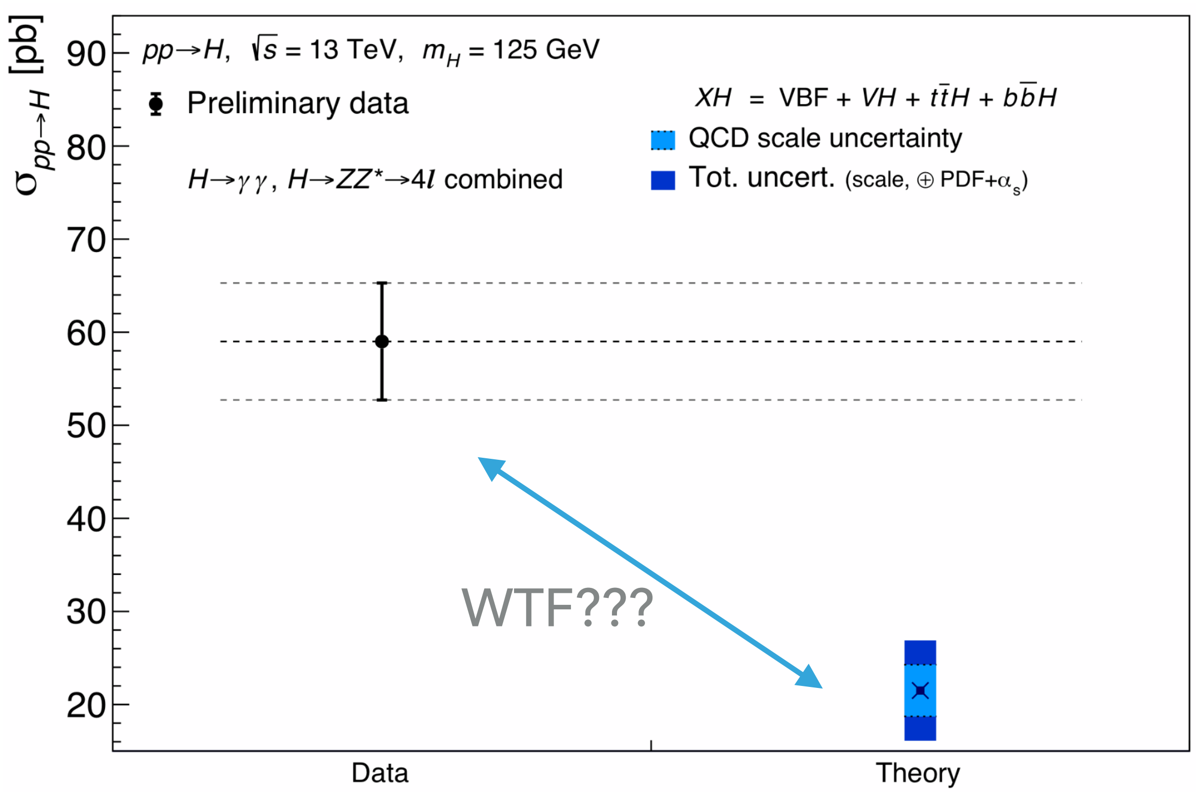
ASCONA 9/5/18

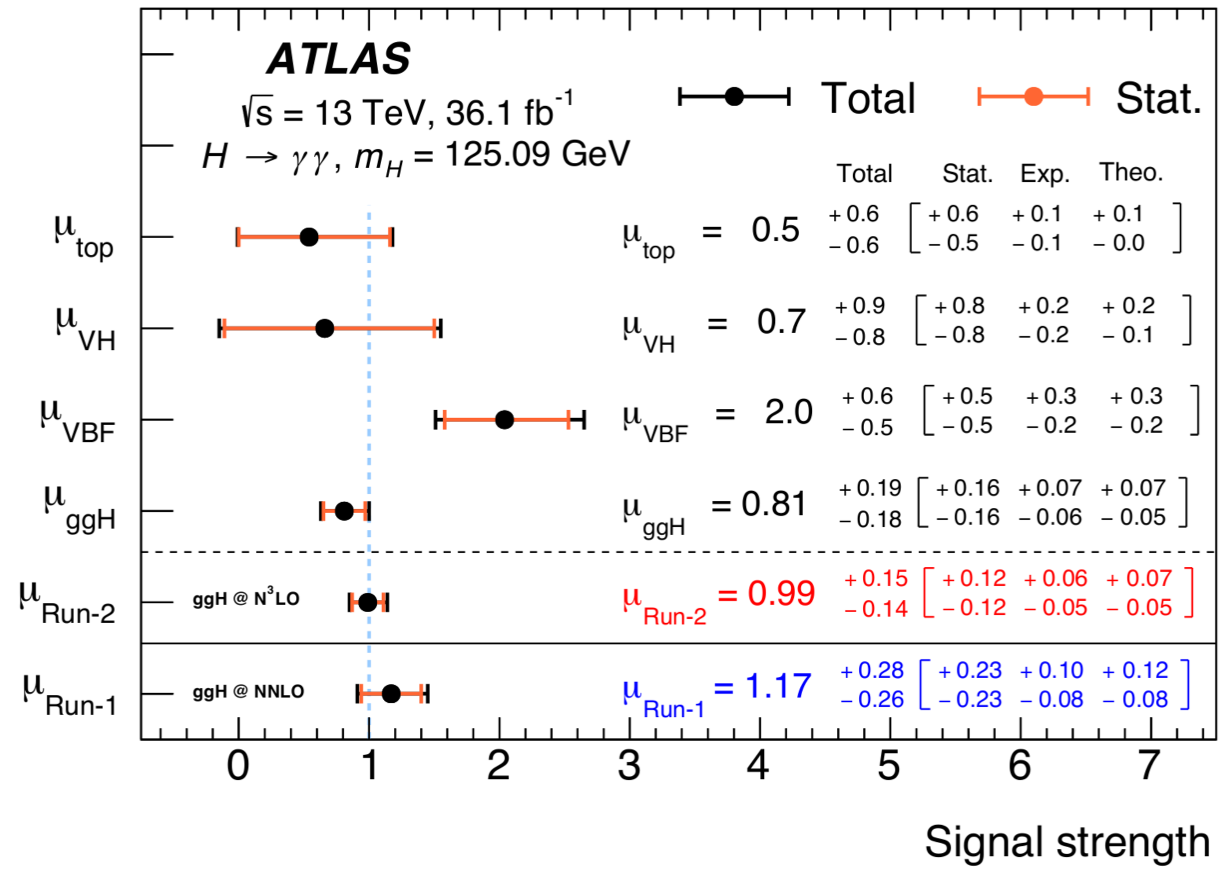
SLAC

FALKO DULAT

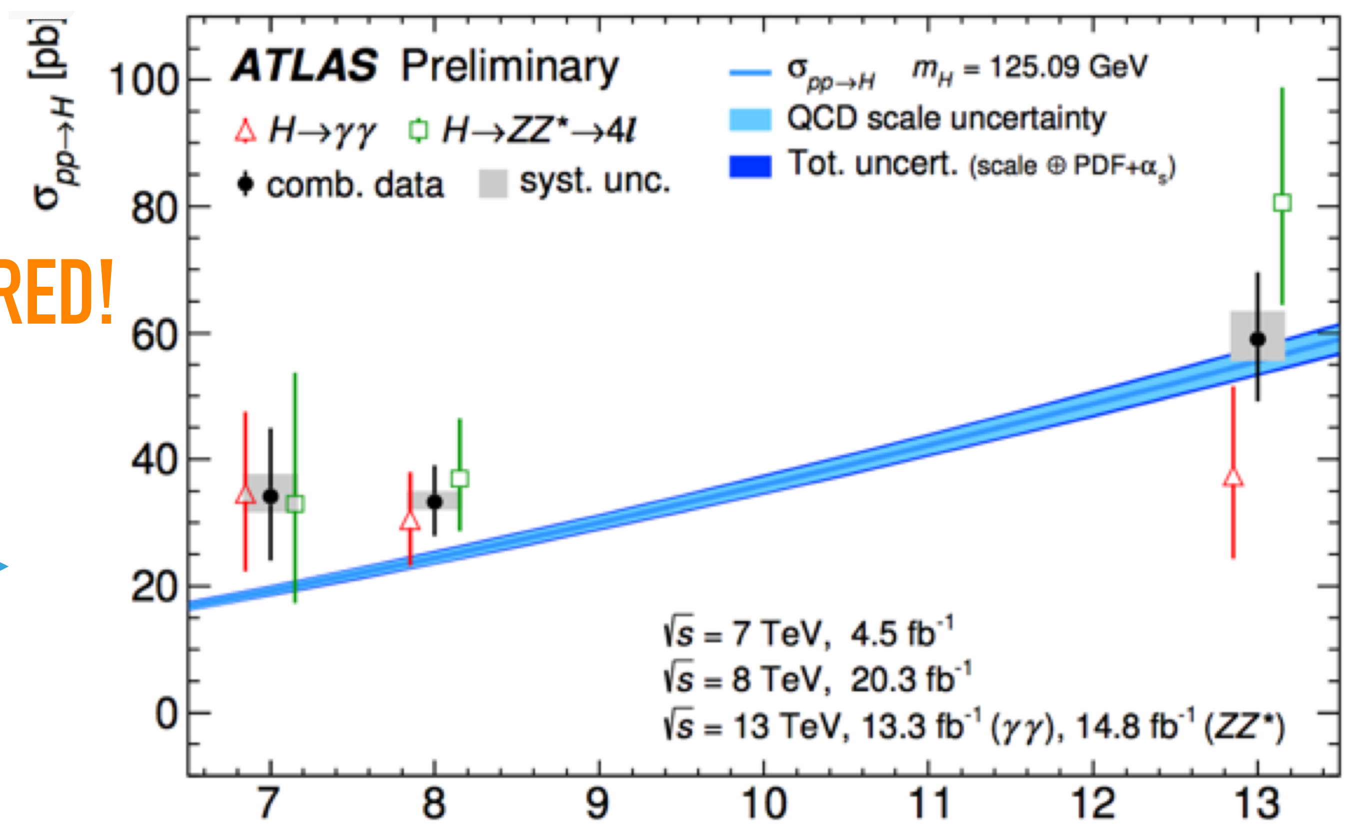
ADVENTURES IN PHASE SPACE INTEGRATION

ELLIPTICS IN HIGGS PRODUCTION

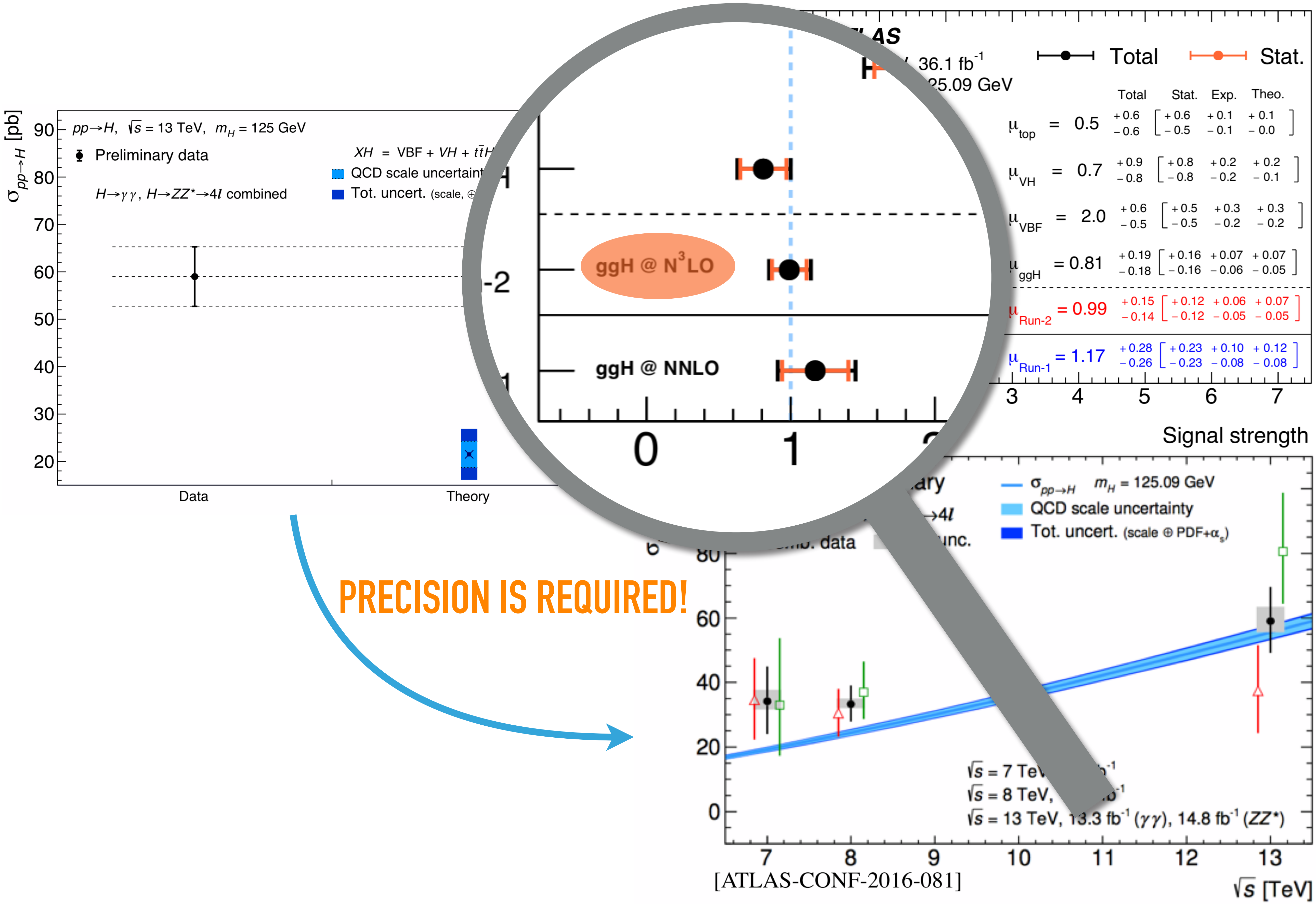




PRECISION IS REQUIRED!



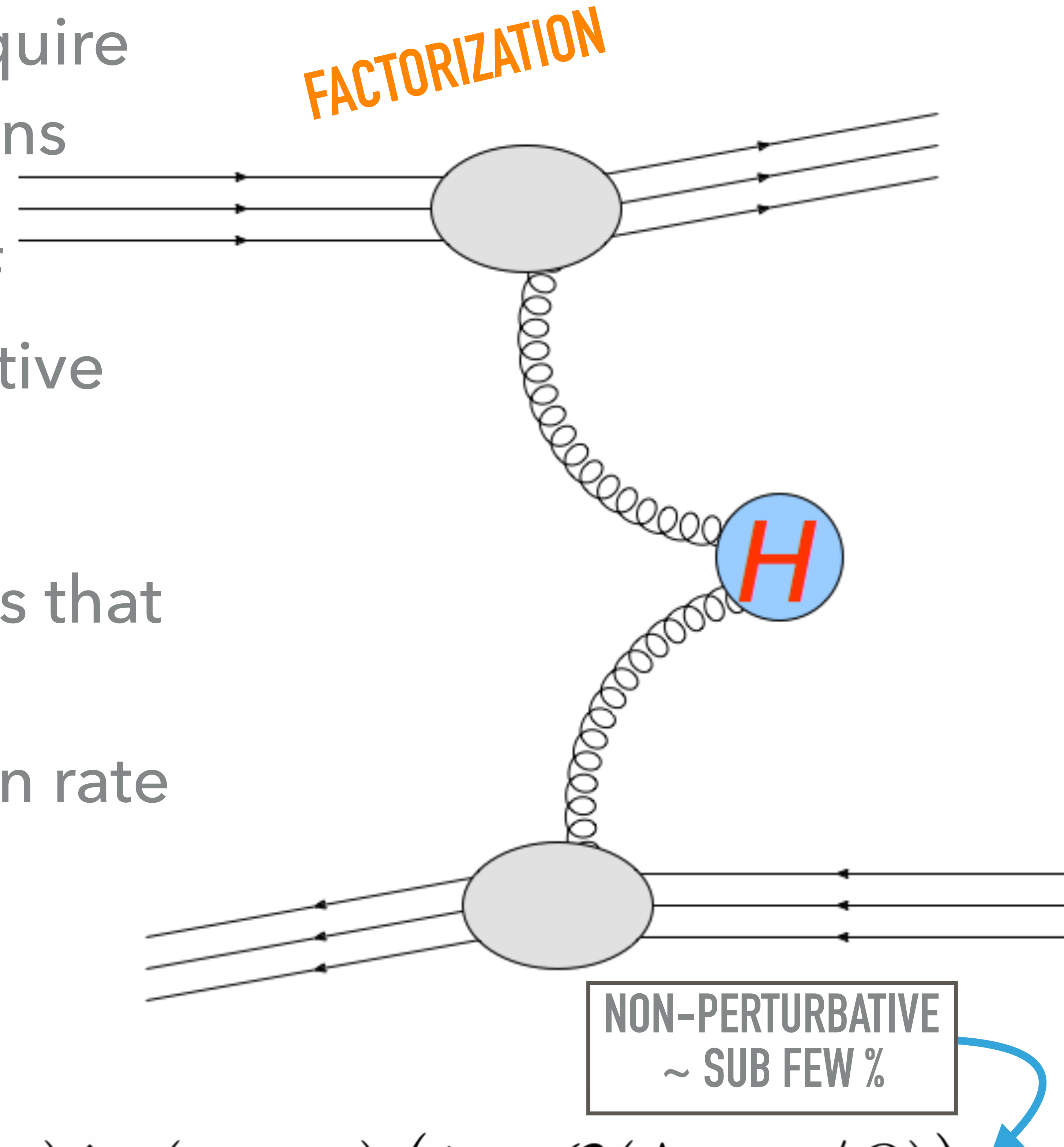
[ATLAS-CONF-2016-081]



PRECISION IS REQUIRED!

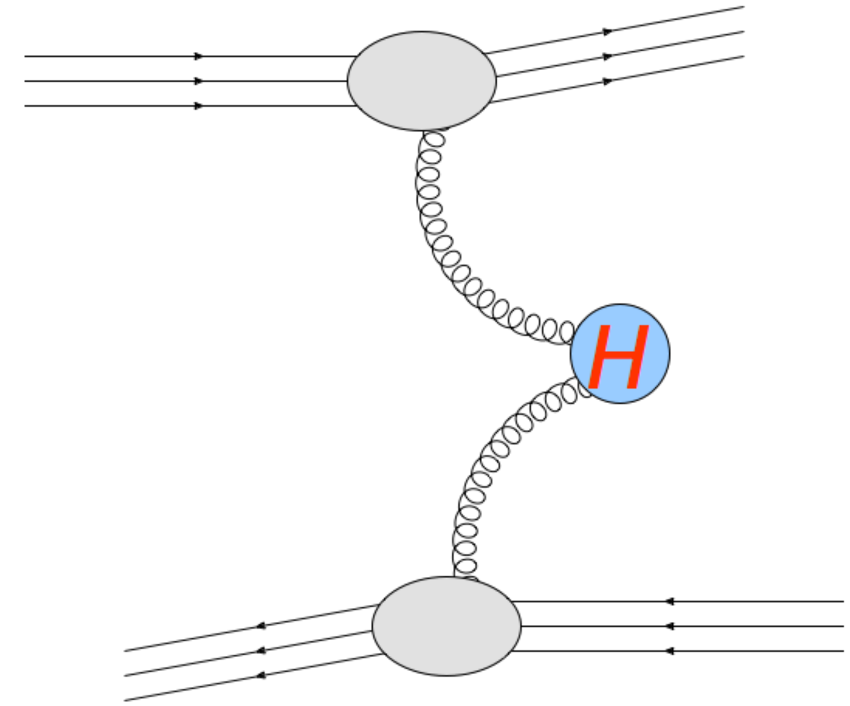
[ATLAS-CONF-2016-081]

- Predictions for the LHC require calculations of cross sections
- Perturbative description of scattering of non-perturbative protons
- Specify certain observables that we want to measure: total production rate, production rate as a function of transverse momentum, etc.

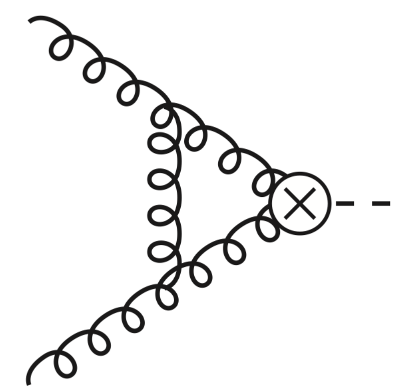


$$\sigma = \int dx_1 dx_2 f_i(x_1) f_j(x_2) \hat{\sigma}_{ij}(x_1, x_2) \left(1 + \mathcal{O}(\Lambda_{\rm QCD}/Q)\right)$$

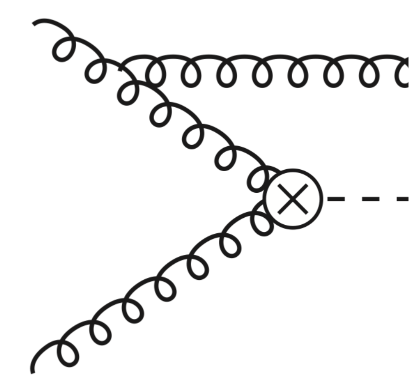
- Quantum mechanics requires averaging over unobserved degrees of freedom



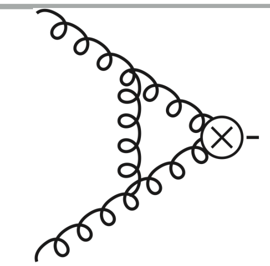
- Integrate over the momenta of unmeasured particles
- Two types of unresolved momentum integrals:



  - Loop integrals: Virtual particles, off-shell momenta $$\int_{\mathbb{R}^{3,1}} \frac{d^4\ell}{\ell^2(\ell+p_1)^2\dots}$$



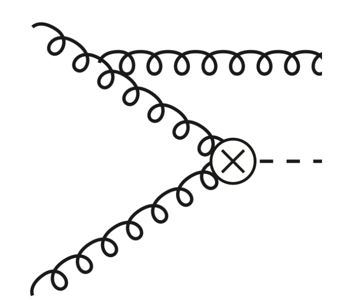
  - Phase-space integrals: Real particles, physical momentum constraints, on-shell momenta $$\int \mathrm{dLips}(\ell)|\mathcal{M}(\ell)|^2 = \int_{\mathbb{R}^{3,1}} d^4\ell\,\delta^+(\ell^2-m^2)\,(\dots)$$



- Integrals require parametrization in unconstrained variables
- Loop integrals: Feynman parametrization, unconstrained integrals over Feynman parameters

$$\int_{\mathbb{R}^{(d-1),1}} \frac{d^d\ell_1 \, \ldots \, d^d\ell_L}{\ell^2(\ell+p_1)^2 \ldots} \propto \int [d^n\vec{x}] \frac{\mathcal{U}^{N_\nu - (L+1)\, d/2}}{\mathcal{F}^{N_\nu - L\, d/2}}$$

- Phase space integrals: On-shell delta-functions introduce non-linear constraints:
- No-generally applicable parametrization to remove non-linear constraints



- Can parametrize on-shell particles with energies and angles, but introduces non-linear gram determinant constraints

$$d\Phi_{N-1}(D) = \mathcal{N}_{N-2}(D)\bar{z}^{(N-2)(D-2)-1}\left(\prod_{\substack{1\le i,j\le N\\ i\neq j,(i,j)\neq(1,2)}} ds_{ij}\right) \times \delta\left(1-\sum_{i=3}^{N}(s_{1i}+s_{2i})+(1-z)\sum_{i=3}^{N}\sum_{j=3}^{i-1}s_{ij}\right) G_N(\{s_{ij}\})^{\frac{D-N-1}{2}}\Theta[G_N(\{s_{ij}\})]$$

$$G_N(\{s_{ij}\}) = \det(s_{1i}s_{2j}+s_{1j}s_{2i}-s_{ij})_{3\le i,j\le N}$$

[Anastasiou, Duhr, FD, Mistlberger]

- Remaining delta function introduces non-linear relation between all integration variables

- Vanishes in the soft limit

- Possible to find phase space parameterizations in specific cases, but usually algebraic

- E.g. parametrization for 2 -> 3

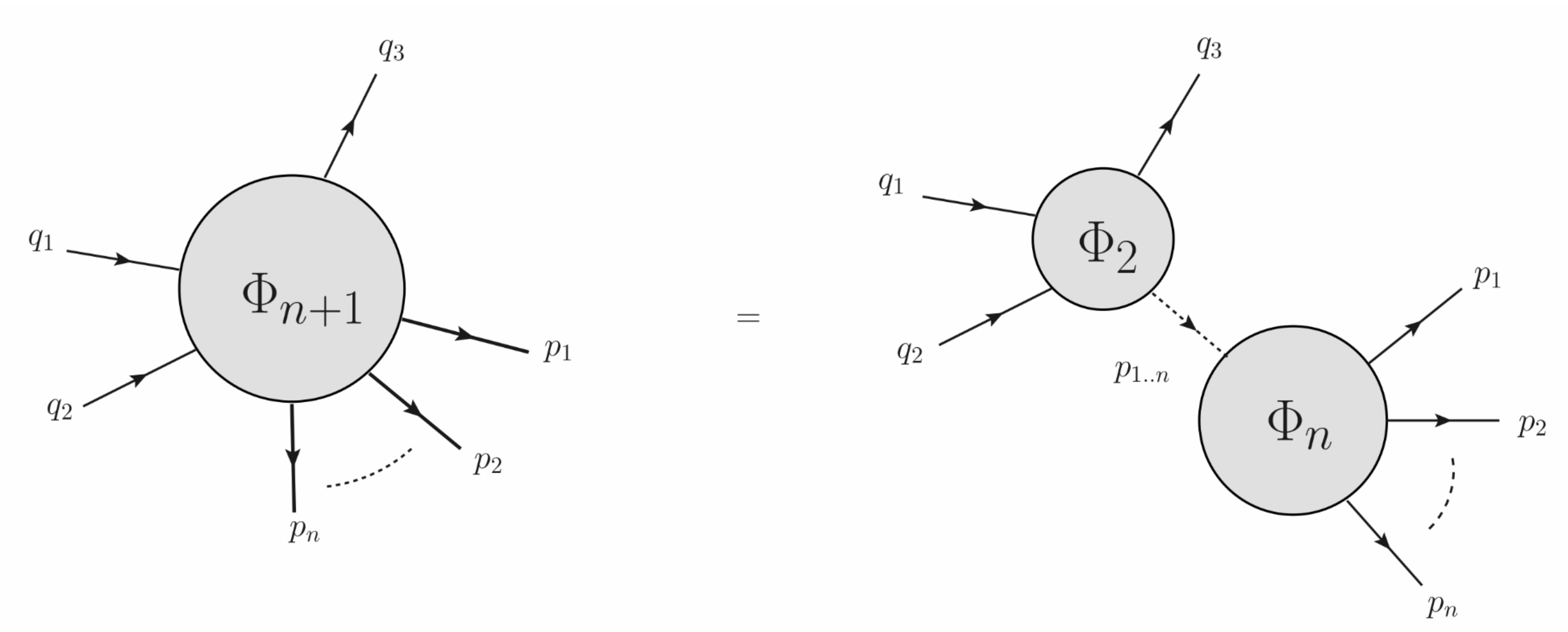
$$
\begin{aligned}
s_{34} &= \frac{s\bar{z}^2 x_1 \bar{x}_1 x_2}{z + x_1 \bar{z}} \\
s_{134} &= -s\bar{z}x_1 \\
s_{234} &= -s\bar{z}\bar{x}_1 \left[\frac{z + x_1 \bar{x}_2 \bar{z}}{z + x_1 \bar{z}}\right] \\
s_{23} &= -s\bar{z}\bar{x}_1 x_3 \\
s_{24} &= -s\bar{z}\bar{x}_1 \bar{x}_3
\end{aligned}
$$

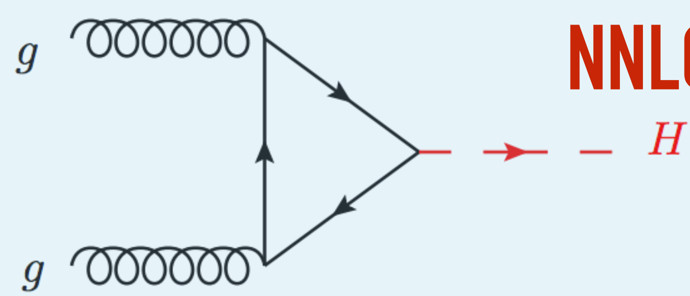
$$
\int d\Phi_3 = \frac{(2\pi)^{-3+2\epsilon}}{16\Gamma(1-2\epsilon)} \int_0^1 dx_1 dx_2 dx_3 dx_4 \left(\frac{s\bar{z}^3 x_1 \bar{x}_1}{z + x_1 \bar{z}}\right) \left(\frac{s^2 \bar{z}^4 x_1^2 \bar{x}_1^2 x_2 \bar{x}_2 x_3 \bar{x}_3 \sin^2(\pi x_4)}{z + x_1 \bar{z}}\right)^{-\epsilon}
$$

[Herzog]

$$
\begin{aligned}
s_{13} &= -s\bar{z}x_1 \left[x_3 \bar{x}_2 + \frac{x_2 \bar{x}_3}{z + x_1 \bar{z}} - 2\cos(\pi x_4) \sqrt{\frac{x_2 \bar{x}_2 x_3 \bar{x}_3}{z + x_1 \bar{z}}}\right] \\
s_{14} &= -s\bar{z}x_1 \left[\bar{x}_3 \bar{x}_2 + \frac{x_2 x_3}{z + x_1 \bar{z}} + 2\cos(\pi x_4) \sqrt{\frac{x_2 \bar{x}_2 x_3 \bar{x}_3}{z + x_1 \bar{z}}}\right]
\end{aligned}
$$

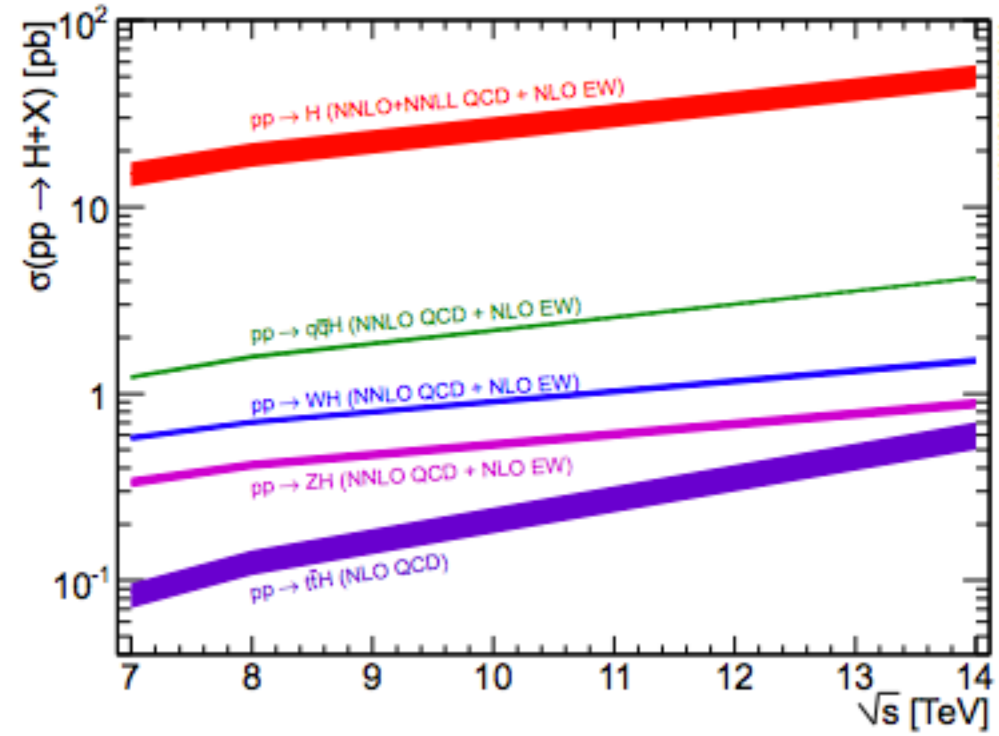
- Phase space integrals often not discussed in the amplitudes / multiloop community
- Come with their own set of challenges
- Phase space integrals can be elliptic too!
- Phase space integrals are often done numerically





87%

N3LO INCLUSIVE
NNLO DIFFERENTIAL



N3LO INCLUSIVE
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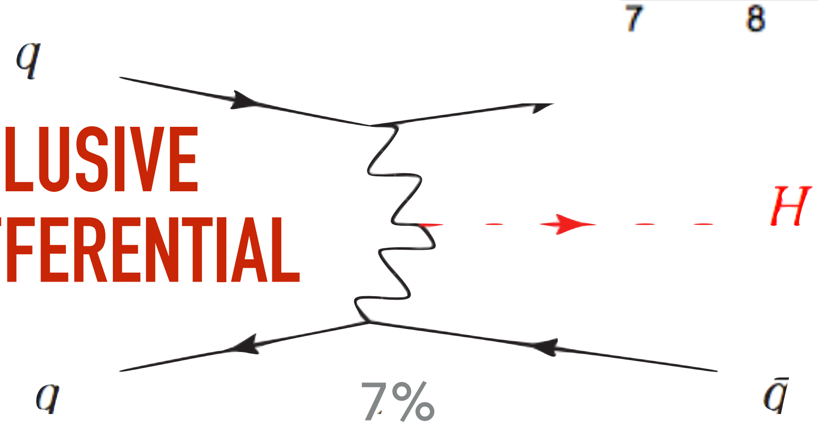
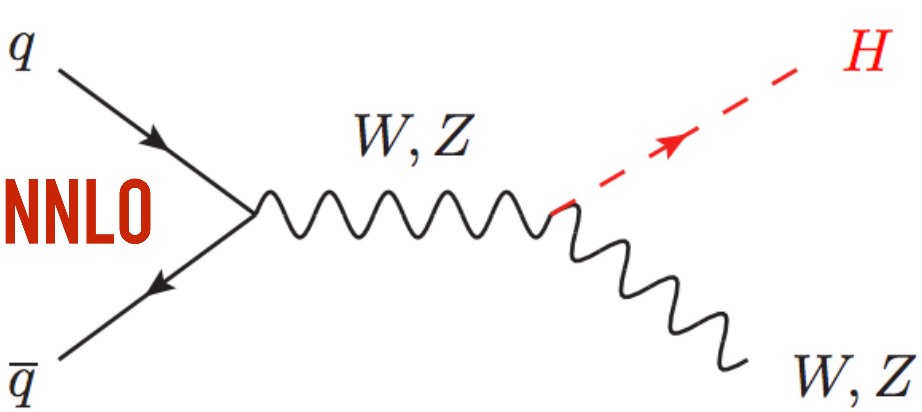


7%

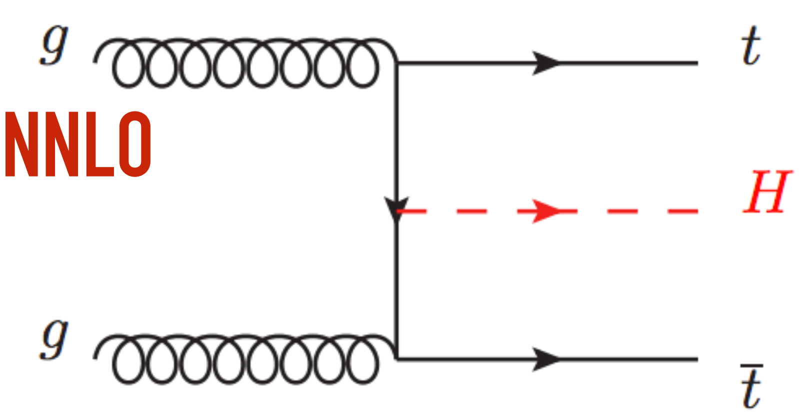
LHC@14TEV

NNLO



3%

NNLO



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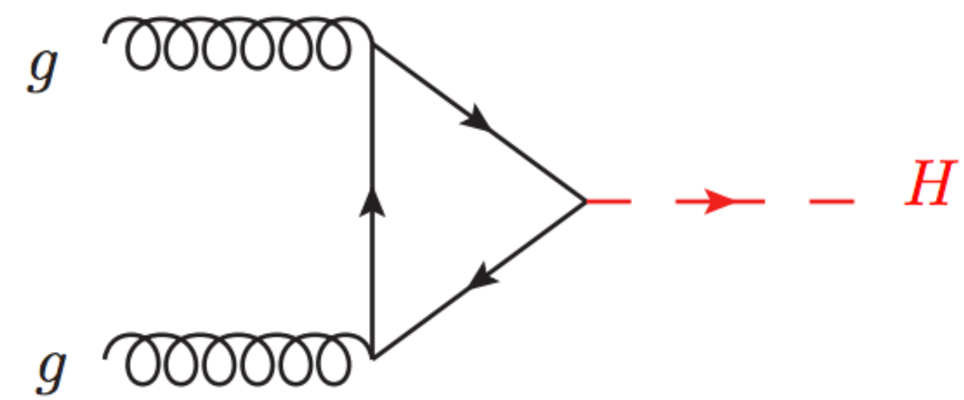
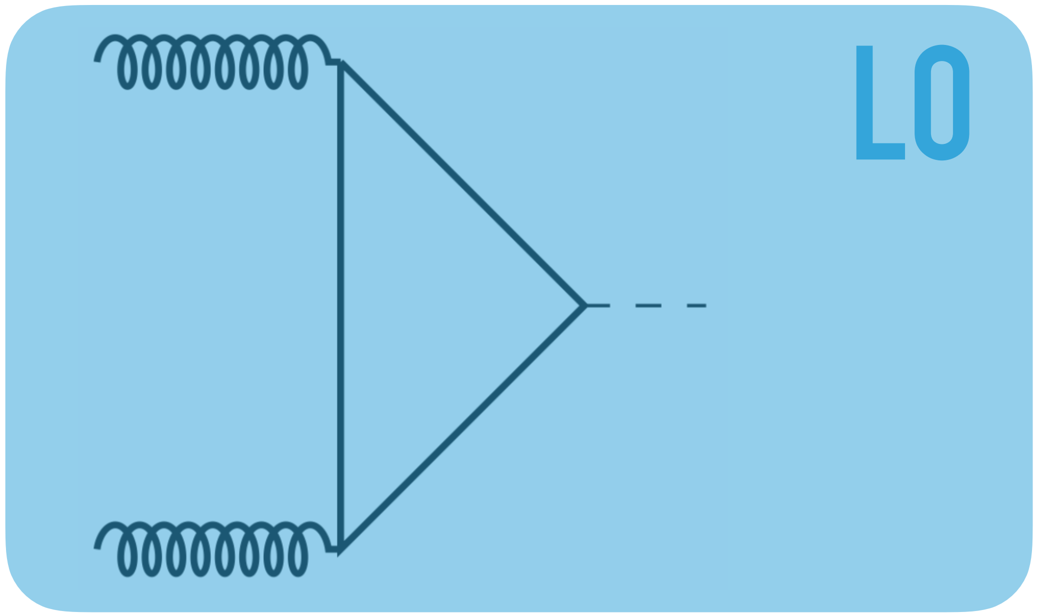
$$\hat{\sigma} = \alpha_s^2 \sigma^{\mathrm{LO}} + \alpha_s^3 \sigma^{\mathrm{NLO}} + \alpha_s^4 \sigma^{\mathrm{NNLO}} + \alpha_s^5 \sigma^{\mathrm{N3LO}}$$

- Goal: Analytic calculation of the N3LO gluon fusion cross section
- Many possible infrared (soft and collinear) and ultraviolet divergences
- Dimensional regularization used to render integrals finite
- Requires analytic calculation

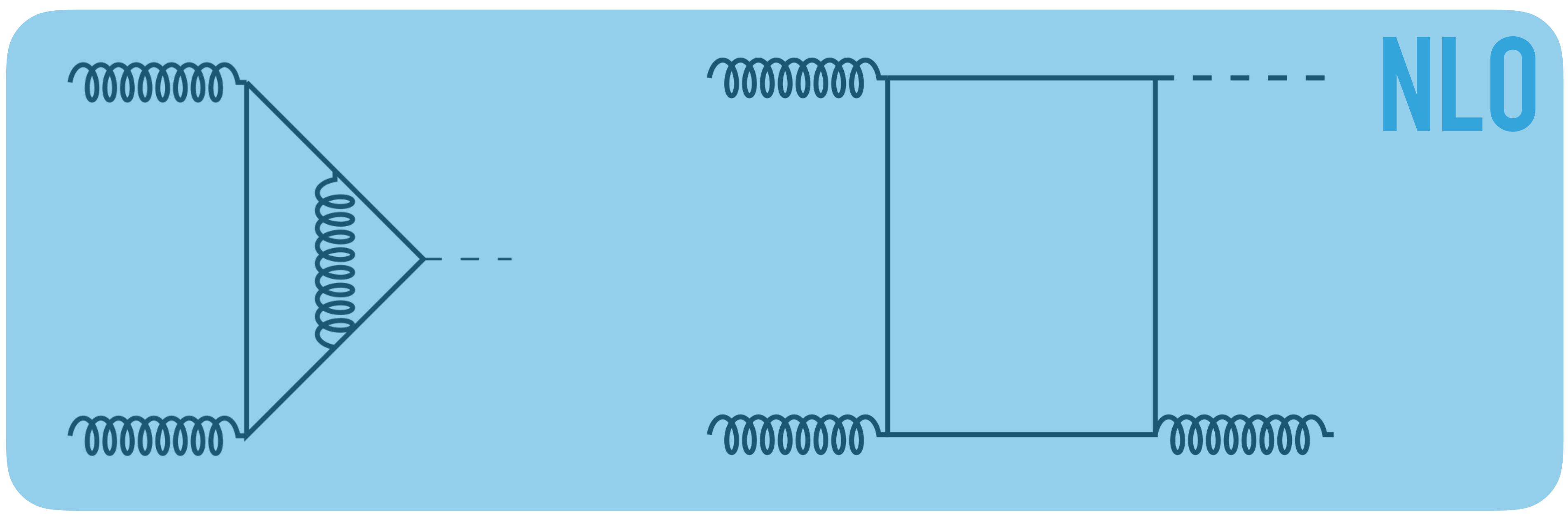


[Anastasiou, Duhr, FD, Furlan, Gehrmann, Herzog, Lazopoulos, Mistlberger; Mistlberger]

$$\hat{\sigma} = \alpha_s^2 \sigma^{\mathrm{LO}} + \alpha_s^3 \sigma^{\mathrm{NLO}} + \alpha_s^4 \sigma^{\mathrm{NNLO}} + \alpha_s^5 \sigma^{\mathrm{N3LO}}$$



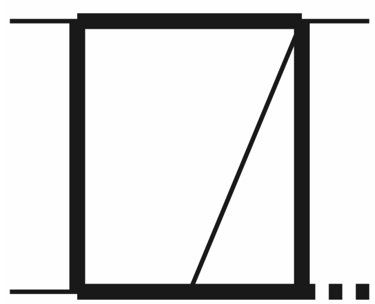
- NLO computed in the early 90s

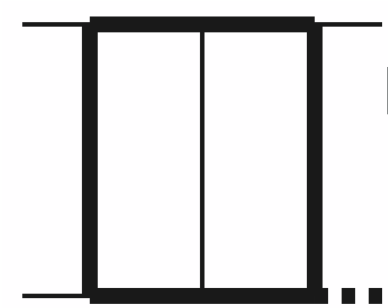


[Djouadi, Spira, Zerwas]

$$\hat{\sigma} = \alpha_s^2 \sigma^{\mathrm{LO}} + \alpha_s^3 \sigma^{\mathrm{NLO}} + \alpha_s^4 \sigma^{\mathrm{NNLO}} + \alpha_s^5 \sigma^{\mathrm{N3LO}}$$

- NNLO corrections are not known in closed form
- Two-loop Higgs+3-parton amplitudes involve elliptic topologies





[Bonciani, Del Duca, Frellesvig, Henn, Moriello, Smirnov]

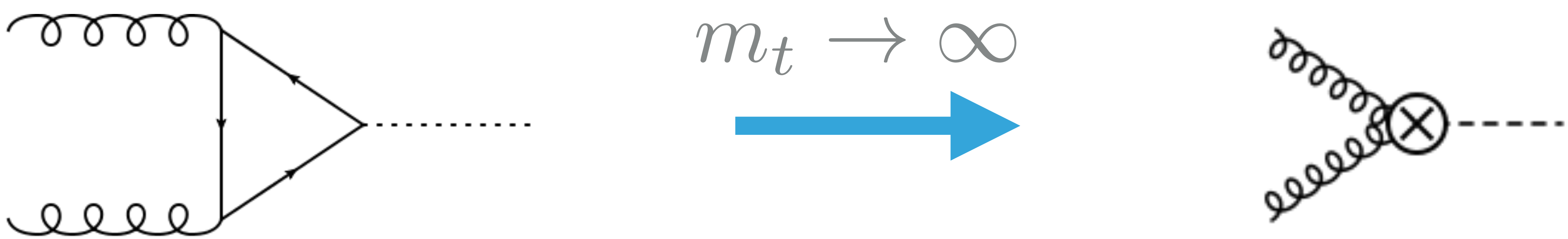
- Second order differential equation:

$$\partial_\alpha^2 h_1^{(i)}(\alpha) + p_1(\alpha)\,\partial_\alpha h_1^{(i)}(\alpha) + q_1(\alpha)\,h_1^{(i)}(\alpha) = r_1^{(i)}(\alpha)$$

$$y_1(\alpha) = K\left(\frac{1}{2} - \frac{k(\alpha)}{2}\right), \qquad y_2(\alpha) = K\left(\frac{1}{2} + \frac{k(\alpha)}{2}\right)$$

- Solutions are integrals over products of complete elliptic integrals and polylogarithms $\int_0^1 \mathcal{G}(t)\mathcal{E}^{(\sigma)}(t)\tilde{K}_i^{(-\sigma)}(t)dt$

- Many examples of elliptic integrals with internal masses known in the literature
- Let’s consider integrals without integral masses:
- Higgs production in heavy top approximation



$$\hat{\sigma} = \alpha_s^2 \sigma^{\mathrm{LO}} + \alpha_s^3 \sigma^{\mathrm{NLO}} + \alpha_s^4 \sigma^{\mathrm{NNLO}} + \alpha_s^5 \sigma^{\mathrm{N3LO}}$$

✓ ✓ ✓ ✓

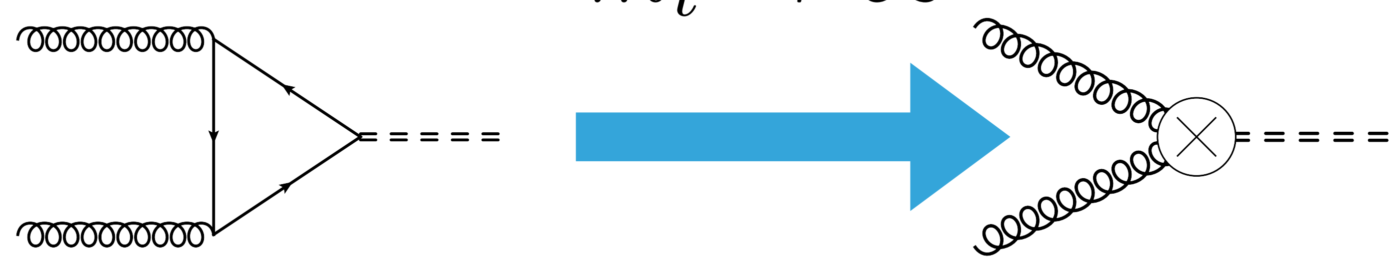
[Anastasiou, Duhr, FD, Furlan, Gehrmann, Herzog, Lazopoulos, Mistlberger; Mistlberger]

$$\frac{d^2\hat{\sigma}_{ij}}{dY\,dp_T^2} \sim \sum_X \int d\Phi_n \left|\mathcal{M}_{ij\to H+X}\right|^2$$

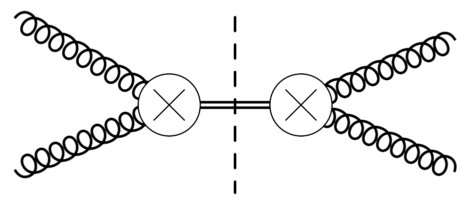
$$\mathcal{L} = \mathcal{L}_{QCD,5} - \frac{1}{4v} C_1\, H\, G^a_{\mu\nu}\, G_a^{\mu\nu}$$

$$m_t \to \infty$$

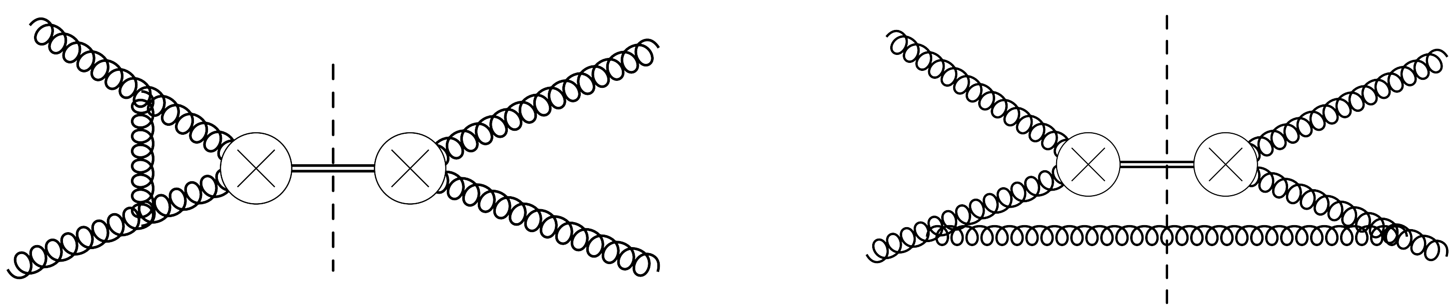
$$z = \frac{m_h^2}{s}$$



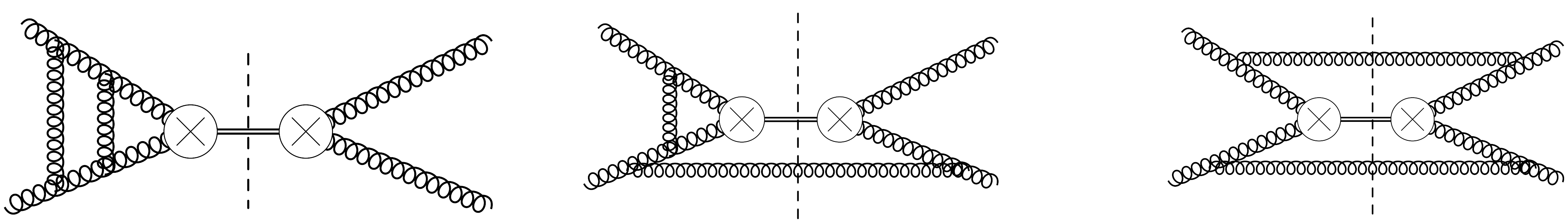
LO:



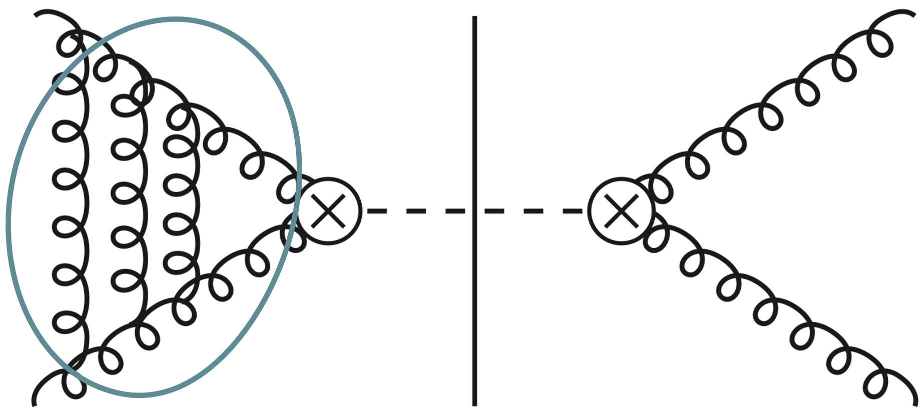
NLO:



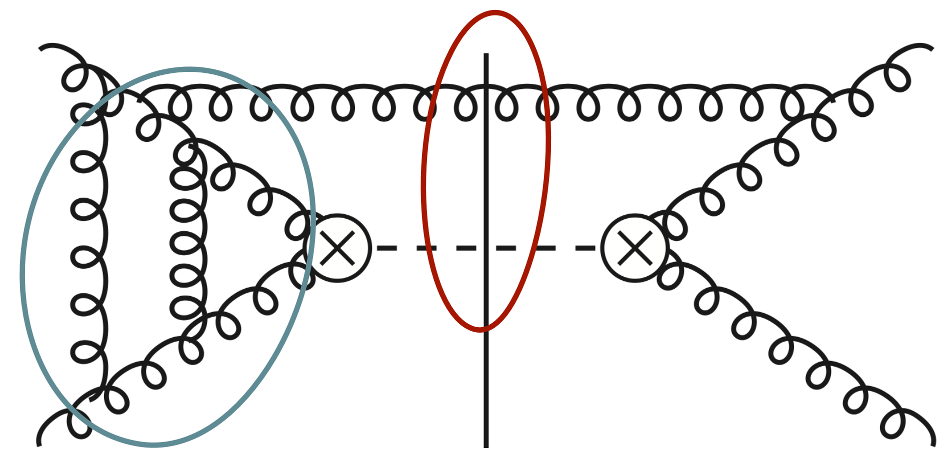
NNLO:



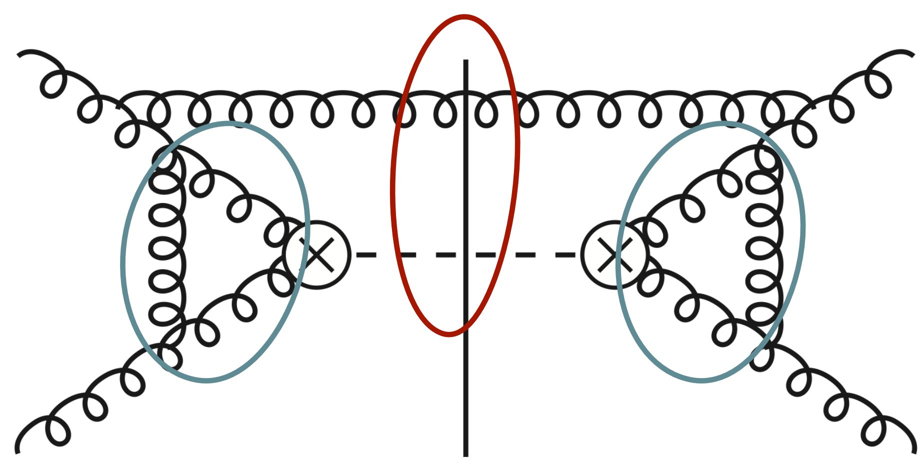
$$\frac{d^2\hat{\sigma}_{ij}}{dY\,dp_T^2} \sim \sum_X \int d\Phi_n \left|\mathcal{M}_{ij\to H+X}\right|^2$$



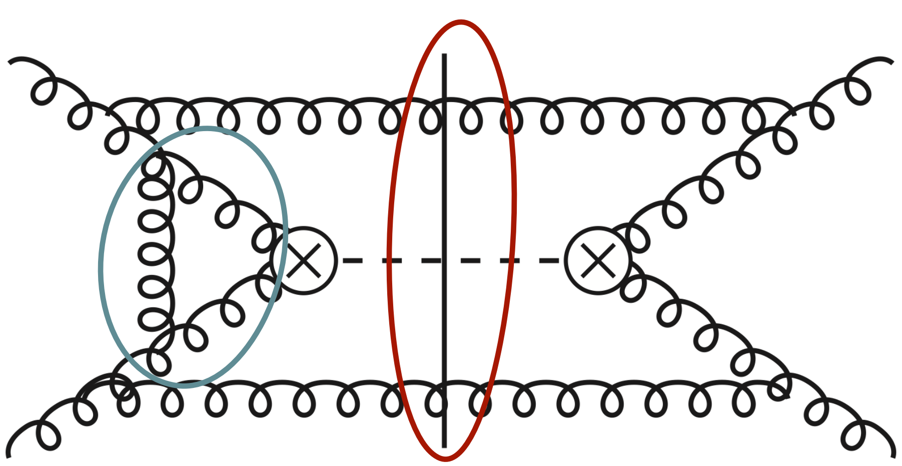
Triple virtual



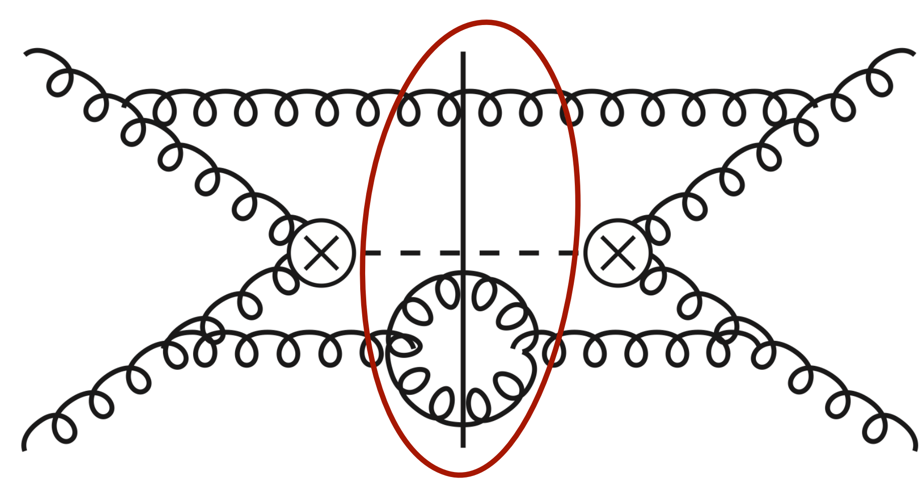
Double-virtual real



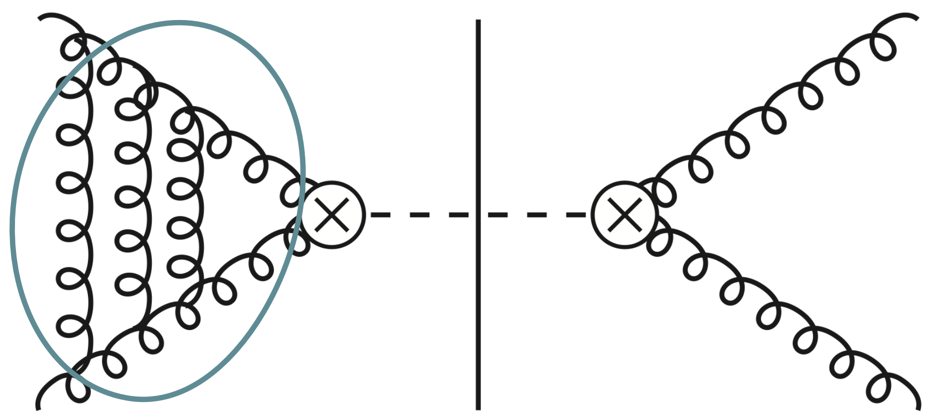
Real-virtual²



Double-real virtual



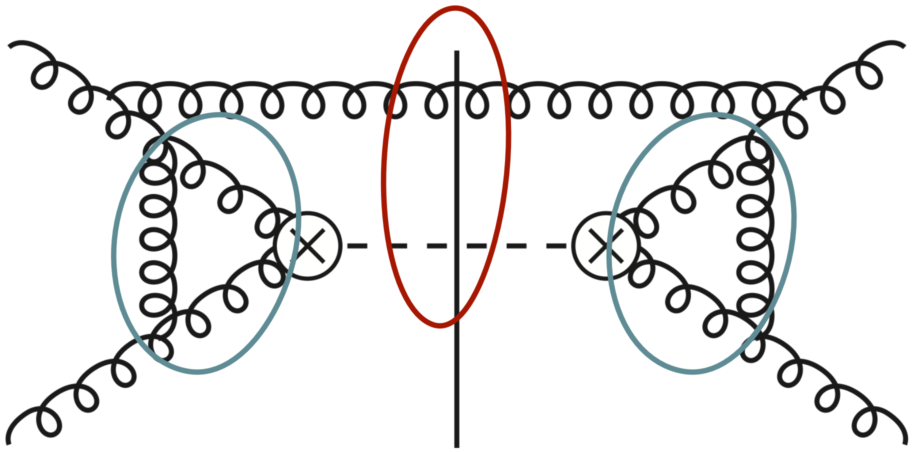
Triple real



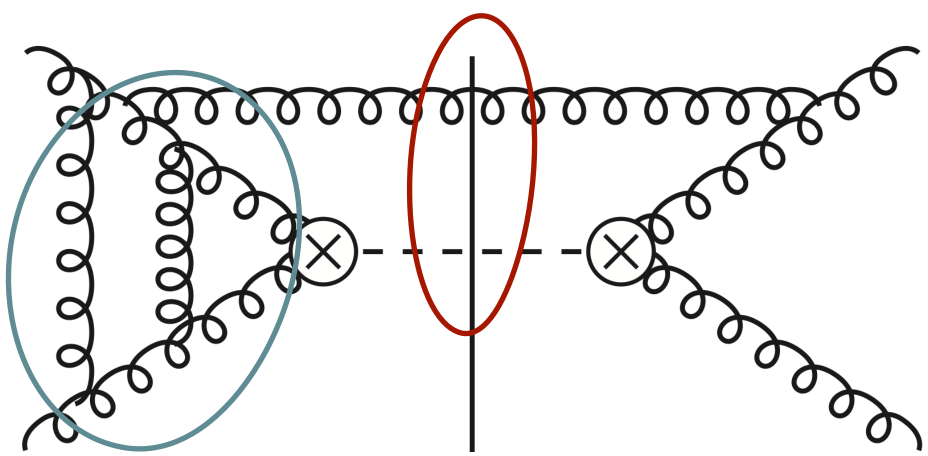
Triple virtual

- Massless three-point function
- No kinematic dependence
- Loop integrals evaluate to zeta values

[Gehrmann, Glover, Huber, Ikizlerli, Studerus]



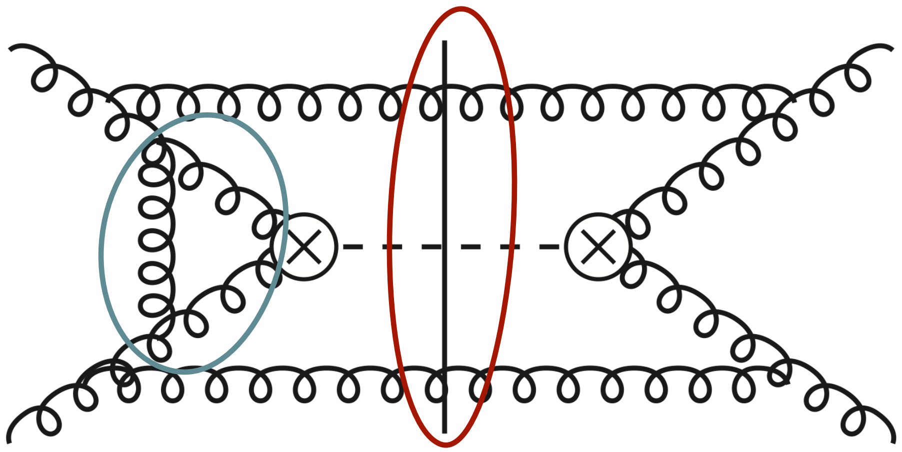
Real-virtual²



Double-virtual real

- Interference of two one-loop four point amplitudes and phase space integral over a two-loop four-point amplitude.
- Combined phase space and loop integrals evaluate to HPLs with indices {0,1,-1}.

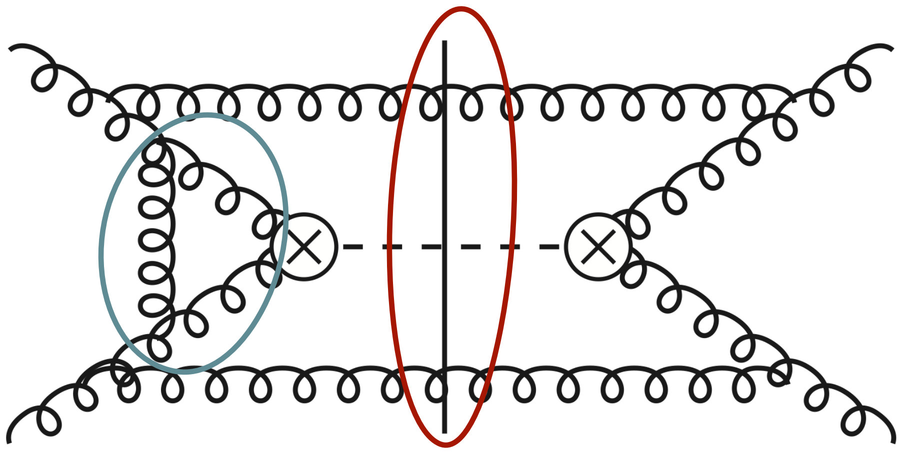
[Anastasiou, Duhr, FD, Herzog, Mistlberger]


Double-real virtual

- Higgs+4-parton phase space integral over one-loop five-point amplitude
- Direct integration unfeasible

$$\int d\Phi_3 = \frac{(2\pi)^{-3+2\epsilon}}{16\Gamma(1-2\epsilon)} \int_0^1 dx_1 dx_2 dx_3 dx_4 \left(\frac{s\bar{z}^3 x_1 \bar{x}_1}{z + x_1 \bar{z}}\right) \left(\frac{s^2 \bar{z}^4 x_1^2 \bar{x}_1^2 x_2 \bar{x}_2 x_3 \bar{x}_3 \sin^2(\pi x_4)}{z + x_1 \bar{z}}\right)^{-\epsilon}$$

$$\begin{aligned} s_{13} &= -s\bar{z}x_1 \left[x_3 \bar{x}_2 + \frac{x_2 \bar{x}_3}{z + x_1 \bar{z}} - 2\cos(\pi x_4) \sqrt{\frac{x_2 \bar{x}_2 x_3 \bar{x}_3}{z + x_1 \bar{z}}} \right] \\ s_{14} &= -s\bar{z}x_1 \left[\bar{x}_3 \bar{x}_2 + \frac{x_2 x_3}{z + x_1 \bar{z}} + 2\cos(\pi x_4) \sqrt{\frac{x_2 \bar{x}_2 x_3 \bar{x}_3}{z + x_1 \bar{z}}} \right] \end{aligned}$$



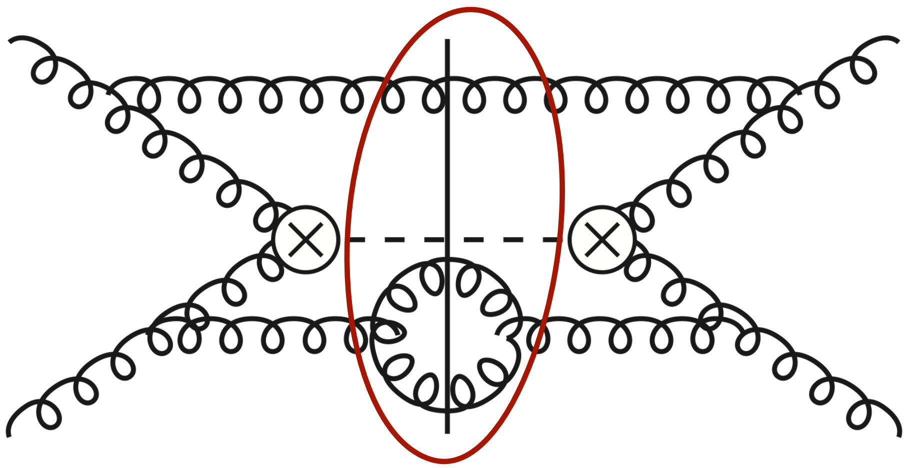
Double-real virtual

- Differential equations in the Higgs mass in canonical form
- Decoupled order-by-order in epsilon
- Algebraic alphabet

$$\mathfrak{A} = \left\{ z, 1-z, 1+z, 1+\sqrt{z}, 1+\sqrt{1+4z}, 2-z+\sqrt{z(z-4)} \right\}$$

- Differential equations solved in terms of Chen-iterated integrals in z
- Practical evaluation: Expand Chen iterated integrals in around z=1 to arbitrary order

[Anastasiou, Duhr, FD, Herzog, Mistlberger; Mistlberger]


Triple real

- Higgs+5-parton phase space integral over tree amplitudes
- Direct integration impossible in closed form

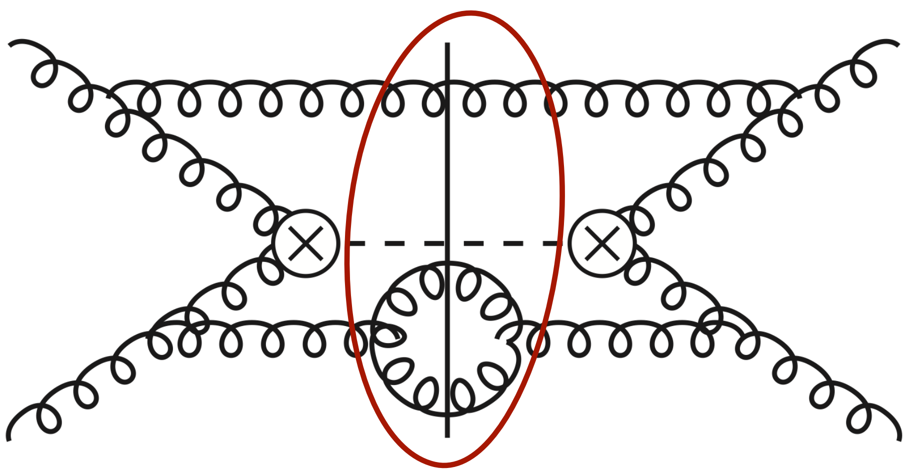
$$d\Phi_{H+m} = \frac{d^d p_h}{(2\pi)^d}(2\pi)\delta_+(p_h^2 - m_h^2)(2\pi)^d \delta^d\left(p_1 + p_2 + p_h + \sum_{i=3}^{m+2} p_i\right)\prod_{i=3}^{m+2}\frac{d^d p_i}{(2\pi)^d}(2\pi)\delta_+(p_i^2)$$

- Possible to derive differential equations for phase space integrals

$$\delta_+(p^2 - m^2) \to \left[\frac{1}{p^2 - m^2}\right]_c$$

- Treat delta-functions as residues of propagators
- Differential equation not in canonical form, but can be expanded around z=1

[Anastasiou, Duhr, FD, Herzog, Mistlberger; Mistlberger]



Triple real

- 550 master integrals for RRR
- System of differential equations:

$$\frac{\partial}{\partial z}\vec{I}(z) = A(z,\epsilon)\vec{I}(z)$$

- Goal is to find a transformation such that

[Mistlberger]

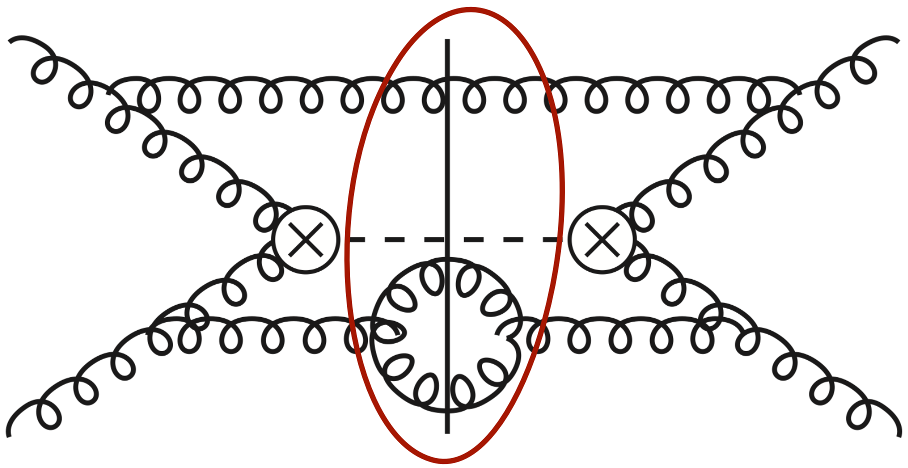
$$\vec{I}(z) = T\vec{I}'(z).$$

$$\epsilon A'(z,\epsilon) = T^{-1}A(z,\epsilon)T - T^{-1}\frac{\partial}{\partial z}T.$$

$$\frac{\partial}{\partial z}\vec{I}'(z) = \epsilon A'(z,\epsilon)\vec{I}'(z). \qquad \lim_{\epsilon\to 0} A'(z,\epsilon) = \text{const.}$$

- System can then be solved order-by-order in epsilon

$$\vec{I}'(z) = \left[\mathbb{I} + \epsilon\int^z dz' A'(z',\epsilon) + \epsilon^2 \int^z dz' \int^{z'} dz'' A'(z',\epsilon)A'(z'',\epsilon) + \ldots\right]\vec{I}'_0.$$



Triple real

- Boundary constant $\vec{I}_0'$ is determined by an expansion around z=1 (soft expansion).

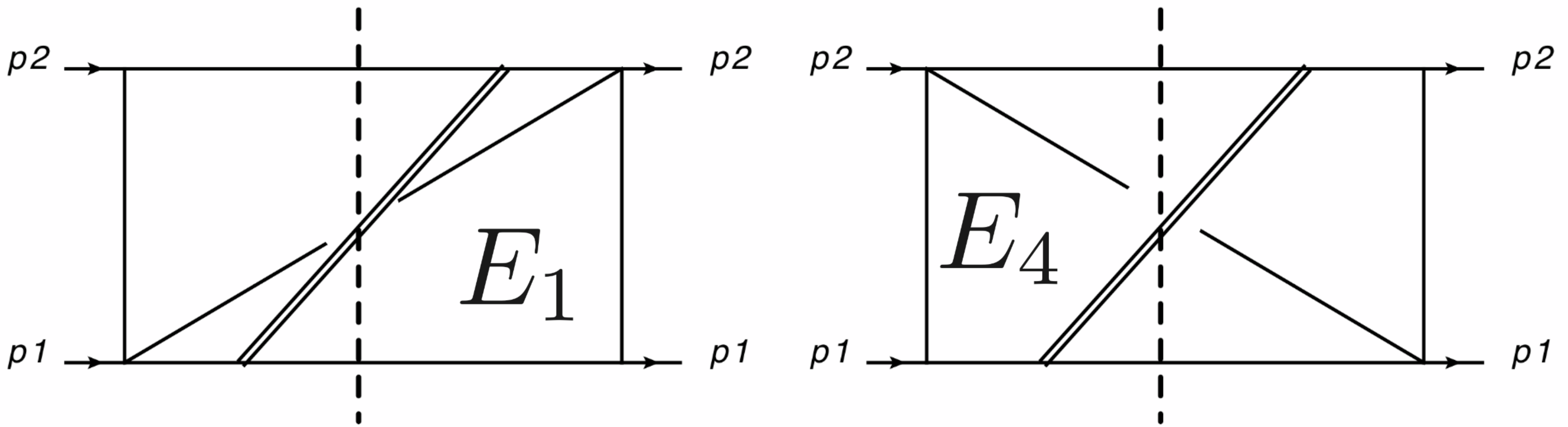
- Main work in solving the system is finding the transformation T.

- Algorithmic methods exist when T is rational in z and epsilon. [Barkatou, Pflügel; Moser; Lee]

- Some sub-systems are algebraic in z.

- Necessary to find a transformation to rationalize before algorithm can be applied.

[Mistlberger]



- There is a 4x4 system that cannot be solved this way.

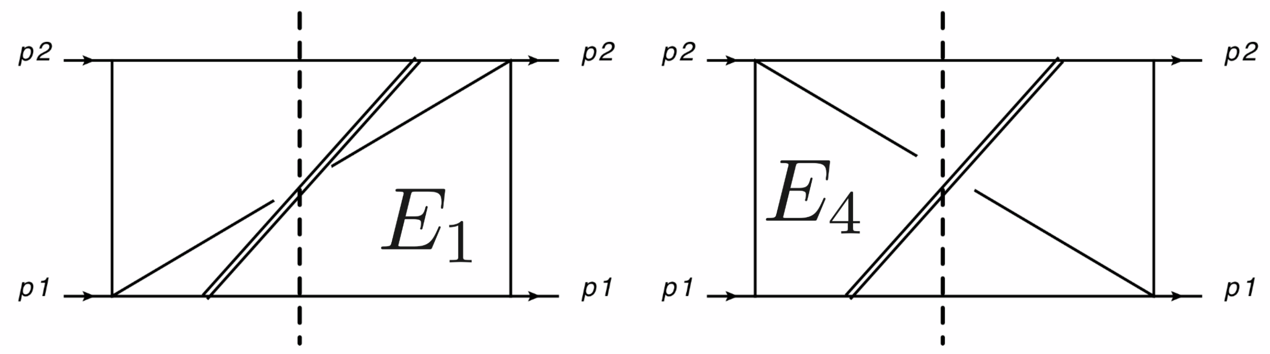
$$E_i = \int d\Phi_{H+3} \frac{n_i}{p^2_{145} p^2_{235} p^2_{1245} p^2_{1235}}$$

$$\begin{aligned} n_1 &= \frac{zs^3}{\epsilon(p^2_{12345} - sz)}. \\ n_2 &= -\frac{s}{16}(p^2_{14} + p^2_{23} + p^2_{35}). \\ n_3 &= -\frac{s}{16}(p^2_{23} + p^2_{35}). \\ n_4 &= \frac{s^2}{\epsilon}. \end{aligned}$$

$$\frac{\partial}{\partial z}\vec{E} = A_0(z)\vec{E} + \epsilon A_1(z,\epsilon)\vec{E} + \vec{y}(z)$$

$$A_0(z) = \begin{pmatrix} \frac{11-2z}{z^2-11z-1} & 0 & 0 & \frac{3-z}{z^2-11z-1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{z} & 0 & 0 & 0 \end{pmatrix}$$

[Mistlberger]



- There is a 4x4 system that cannot be solved this way.
- For ep=0 the system becomes a coupled 2x2 system

$$\frac{\partial}{\partial z}\begin{pmatrix} E_4^0 \\ E_1^0 \end{pmatrix} = A_T.\begin{pmatrix} E_4^0 \\ E_1^0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{z} \\ \frac{3-z}{z^2-11z-1} & \frac{11-2z}{z^2-11z-1} \end{pmatrix}\cdot\begin{pmatrix} E_4^0 \\ E_1^0 \end{pmatrix}$$

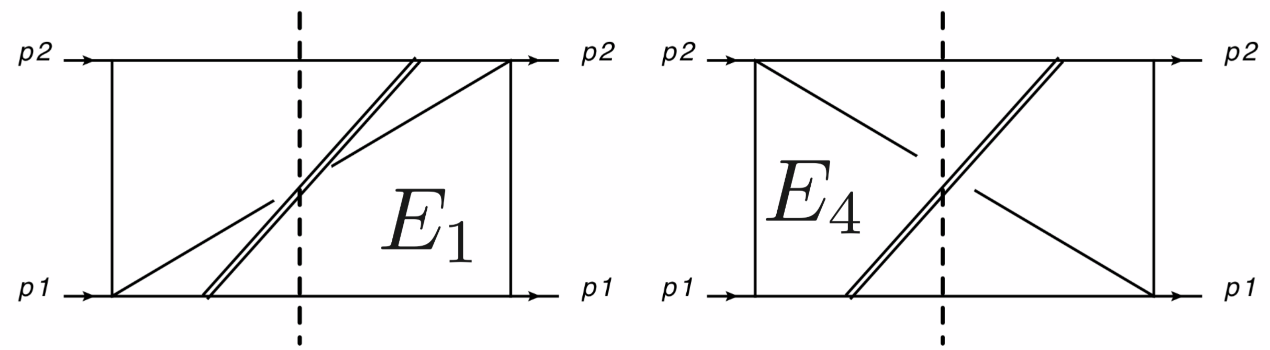
- Finding a transformation that removes the ep=0 part of the system amounts to finding the homogeneous solution.

$$\begin{pmatrix} E_4^0 \\ E_1^0 \end{pmatrix} = T_E.\begin{pmatrix} E_4'^0 \\ E_1'^0 \end{pmatrix} = \begin{pmatrix} t_{11}(z) & t_{12}(z) \\ t_{21}(z) & t_{22}(z) \end{pmatrix}\cdot\begin{pmatrix} E_4'^0 \\ E_1'^0 \end{pmatrix}$$

$$\frac{\partial}{\partial z}\begin{pmatrix} E_4'^0 \\ E_1'^0 \end{pmatrix} = 0.$$

$$\frac{\partial}{\partial z}T_E = A_T.T_E.$$

[Mistlberger]



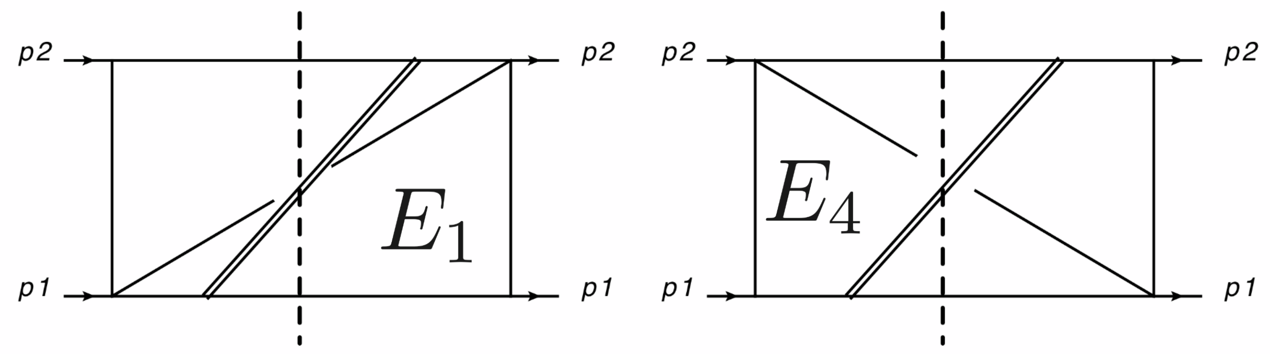
$$\frac{\partial}{\partial z}\begin{pmatrix} E_4^0 \\ E_1^0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{z} \\ \frac{3-z}{z^2-11z-1} & \frac{11-2z}{z^2-11z-1} \end{pmatrix} \cdot \begin{pmatrix} E_4^0 \\ E_1^0 \end{pmatrix}$$

- Coupled 2x2 system can be transformed into a second order differential equation

$$\frac{\partial^2}{\partial z^2}E_4^0 + \frac{(3z^2-22z-1)}{z\,(z^2-11z-1)}\frac{\partial}{\partial z}E_4^0 + \frac{(z-3)}{z\,(z^2-11z-1)}E_4^0 = 0.$$

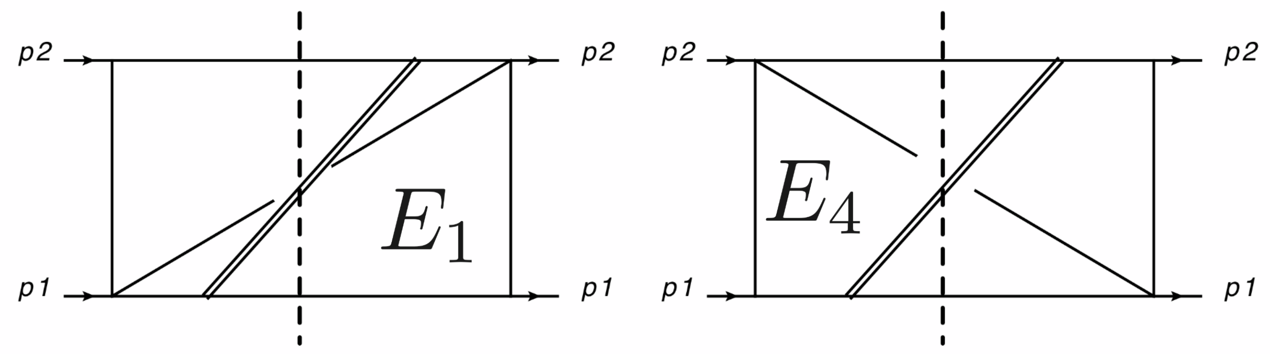
$$E_1^0 = z\frac{\partial}{\partial z}E_4^0.$$

- Differential equation was solved directly by Stefan Weinzierl in terms of complete elliptic integrals



$$\frac{\partial}{\partial z}\begin{pmatrix} E_4^0 \\ E_1^0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{z} \\ \frac{3-z}{z^2-11z-1} & \frac{11-2z}{z^2-11z-1} \end{pmatrix} \cdot \begin{pmatrix} E_4^0 \\ E_1^0 \end{pmatrix}$$

- Alternative: The leading singularity of a Feynman integral has to satisfy the same homogeneous differential equation as the full Feynman integral
- Compute leading singularity and normalize Feynman integral to have unit leading singularity
- System of differential equations should decouple order by order in ep.
- This normalization will not be algebraic.

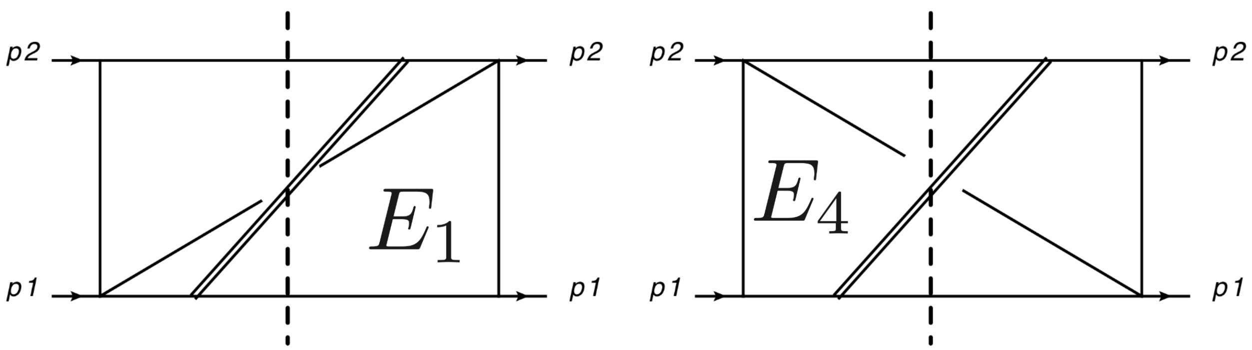


$$\frac{\partial}{\partial z}\begin{pmatrix} E_4^0 \\ E_1^0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{z} \\ \frac{3-z}{z^2-11z-1} & \frac{11-2z}{z^2-11z-1} \end{pmatrix} \cdot \begin{pmatrix} E_4^0 \\ E_1^0 \end{pmatrix}$$

- The Feynman integrals are dimension 3x4 - 4 = 8
- It is only possible to take a codimension 7 residue

$$\text{Leading Singularity } (E_4) \sim \int dx \frac{\theta\left((x-z)\left(x^3 - x^2 z + 2x^2 + 2xz + x - z\right)\right)}{\sqrt{(x-z)\left(x^3 - x^2 z + 2x^2 + 2xz + x - z\right)}}$$

- The root in the denominator has four distinct roots
- The leading singularity is elliptic



$$\frac{\partial}{\partial z}\begin{pmatrix} E_4^0 \\ E_1^0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{z} \\ \frac{3-z}{z^2-11z-1} & \frac{11-2z}{z^2-11z-1} \end{pmatrix} \cdot \begin{pmatrix} E_4^0 \\ E_1^0 \end{pmatrix}$$

$$\text{Leading Singularity } (E_4) \sim \int dx \frac{\theta\left((x-z)\left(x^3-x^2z+2x^2+2xz+x-z\right)\right)}{\sqrt{(x-z)\left(x^3-x^2z+2x^2+2xz+x-z\right)}}$$

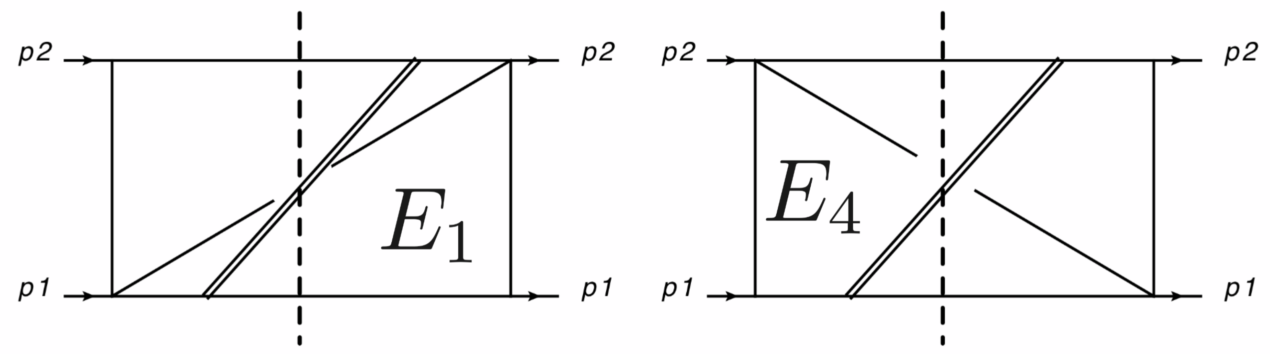
- The leading singularity can be computed in terms of complete elliptic integrals

$$\begin{aligned} I_1 &= \int_{r_2}^{r_3} dx \frac{1}{\sqrt{(x-r_1)(x-r_2)(x-r_3)(x-r_4)}} \\ &= \frac{2}{\sqrt{(r_4-r_2)(r_3-r_1)}} K(1-m). \\ I_2 &= \int_{r_3}^{r_4} dx \frac{1}{\sqrt{(x-r_1)(x-r_2)(x-r_3)(x-r_4)}} \\ &= \frac{2}{\sqrt{(r_4-r_2)(r_3-r_1)}} K(m). \end{aligned}$$

$$\varpi = \begin{pmatrix} I_1(z) & I_2(z) \\ z\partial_z I_1(z) & z\partial_z I_2(z) \end{pmatrix}$$

$$t_{ij}(z) = c_1 I_1 + c_2 I_2 + c_3 z \frac{\partial}{\partial z} I_1 + c_4 z \frac{\partial}{\partial z} I_2, \qquad c_i \in \mathbb{C}.$$

- The coefficients can be determined by equating expansions but are complex and unwieldy.



$$\begin{pmatrix} E_4^0 \\ E_1^0 \end{pmatrix} = T_E . \begin{pmatrix} E_4'^0 \\ E_1'^0 \end{pmatrix} = \begin{pmatrix} t_{11}(z) & t_{12}(z) \\ t_{21}(z) & t_{22}(z) \end{pmatrix} . \begin{pmatrix} E_4'^0 \\ E_1'^0 \end{pmatrix}$$
$$\frac{\partial}{\partial z} \begin{pmatrix} E_4'^0 \\ E_1'^0 \end{pmatrix} = 0.$$
$$\frac{\partial}{\partial z} T_E = A_T . T_E .$$

- The only obstruction to solving the entire system is the need for a non-algebraic transformation to decouple the system in the ep=0 limit.

- The homogeneous solution of the 2x2 system is such a transformation.

[Mistlberger]

- By definition, the rotated system is decoupled order-by-order:

$$E_1 = t_{22} E_1' + t_{21} E_4',$$
$$E_4 = t_{11} E_4' + t_{12} E_1',$$

- The price to pay is the introduction of integrals over the unknown functions $t_{ij}(z)$

$$\vec{I}'(z) = \left[\mathbb{I} + \epsilon \int^z dz' A'(z',\epsilon) + \epsilon^2 \int^z dz' \int^{z'} dz'' A'(z',\epsilon) A'(z'',\epsilon) + \ldots \right] \vec{I}_0'$$

$$\vec{I}'(z) = \left[\mathbb{I} + \epsilon \int^{z} dz' A'(z',\epsilon) + \epsilon^2 \int^{z} dz' \int^{z'} dz'' A'(z',\epsilon) A'(z'',\epsilon) + \dots \right] \vec{I}'_0$$

$$\begin{aligned} E_1 &= t_{22} E'_1 + t_{21} E'_4, \\ E_4 &= t_{11} E'_4 + t_{12} E'_1, \end{aligned}$$

- Such a solution is useless unless we can evaluate the functions $t_{ij}(z)$
- Use differential equations to obtain power series for the $t_{ij}(z)$

$$t_{ij}(z) = \sum_{n=0}^{\infty} \bar{z}^n b_{ij}^{(n)} \qquad t_{ij}(z) = \sum_{n=0}^{\infty} z^n c_{ij}^{(n)} + \log(z) \sum_{n=0}^{\infty} d_{ij}^{(n)} z^n.$$

$$T_E = e^{-\log(\bar{z}) \lim_{z\to 1} \bar{z} A_T} \cdot \begin{pmatrix} t_{11}^1 & t_{12}^1 \\ t_{21}^1 & t_{22}^1 \end{pmatrix} = \begin{pmatrix} t_{11}^1 & t_{12}^1 \\ t_{21}^1 & t_{22}^1 \end{pmatrix} + \mathcal{O}(\bar{z}^1).$$

$$T_E = e^{\log(z) \lim_{z\to 0} z A_T} \cdot \begin{pmatrix} t_{11}^0 & t_{12}^0 \\ t_{21}^0 & t_{22}^0 \end{pmatrix} = \begin{pmatrix} t_{11}^0 & t_{12}^0 \\ t_{21}^0 & t_{22}^0 \end{pmatrix} + \log(z) \begin{pmatrix} t_{21}^0 & t_{22}^0 \\ 0 & 0 \end{pmatrix} + \mathcal{O}(z^1)$$

- Plugging in the Ansatz yields difference equations for the coefficients

$$b_{11}^{(n+2)} = \frac{(n+1) b_{11}^{(n+1)}}{n+2} - \frac{b_{21}^{(n+1)}}{n+2}.$$

$$b_{21}^{(n+2)} = \frac{b_{11}^{(n)}}{11(n+2)} + \frac{2 b_{11}^{(n+1)}}{11(n+2)} + \frac{1}{11} b_{21}^{(n)} + \frac{9}{11} b_{21}^{(n+1)}$$

$$\begin{aligned} c_{11}^{(n+2)} &= \frac{c_{11}^{(n)}}{(n+2)^2} - \frac{3c_{11}^{(n+1)}}{(n+2)^2} + \frac{c_{21}^{(n)}}{n+2} - \frac{11 c_{21}^{(n+1)}}{n+2} - \frac{2 d_{11}^{(n)}}{(n+2)^3} \\ &+ \frac{6 d_{11}^{(n+1)}}{(n+2)^3} - \frac{d_{21}^{(n)}}{(n+2)^2} + \frac{11 d_{21}^{(n+1)}}{(n+2)^2}. \\ c_{21}^{(n+2)} &= \frac{c_{11}^{(n)}}{n+2} - \frac{3 c_{11}^{(n+1)}}{n+2} + c_{21}^{(n)} - 11 c_{21}^{(n+1)} - \frac{d_{11}^{(n)}}{(n+2)^2} + \frac{3 d_{11}^{(n+1)}}{(n+2)^2}. \\ d_{11}^{(n+2)} &= \frac{d_{11}^{(n)}}{(n+2)^2} - \frac{3 d_{11}^{(n+1)}}{(n+2)^2} + \frac{d_{21}^{(n)}}{n+2} - \frac{11 d_{21}^{(n+1)}}{n+2}. \\ d_{21}^{(n+2)} &= \frac{d_{11}^{(n)}}{n+2} - \frac{3 d_{11}^{(n+1)}}{n+2} + d_{21}^{(n)} - 11 d_{21}^{(n+1)}. \end{aligned}$$

[Mistlberger]

$$b_{11}^{(n+2)} = \frac{(n+1)b_{11}^{(n+1)}}{n+2} - \frac{b_{21}^{(n+1)}}{n+2}.$$

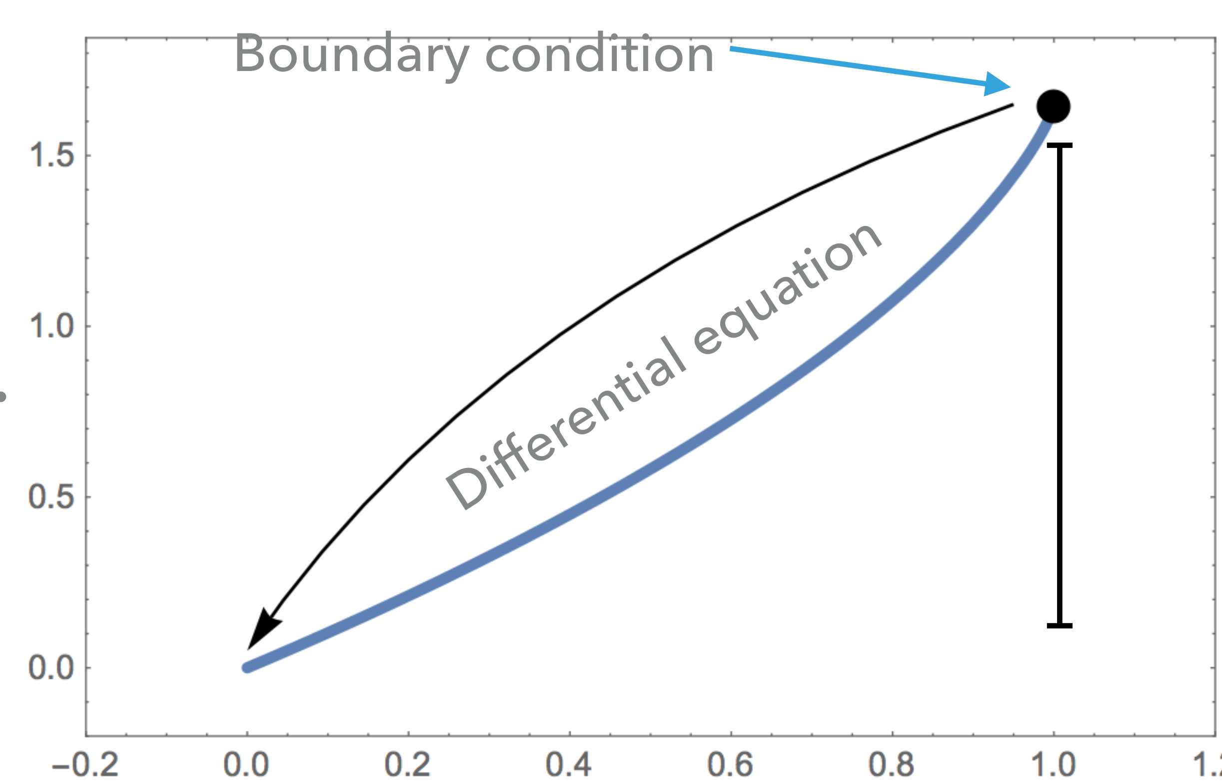
$$b_{21}^{(n+2)} = \frac{b_{11}^{(n)}}{11(n+2)} + \frac{2b_{11}^{(n+1)}}{11(n+2)} + \frac{1}{11}b_{21}^{(n)} + \frac{9}{11}b_{21}^{(n+1)}$$

$$\begin{aligned} c_{11}^{(n+2)} &= \frac{c_{11}^{(n)}}{(n+2)^2} - \frac{3c_{11}^{(n+1)}}{(n+2)^2} + \frac{c_{21}^{(n)}}{n+2} - \frac{11c_{21}^{(n+1)}}{n+2} - \frac{2d_{11}^{(n)}}{(n+2)^3} \\ &+ \frac{6d_{11}^{(n+1)}}{(n+2)^3} - \frac{d_{21}^{(n)}}{(n+2)^2} + \frac{11d_{21}^{(n+1)}}{(n+2)^2}. \\ c_{21}^{(n+2)} &= \frac{c_{11}^{(n)}}{n+2} - \frac{3c_{11}^{(n+1)}}{n+2} + c_{21}^{(n)} - 11c_{21}^{(n+1)} - \frac{d_{11}^{(n)}}{(n+2)^2} + \frac{3d_{11}^{(n+1)}}{(n+2)^2}. \\ d_{11}^{(n+2)} &= \frac{d_{11}^{(n)}}{(n+2)^2} - \frac{3d_{11}^{(n+1)}}{(n+2)^2} + \frac{d_{21}^{(n)}}{n+2} - \frac{11d_{21}^{(n+1)}}{n+2}. \\ d_{21}^{(n+2)} &= \frac{d_{11}^{(n)}}{n+2} - \frac{3d_{11}^{(n+1)}}{n+2} + d_{21}^{(n)} - 11d_{21}^{(n+1)}. \end{aligned}$$

- The difference equations can be solved to any required order to obtain power series solutions for the DE.

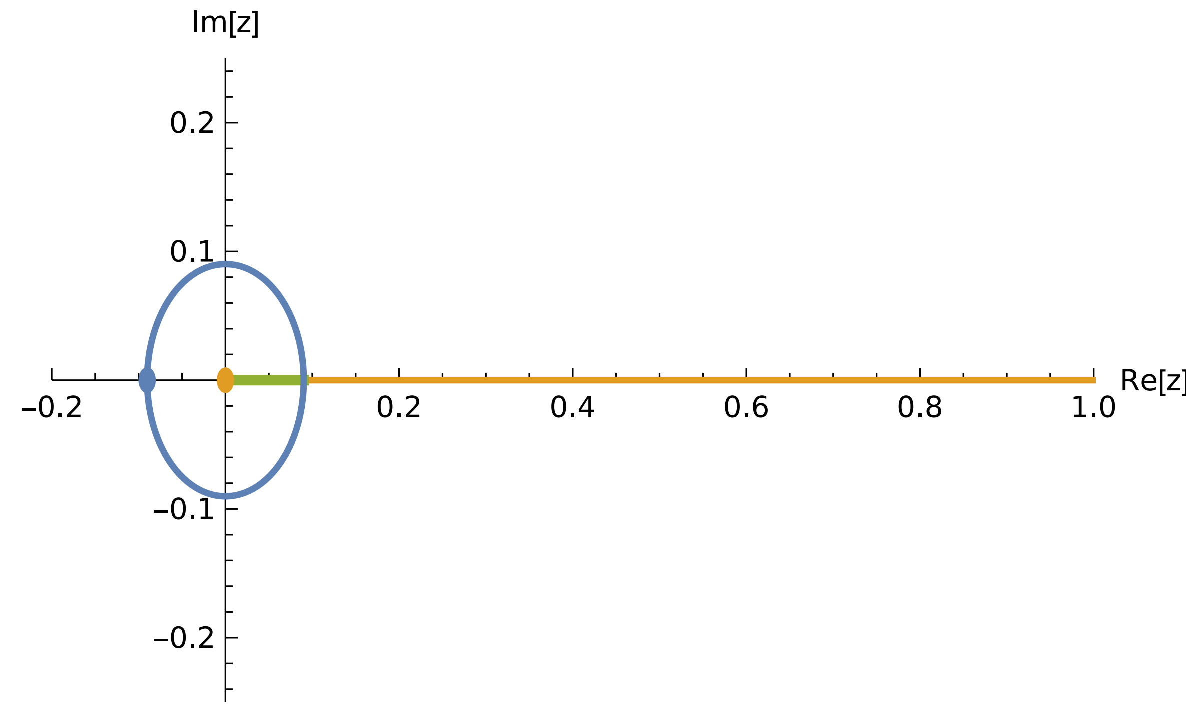
- The boundary conditions for the b coefficients can be determined from the knowledge of the system at z=1.

- Hard to compute the associator to determine the boundary values at z=0.



[Mistlberger]

$$z = 0.$$
$$z = \frac{1}{2}\left(11 - 5\sqrt{5}\right) \sim -0.09.$$
$$z = \frac{1}{2}\left(11 + 5\sqrt{5}\right) \sim 11.09.$$



- Expansion around z=1 has a radius of convergence of 1
- Expansion around z=0 has a radius of convergence of ~0.09
- In the interval (0,0.09) the two expansions overlap.
- Approximation of the associator can be obtained by matching both expansions at a point in the interval.
- Possible to evaluate the functions $t_{ij}(z)$ to arbitrary precision.

[Mistlberger]

- The system of differential equations is decoupled order-by-order in epsilon.
- We can evaluate the homogeneous solutions to arbitrary precision.

$$\vec{I}'(z) = \left[\mathbb{I} + \epsilon \int^{z} dz' A'(z',\epsilon) + \epsilon^2 \int^{z} dz' \int^{z'} dz'' A'(z',\epsilon) A'(z'',\epsilon) + \dots \right] \vec{I}'_0$$

- The system can now be solved order-by-order in terms of Chen iterated integrals.

$$J(\vec{\omega},z) = J(\omega_n(z),\dots,\omega_1(z),z) = \int_0^z dz' \omega_n(z') J(\omega_{n-1}(z'),\dots,\omega_1(z'),z').$$

$$\Big\{1, \frac{1}{1-z}, \frac{1}{z}, \frac{1}{z+1}, \frac{1}{\sqrt{z}}, \frac{1}{\sqrt{4-z}\sqrt{z}}, \frac{\sqrt{z}}{1-z}, \frac{1}{\sqrt{z}\sqrt{z+4}}, \frac{\sqrt{z}}{\sqrt{z+4}}, \frac{1}{\sqrt{4z+1}}, \frac{\sqrt{4z+1}}{z},$$
$$t_{11}, t_{12}, t_{21}, t_{22}, \frac{t_{11}}{1-z}, \frac{t_{11}}{z}, \frac{t_{11}}{z+1}, \frac{t_{12}}{1-z}, \frac{t_{12}}{z}, \frac{t_{12}}{z+1}, \frac{t_{21}}{z}, \frac{t_{22}}{z}\Big\}.$$

[Mistlberger]

$$J(\vec{\omega}, z) = J(\omega_n(z), \ldots, \omega_1(z), z) = \int_0^z dz' \omega_n(z') J(\omega_{n-1}(z'), \ldots, \omega_1(z'), z')$$

- Chen iterated integrals can be shuffle regulated in the usual fashion

$$J\Big(\omega_n(z), \ldots, \omega_1(z), \tfrac{1}{z}; z\Big) = \log(z) J\Big(\omega_n(z), \ldots, \omega_1(z); z\Big) - J\Big(\omega_n(z), \ldots, \omega_2(z), \tfrac{1}{z}, \omega_1(z); z\Big) + \ldots$$

- Letters that are divergent for z=0 are regulated as

$$J\left(\omega_n(z), \ldots, \omega_1(z), \frac{f(z)}{z}; z\right) = J\left(\omega_n(z), \ldots, \omega_1(z), \frac{f(z)-f(0)}{z}; z\right) + f(0) J\left(\omega_n(z), \ldots, \omega_1(z), \frac{1}{z}; z\right)$$

[Mistlberger]

$$J(\vec{\omega}, z) = J(\omega_n(z), \ldots, \omega_1(z), z) = \int_0^z dz' \omega_n(z') J(\omega_{n-1}(z'), \ldots, \omega_1(z'), z')$$

$$\left\{1, \frac{1}{1-z}, \frac{1}{z}, \frac{1}{z+1}, \frac{1}{\sqrt{z}}, \frac{1}{\sqrt{4-z}\sqrt{z}}, \frac{\sqrt{z}}{1-z}, \frac{1}{\sqrt{z}\sqrt{z+4}}, \frac{\sqrt{z}}{\sqrt{z+4}}, \frac{1}{\sqrt{4z+1}}, \frac{\sqrt{4z+1}}{z},\right.$$
$$\left. t_{11}, t_{12}, t_{21}, t_{22}, \frac{t_{11}}{1-z}, \frac{t_{11}}{z}, \frac{t_{11}}{z+1}, \frac{t_{12}}{1-z}, \frac{t_{12}}{z}, \frac{t_{12}}{z+1}, \frac{t_{21}}{z}, \frac{t_{22}}{z}\right\}.$$

- The iterated integrals are not pure
- The iterated integrals fulfill more identities than just shuffle

$$\sum_i c_i a_i(\bar{z}) J(\vec{\omega}_i, \bar{z}) = 0, \qquad , c_i \in \mathbb{Q},$$

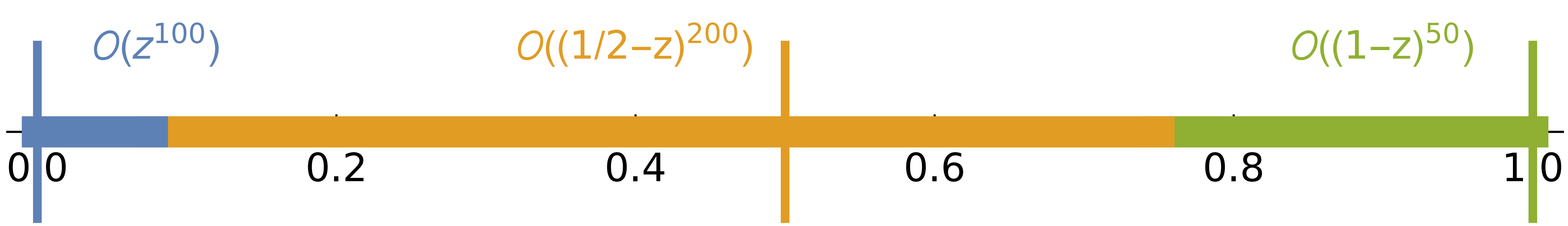
- The coefficients can be determined, by expanding the iterated integrals and prefactors to sufficiently high order in z and demanding that every power in z vanishes separately

$$\begin{aligned}
J\left(t_{11}, \frac{t_{12}}{1-\bar{z}}, \frac{1}{1-\bar{z}}\right) &= J\left(t_{12}, \frac{t_{11}}{1-\bar{z}}, \frac{1}{1-\bar{z}}\right) - J\left(t_{21}, \frac{t_{12}}{1-\bar{z}}, \frac{1}{1-\bar{z}}\right) \\
&- \frac{11}{5} J\left(\frac{t_{21}}{1-\bar{z}}, \frac{t_{12}}{1-\bar{z}}, \frac{1}{1-\bar{z}}\right) + J\left(t_{22}, \frac{t_{11}}{1-\bar{z}}, \frac{1}{1-\bar{z}}\right) \\
&+ \frac{11}{5} J\left(\frac{t_{22}}{1-\bar{z}}, \frac{t_{11}}{1-\bar{z}}, \frac{1}{1-\bar{z}}\right) + \frac{1}{5}(5\bar{z}-16)\, t_{11} J\left(\frac{t_{12}}{1-\bar{z}}, \frac{1}{1-\bar{z}}\right) \\
&- \frac{1}{5}(5\bar{z}-16)\, t_{12} J\left(\frac{t_{11}}{1-\bar{z}}, \frac{1}{1-\bar{z}}\right).
\end{aligned}$$

[Mistlberger]

- With this all ingredients are in place to cross section
- Differential equations are solved in terms of Chen iterated integrals with algebraic and non-algebraic letters
- Iterated integrals can be regulated and identities can be resolved by match power series
- How do we numerically evaluate the result?
- In principle each length-n iterated integral can be evaluated as an n-dimensional integral (with eg. Monte Carlo)
- Not very fast, stable or efficient :(

- More efficient to derive series expansions around several points
- Critical points of the cross section: $\left\{\frac{1}{2}\left(11+5\sqrt{5}\right), 4, 1, 0, \frac{1}{2}\left(11-5\sqrt{5}\right), -\frac{1}{4}, -1, -4\right\} \sim \left\{11.0902, 4, 1, 0, -0.0901699, -\frac{1}{4}, -1, -4\right\}.$
- Derive expansions around $\left\{z=0, z=\frac{1}{2}, z=1\right\}$
- Associated radii of convergence $\left\{r_0 = \left|\frac{1}{2}(11-5\sqrt{5})\right| \sim 0.09,\ r_{1/2} = \frac{1}{2},\ r_1 = 1\right\}$
- Sufficient to cover the entire interval (0,1)
- Allows for relative precision better than 10^{-10}



- Many sources of elliptic structures in Higgs production
  - Massive internal lines in the full standard model
  - Complicated massless phase space integrals
- Possible to solve large systems of DEs with elliptic subsystems
- Crucial to approximate associators by matching series expansions
- This technique does not actually rely on knowledge about elliptic functions, maybe generalizable to higher functions?
- Even more elliptic structures in Higgs production if we introduce more constraints (differential Higgs)