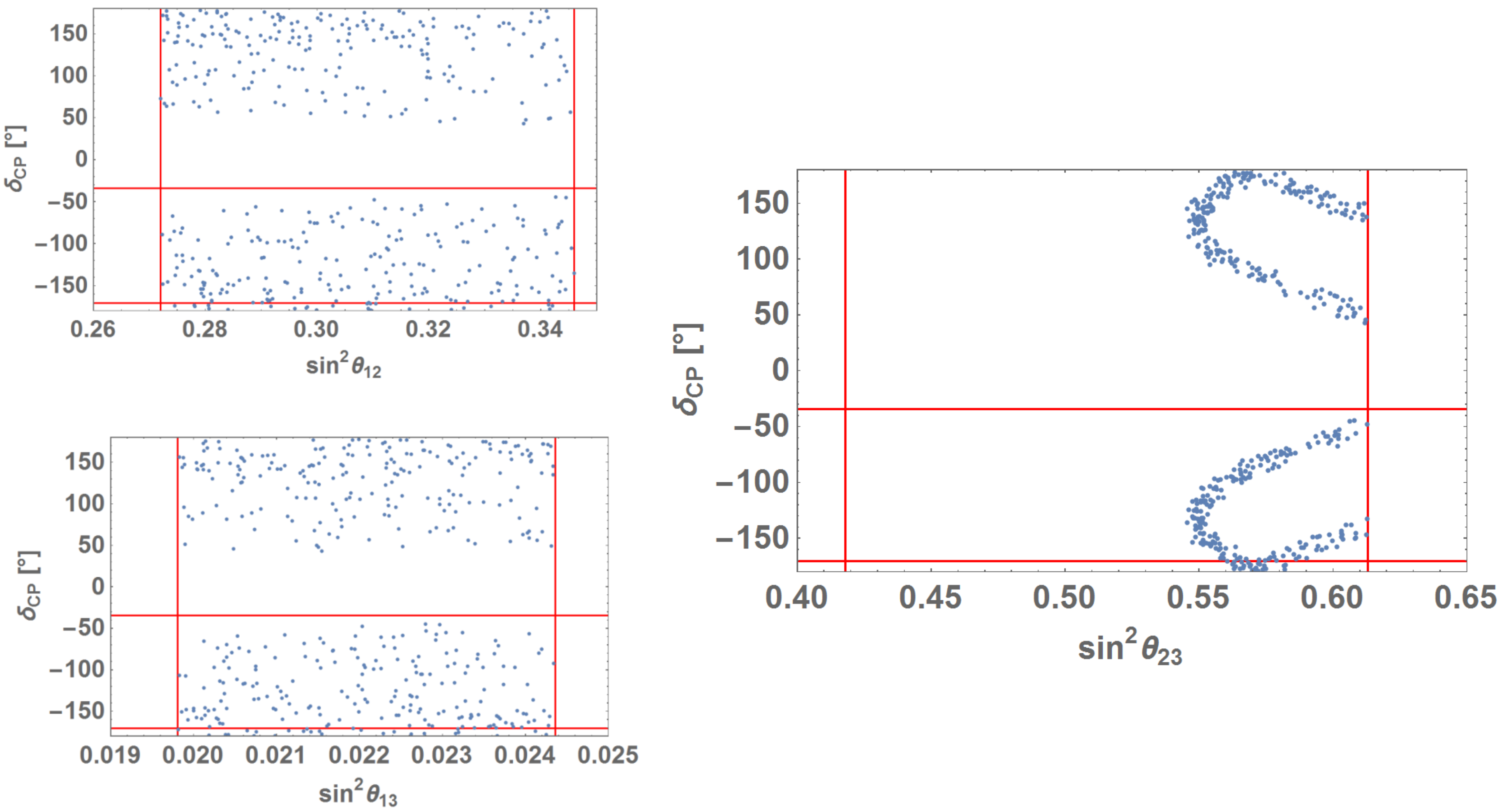
Input observables: 4 reals

$$\frac{\Delta m^2_{\mathrm{sol}}}{\Delta m^2_{\mathrm{atm}}}, \theta_{12}, \theta_{13}, \theta_{23}$$

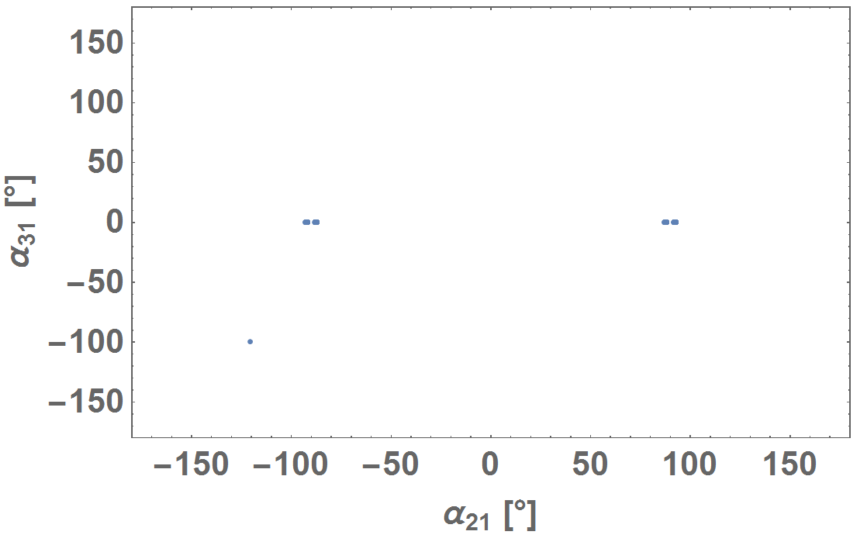
Prediction:

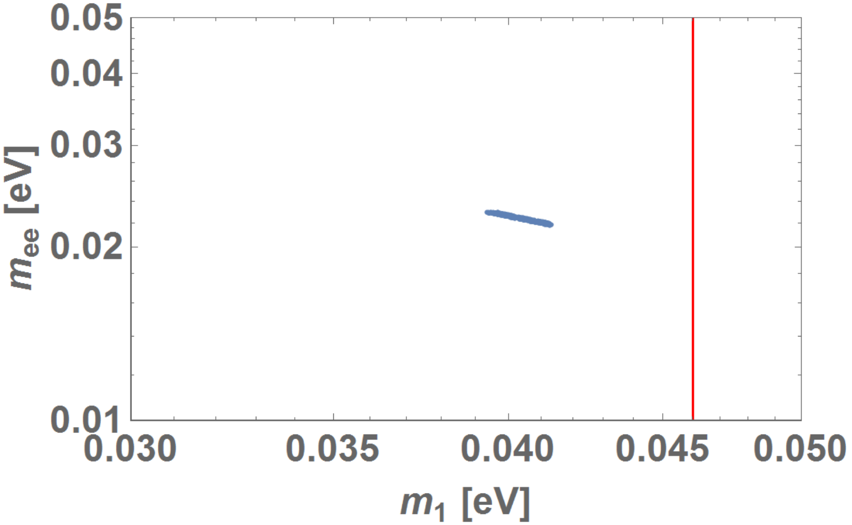
$$\delta_{CP}, \alpha_{12}, \alpha_{13}$$

Extended models in A_4 (Preliminary)



Extended models in A_4 (Preliminary)





S_3 model

- Rep. of S_3 : $1, 1', 2$
- Flavons $\phi^{(1)}, \phi^{(2)}$ take VEV $\langle\phi^{(1)}\rangle = u_1, \langle\phi^{(2)}\rangle = (u_2, 0)$: assumption
- No right-handed neutrino

	e_1^c	e_2^c	e_3^c	$L^{(1)}$	$L^{(2)}$	H_d	H_u	$\phi^{(1)}$	$\phi^{(2)}$
$SU(2)_L \times U(1)_Y$	$(1,+1)$	$(1,+1)$	$(1,+1)$	$(2,-1/2)$	$(2,-1/2)$	$(2,-1/2)$	$(2,+1/2)$	$(1,0)$	$(1,0)$
S_3	1	1	$1'$	1	2	1	1	1	2
n	-3	-4	-4	1	1	0	0	2	3

Superpotential

◆ Charged lepton mass term: diagonalized

$$w_e = \beta_1 e_1^c H_d \left(L^{(1)}\phi^{(1)}\right)_1 + \beta_2 e_2^c H_d \left(L^{(2)}\phi^{(2)}\right)_1 - \beta_3 e_3^c H_d \left(L^{(2)}\phi^{(2)}\right)_{1'}$$

◆ Neutrino mass term:

$$w_\nu = \frac{HH}{\Lambda}\left[d\left(L^{(2)}L^{(2)}\right)_2 Y^{(2)} + a\left(L^{(1)}L^{(2)}\right)_2 Y^{(2)} + b\left(L^{(1)}L^{(1)}\right)_1 Y^{(1)} + c\left(L^{(2)}L^{(2)}\right)_1 Y^{(1)}\right]$$

Modular forms with weight 2

$$Y^{(1)} = Y$$
$$Y^{(2)} = (Y_1, Y_2)$$

S_3 model

Three ways of generation assignment

Model	$L^{(1)}$	$L^{(2)}$	m_ν
1	L_3	(L_1, L_2)	$\begin{pmatrix} dY_2 & dY_1 & aY_1 \\ dY_1 & -dY_2 & aY_2 \\ aY_1 & aY_2 & 0 \end{pmatrix} + \begin{pmatrix} cY & 0 & 0 \\ 0 & cY & 0 \\ 0 & 0 & bY \end{pmatrix}$
2	L_2	(L_1, L_3)	$\begin{pmatrix} dY_2 & aY_1 & dY_1 \\ aY_1 & 0 & aY_2 \\ dY_1 & aY_2 & -dY_2 \end{pmatrix} + \begin{pmatrix} cY & 0 & 0 \\ 0 & bY & 0 \\ 0 & 0 & cY \end{pmatrix}$
3	L_1	(L_2, L_3)	$\begin{pmatrix} 0 & aY_1 & aY_2 \\ aY_1 & -dY_2 & dY_1 \\ aY_2 & dY_1 & dY_2 \end{pmatrix} + \begin{pmatrix} bY & 0 & 0 \\ 0 & cY & 0 \\ 0 & 0 & cY \end{pmatrix}$

4 complex parameters : a, b, c, τ（excluding an overall coefficient）

$Y_1(\tau), Y_2(\tau)$ are modular forms

5 experimental values : the ratio of mass differences 1, mixing angles 3, (CP phase 1)

S_3 model

Numerical results

Model	$\frac{r}{10^{-2}}$	$\frac{\sin^2\theta_{12}}{10^{-1}}$	$\frac{\sin^2\theta_{13}}{10^{-2}}$	$\frac{\sin^2\theta_{23}}{10^{-1}}$	$\frac{\delta_{CP}}{\pi}$	$\frac{\alpha_2}{\pi}$
Best fit	2.94	2.97	2.16	5.89	1.31	
1	2.91	2.79	2.37	5.95	1.79	0.26
2	2.93	2.94	2.30	5.81	0.37	0.16
3	2.89	3.02	2.16	5.89	0.48	0.14

Model	τ
1	$-0.507 + 0.781i$
2	$0.480 + 1.052i$
3	$-0.507 + 0.960i$

4:Summary

- Modular sym. can be originated from torus compactification
- In 4D effective theories of string, modular sym. can be broken into finite discrete symmetry in some setup

- We assumed flavor sym. of leptons come from modular group, and performed numerical study on PMNS matrix
- Coupling constant is non-trivial rep. of flavor sym. and modular form

- We studied A_4 and S_3 models
- ➢ A_4 model : Mass matrix is strongly restricted
  We found realistic solutions in NO

- ➢ S_3 model : There are more free parameters than A_4 model
  We found realistic solution in IO

Kinetic Term

Modular transformation:

$$\tau \to \frac{a\tau + b}{c\tau + d}, \qquad ad - bc = 1$$

$$f(\tau) \to (c\tau + d)^k f(\tau)$$

$$\phi_i' \to (c\tau + d)^{-k_i} \phi_i$$

Kinetic term is invariant

$$\mathcal{L}_{\mathrm{KE}} = \frac{\left|\partial_\mu \tau\right|^2}{\langle -i\tau + i\bar{\tau}\rangle^2} + \frac{\left|\partial_\mu \phi^{(I)}\right|^2}{\langle -i\tau + i\bar{\tau}\rangle^{k_I}}$$

Kinetic Term

Kinetic term of the modulus τ $$\frac{\left|\partial_\mu \tau\right|^2}{\langle -i\tau + i\bar{\tau}\rangle^2}$$

Modular transformation $$\tau' = \frac{a\tau + b}{c\tau + d}, ad - bc = 1$$

- numerator

$$\partial_\mu \tau' = \frac{(a\partial_\mu \tau)(c\tau + d) - (a\tau + b)(c\partial_\mu \tau)}{(c\tau + d)^2} = \frac{(ad - bc)\partial_\mu \tau}{(c\tau + d)^2} = \frac{\partial_\mu \tau}{(c\tau + d)^2}$$

- denominator

$$\tau' - \bar{\tau}' = \frac{(a\tau + b)(c\bar{\tau} + d) - (a\bar{\tau} + b)(c\tau + d)}{|c\tau + d|^2} = \frac{(ad - bc)(\tau - \bar{\tau})}{|c\tau + d|^2} = \frac{\tau - \bar{\tau}}{|c\tau + d|^2}$$

$$\Rightarrow \quad \frac{\left|\partial_\mu \tau'\right|^2}{\langle -i\tau' + i\bar{\tau}'\rangle^2} = \frac{\left|\partial_\mu \tau\right|^2}{\langle -i\tau + i\bar{\tau}\rangle^2}$$ Modular invariant

Modular Form

How to find the concrete form of modular form with weight 2 and non-trivial rep. of $\Gamma(N)$

- Suppose functions $f_i(\tau)$ to be modular forms with weight k_i
- Also suppose $\sum_i k_i = 0$

⇒ $\frac{d}{d\tau}\sum_i \log f_i(\tau)$ is a modular form with weight 2

Proof

Modular transformation: $\tau' = \frac{a\tau + b}{c\tau + d}, ad - bc = 1$

$$\frac{d}{d\tau'} = \frac{d\tau}{d\tau'}\frac{d}{d\tau} = (c\tau + d)^2 \frac{d}{d\tau}, \qquad f_i(\tau') = (c\tau + d)^{k_i} f_i(\tau)$$

⇓

$$\frac{d}{d\tau'}\sum_i \log f_i(\tau') = (c\tau + d)^2 \frac{d}{d\tau}\sum_i [\log f_i(\tau) + \underbrace{k_i(c\tau + d)]}_{=0}$$

$$= (c\tau + d)^2 \frac{d}{d\tau}\sum_i \log f_i(\tau)$$

➢ When we find a set of $f_i(\tau)$,
we can construct modular form with weight 2

Modular Form

Dedekind Eta function is convenient to construct modular function.

S_3 doublet

$\eta(2\tau), \eta\left(\frac{\tau}{2}\right), \eta\left(\frac{\tau+1}{2}\right)$ are closed with S- and T-transf.

S-transf.

$$\eta(2\tau) \to \sqrt{\frac{-i\tau}{2}}\eta(\tau/2),$$
$$\eta(\tau/2) \to \sqrt{-i3\tau}\eta(2\tau),$$
$$\eta((\tau+1)/2) \to e^{-i\pi/12}\sqrt{-i\tau}\eta((\tau+1)/2).$$

T-transf.

$$\eta(2\tau) \to e^{i\pi/6}\eta(2\tau),$$
$$\eta(\tau/2) \to \eta((\tau+1)/2),$$
$$\eta((\tau+1)/2) \to e^{i\pi/12}\eta(\tau/2).$$

⇒ $$Y(\alpha,\beta,\gamma|\tau) = \frac{d}{d\tau}\left(\alpha\log\eta(\tau/2) + \beta\log\eta((\tau+1)/2) + \gamma\log\eta(2\tau)\right)$$

with $\alpha+\beta+\gamma=0$ is a modular form with weight 2.

Modular Form

Conditions for α, β, γ

$$\begin{pmatrix} Y_1(-1/\tau) \\ Y_2(-1/\tau) \end{pmatrix} = \tau^2 \rho(S) \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \end{pmatrix}, \qquad \begin{pmatrix} Y_1(\tau+1) \\ Y_2(\tau+1) \end{pmatrix} = \rho(T) \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \end{pmatrix}.$$

$$\rho(S) = \frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}, \qquad \rho(T) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$\Rightarrow$ $$Y_1(\tau) = cY(1, 1, -2|\tau), \qquad Y_2(\tau) = \sqrt{3} cY(1, -1, 0|\tau)$$

Modular Form

A_4 triplet (weight 2)   Representations of A_4: $1, 1', 1'', 3$

$$Y_1(\tau) = \frac{i}{2\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)} - \frac{27\eta'(3\tau)}{\eta(3\tau)}\right)$$

$$Y_2(\tau) = \frac{-i}{\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega^2\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right)$$

$$Y_3(\tau) = \frac{-i}{\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega^2\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right)$$

Multiplied by overall coefficient

S_3 doublet (weight 2)   Representations of S_3: $1, 1', 2$

$$Y_1(\tau) = \frac{i}{4\pi}\left(\frac{\eta'(\tau/2)}{\eta(\tau/2)} + \frac{\eta'((\tau+1)/2)}{\eta((\tau+1)/2)} - \frac{8\eta'(2\tau)}{\eta(2\tau)}\right)$$

$$Y_2(\tau) = \frac{\sqrt{3}i}{4\pi}\left(\frac{\eta'(\tau/2)}{\eta(\tau/2)} - \frac{\eta'((\tau+1)/2)}{\eta((\tau+1)/2)}\right)$$

Multiplied by overall coefficient

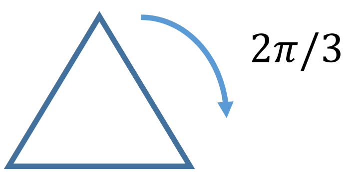
When we use more complicated compactification than torus, modular group Γ can be (partially) broken.

Torus T^2 → Modular group Γ

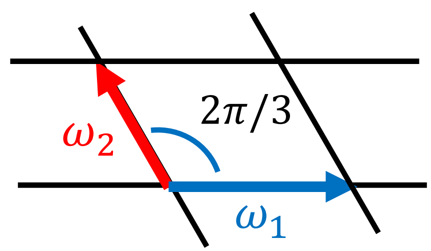
↓ More complicated geometry ↓ break

e.g.) toroidal orbifold T^2/Z_N → Subgroup of Γ

Z_N: rotational sym. of order N



Z_3



With identification of ↑ and →

T^2/Z_3 orbifold

S_3 model

S_3 model: Inverted Ordering

$$m_\nu = U^* \begin{pmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{pmatrix} U^\dagger = m_1 m^{(1)} + e^{-i\alpha_2} m_2 m^{(2)} + e^{-i\alpha_3} m_3 m^{(3)}$$
$$\simeq m_1 m^{(1)} + e^{-i\alpha_2} m_2 m^{(2)}$$

⇒ The value of α_2 is a prediction of the model.