Elliptic Polylogarithms and Feynman Integrals

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CERN TH

Elliptic Integrals in Mathematics and Physics,
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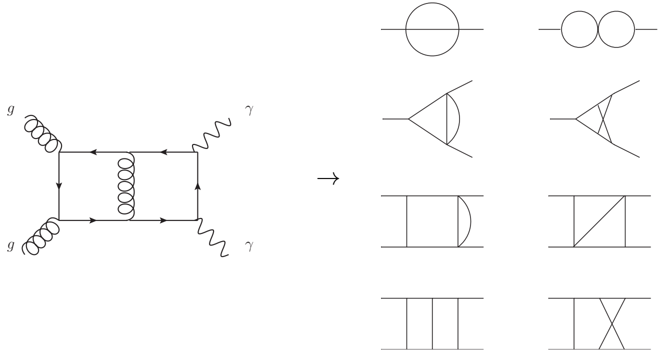
Based on collaboration with *A. Primo*, *E. Remiddi*,
J. Broedel, *C. Duhr*, *F. Dulat*, *B. Penante*

Feynman integrals are a useful representation for scattering amplitudes



Their analytic structure is dictated by *unitarity* and encoded in **special functions** used for their evaluation

Many F.I.s are expressible as Multiple PolyLogarithms (MPLs)

$$G(0;x) = \ln(x)\,, \qquad G(a;x) = \ln\left(1 - \frac{x}{a}\right) \quad \text{for} \quad a \neq 0$$

$$G(\underbrace{0,\ldots,0}_{n};x) = \frac{1}{n!}\ln^n(x)\,, \qquad G(a,\vec{w};x) = \int_0^x \frac{dy}{y-a}\,G(\vec{w};y)\,.$$

[E.Remiddi, J.Vermaseren '99; T. Gehrmann, E.Remiddi '00; Goncharov et al '00; Duhr, Gangl, Rhodes '13; ...]

A bit more "mathematically":

- Space of functions generated by integrating **rational functions** on the **Riemann sphere** $\mathbb{CP}^1$

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- Space of functions generated by integrating **rational functions** on the **Riemann sphere** $\mathbb{CP}^1$

Why is this the case? – **The differential equations method**

[Kotikov '90, Remiddi '97, **Gehrmann-Remiddi '00**,..., **J. Henn '13**; C. Papadopoulos '14]

⇓

Direct consequence of **Integration-by-parts (IBPs)** identities in d-dimensions!

$$\int \prod_{j=1}^{l} \frac{d^d\, k_j}{(2\pi)^d} \left(\frac{\partial}{\partial k_j^\mu}\, v_\mu\, \frac{S_1^{\sigma_1} \ldots S_s^{\sigma_s}}{D_1^{\alpha_1} \ldots D_n^{\alpha_n}} \right) = 0\,, \qquad v^\mu = k_j^\mu, p_k^\mu$$

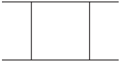
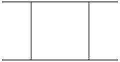
Reduced to N master integrals, $I_i(d; x_k)$ with $i = 1, \ldots, N$.

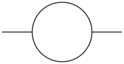
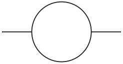
⇓

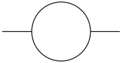
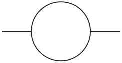
Differentiating the masters and using the **IBPs** we get a system of N coupled differential equations

$$\frac{\partial}{\partial\, x_k} I_i(d; x_k) = \sum_{j=1}^{N} c_{ij}(d; x_k)\, I_j(d; x_k)\,.$$

They look more or less like this:

$s\frac{\partial}{\partial s}$  $=\ \epsilon$ 

$+\frac{1-2\epsilon}{s+t}\Bigg[\frac{1}{s+t+u}$  $-\frac{1}{u}$  $\Bigg]$

$+\frac{1-2\epsilon}{s+u}\Bigg[\frac{1}{s+t+u}$  $-\frac{1}{t}$  $\Bigg]$

The coefficients are always **rational functions**! $\rightarrow$ If <u>first order</u>, we already see why we get iterated integrals over rational functions (ϵ expansion!)

Quite in general, differential equations are in <u>block form</u>

$$I_j(d;x_k) = (m_j(d;x_k)\,,\ \ sub_j(d;x_k))$$

$$\Downarrow$$

$$\frac{\partial}{\partial x_r} m_i(d;x_k) = \sum_{j=1}^{N} \underbrace{h_{ij}(d;x_k)}_{\Downarrow} m_j(d;x_k) + \sum_{j=1}^{M} nh_{ij}(d;x_k)\, sub_j(d;x_k)\,.$$

homogeneous piece is MAIN source of complexity

$$\Downarrow$$

Loosely speaking (caveat: square roots!):

- if decoupled as $d \to 4$, **MPLs**

- if coupled as $d \to 4$, **Elliptic...?**

As it turns out, it is often possible to choose a basis of MIs such that their "polylogarithmic" nature becomes manifest $\rightarrow$ integrals with unit leading singularities satisfy differential equations in **canonical form** [J. Henn '13]

$$\frac{\partial}{\partial x_r} \vec{m}(d; x_k) = (d-4)\, A(x_k)\, \vec{m}(d; x_k)$$

where $A(x_k)$ are differentials of logarithms[1], for every x_r, i.e. in differential form:

$$d\vec{m}(d; x_k) = (d-4)\, dB(x_k)\, \vec{m}(d; x_k)$$

Now, as we very well know, this is not the end of the story...

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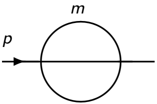
Where the elliptic story begins, a.k.a. **the sunrise graph:**



$$= \int \frac{d^d k \; d^d l}{(k^2 - m^2)(l^2 - m^2)((k - l - p)^2 - m^2)}$$

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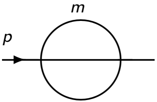

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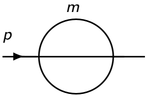
In the language above, one finds that it fulfils an **irreducible second order differential equation** [Broadhurst, Fleischer, Tarasov '93][Remiddi, Laporta '04] [Mueller-Stach, Weinzierl, Zayadeh '12]

$$\left(\frac{d^2}{d\, s^2} + A(d;s) \frac{d}{d\, s} + B(d;s) \right)$$



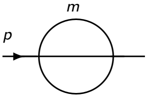
$$+ \; G(d;s)$$

$$= 0$$

Sunrise written as **iterated integrals** over the its homogeneous solutions

Homogeneous solutions given by the **maximal cut**:
[S.Laporta, E.Remiddi '04] [A.Primo, L.Tancredi '16, '17]

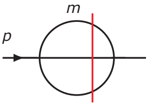
$$\left(\frac{d^2}{d\,s^2} + A(d;s)\,\frac{d}{d\,s} + B(d;s)\right)$$



$$+\,G(d;s)$$

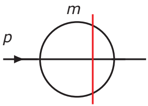
$$=0$$

Cut $\rightarrow$ $$\left(\frac{d^2}{d\,s^2} + A(d;s)\,\frac{d}{d\,s} + B(d;s)\right)$$



$$=0$$

The cut / imaginary part of the sunrise graph in $d = 2$



$$= \int_{b_2}^{b_3} \frac{db}{\sqrt{(b-b_1)(b-b_2)(b-b_3)(b-b_4)}}$$

$$= \frac{1}{\sqrt{(3m-\sqrt{s})(\sqrt{s}+m)^3}} \, \mathrm{K}\left(\frac{16 m^3 \sqrt{s}}{(3m-\sqrt{s})(\sqrt{s}+m)^3} \right)$$

where $\mathrm{K}(x)$ is the complete elliptic integral of the first kind.

$$\mathrm{K}(x) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)\,(1-x\,t^2)}}$$

By solving its second order differential equation, t he sunrise can be written as iterated integrals over algebraic functions and (products) of complete elliptic integrals... *[see talks. by Weinzierl and Dulat]*

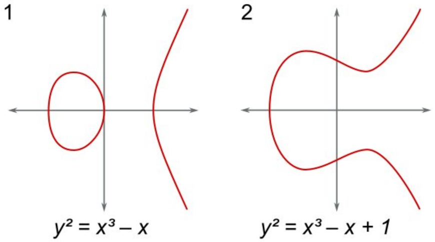
Let's try to make sense of it...

Elliptic curves – some notation

Elliptic curve parametrized by a **quartic** or **cubic** polynomial

$$y^2 = (x - a_1)(x - a_2)(x - a_3)(x - a_4)\,, \qquad y^2 = (x - e_1)(x - e_2)(x - e_3)$$

The two representations are equivalent (up to sending one point to infinity)



Elliptic curves – some notation

Quartic case

$$y^2 = (x - a_1)(x - a_2)(x - a_3)(x - a_4)\,,$$

The elliptic curve is characterised by two periods

$$\omega_1 = 2c_4 \int_{a_2}^{a_3} \frac{dx}{y} = 2\mathrm{K}(\lambda)\,, \qquad \omega_2 = 2c_4 \int_{a_1}^{a_2} \frac{dx}{y} = 2\,i\,\mathrm{K}(1-\lambda)$$

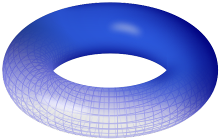
with

$$\lambda = \frac{(a_1 - a_4)(a_2 - a_3)}{(a_1 - a_3)(a_2 - a_4)}\,, \quad c_4 = \frac{1}{2}\sqrt{(a_1 - a_3)(a_2 - a_4)}\,.$$

We usually choose $\omega_1 \in \mathbb{R}$ and $\omega_2 \in i\,\mathbb{R}$, and define

$$\tau = \frac{\omega_2}{\omega_1}$$

Elliptic curves are also equivalent to genus 1 surface: *Complex Torus*



Take a complex lattice

$$\Lambda = \{\omega_1\, m + \omega_2\, n \quad : \quad m, n \in \mathbb{Z}\}$$

$\omega_{1,2}$ are called the *periods* on the lattice

$$\textbf{Complex Torus} \sim \mathbb{C}/\Lambda$$

Weierstrass $\wp(z)$ function, *doubly periodic* on the torus

$$\wp(z) = \frac{1}{z^2} + \sum_{n^2+m^2\neq 0} \left[\frac{1}{(z + n\omega_1 + m\omega_2)^2} - \frac{1}{(m\omega_1 + n\omega_2)^2} \right]$$

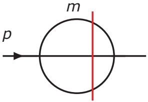
One finds $z \to [x, y, 1] \equiv [\wp(z), \wp'(z), 1]$

$$[\wp'(z)]^2 = 4\wp(z)^3 + g_2\wp(z)^2 - g_3 \quad \Rightarrow \quad y^2 = 4x^3 + g_2 x^2 - g_3$$

To move from the elliptic curve to the Torus, (a variant of) **Abel map:**

$$z_x = \frac{c_4}{\omega_1} \int_{a_1}^{x} \frac{dt}{\sqrt{P_4(t)}}$$

A toy model: the imaginary part of the sunrise in $d = 2 - 2\epsilon$ dimensions



$$= \int_{4m^2}^{(\sqrt{s}-m)^2} \frac{db}{\sqrt{P_4(b,s)}} \left(\frac{P_4(b,s)}{s\,b}\right)^{-\epsilon}$$

with

$$P_4(b,s) = (b-b_1)(b-b_2)(b-b_3)(b-b_4)\,.$$

These integrals (and generalisations thereof) appear in many other Feynman integrals. How do we make sense of them?

The **imaginary part** of the sunrise is a good place to see how these functions show up in different guises... [E. Remiddi, L. Tancredi '17]

An example:

$$F(s,m^2) = \int_{4m^2}^{(\sqrt{s}-m)^2} \frac{db}{\sqrt{P_4(b,s)}} \log b$$

1. Integral over the root of a quartic polynomial: iterated integrals of rational functions over an elliptic curve

2. It also satisfies a non-homogeneous second-order differential equation in s

$$\left[\frac{d^2}{ds^2} + \left(\frac{1}{s} + \frac{1}{s-1} + \frac{1}{s-9}\right)\frac{d}{ds} - \left(\frac{1}{3s} + \frac{1}{4(s-1)} + \frac{1}{12(s-9)}\right)\right] F(s) = R(s)\,, \qquad m^2 = 1$$

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We would like to

a. Understand these two representations

b. Figure out how to connect them (i.e., freely go from one to the other)

$$\Downarrow$$

Let us see what we can say about the functions defined by these repeated integrals...

$$\int_{4m^2}^{(\sqrt{s}-m)^2} \frac{db}{\sqrt{P_4(b,s)}} \log b \quad \ldots$$

1. Natural language: **iterated integrals** of rational functions on an **elliptic curve** *(Generalization of MPLs, iterated integrals on the Riemann sphere...)*

What are rational functions on the elliptic curve?

A rational function on the elliptic curve is a function $R(x, y)$ subject to the constraint $y = \sqrt{P(x)}$

$$R(x,y) = \frac{p_1(x) + p_2(x)\,y}{q_1(x) + q_2(x)\,y} = \frac{p_1(x) + p_2(x)\sqrt{P(x)}}{q_1(x) + q_2(x)\sqrt{P(x)}} = R_1(x) + \frac{1}{\sqrt{P(x)}}R_2(x)$$

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Given an **elliptic curve** $y^2 = P(x)$, with $P(x)$ (cubic polynomial for simplicity), let us study iterated integrals of rational functions on the curve.

$$\int dx \left(R_1(x) + \frac{1}{\sqrt{P(x)}} R_2(x) \right) = \ ?$$

After **partial fractioning**, one clearly ends up with

$$\int \frac{dx}{(x-c_i)^k} \qquad \text{from} \quad R_1(x)$$

$$\int \frac{dx}{y} x^k , \quad \int \frac{dx}{y(x-c_i)^k} , \qquad \text{from} \quad \frac{1}{y} R_2(x)$$

Integration by parts reduce everything to 4 kernels

$$\int \frac{dx}{(x-c_i)} , \quad \int \frac{dx}{y} , \quad \int \frac{x\, dx}{y} , \quad \int \frac{dx}{y(x-c_i)}$$

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MPLs have one more property: integration kernels with **simple poles**!

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$$\int \frac{x\,dx}{y} \sim -\int du \left(\frac{2}{u^2} + \mathcal{O}(u^0) \right) \quad \to \quad \underline{\text{double pole}} \text{ at infinity!}$$

Choose instead its **primitive**!

$$Z_3(x) \sim \int^x \frac{x\,dx}{y} \sim \frac{1}{u} \qquad u = \frac{1}{\sqrt{x}} \quad \to \quad \textit{Trascendental Kernel!}$$

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Fundamental differences with MPLs:

- Impossible to find basis of kernels which are **algebraic** and only with **simple poles**.

- We need **infinite tower** of integration kernels to span the whole space!

Integration Kernels (for "cubic" model) [J. Broedel, C. Duhr, F. Dulat, L. Tancredi '17]

$$\begin{aligned}
\varphi_0(0,x) &= \frac{c_3}{y}\,, \\
\varphi_1(c,x) &= \frac{1}{x-c}\,, \qquad \varphi_{-1}(c,x) = \frac{y_c}{y(x-c)}\,, \qquad \varphi_1(\infty,x) = \frac{c_3}{y}\,Z_3(x)\,, \\
\varphi_n(c,x) &= \left(\frac{1}{x-c} + \frac{c_3}{2y}\,Z_3(x)\right)\,Z_3^{(n-1)}(x)\,, \\
\varphi_{-n}(c,x) &= \frac{y_c}{y(x-c)}\,Z_3^{(n-1)}(x)\,, \qquad \varphi_n(\infty,x) = \frac{c_3}{y}\,Z_3^{(n)}(x)\,.
\end{aligned}$$

$$\mathrm{E}_3\left(\begin{smallmatrix} n_1 & \cdots & n_k \\ c_1 & \cdots & c_k \end{smallmatrix};x\right) = \int_0^x dt\,\varphi_{n_1}(c_1,t)\,\mathrm{E}_3\left(\begin{smallmatrix} n_2 & \cdots & n_k \\ c_2 & \cdots & c_k \end{smallmatrix};t\right)\,,$$

With the usual properties (shuffle...)

$$\mathrm{E}_3(\vec{c};x)\,\mathrm{E}_3(\vec{d};x) = \sum_{\vec{w}\in\vec{c}\sqcup\!\sqcup\vec{d}} \mathrm{E}_3(\vec{w};x)\,, \qquad \vec{c} = \left(\begin{smallmatrix} n_1 & \cdots & n_k \\ c_1 & \cdots & c_k \end{smallmatrix}\right)\,, \quad \text{same for } \vec{d}$$

Very same construction can be repeated using quartic representation of an elliptic curve

$$y^2 = P(x)\,, \qquad P(x) = (x-a_1)(x-a_2)(x-a_3)(x-a_4)$$

$$\begin{aligned}
\psi_0(0,x) &= \frac{c_4}{y}\,, \\
\psi_1(c,x) &= \frac{1}{x-c}\,, \qquad \psi_{-1}(c,x) = \frac{y_c}{y(x-c)}\,, \\
\psi_1(\infty,x) &= \frac{c_4}{y}\,Z_4(x)\,, \qquad \psi_{-1}(\infty,x) = \frac{x}{y}\,, \\
\psi_{-n}(\infty,x) &= \frac{x}{y}\,Z_4^{(n-1)}(x) - \frac{\delta_{n2}}{c_4}\,, \\
\psi_n(c,x) &= \frac{1}{x-c}\,Z_4^{(n-1)}(x) - \delta_{n2}\,\Phi_4(x)\,, \\
\psi_n(\infty,x) &= \frac{c_4}{y}\,Z_4^{(n)}(x)\,, \qquad \psi_{-n}(c,x) = \frac{y_c}{y(x-c)}\,Z_4^{(n-1)}(x)\,,
\end{aligned}$$

$$\mathrm{E}_4\left(\begin{smallmatrix} n_1 & \dots & n_k \\ c_1 & \dots & c_k \end{smallmatrix}; x\right) = \int_0^x dt\,\psi_{n_1}(c_1,t)\,\mathrm{E}_4\left(\begin{smallmatrix} n_2 & \dots & n_k \\ c_2 & \dots & c_k \end{smallmatrix}; t\right)\,,$$

Most of the kernels not needed in physical applications...

$$\begin{aligned}
\psi_0(0,x) &= \frac{c_4}{y}\,, \\
\psi_1(c,x) &= \frac{1}{x-c}\,, \qquad \psi_{-1}(c,x) = \frac{y_c}{y(x-c)}\,, \\
\psi_1(\infty,x) &= \frac{c_4}{y}\,Z_4(x)\,, \qquad \psi_{-1}(\infty,x) = \frac{x}{y}\,,
\end{aligned}$$

They contain MPLs as a trivial subset

$$\mathrm{E}_4\left(\begin{smallmatrix} 1 & \dots & 1 \\ c_1 & \dots & c_k \end{smallmatrix};x\right) = G(c_1,\dots,c_k;x)\,,$$

Indeed, in this language, integrals above become straightforward

$$F(s) = \int_{4m^2}^{(\sqrt{s}-m)^2} \frac{db}{\sqrt{P_4(b,s)}} \log(b)$$
$$= \int_{4m^2}^{(\sqrt{s}-m)^2} \frac{db}{\sqrt{P_4(b,s)}} \mathrm{E}_4\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}; b\right)$$
$$= \frac{2}{\sqrt{(\sqrt{s}-m)^2(\sqrt{s}+3m)}} \left\{ \mathrm{E}_4\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}; (\sqrt{s}-m)^2\right) - \mathrm{E}_4\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}; 4m^2\right) \right\}$$

And similarly for all other integrals appearing in the imaginary part of the sunrise graph, at **every order in** ϵ.

We can prove that our functions are equivalent to the **elliptic polylogarithms** introduced in the mathematics and string theory literature
[F. Brown, A. Levin, '11] [J. Broedel, C. Mafra, N. Matthes, O. Schlotterer '14]

Iterated integrals build on the **Torus** – two equivalent representations

1. $$\widetilde{\Gamma}\left(\begin{smallmatrix} n_1 & \cdots & n_k \\ c_1 & \cdots & c_k \end{smallmatrix}; x\right) = \int_0^x dt\, g^{(n_1)}(t - c_1)\, \widetilde{\Gamma}\left(\begin{smallmatrix} n_2 & \cdots & n_k \\ c_2 & \cdots & c_k \end{smallmatrix}; t\right)$$

2. $$\Gamma\left(\begin{smallmatrix} n_1 & \cdots & n_k \\ c_1 & \cdots & c_k \end{smallmatrix}; x\right) = \int_0^x dt\, f^{(n_1)}(t - c_1)\, \Gamma\left(\begin{smallmatrix} n_2 & \cdots & n_k \\ c_2 & \cdots & c_k \end{smallmatrix}; t\right)$$

1. The integration kernels are defined through generating function

$$\widetilde{\Gamma}\left(\begin{smallmatrix} n_1 & \cdots & n_k \\ c_1 & \cdots & c_k \end{smallmatrix} ; x \right) = \int_0^x dt\, g^{(n_1)}(t - c_1)\, \widetilde{\Gamma}\left(\begin{smallmatrix} n_2 & \cdots & n_k \\ c_2 & \cdots & c_k \end{smallmatrix} ; t \right)$$

$$F(z, \alpha, \tau) = \frac{1}{\alpha} \sum_{n \geq 0} g^{(n)}(z, \tau)\, \alpha^n = \frac{\theta_1'(0, \tau)\theta_1(z + \alpha, \tau)}{\theta_1(z, \tau)\theta_1(\alpha, \tau)}$$

$\theta_1(z, \tau)$ is the odd Jacobi theta function.
Kernels are **holomorphic** but not doubly periodic

$$g^{(1)}(z + \omega_1, \tau) = g^{(1)}(z, \tau)\,, \qquad g^{(1)}(z + \omega_2, \tau) = g^{(1)}(z, \tau) - \frac{2\pi i}{\omega_1}$$

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2. Similarly, for the other representation

$$\Gamma\left(\begin{smallmatrix} n_1 & \cdots & n_k \\ c_1 & \cdots & c_k \end{smallmatrix}; x\right) = \int_0^x dt\, f^{(n_1)}(t - c_1)\, \Gamma\left(\begin{smallmatrix} n_2 & \cdots & n_k \\ c_2 & \cdots & c_k \end{smallmatrix}; t\right)$$

$$\frac{1}{\alpha}\sum_{n\geq 0} g^{(n)}(z,\tau)\,\alpha^n = \exp\left[2\pi i\alpha\frac{\mathrm{Im}z}{\mathrm{Im}\tau}\right] F(z,\alpha,\tau)$$

Kernels are related

$$f^{(n)} = \sum_{k\geq 0}^{n} \frac{1}{k!}\left[2\pi i\frac{\mathrm{Im}z}{\mathrm{Im}\tau}\right]^k g^{(n-k)}(z,\tau) \quad \rightarrow \quad f^{(1)}(z,\tau) = g^{(1)}(z,\tau) + 2\pi i\,\frac{\mathrm{Im}z}{\mathrm{Im}\tau}$$

Kernels are NOT holomorphic, and instead **doubly periodic**

$$f^{(n)}(z+\omega_1,\tau) = f^{(n)}(z,\tau)\,, \qquad f^{(n)}(z+\omega_2,\tau) = f^{(n)}(z,\tau)$$

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Kernels are NOT holomorphic, and instead **doubly periodic**

$$f^{(n)}(z+\omega_1,\tau) = f^{(n)}(z,\tau)\,, \qquad f^{(n)}(z+\omega_2,\tau) = f^{(n)}(z,\tau)$$

Back to the **toy model** above: simply written in terms of these functions

$$F(s) = \frac{2\omega_1}{\sqrt{(\sqrt{s}-1)^2(\sqrt{s}+3)}}$$
$$\times \Bigg\{ 2\log 2\,\widetilde{\Gamma}\left(\begin{smallmatrix}0\\0\end{smallmatrix};\tfrac{\tau}{2}\right) - 2\log\left(\sqrt{s}-1\right)\widetilde{\Gamma}\left(\begin{smallmatrix}0\\0\end{smallmatrix};\tfrac{1+\tau}{2}\right)$$
$$+\widetilde{\Gamma}\left(\begin{smallmatrix}1 & 0\\ -\frac{1}{3} & 0\end{smallmatrix};\tfrac{\tau}{2}\right) - \widetilde{\Gamma}\left(\begin{smallmatrix}1 & 0\\ -\frac{1}{3} & 0\end{smallmatrix};\tfrac{1+\tau}{2}\right) + \widetilde{\Gamma}\left(\begin{smallmatrix}1 & 0\\ \frac{1}{3} & 0\end{smallmatrix};\tfrac{\tau}{2}\right) - \widetilde{\Gamma}\left(\begin{smallmatrix}1 & 0\\ \frac{1}{3} & 0\end{smallmatrix};\tfrac{1+\tau}{2}\right)$$
$$-2\widetilde{\Gamma}\left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix};\tfrac{\tau}{2}\right) + 2\widetilde{\Gamma}\left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix};\tfrac{1+\tau}{2}\right) \Bigg\}, \qquad \text{with } m^2 = 1$$

And similarly for all other integrals appearing in the imaginary part of the sunrise graph, at every order in ϵ.

Something interesting: result above written as $\widetilde{\Gamma}\left(\begin{smallmatrix} n_1 & \dots & n_k \\ z_1 & \dots & z_k \end{smallmatrix}; x\right)$ where all arguments are "rational points" in τ (everything depends only on τ!)

$$\frac{r}{6} + \frac{s}{6}\tau \qquad r, s \in \mathbb{Z}\,.$$

It turns out, quite in general, that in this case the integration kernels of the Γ polylogarithms transform nicely under **modular transformations**!

$$\Downarrow$$

This implies that eMPLs evaluated at rational points can be written as linear combinations of **iterated integrals over Eisenstein series**

$$I\left(\begin{smallmatrix} n_1 & N_1 \\ r_1 & s_1 \end{smallmatrix} \Big| \dots \Big| \begin{smallmatrix} n_k & N_k \\ r_k & s_k \end{smallmatrix}; \tau\right) \equiv \int_{i\infty}^{\tau} d\tau'\, h^{(n_1)}_{N_1,r_1,s_1}(\tau')\, I\left(\begin{smallmatrix} n_2 & N_2 \\ r_2 & s_2 \end{smallmatrix} \Big| \dots \Big| \begin{smallmatrix} n_k & N_k \\ r_k & s_k \end{smallmatrix}; \tau'\right)\,,$$

$$h^{(n)}_{N,r,s}(\tau) = \sum_{k=0}^{n} \frac{(2\pi i s)^k}{k!\, N^k} g^{(n-k)}\left(\frac{r}{N} + \tau\frac{s}{N}; \tau\right) = \sum_{\substack{(\alpha,\beta)\in\mathbb{Z}^2 \\ (\alpha,\beta)\neq(0,0)}} \frac{e^{-2\pi i (s\alpha - r\beta)/N}}{(\alpha+\beta\tau)^n}\,.$$

Something interesting: result above written as $\widetilde{\Gamma}\left(\begin{smallmatrix} n_1 & \dots & n_k \\ z_1 & \dots & z_k \end{smallmatrix} ; x\right)$ where all arguments are "rational points" in τ (everything depends only on τ!)

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$$h^{(n)}_{N,r,s}(\tau) = \sum_{k=0}^{n} \frac{(2\pi i s)^k}{k!\, N^k} g^{(n-k)}\left(\frac{r}{N} + \tau \frac{s}{N}; \tau \right) = \sum_{\substack{(\alpha,\beta)\in\mathbb{Z}^2 \\ (\alpha,\beta)\neq(0,0)}} \frac{e^{-2\pi i (s\alpha - r\beta)/N}}{(\alpha + \beta\tau)^n}\,.$$

As an example, take the integral

$$\mathcal{I}(s) = \int_{4m^2}^{(\sqrt{s}-m)^2} \frac{db}{\sqrt{P_4(b,s)}} \ln\left(b - 4m^2\right)$$

repeating the exercise above we find (again $m = 1$ for simplicity):

$$\mathcal{I}(s) = \frac{1}{2\pi i}\, I\left(\begin{smallmatrix} 2 & 6 \\ 2 & 0 \end{smallmatrix}; \tau\right) - \frac{1}{2\pi i} I\left(\begin{smallmatrix} 2 & 6 \\ 0 & 3 \end{smallmatrix}; \tau\right) + \frac{i\pi}{4}\tau + \log 2 + \frac{1}{2}\log 3\,.$$

I.e. iterated integrals of Eisenstein series for $\Gamma(6)$! Similarly for all integrals in the imaginary part of the sunrise graph.

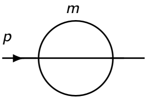
Inspired by the integrals appearing in the imaginary part of the sunrise graph, we defined a general class of functions and found it to be equivalent to the elliptic polylogarithms of the math literature.

But are they useful for something else except the imaginary part of the sunrise?

Indeed, many examples both from the math and from the physics world turn out to be expressible in terms of these functions

Start with the complete **two-loop massive sunrise graph** ($d = 2$)

$$S_{111}(p^2, m^2) =$$



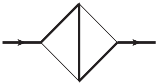
$$= \frac{1}{m^2 - p^2} \frac{1}{c_4} \left[\frac{1}{c_4} \mathrm{E}_4\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}; 1\right) - 2\mathrm{E}_4\left(\begin{smallmatrix} 0 & -1 \\ 0 & \infty \end{smallmatrix}; 1\right) + \mathrm{E}_4\left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}; 1\right) \right.$$

$$\left. + \mathrm{E}_4\left(\begin{smallmatrix} 0 & -1 \\ 0 & 1 \end{smallmatrix}; 1\right) - \mathrm{E}_4\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}; 1\right) \right]$$

Result can be obtained straightforwardly by **direct integration** over Feynman parameters, and extended to **higher orders** in ϵ. Similar results can be obtained by using a dispersion relation, for equal and **different masses**!

First "generalisation", the **kite** integral, finite in $d = 2$

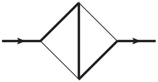
$$K_{11111}(p^2, m^2) =$$



$$
\begin{aligned}
&= \frac{1}{z}\Big[2\pi^2 G(0,z) - 2\pi^2 G(1,z) + 3G(0,0,0,z) - 6G(0,1,0,z) - 24\zeta(3) \\
&\quad +12G(0,1,1,z) - 3G(1,0,0,z) - 6G(1,0,1,z) + 6G(1,1,0,z) + \ldots\Big] \\
&+ \frac{1+z}{(a_1-a_3)^2(1-z)z}\Big[\mathrm{E}_4\left(\begin{smallmatrix}0&0&1\\0&0&1\end{smallmatrix};1\right) + \mathrm{E}_4\left(\begin{smallmatrix}0&1&0\\0&1&0\end{smallmatrix};1\right) - \mathrm{E}_4\left(\begin{smallmatrix}1&0&0\\0&0&0\end{smallmatrix};1\right)\Big] \\
&+ \frac{1+z}{(a_1-a_3)(1-z)z}\Big[\mathrm{E}_4\left(\begin{smallmatrix}0&-1&1\\0&0&1\end{smallmatrix};1\right) + \mathrm{E}_4\left(\begin{smallmatrix}0&-1&1\\0&1&1\end{smallmatrix};1\right) \\
&\qquad\qquad +2\mathrm{E}_4\left(\begin{smallmatrix}0&-1&1\\0&\infty&1\end{smallmatrix};1\right) + \mathrm{E}_4\left(\begin{smallmatrix}0&1&1\\0&0&1\end{smallmatrix};1\right) + \ldots\Big] + 79 \text{ more } \mathrm{E}_4 s
\end{aligned}
$$

Also here, result obtained by **direct integration**!

First (small) generalisation, the **kite** integral (contains the sunrise), finite in $d = 2$

$K_{11111}(p^2, m^2) =$ 

$$= \frac{1}{z}\Big[2\pi^2 G(0,z) - 2\pi^2 G(1,z) + 3G(0,0,0,z) - 6G(0,1,0,z) - 24\zeta(3)$$
$$+12G(0,1,1,z) - 3G(1,0,0,z) - 6G(1,0,1,z) + 6G(1,1,0,z) + \ldots\Big]$$
$$+ \frac{1+z}{(a_1-a_3)^2(1-z)z}\left[\mathrm{E}_4\left(\begin{smallmatrix}0&0&1\\0&0&1\end{smallmatrix};1\right) + \mathrm{E}_4\left(\begin{smallmatrix}0&1&0\\0&1&0\end{smallmatrix};1\right) - \mathrm{E}_4\left(\begin{smallmatrix}1&0&0\\0&0&0\end{smallmatrix};1\right)\right]$$
$$+ \frac{1+z}{(a_1-a_3)(1-z)z}\Big[\mathrm{E}_4\left(\begin{smallmatrix}0&-1&1\\0&0&1\end{smallmatrix};1\right) + \mathrm{E}_4\left(\begin{smallmatrix}0&-1&1\\0&1&1\end{smallmatrix};1\right)$$
$$+2\mathrm{E}_4\left(\begin{smallmatrix}0&-1&1\\0&\infty&1\end{smallmatrix};1\right) + \mathrm{E}_4\left(\begin{smallmatrix}0&1&1\\0&0&1\end{smallmatrix};1\right) + \ldots\Big] + 79 \text{ more } \mathrm{E}_4 s$$

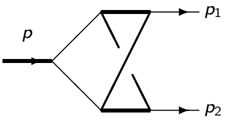
Definitely does not look **pure**! See Brenda's talk...!

Note that:

With this formalism it is simple to show that sunrise (and the kite) can be written as iterated integrals over modular forms of $\Gamma(6)$! [Adams, Weinzierl '17], [Broedel, Duhr, Dulat, Penante, Tancredi '18]

1. Go from $\mathrm{E}_4 \to \Gamma$ functions (entirely algorithmic)
2. Check that one obtains only eMPLs evaluated at rational points!

A more interesting example: [M. Czakon, A. Mitov '08; A. von Manteuffel, L. Tancredi '17]

$T(d; p^2, m^2) =$ 

- $p_1^2 = p_2^2 = 0$, four massive lines
- $a = -m^2/p^2$
- 2 master integrals, $T_1(a)$, $T_2(a)$
- Satisfy 2 coupled diff. eqs
- Needed for NNLO $\gamma\gamma$, $t\bar{t}$, ...

Again, it can be computed in terms of E_4 by direct integration over Feynman parameters.

Convenient to use **Feynman parameters** (the integral is finite in $d = 4$!)

$$T(4; a) = \int \frac{d^4 k \, d^4 l}{(k^2 - m^2)(l^2 - m^2)((k - p_1)^2 - m^2)((l - p_2)^2 - m^2)(k - l - p_1)^2 (l - k - p_2)^2}$$
$$= \int_0^1 \prod_{i=1}^{6} dx_i \; \delta\left(1 - \sum_{i=1}^{6} x_i\right) \frac{1}{[F(x_1, x_2, x_3, x_4, x_5, x_6, p^2, m^2)]^2}$$

with

$$F(x_1, x_2, x_3, x_4, x_5, x_6, p^2, m^2) =$$
$$\Big(p^2 \left[\left((x_3 + x_4)\, x_5 + x_2\,(x_3 + x_5)\right) x_6 + x_1 x_5\,(x_4 + x_6)\right] + m^2\,(x_1 + x_2 + x_3 + x_4)$$
$$\times\,(x_3 x_4 + x_5 x_4 + x_6 x_4 + x_3 x_5 + x_3 x_6 + x_2\,(x_3 + x_5 + x_6) + x_1\,(x_2 + x_4 + x_5 + x_6))\Big)$$

We can perform all integrations in terms of E_4 functions using Cheng-Wu theorem [M. Hidding, F. Moriello '17]

$$
\begin{aligned}
T(4,a) = \frac{2a^2}{c_4^2}\Bigg[& 5\mathrm{E}_4\left(\begin{smallmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & r_{mp} \end{smallmatrix};1\right) + 5\mathrm{E}_4\left(\begin{smallmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & r_{pp} \end{smallmatrix};1\right) + 5\mathrm{E}_4\left(\begin{smallmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & r_{mp} \end{smallmatrix};1\right) + 5\mathrm{E}_4\left(\begin{smallmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & r_{pp} \end{smallmatrix};1\right) \\
& - 3\mathrm{E}_4\left(\begin{smallmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & r_{mm} & 0 \end{smallmatrix};1\right) - 3\mathrm{E}_4\left(\begin{smallmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & r_{mm} & 1 \end{smallmatrix};1\right) - 3\mathrm{E}_4\left(\begin{smallmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & r_{pm} & 0 \end{smallmatrix};1\right) - 3\mathrm{E}_4\left(\begin{smallmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & r_{pm} & 1 \end{smallmatrix};1\right) \\
& + 3\log a\left(\mathrm{E}_4\left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & r_{mm} \end{smallmatrix};1\right) + \mathrm{E}_4\left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & r_{pm} \end{smallmatrix};1\right)\right)\Bigg] \\
- \frac{4a^2}{c_4}\Bigg[& 5\mathrm{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 0 & r_{mp} \end{smallmatrix};1\right) + 5\mathrm{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 0 & r_{pp} \end{smallmatrix};1\right) + 5\mathrm{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 1 & r_{mp} \end{smallmatrix};1\right) + 5\mathrm{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 1 & r_{pp} \end{smallmatrix};1\right) \\
& - 3\mathrm{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & r_{mm} & 0 \end{smallmatrix};1\right) - 3\mathrm{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & r_{mm} & 1 \end{smallmatrix};1\right) - 3\mathrm{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & r_{pm} & 0 \end{smallmatrix};1\right) - 3\mathrm{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & r_{pm} & 1 \end{smallmatrix};1\right) \\
& + 3\log a\left(\mathrm{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 \\ 0 & \infty & r_{mm} \end{smallmatrix};1\right) + \mathrm{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 \\ 0 & \infty & r_{pm} \end{smallmatrix};1\right)\right)\Bigg]
\end{aligned}
$$

with

$$c_4 = \frac{1+\sqrt{1-16a}}{4}$$

$$r_{mm} = \frac{1-\sqrt{1-4a}}{2} \quad r_{mp} = \frac{1-\sqrt{1+4a}}{2} \quad r_{pm} = \frac{1+\sqrt{1-4a}}{2} \quad r_{pp} = \frac{1+\sqrt{1+4a}}{2}$$

Note that:

In this case, it does **NOT** seem to be possible to write this integral as iterated integrals over modular forms. The corresponding Γ eMPLs are **NOT** evaluated at rational points (extra threshold coming from subtopologies!)

We can perform all integrations in terms of E_4 functions using Cheng-Wu theorem [M. Hidding, F. Moriello '17]

$$
\begin{aligned}
T(4,a) = \frac{2a^2}{c_4^2}\Bigg[& 5\mathrm{E}_4\left(\begin{smallmatrix}0&0&1&1\\0&0&0&r_{mp}\end{smallmatrix};1\right) + 5\mathrm{E}_4\left(\begin{smallmatrix}0&0&1&1\\0&0&0&r_{pp}\end{smallmatrix};1\right) + 5\mathrm{E}_4\left(\begin{smallmatrix}0&0&1&1\\0&0&1&r_{mp}\end{smallmatrix};1\right) + 5\mathrm{E}_4\left(\begin{smallmatrix}0&0&1&1\\0&0&1&r_{pp}\end{smallmatrix};1\right) \\
& - 3\mathrm{E}_4\left(\begin{smallmatrix}0&0&1&1\\0&0&r_{mm}&0\end{smallmatrix};1\right) - 3\mathrm{E}_4\left(\begin{smallmatrix}0&0&1&1\\0&0&r_{mm}&1\end{smallmatrix};1\right) - 3\mathrm{E}_4\left(\begin{smallmatrix}0&0&1&1\\0&0&r_{pm}&0\end{smallmatrix};1\right) - 3\mathrm{E}_4\left(\begin{smallmatrix}0&0&1&1\\0&0&r_{pm}&1\end{smallmatrix};1\right) \\
& + 3\log a\left(\mathrm{E}_4\left(\begin{smallmatrix}0&0&1\\0&0&r_{mm}\end{smallmatrix};1\right) + \mathrm{E}_4\left(\begin{smallmatrix}0&0&1\\0&0&r_{pm}\end{smallmatrix};1\right)\right)\Bigg] \\
- \frac{4a^2}{c_4}\Bigg[& 5\mathrm{E}_4\left(\begin{smallmatrix}0&-1&1&1\\0&\infty&0&r_{mp}\end{smallmatrix};1\right) + 5\mathrm{E}_4\left(\begin{smallmatrix}0&-1&1&1\\0&\infty&0&r_{pp}\end{smallmatrix};1\right) + 5\mathrm{E}_4\left(\begin{smallmatrix}0&-1&1&1\\0&\infty&1&r_{mp}\end{smallmatrix};1\right) + 5\mathrm{E}_4\left(\begin{smallmatrix}0&-1&1&1\\0&\infty&1&r_{pp}\end{smallmatrix};1\right) \\
& - 3\mathrm{E}_4\left(\begin{smallmatrix}0&-1&1&1\\0&\infty&r_{mm}&0\end{smallmatrix};1\right) - 3\mathrm{E}_4\left(\begin{smallmatrix}0&-1&1&1\\0&\infty&r_{mm}&1\end{smallmatrix};1\right) - 3\mathrm{E}_4\left(\begin{smallmatrix}0&-1&1&1\\0&\infty&r_{pm}&0\end{smallmatrix};1\right) - 3\mathrm{E}_4\left(\begin{smallmatrix}0&-1&1&1\\0&\infty&r_{pm}&1\end{smallmatrix};1\right) \\
& + 3\log a\left(\mathrm{E}_4\left(\begin{smallmatrix}0&-1&1\\0&\infty&r_{mm}\end{smallmatrix};1\right) + \mathrm{E}_4\left(\begin{smallmatrix}0&-1&1\\0&\infty&r_{pm}\end{smallmatrix};1\right)\right)\Bigg]
\end{aligned}
$$

Again, simple result, but definitely not pure! (see Brenda's talk!)

This result is particularly important for “practical applications”, because it is the **first example of a realistic family of Feynman integrals** that can be expressed in terms of elliptic polylogarithms! (relevant for $t\bar{t}$, $\gamma\gamma$, Hj, HH,... production)

$$\Downarrow$$

It is a first step! Many other examples will follow soon:
two- three- and four- point functions at two loops!

At least one loose end:

All examples that I showed you have been computed by **direct integration** over Feynman parameters (or dispersion relations)!

How do we relate these functions to what comes out from solving the corresponding **differential equations**? Still work in progress! Stay tuned!

THANKS!