Elliptic curves with a plural “s”

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I.: **Periodic functions and periods**

II: **No elliptic curves**

III: **One elliptic curve**

IV: **Several elliptic curves**

Part I

Periodic functions and periods

Periodic functions

Let us consider a non-constant meromorphic function f of a complex variable z.

A period ω of the function f is a constant such that for all z:

$$f(z+\omega) = f(z)$$

The set of all periods of f forms a lattice, which is either

- trivial (i.e. the lattice consists of $\omega = 0$ only),
- a simple lattice, $\Lambda = \{n\omega \mid n \in \mathbb{Z}\}$,
- a double lattice, $\Lambda = \{n_1\omega_1 + n_2\omega_2 \mid n_1, n_2 \in \mathbb{Z}\}$.

Examples of periodic functions

- Singly periodic function: Exponential function

$$\exp(z)\,.$$

$\exp(z)$ is periodic with peridod $\omega = 2\pi i$.

- Doubly periodic function: Weierstrass's $\wp$-function

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}} \left(\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right), \qquad \Lambda = \{n_1\omega_1 + n_2\omega_2 | n_1, n_2 \in \mathbb{Z}\}\,,$$

$$\mathrm{Im}(\omega_2/\omega_1) \neq 0.$$

$\wp(z)$ is periodic with periods ω_1 and ω_2.

Inverse functions

The corresponding inverse functions are in general multivalued functions.

- For the exponential function $x = \exp(z)$ the inverse function is the logarithm

$$z = \ln(x) .$$

- For Weierstrass's elliptic function $x = \wp(z)$ the inverse function is an elliptic integral

$$z = \int\limits_x^\infty \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}}, \qquad g_2 = 60 \sum_{\omega \in \Lambda \backslash \{0\}} \frac{1}{\omega^4}, \quad g_3 = 140 \sum_{\omega \in \Lambda \backslash \{0\}} \frac{1}{\omega^6}.$$

Periods as integrals over algebraic functions

In both examples the periods can be expressed as integrals involving only algebraic functions.

- Period of the exponential function:

$$2\pi i \quad = \quad 2i \int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}}.$$

- Periods of Weierstrass's $\wp$-function: Assume that g_2 and g_3 are two given algebraic numbers. Then

$$\omega_1 = 2\int_{t_1}^{t_2} \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}}, \qquad \omega_2 = 2\int_{t_3}^{t_2} \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}},$$

where t_1, t_2 and t_3 are the roots of the cubic equation $4t^3 - g_2 t - g_3 = 0$.

Numerical periods

Kontsevich and Zagier suggested the following generalisation:

A numerical period is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients, over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients.

Remarks:

- One can replace "rational" with "algebraic".
- The set of all periods is countable.
- Example: $\ln 2$ is a numerical period.

$$\ln 2 \quad = \quad \int_1^2 \frac{dt}{t}.$$

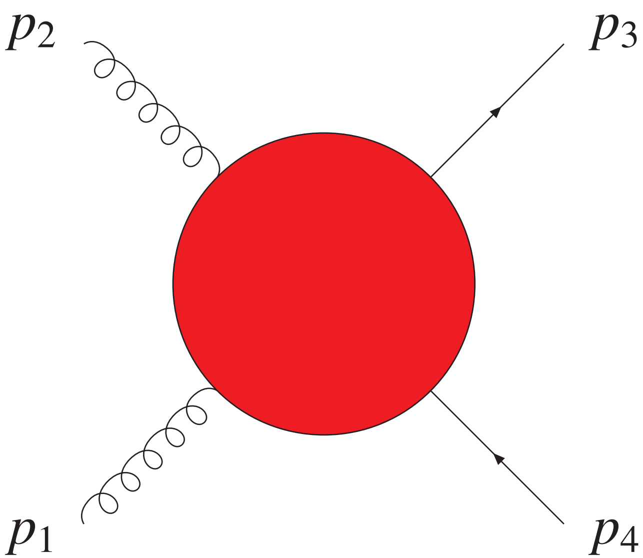
Part II

No elliptic curves

(Introduction to Feynman integrals)

Scattering amplitudes

For a theoretical description we need to know the scattering amplitude:

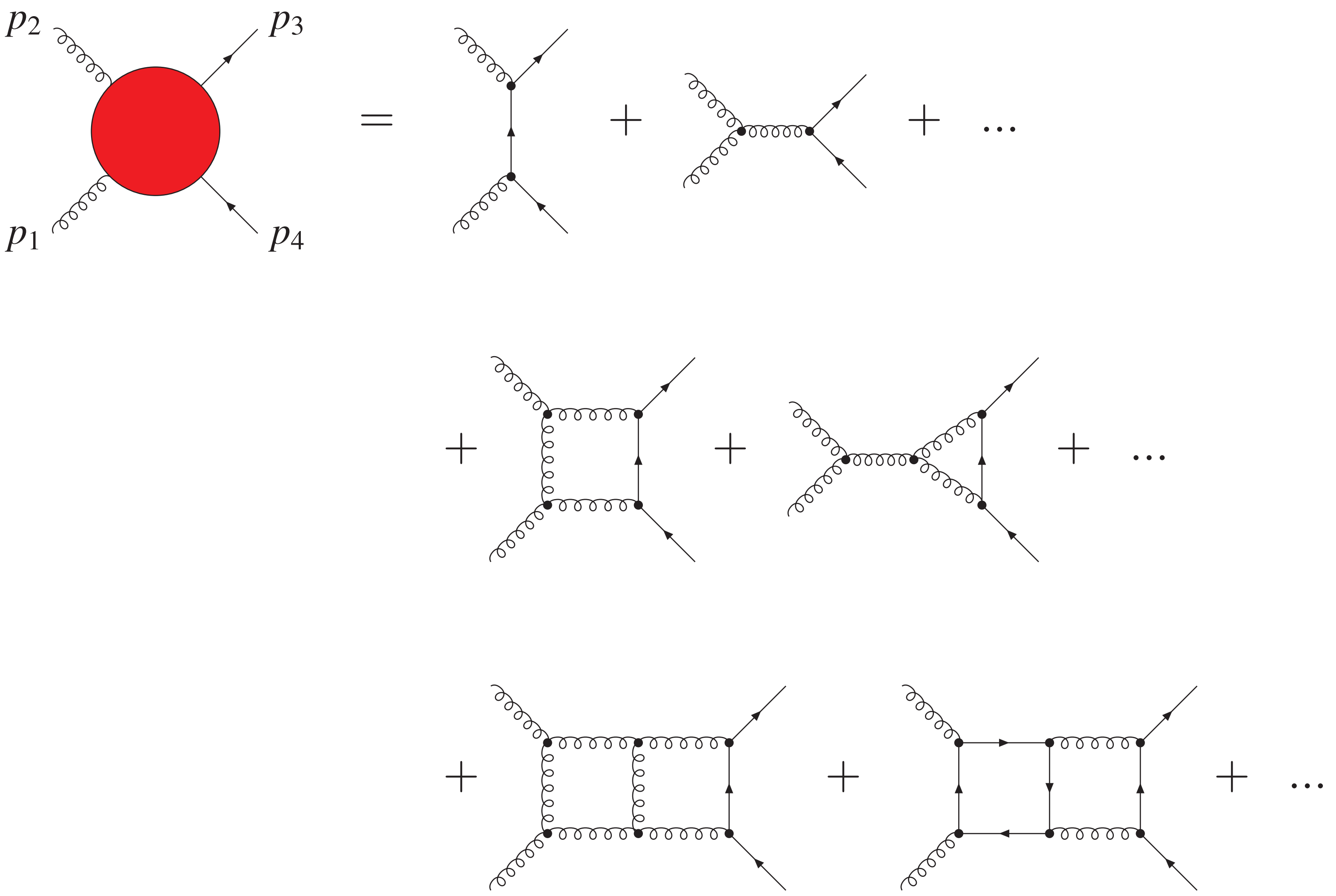


N_{ext} external particles with momenta p_1, ..., $p_{N_{\mathrm{ext}}}$.

Momentum conservation: $p_1 + ... + p_{N_{\mathrm{ext}}} = 0$.

Feynman diagrams

We may compute the scattering amplitude within perturbation theory:

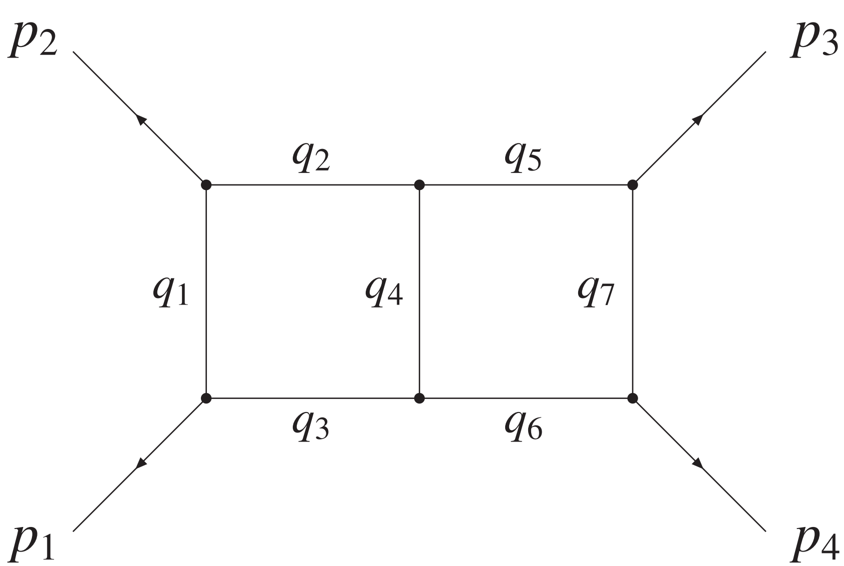


$\}\ O(g^2)$

$\}\ O(g^4)$

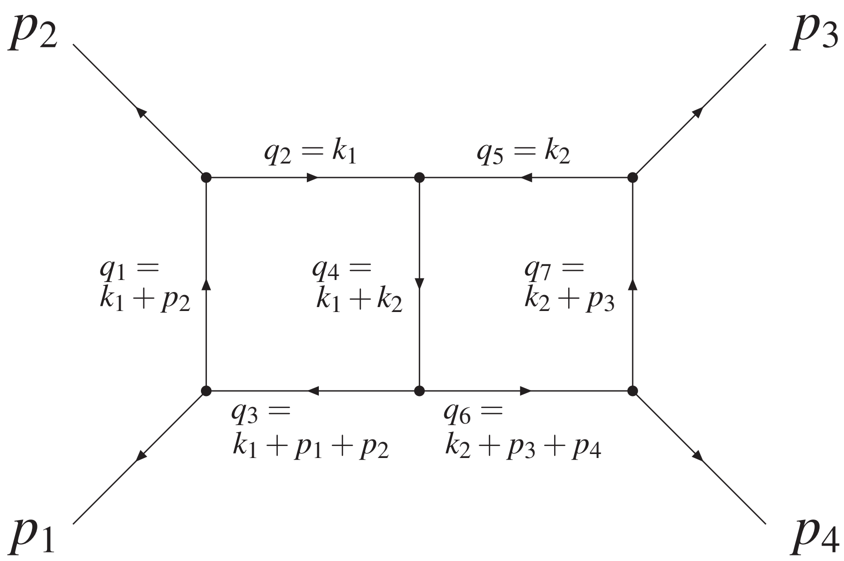
$\}\ O(g^6)$

Feynman integrals



- Dimensional regularisation:
  - Work in D space-time dimensions.
  - Set $D = 4 - 2\varepsilon$.
- Consider a graph G with
  - N_{ext} external legs
  - n internal edges
  - l loops (= first Betti number)
- To each internal edge e_j associate
  - a momentum q_j (a D-dimensional vector)
  - a mass m_j

Feynman integrals



- Choose an orientation for each internal edge
- Choose l independent loop momenta k_1, ..., k_l.
- Impose momentum conservation at each vertex.
- This gives

$$q_i = \sum_{j=1}^{l} \lambda_{ij} k_j + \sum_{j=1}^{N_{\text{ext}}} \sigma_{ij} p_j,$$

$$\lambda_{ij}, \sigma_{ij} \in \{-1, 0, 1\}.$$

Feynman rules

Each part in a Feynman graph corresponds to a mathematical expression.
In the simplest version:

- Edge:
$$\frac{1}{q^2 - m^2}$$

- Vertex:
$$1$$

- External line:
$$1$$

- For each internal momentum not constrained by momentum conservation
$$\int \frac{d^D k}{(2\pi)^D}$$

Feynman integrals

Associate to a Feynman graph G with N_{ext} external lines, n internal lines and l loops the set of Feynman integrals

$$I_{\nu_1 \nu_2 \ldots \nu_n} = \left(\mu^2\right)^{\nu - lD/2} \int \frac{d^D k_1}{(2\pi)^D} \ldots \frac{d^D k_l}{(2\pi)^D} \prod_{j=1}^{n} \frac{1}{\left(q_j^2 - m_j^2\right)^{\nu_j}},$$

with $\nu_j \in \mathbb{Z}$ and $\nu = \nu_1 + \ldots + \nu_n$.

The arbitrary scale μ makes the Feynman integral dimensionless. We may choose μ^2 to be given as an expression of $p_i \cdot p_j$ and m_j^2.

Feynman integrals

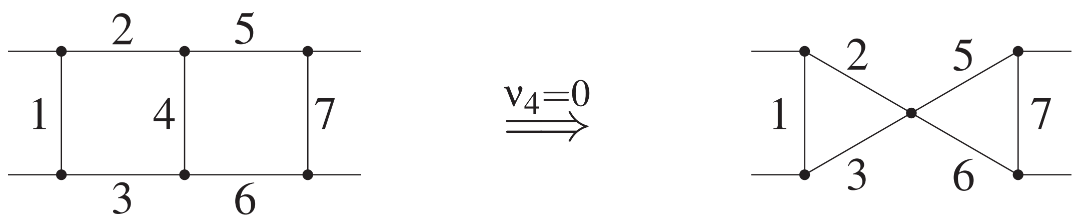
- $I_{\nu_1\nu_2...\nu_n}$ is a function of D and the kinematic variables $p_i \cdot p_j$ and m_j^2.
- How many independent kinematic variables are there?
  - $\frac{1}{2}N_{\text{ext}}\left(N_{\text{ext}}-1\right)$ invariants $p_i \cdot p_j$,
  - n internal masses m_j^2.
- $I_{\nu_1\nu_2...\nu_n}$ depends only on ratios of kinematic variables. Set

$$N_B = \frac{1}{2}N_{\text{ext}}\left(N_{\text{ext}}-1\right)+n-1$$

- kinematic base manifold: $M = \mathbb{P}^{N_B}\left(\mathbb{C}\right)$
- $I_{\nu_1\nu_2...\nu_n}$ is a function on $\mathbb{C} \times \mathbb{P}^{N_B}\left(\mathbb{C}\right)$ with $D \in \mathbb{C}$ and $x \in \mathbb{P}^{N_B}\left(\mathbb{C}\right)$.

Pinching of propagators

If for some exponent we have $\nu_j = 0$, the corresponding propagator is absent and the topology simplifies:



Integration by parts

Within dimensional regularisation we have for any loop momentum k_i and $v \in \{p_1, ..., p_{N_{\text{ext}}}, k_1, ..., k_l\}$

$$\int \frac{d^D k_1}{(2\pi)^D} \cdots \frac{d^D k_l}{(2\pi)^D} \frac{\partial}{\partial k_i^\mu} v^\mu \prod_{j=1}^{n} \frac{1}{\left(q_j^2 - m_j^2\right)^{\nu_j}} = 0.$$

Working out the derivatives leads to relations among integrals with different sets of indices $(\nu_1, ..., \nu_n)$.

This allows us to express most of the integrals in terms of a few master integrals.

Tkachov '81, Chetyrkin '81

Laporta's algorithm

Expressing all integrals in terms of the master integrals requires to solve a rather large linear system of equations.

This system has a block-triangular structure, originating from subtopologies.

Order the integrals by complexity (more propagators $\Rightarrow$ more difficult)

Solve the system bottom-up, re-using the results for the already solved sectors.

Laporta '01

Master integrals

Let us denote the number of master integrals by N_{master}.

The integrands of the master integrals span the cohomology group

$$\frac{d^D k_1}{(2\pi)^D} \cdots \frac{d^D k_l}{(2\pi)^D} \prod_{j=1}^{n} \frac{1}{\left(q_j^2 - m_j^2\right)^{\nu_j}} \quad \mathrm{mod}\ (\text{exact forms})$$

Denote by F the vector space spanned by the master integrals.
Clearly, $\dim F = N_{\mathrm{master}}$.

This defines the fibre F.

Differential equations

Let x_k be a kinematic variable. Let $I_i \in \{I_1, \dots, I_{N_{\mathrm{master}}}\}$ be a master integral. Carrying out the derivative

$$\frac{\partial}{\partial x_k} I_i$$

under the integral sign and using integration-by-parts identities allows us to express the derivative as a linear combination of the master integrals.

$$\frac{\partial}{\partial x_k} I_i = \sum_{j=1}^{N_{\mathrm{master}}} a_{ij} I_j$$

(Kotikov '90, Remiddi '97, Gehrmann and Remiddi '99)

Differential equations

Let us formalise this:

$\vec{I} = (I_1, ..., I_{N_{\mathrm{master}}})$, set of master integrals,

$\vec{x} = (x_1, ..., x_{N_B})$, set of kinematic variables the master integrals depend on.

We obtain a system of differential equations of Fuchsian type

$$d\vec{I} + A\vec{I} = 0,$$

where A is a matrix-valued one-form

$$A = \sum_{i=1}^{N_B} A_i dx_i.$$

The matrix-valued one-form A satisfies the integrability condition

$$dA + A \wedge A = 0 \qquad \text{(flat Gauß-Manin connection).}$$

Computation of Feynman integrals reduced to solving differential equations!

The ε-form of the differential equation

If we change the basis of the master integrals $\vec{J} = U\vec{I}$, the differential equation becomes

$$(d + A')\vec{J} = 0, \qquad A' = UAU^{-1} + U dU^{-1}$$

Suppose one finds a transformation matrix U, such that

$$A' \quad = \quad \varepsilon \sum_j C_j \, d \ln p_j(\vec{x}),$$

where

- ε appears only as prefactor,
- C_j are matrices with constant entries,
- $p_j(\vec{x})$ are polynomials in the external variables,

then the system of differential equations is easily solved in terms of multiple polylogarithms.

Henn ’13

Transformation to the ε-form

We may

- perform a rational / algebraic transformation on the kinematic variables

$$(x_1, \dots, x_{N_B}) \quad \rightarrow \quad (x'_1, \dots, x'_{N_B}),$$

  often done to absorb square roots.

- change the basis of the master integrals

$$\vec{I} \quad \rightarrow \quad U\vec{I},$$

  where U is rational in the kinematic variables

Henn '13; Gehrmann, von Manteuffel, Tancredi, Weihs '14; Argeri et al. '14; Lee '14; Meyer '16; Prausa '17; Gituliar, Magerya '17; Lee, Pomeransky '17;

Multiple polylogarithms

Definition based on nested sums:

$$\text{Li}_{m_1,m_2,\ldots,m_k}(x_1,x_2,\ldots,x_k) = \sum_{n_1>n_2>\ldots>n_k>0}^{\infty} \frac{x_1^{n_1}}{n_1^{m_1}} \cdot \frac{x_2^{n_2}}{n_2^{m_2}} \cdot \ldots \cdot \frac{x_k^{n_k}}{n_k^{m_k}}$$

Definition based on iterated integrals:

$$G(z_1,\ldots,z_k;y) = \int_0^y \frac{dt_1}{t_1 - z_1} \int_0^{t_1} \frac{dt_2}{t_2 - z_2} \ldots \int_0^{t_{k-1}} \frac{dt_k}{t_k - z_k}$$

Conversion:

$$\text{Li}_{m_1,\ldots,m_k}(x_1,\ldots,x_k) = (-1)^k G_{m_1,\ldots,m_k}\left(\frac{1}{x_1}, \frac{1}{x_1 x_2}, \ldots, \frac{1}{x_1 \ldots x_k}; 1\right)$$

Short hand notation:

$$G_{m_1,\ldots,m_k}(z_1,\ldots,z_k;y) = G(\underbrace{0,\ldots,0}_{m_1-1}, z_1, \ldots, z_{k-1}, \underbrace{0\ldots,0}_{m_k-1}, z_k; y)$$

Example

Let us consider a simple example: One integral I in one variable x with boundary condition $I(0) = 1$. Consider the differential equation

$$(d + A) I = 0, \qquad A = -\varepsilon \, d \ln(x-1).$$

Note that

$$d \ln(x-1) = \frac{dx}{x-1}$$

and

$$I(x) = 1 + \varepsilon G(1;x) + \varepsilon^2 G(1,1;x) + \varepsilon^3 G(1,1,1;x) + \ldots$$

Iterated integrals

For ω_1, ..., ω_k differential 1-forms on a manifold M and $\gamma : [0,1] \to M$ a path, write for the pull-back of ω_j to the interval $[0,1]$

$$f_j(\lambda)\, d\lambda \quad = \quad \gamma^* \omega_j.$$

The iterated integral is defined by (Chen '77)

$$I_\gamma(\omega_1, \ldots, \omega_k; \lambda) \quad = \quad \int_0^\lambda d\lambda_1 f_1(\lambda_1) \int_0^{\lambda_1} d\lambda_2 f_2(\lambda_2) \ldots \int_0^{\lambda_{k-1}} d\lambda_k f_k(\lambda_k).$$

Example 1: Multiple polylogarithms (Goncharov '98)

$$\omega_j \quad = \quad \frac{d\lambda}{\lambda - z_j}.$$

Example 2: Iterated integrals of modular forms (Brown '14): $f_j(\tau)$ a modular form,

$$\omega_j \quad = \quad 2\pi i\, f_j(\tau)\, d\tau.$$

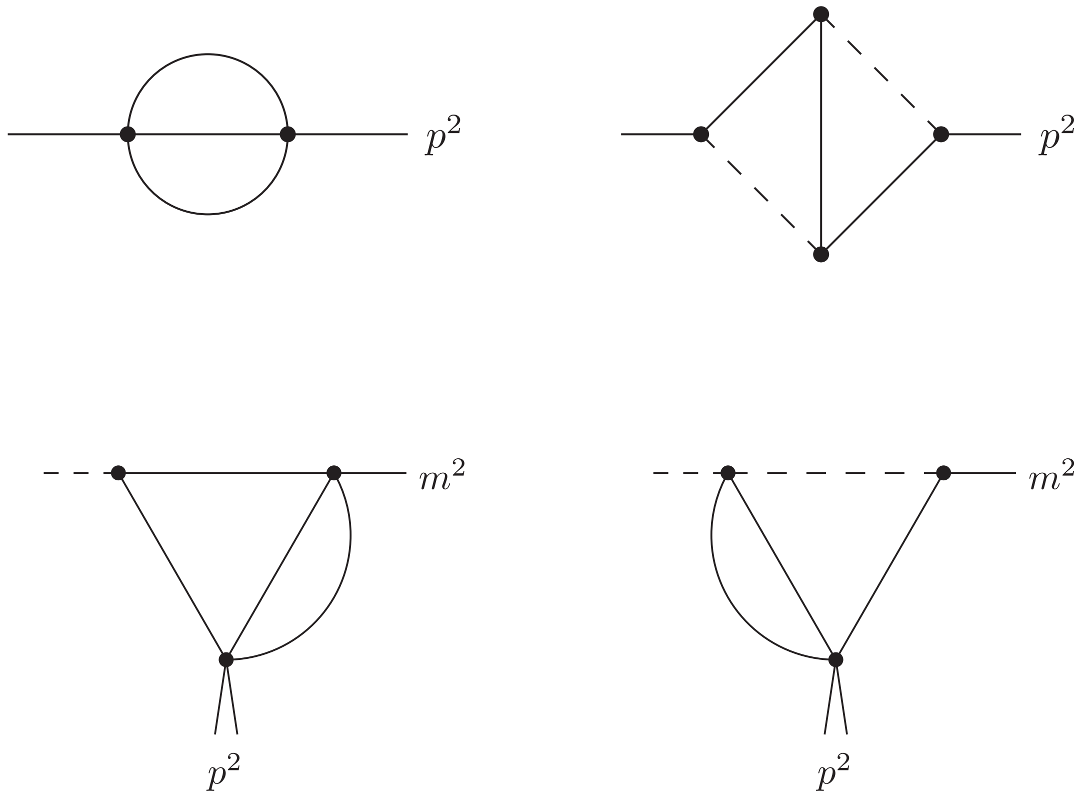
Part III

One elliptic curve

(Feynman integrals beyond multiple polylogarithms)

Single-scale Feynman integrals beyond multiple polylogarithms

Not all Feynman integrals are expressible in terms of multiple polylogarithms!



The Picard-Fuchs operator

Let I be one of the master integrals $\{I_1, ..., I_{N_{\mathrm{master}}}\}$. Choose a path $\gamma : [0,1] \to M$ and study the integral I as a function of the path parameter λ.

Instead of a system of N_{master} first-order differential equations

$$(d + A)\vec{I} \quad = \quad 0,$$

we may equivalently study a single differential equation of order N_{master}

$$\sum_{j=0}^{N_{\mathrm{master}}} p_j(\lambda) \frac{d^j}{d\lambda^j} I \quad = \quad 0.$$

We may work modulo sub-topologies and ε-corrections:

$$L = \sum_{j=0}^{r} p_j(\lambda) \frac{d^j}{d\lambda^j} : \qquad L\, I = 0 \quad \mathrm{mod} \; (\text{sub-topologies, } \varepsilon\text{-corrections})$$

Factorisation of the Picard-Fuchs operator

Suppose the differential operator factorises into linear factors:

$$L = \left(a_r(\lambda)\frac{d}{d\lambda} + b_r(\lambda)\right) \dots \left(a_2(\lambda)\frac{d}{d\lambda} + b_2(\lambda)\right)\left(a_1(\lambda)\frac{d}{d\lambda} + b_1(\lambda)\right)$$

Iterated first-order differential equation.

Denote homogeneous solution of the j-th factor by

$$\psi_j(\lambda) = \exp\left(-\int_0^\lambda d\kappa\, \frac{b_j(\kappa)}{a_j(\kappa)}\right).$$

Full solution given by iterated integrals

$$C_1\psi_1(\lambda) + C_2\psi_1(\lambda)\int_0^\lambda d\lambda_1 \frac{\psi_2(\lambda_1)}{a_1(\lambda_1)\psi_1(\lambda_1)} + C_3\psi_1(\lambda)\int_0^\lambda d\lambda_1 \frac{\psi_2(\lambda_1)}{a_1(\lambda_1)\psi_1(\lambda_1)}\int_0^{\lambda_1} d\lambda_2 \frac{\psi_3(\lambda_2)}{a_2(\lambda_2)\psi_2(\lambda_2)} + \dots$$

Multiple polylogarithms are of this form.

Picard-Fuchs operator: Beyond linear factors

Suppose the differential operator

$$\sum_{j=0}^{r} p_j(\lambda) \frac{d^j}{d\lambda^j}$$

does not factor into linear factors.

The next more complicate case:
The differential operator contains one irreducible second-order differential operator

$$a_j(\lambda) \frac{d^2}{d\lambda^2} + b_j(\lambda) \frac{d}{d\lambda} + c_j(\lambda)$$

An example from mathematics: Elliptic integral

The differential operator of the second-order differential equation

$$\left[k\left(1-k^2\right)\frac{d^2}{dk^2}+\left(1-3k^2\right)\frac{d}{dk}-k\right]f(k) \quad = \quad 0$$

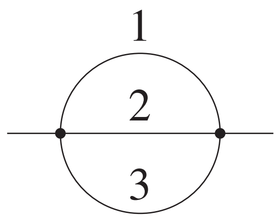
is irreducible.

The solutions of the differential equation are $K(k)$ and $K(\sqrt{1-k^2})$, where $K(k)$ is the complete elliptic integral of the first kind:

$$K(k) \quad = \quad \int\limits_0^1 \frac{dx}{\sqrt{\left(1-x^2\right)\left(1-k^2x^2\right)}}.$$

An example from physics: The two-loop sunrise integral

$$S_{\nu_1\nu_2\nu_3}(D,x) \quad =$$



Picard-Fuchs operator for $S_{111}(2,x)$:

$$L \quad = \quad x(x-1)(x-9)\frac{d^2}{dx^2} + \left(3x^2 - 20x + 9\right)\frac{d}{dx} + (x-3)$$

(Broadhurst, Fleischer, Tarasov ’93)

Irreducible second-order differential operator.

Picard-Fuchs operator for the periods of a family of elliptic curves.

The elliptic curve

How to get the elliptic curve?

- From the Feynman graph polynomial:

$$-x_1 x_2 x_3 x + (x_1 + x_2 + x_3)(x_1 x_2 + x_2 x_3 + x_3 x_1) = 0$$

- From the maximal cut:

$$v^2 - (u-x)(u-x+4)\left(u^2+2u+1-4x\right) = 0$$

Baikov '96; Lee '10; Kosower, Larsen, '11; Caron-Huot, Larsen, '12; Frellesvig, Papadopoulos, '17; Bosma, Sogaard, Zhang, '17; Harley, Moriello, Schabinger, '17

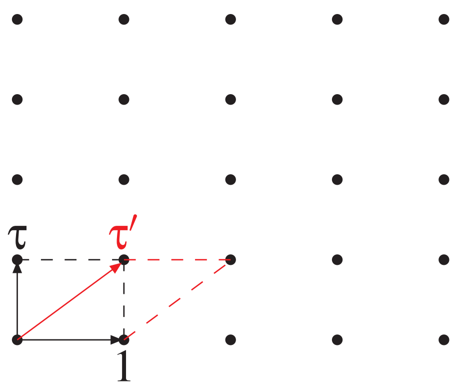
The periods ψ_1, ψ_2 of the elliptic curve are solutions of the homogeneous differential equation.

Adams, Bogner, S.W., '13; Primo, Tancredi, '16

$$\text{Set} \quad \tau = \frac{\psi_2}{\psi_1}, \qquad q = e^{2i\pi\tau}.$$

Bases of lattices

The periods ψ_1 and ψ_2 generate a lattice. Any other basis as good as (ψ_2, ψ_1).
Convention: Normalise $(\psi_2, \psi_1) \to (\tau, 1)$ where $\tau = \psi_2/\psi_1$.



Change of basis: $$\begin{pmatrix} \psi_2' \\ \psi_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix},$$

Transformation should be invertible: $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}\,(2, \mathbb{Z})\,,$$

In terms of τ and τ': $$\tau' = \frac{a\tau + b}{c\tau + d}$$

The ε-form of the differential equation for the sunrise

It is not possible to obtain an ε-form by a rational/algebraic change of variables and/or a rational/algebraic transformation of the basis of master integrals.

However by factoring off the (non-algebraic) expression ψ_1/π from the master integrals in the sunrise sector one obtains an ε-form:

$$I_1 = 4\varepsilon^2 S_{110}\left(2-2\varepsilon,x\right), \quad I_2 = -\varepsilon^2 \frac{\pi}{\psi_1} S_{111}\left(2-2\varepsilon,x\right), \quad I_3 = \frac{1}{\varepsilon}\frac{1}{2\pi i}\frac{d}{d\tau} I_2 + \frac{1}{24}\left(3x^2-10x-9\right)\frac{\psi_1^2}{\pi^2} I_2.$$

If in addition one makes a (non-algebraic) change of variables from x to τ, one obtains

$$\frac{d}{d\tau}\vec{I} \quad = \quad \varepsilon\, A(\tau)\, \vec{I},$$

where $A(\tau)$ is an ε-independent 3×3-matrix whose entries are modular forms.

The ε-form of the differential equation for the sunrise

The matrix $A(\tau)$ is given by

$$A(\tau) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -f_2(\tau) & 1 \\ \frac{1}{4}f_3(\tau) & f_4(\tau) & -f_2(\tau) \end{pmatrix},$$

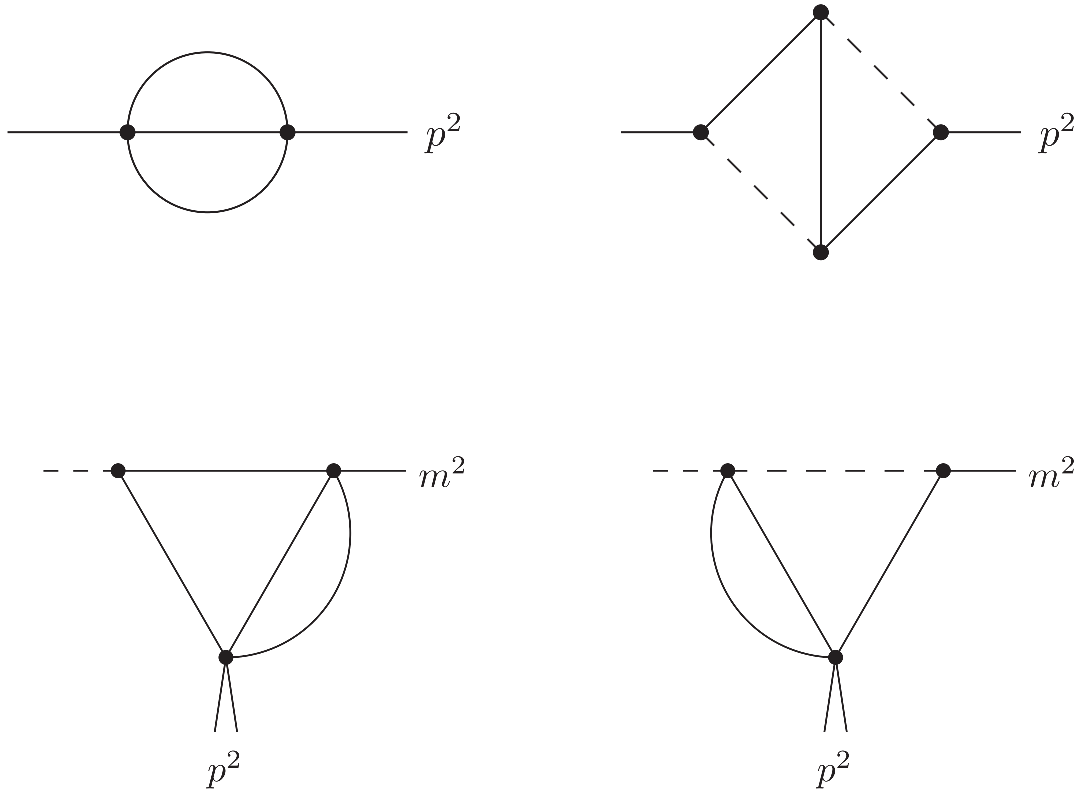
where f_2, f_3 and f_4 are modular forms of $\Gamma_1(6)$ of modular weight 2, 3 and 4, respectively.

I_1, I_2 and I_3 are expressed as iterated integrals of modular forms to all orders in ε.

Adams, S.W., ’17, ’18

Feynman integrals evaluating to iterated integrals of modular forms

This applies to a wider class of Feynman integrals:



Congruence subgroups

Apart from $\mathrm{SL}_2(2,\mathbb{Z})$ we may also look at congruence subgroups, for example

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}_2(\mathbb{Z}) : c \equiv 0 \bmod N \right\}$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}_2(\mathbb{Z}) : a,d \equiv 1 \bmod N,\ c \equiv 0 \bmod N \right\}$$

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}_2(\mathbb{Z}) : a,d \equiv 1 \bmod N,\ b,c \equiv 0 \bmod N \right\}$$

Modular forms for congruence subgroups: Require "nice" transformation properties only for subgroup Γ (plus holomorphicity on $\mathbb{H}$ and at the cusps).

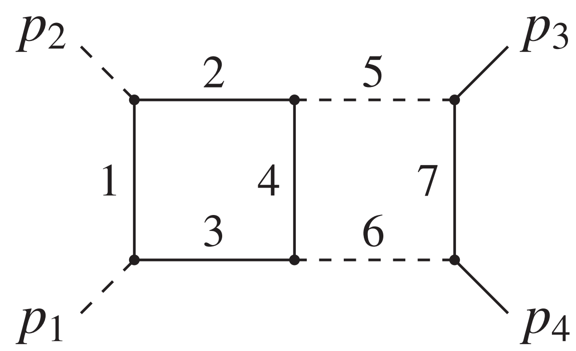
Part IV

Several elliptic curves

(An example from top-pair production)

Kinematics

$$I_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7}\left(D,\frac{s}{m^2},\frac{t}{m^2}\right) = \left(m^2\right)^{\sum\limits_{j=1}^{7}\nu_j - D} \int \frac{d^Dk_1}{(2\pi)^D}\frac{d^Dk_2}{(2\pi)^D}\prod_{j=1}^{7}\frac{1}{P_j^{\nu_j}},$$



$$p_1^2 = p_2^2 = 0, \qquad p_3^2 = p_4^2 = m^2,$$

$$s = (p_1 + p_2)^2, \qquad t = (p_2 + p_3)^2.$$

Picard-Fuchs operator of elliptic curves

- Sunrise integral: An elliptic curve can be obtained either from
  - Feynman graph polynomial
  - maximal cut

  The periods ψ_1, ψ_2 are the solutions of the homogeneous differential equations.

  Adams, Bogner, S.W., '13, '14

- In general: The maximal cuts are solutions of the homogeneous differential equations.

  Primo, Tancredi, '16

Search for Feynman integrals, whose maximal cuts are periods of an elliptic curve.

Maximal cuts

Maximal cut: For a Feynman integral

$$I_{\nu_1\nu_2\ldots\nu_n} = \left(\mu^2\right)^{\nu - lD/2} \int \frac{d^Dk_1}{(2\pi)^D} \cdots \frac{d^Dk_l}{(2\pi)^D} \prod_{j=1}^{n} \frac{1}{P_j^{\nu_j}}$$

take the n-fold residue at

$$P_1 = \ldots = P_n = 0$$

of the integrand and integrate over the remaining $(lD-n)$ variables along a contour $\mathcal{C}$.

Maximal cuts

Sunrise :

$$\mathrm{MaxCut}_{\mathcal{C}}\, I_{1001001}\,(2-2\varepsilon) =$$

$$\frac{um^2}{\pi^2}\int_{\mathcal{C}} \frac{dP}{(P-t)^{\frac{1}{2}}\,(P-t+4m^2)^{\frac{1}{2}}\,(P^2+2m^2P-4m^2t+m^4)^{\frac{1}{2}}} + O(\varepsilon)\,.$$

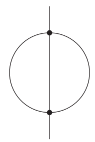
Double box :

$$\mathrm{MaxCut}_{\mathcal{C}}\, I_{1111111}\,(4-2\varepsilon) =$$

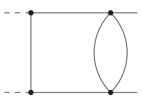
$$\frac{um^6}{4\pi^4 s^2}\int_{\mathcal{C}} \frac{dP}{(P-t)^{\frac{1}{2}}\,(P-t+4m^2)^{\frac{1}{2}}\left(P^2+2m^2P-4m^2t+m^4-\frac{4m^2\left(m^2-t\right)^2}{s}\right)^{\frac{1}{2}}} + O(\varepsilon)\,.$$

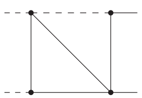
Three elliptic curves

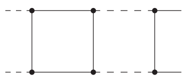
$$E^{(a)} \quad : \quad w^2 = (z-t)\left(z-t+4m^2\right)\left(z^2+2m^2z-4m^2t+m^4\right)$$



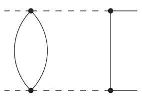
$$E^{(b)} \quad : \quad w^2 = (z-t)\left(z-t+4m^2\right)\left(z^2+2m^2z-4m^2t+m^4-\frac{4m^2\left(m^2-t\right)^2}{s}\right)$$







$$E^{(c)} \quad : \quad w^2 = (z-t)\left(z-t+4m^2\right)\left(z^2+\frac{2m^2\left(s+4t\right)}{\left(s-4m^2\right)}z+\frac{sm^2\left(m^2-4t\right)-4m^2t^2}{s-4m^2}\right)$$



Remarks

- $E^{(a)}$ gives rise to iterated integrals of modular forms of $\Gamma_1(6)$.
- For $s \to \infty$ the curves $E^{(b)}$ and $E^{(c)}$ degenerate to $E^{(a)}$.
- If we would have only one curve, we expect that the result can be written in elliptic polylogarithms.
- We have three elliptic curves.

Results

The differential equation for the master integrals can be brought into the form

$$d\vec{I} = \varepsilon A \vec{I},$$

where A is independent of ε.

The Laurent expansion in ε of all master integrals can be computed systematically to all orders in ε in terms of iterated integrals.

The solution

- reduces to multiple polylogarithms for $t = m^2$ and
- reduces to iterated integrals of modular forms of $\Gamma_1(6)$ for $s = \infty$.

Adams, Chaubey, S.W., '18

Conclusions

- Loop integrals with masses important for top, W/Z- and H-physics.
- May involve elliptic sectors from two loops onwards.
- There is a class of Feynman integrals evaluating to iterated integrals of modular forms.
- The planar double box integral relavant to $t\bar{t}$-production with a closed top loop depends on two variables and involves several elliptic sub-sectors. More than one elliptic curve occurs. Results expressed in terms of Chen's iterated integrals.
- We may expect more results in the near future.