# Elliptic Purity

Brenda Penante

In collaboration with
J. Broedel, C. Duhr, F. Dulat, L. Tancredi

Elliptic functions in mathematics and physics

Ascona

Sep 6th, 2018

# We just heard from Lorenzo:

- Interplay between DE approach and direct integration of Feynman integrals for MPLs

- In the elliptic case, we are not quite there yet

- Elliptic Polylogarithms (eMPLs) on the elliptic curve

- Several examples of elliptic FIs and their representation in terms of eMPLs

Another notion from the MPL world: Purity

Examples of elliptic FIs didn't look pure

This talk: Notion of Purity in the elliptic case

# What do you mean "Pure"?

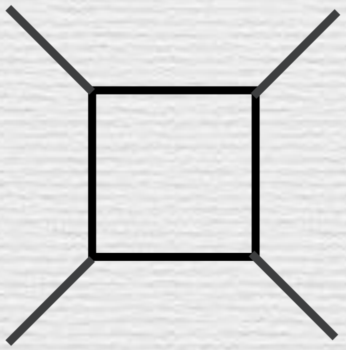
- Definition based on residues

— Arkani-Hamed, Bourjaily
Cachazo, Trnka '12 —

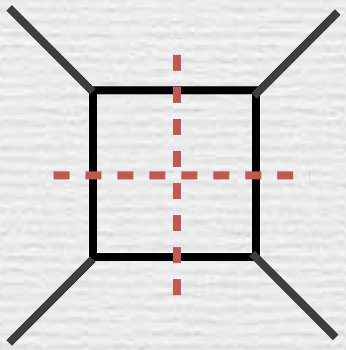
*An integral is pure if all of its non-vanishing residues are the same up to a sign*

"Integrals with unit leading singularity"

- Ex: 4-mass box



$$= \frac{2}{st}\left[\frac{1}{\epsilon^2} - \frac{\log(st)}{\epsilon} + \log(-s)\log(-t) - \frac{2\pi^2}{3}\right]$$



$$= \pm\frac{1}{st}$$

# What do you mean "Pure"?

- Definition based on residues

— Arkani-Hamed, Bourjaily
Cachazo, Trnka '12 —

*An integral is pure if all of its non-vanishing residues are the same up to a sign*

"Integrals with unit leading singularity"

- Prototypical example: MPLs

$$G(a_1, \ldots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t)$$

- Weight of an MPL = number of integrations

$$G(a_1,\ldots,a_n;z) = \int_0^z \frac{dt}{t-a_1} G(a_2,\ldots,a_n;t) \qquad a_i \in \mathbb{C}$$

$$G(\underbrace{0,\ldots,0}_{n \text{ times}};z) = \frac{1}{n!}\log^n z \qquad\qquad G(;z) \equiv 1$$

Weight $n$

$$G(a_1,\ldots,a_n;z)$$

$$\zeta_n = -G(\vec{0}_{n-1},1;1)$$

$$i\pi = \log(-1)$$

# What do you mean "Pure"?

- Definition based on total differential — Henn '13 —

\# of integrations

*A pure function of weight n is a function whose total derivative can be expressed in terms of pure functions of weight n-1 (times algebraic factors)*

algebraic

$$dG(a_1, \ldots, a_n; z) = \sum_{i=1}^{n} G(a_1, \ldots, \hat{a}_i, \ldots, a_n; z)\, d\log \frac{a_{i-1} - a_i}{a_{i+1} - a_i}$$

weight $n$

weight $n-1$

# What do you mean "Pure"?

- We seek to generalise the following to the elliptic case:

*A function is called pure if it is unipotent and it has at most logarithmic singularities.*

(Unipotent: total diff has no homogeneous term)

Log singularities

$$dG(a_1, \ldots, a_n; z) = \sum_{i=1}^{n} G(a_1, \ldots, \hat{a}_i, \ldots, a_n; z)\, d\log \frac{a_{i-1} - a_i}{a_{i+1} - a_i}$$

Pure

Unipotent

# Why bother?

- Meaning not entirely understood even in the MPL case
- Nevertheless, shows underlying structure
  Eg. N=4 SYM:

  anomalous dimensions, amplitudes,
  certain form factors, etc

  L-loops $\leftrightarrow$ Weight 2L functions
  "Uniform transcendentality"

- Organisational principle:
  functional identities among functions of fixed weight
- "Maximal transcendentality principle"
  — Kotikov, Lipatov, Onishchenko, Velizhanin '04 —

# Why bother?

Total differential: $dG(a_1,\ldots,a_n;z) = \sum_{i=1}^{n} G(a_1,\ldots,\hat{a}_i,\ldots,a_n;z)\, d\log\frac{a_{i-1}-a_i}{a_{i+1}-a_i}$

Symbol: $\mathcal{S}(G(a_1,\ldots,a_n;z)) = \sum_{i=1}^{n} \mathcal{S}(G(a_1,\ldots,\hat{a}_i,\ldots,a_n;z)) \otimes \frac{a_{i-1}-a_i}{a_{i+1}-a_i}$

— Goncharov, Spradlin, Vergu, Volovich '10 —

Length $n$ ⟶ $n$ -fold tensor product

- Taming analytical expressions, functional identities
- Symbol bootstrap with MPL ansatz in N=4 SYM

— Caron-Huot, Dixon, Drummond, Duhr, Harrington, Henn, McLeod, Papathanaseou, Pennington, Spradlin, von Hippel —

# Why bother?

Differential equations in canonical form

Matrix of “dlog” forms

$$dF = \varepsilon\, dA\, F$$

Vector of master integrals

For MPLs: natural solution in terms of pure functions G

To-do: develop a general framework also for elliptic integrals

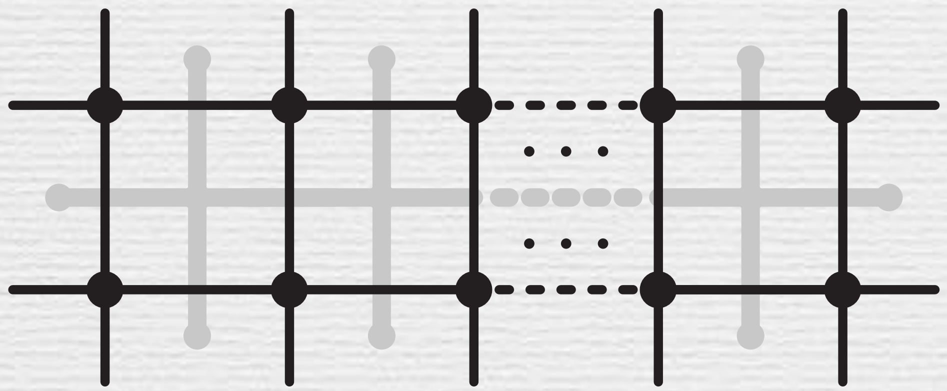
[Weinzierl & Tancredi’s talks]

# The real world: N=4 Super Yang-Mills

Conjecturally of uniform (maximal) weight

Elliptic integrals (and beyond) are known to appear:

— Caron-Huot, Larsen '12 / Nandan, Paulos, Spradlin, Volovich '14 / Bourjaily, McLeod, Spradlin, von Hippel, Wilhelm '17 —

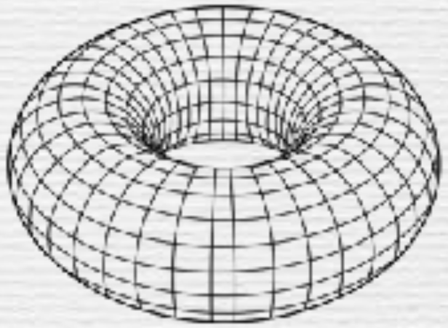


The elliptic double box, and more generally traintracks

— Bourjaily, He, McLeod, von Hippel, Wilhelm '18 —

We'd like to give an elliptic meaning to these statements!

# Elliptic Polylogarithms



Feynman integrals which evaluate to elliptic integrals can be represented as eMPLs on the complex torus:

— Brown, Levin '11, Broedel, Mafra, Matthes, Schlotterer '14 —

$$\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & \dots & n_k \\ z_1 & \dots & z_k \end{smallmatrix}; z\right) = \int_0^z dz' g^{(n_1)}(z'-z_1)\tilde{\Gamma}\left(\begin{smallmatrix} n_2 & \dots & n_k \\ z_2 & \dots & z_k \end{smallmatrix}; z\right)$$

holomorphic, ~~double periodic~~

$$\Gamma\left(\begin{smallmatrix} n_1 & \dots & n_k \\ z_1 & \dots & z_k \end{smallmatrix}; z, \tau\right) = \int_0^z dz'\, f^{(n_1)}(z'-z_1,\tau)\,\Gamma\left(\begin{smallmatrix} n_2 & \dots & n_k \\ z_2 & \dots & z_k \end{smallmatrix}; z', \tau\right)$$

~~holomorphic~~, double periodic

Or sometimes as iterated integrals over modular forms

— Brown '14 —

$$I\left(\begin{smallmatrix} n_1 & N_1 \\ r_1 & s_1 \end{smallmatrix} \Big| \dots \Big| \begin{smallmatrix} n_k & N_k \\ r_k & s_k \end{smallmatrix}; \tau\right) = \int_{i\infty}^{\tau} d\tau'\, h^{(n_1)}_{N_1,r_1,s_1}(\tau')\, I\left(\begin{smallmatrix} n_2 & N_2 \\ r_2 & s_2 \end{smallmatrix} \Big| \dots \Big| \begin{smallmatrix} n_k & N_k \\ r_k & s_k \end{smallmatrix}; \tau'\right)$$

holomorphic, double periodic

rational points $\quad z_{r,s} = \dfrac{r}{N} + \dfrac{s}{N}\tau$

# Elliptic Polylogarithms

— Broedel, Duhr, Dulat, Tancredi '17 —

Feynman integrals can equivalently be represented as eMPLs defined on the elliptic curve

$$y^2 = (x-a_1)(x-a_2)(x-a_3)(x-a_4) \equiv P_4(x)$$

$$\mathrm{E}_4\left(\begin{smallmatrix} n_1 & \dots & n_k \\ c_1 & \dots & c_k \end{smallmatrix}; x, \vec{a}\right) = \int_0^x dt\, \psi_{n_1}(c_1, t, \vec{a})\, \mathrm{E}_4\left(\begin{smallmatrix} n_2 & \dots & n_k \\ c_2 & \dots & c_k \end{smallmatrix}; t, \vec{a}\right)$$

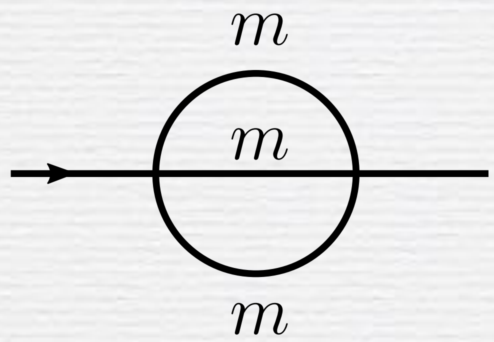
$n_i \in \mathbb{N}$

$c_i \in \mathbb{C} \cup \{\infty\}$

Roots: $\vec{a} = (a_1, a_2, a_3, a_4)$

Where does each description stand when it comes to purity?

# Ex: The sunrise




We have several reasons to believe it should be somehow pure:

- If one internal mass vanishes: pure combination of MPLs
- Expressed as iterated integrals over modular forms: pure

— Adams, Weinzierl '17 —

(pure = sum of uniform weight function with rational *numbers* as coefficients)

First master: — Broedel, Duhr, Dulat, Tancredi '17 —

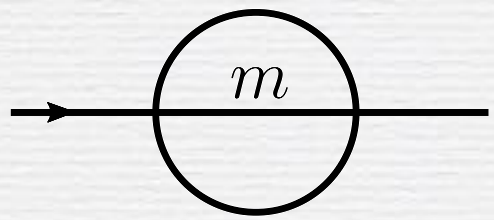
$$c_4 = \frac{1}{2}\sqrt{(a_1 - a_3)(a_2 - a_4)}$$

Roots (functions of $p^2$, $m^2$)

$$S_1(p^2, m^2) = \frac{1}{(m^2+p^2)c_4}\Bigg[\frac{1}{c_4}\mathrm{E}_4\left(\begin{smallmatrix}0 & 0\\ 0 & 0\end{smallmatrix}; 1, \vec{a}\right) - 2\mathrm{E}_4\left(\begin{smallmatrix}0 & -1\\ 0 & \infty\end{smallmatrix}; 1, \vec{a}\right) - \mathrm{E}_4\left(\begin{smallmatrix}0 & -1\\ 0 & 0\end{smallmatrix}; 1, \vec{a}\right) - \mathrm{E}_4\left(\begin{smallmatrix}0 & -1\\ 0 & 1\end{smallmatrix}; 1, \vec{a}\right) - \mathrm{E}_4\left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}; 1, \vec{a}\right)\Bigg] + \mathcal{O}(\epsilon)$$

- Also, pure in terms of $\mathrm{E}_4$ if all masses are different

# Ex: The sunrise




We have several reasons to believe it should be somehow pure:

- If one internal mass vanishes: pure combination of MPLs
- Expressed as iterated integrals over modular forms: pure

— Adams, Weinzierl '17 —

(pure = sum of uniform weight function with rational *numbers* as coefficients)

First master: — Broedel, Duhr, Dulat, Tancredi '17 —

$$c_4 = \frac{1}{2}\sqrt{(a_1 - a_3)(a_2 - a_4)}$$

Roots (functions of $p^2$, $m^2$)

$$S_1(p^2, m^2) = \frac{1}{(m^2+p^2)c_4}\Big[\frac{1}{c_4}\mathrm{E}_4\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix};1,\vec{a}\right) - 2\mathrm{E}_4\left(\begin{smallmatrix}0&-1\\0&\infty\end{smallmatrix};1,\vec{a}\right) - \mathrm{E}_4\left(\begin{smallmatrix}0&-1\\0&0\end{smallmatrix};1,\vec{a}\right)$$
$$- \mathrm{E}_4\left(\begin{smallmatrix}0&-1\\0&1\end{smallmatrix};1,\vec{a}\right) - \mathrm{E}_4\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix};1,\vec{a}\right)\Big] + \mathcal{O}(\epsilon)$$

- Also, pure in terms of $\mathrm{E}_4$ if all masses are different

So, we can use as guiding principle

*An elliptic integral is pure if it is pure when expressed in terms of* $\tilde{\Gamma}$

$=$

*Linear combination of* $\tilde{\Gamma}$ *with coefficients being rational numbers*

Why bother defining yet another version of eMPLs?

What's wrong with the $E_4$ functions?

# Desired properties:

## 1. Pure eMPLs on the elliptic curve

Feynman integrals are more naturally studied on the elliptic curve
(simpler functions of kinematic dof)

## 2. Definite Parity

Integrands are rational functions, result should not depend on choice of branch for the square root $y^2 = P_4(x)$

$$(x, y) \to (x, -y) \qquad \Longleftrightarrow \qquad z \to -z$$

Basis of $\tilde{\Gamma}$ does not have definite parity

$$\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & \cdots & n_k \\ z_1 & \cdots & z_k \end{smallmatrix}; z\right) = \int_0^z dz' g^{(n_1)}(z' - z_1) \tilde{\Gamma}\left(\begin{smallmatrix} n_2 & \cdots & n_k \\ z_2 & \cdots & z_k \end{smallmatrix}; z\right) \qquad g^{(n)}(-z, \tau) = (-1)^n g^{(n)}(z, \tau)$$

# So, what is wrong with the definition of $\mathrm{E}_4$?

— Broedel, Duhr, Dulat, Tancredi '17 —

REVIEW

$$\mathrm{E}_4\left(\begin{smallmatrix} n_1 & \cdots & n_k \\ c_1 & \cdots & c_k \end{smallmatrix}; x, \vec{a}\right) = \int_0^x dt\, \psi_{n_1}(c_1, t, \vec{a})\, \mathrm{E}_4\left(\begin{smallmatrix} n_2 & \cdots & n_k \\ c_2 & \cdots & c_k \end{smallmatrix}; t, \vec{a}\right)$$

kernels such that $\left\{ \begin{array}{l} \int \text{ Keeps you in the same space of functions} \\ \text{At most simple poles} \end{array} \right.$

$$\psi_1(c, x) = \frac{1}{x-c} \quad \psi_0(0, x) = \frac{c_4}{y} \quad \psi_{-1}(c, x) = \frac{y_c}{y(x-c)} \quad \psi_{-1}(\infty, x) = \frac{x}{y}$$

↓

Regular polylogarithms:

$$\mathrm{E}_4\left(\begin{smallmatrix} 1 & \cdots & 1 \\ c_1 & \cdots & c_k \end{smallmatrix}; x\right) = G(c_1, \ldots, c_k; x)\,, \;\; c_i \neq \infty$$

$$c_4 = \frac{1}{2}\sqrt{(a_1 - a_3)(a_2 - a_4)}$$

$$y_c \equiv \sqrt{P_4(c)}$$

Requirement of kernels with at most simple poles imposes an infinite tower of transcendental kernels:

Single pole at $x = \infty$

Double pole at $x = \infty$

$$Z_4(x) \equiv \int_{a_1}^{x} dx' \, \Phi_4(x')$$

$$\Phi_4(x, \vec{a}) \equiv \frac{1}{c_4 \, y} \left( x^2 - \frac{s_1}{2} x + \frac{s_2}{6} \right) + 4 c_4 \frac{\eta_1}{\omega_1} \frac{1}{y}$$

$Z_4^{(n)}(x)$ polynomial in $Z_4$ → Regular everywhere

$$\psi_0(0, x) = \frac{c_4}{y} \,,$$

$$\psi_1(c, x) = \frac{1}{x - c} \,, \qquad \psi_{-1}(c, x) = \frac{y_c}{y(x - c)} \,,$$

$$\psi_1(\infty, x) = \frac{c_4}{y} Z_4(x) \,, \qquad \psi_{-1}(\infty, x) = \frac{x}{y} \,,$$

$$\psi_{-n}(\infty, x) = \frac{x}{y} Z_4^{(n-1)}(x) - \frac{\delta_{n2}}{c_4} \,,$$

$$\psi_n(c, x) = \frac{1}{x - c} Z_4^{(n-1)}(x) - \delta_{n2} \, \Phi_4(x) \,,$$

$$\psi_n(\infty, x) = \frac{c_4}{y} Z_4^{(n)}(x) \,, \qquad \psi_{-n}(c, x) = \frac{y_c}{y(x - c)} Z_4^{(n-1)}(x)$$

Not needed for most physical applications!

# Unipotent?

$\tilde{\Gamma}$ $\qquad\qquad$ $\mathrm{E}_4$

$$d\widetilde{\Gamma}\left(A_1 \cdots A_k; z, \tau\right) = \sum_{p=1}^{k-1} (-1)^{n_{p+1}}\, \widetilde{\Gamma}\left(A_1 \cdots A_{p-1} \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} A_{p+2} \cdots A_k; z, \tau\right) \omega_{p,p+1}^{(n_p + n_{p+1})}$$

$$+ \sum_{p=1}^{k} \sum_{r=0}^{n_p+1} \Bigg[ \binom{n_{p-1} + r - 1}{n_{p-1} - 1} \widetilde{\Gamma}\left(A_1 \cdots A_{p-1}^{[r]}\, \hat{A}_p\, A_{p+1} \cdots A_k; z, \tau\right) \omega_{p,p-1}^{(n_p - r)}$$

$$- \binom{n_{p+1} + r - 1}{n_{p+1} - 1} \widetilde{\Gamma}\left(A_1 \cdots A_{p-1}\, \hat{A}_p\, A_{p+1}^{[r]} \cdots A_k; z, \tau\right) \omega_{p,p+1}^{(n_p - r)} \Bigg]$$

one-forms w/
log singularities

$$A_i^{[r]} \equiv \begin{pmatrix} n_i + r \\ z_i \end{pmatrix} \qquad A_i^{[0]} \equiv A_i$$

$$\omega_{ij}^{(n)} = (dz_j - dz_i)\, g^{(n)}(z_j - z_i, \tau) + \frac{n\, d\tau}{2\pi i}\, g^{(n+1)}(z_j - z_i, \tau)$$

Important: $g^{(n)}(z, \tau)$ have at most simple poles for $z = m + n\tau,\ m, n \in \mathbb{Z}$

# Unipotent?

$\tilde{\Gamma}$ ✓ $\quad\quad$ $\mathrm{E}_4$ ✗

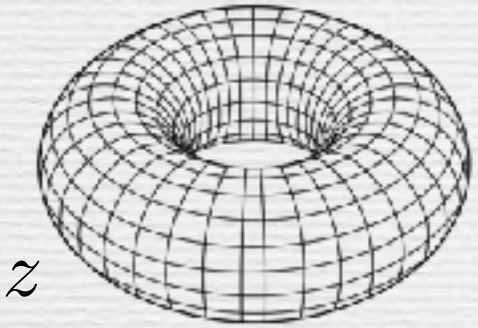
$$d\widetilde{\Gamma}\left(A_1 \cdots A_k; z, \tau\right) = \sum_{p=1}^{k-1} (-1)^{n_{p+1}} \widetilde{\Gamma}\left(A_1 \cdots A_{p-1} \, {}^{0}_{0} \, A_{p+2} \cdots A_k; z, \tau\right) \omega_{p,p+1}^{(n_p+n_{p+1})}$$
$$+ \sum_{p=1}^{k} \sum_{r=0}^{n_p+1} \left[ \binom{n_{p-1}+r-1}{n_{p-1}-1} \widetilde{\Gamma}\left(A_1 \cdots A_{p-1}^{[r]} \, \hat{A}_p \, A_{p+1} \cdots A_k; z, \tau\right) \omega_{p,p-1}^{(n_p-r)} \right.$$
$$\left. - \binom{n_{p+1}+r-1}{n_{p+1}-1} \widetilde{\Gamma}\left(A_1 \cdots A_{p-1} \, \hat{A}_p \, A_{p+1}^{[r]} \cdots A_k; z, \tau\right) \omega_{p,p+1}^{(n_p-r)} \right]$$

→ one-forms w/ log singularities

*A function is called pure if it is unipotent and it has at most logarithmic singularities.*

In order to define a pure version of $\mathrm{E}_4$ we draw inspiration from their relations with $\tilde{\Gamma}$

# Relation between $E_4$ and $\tilde{\Gamma}$



$z$

$$\tau = \frac{\omega_2}{\omega_1}$$

vs. $\quad y^2 = (x - a_1)(x - a_2)(x - a_3)(x - a_4) \equiv P_4(x)$

**Kappa function**

$$\kappa(., \vec{a}) : \mathbb{C}/\Lambda_\tau \to \mathbb{C}$$

$$(c_4 \kappa'(z, \vec{a}))^2 = P_4(\kappa(z, \vec{a}))$$

$$(x, y) = (\kappa(z), c_4 \kappa'(z))$$

→

**Abel's map**

←

$$(x, y) \mapsto z \equiv \frac{c_4}{\omega_1} \int_{a_1}^{x} \frac{dx}{y} \mod \Lambda$$

# Relation between $\mathrm{E}_4$ and $\tilde{\Gamma}$

$\tilde{\Gamma}$ kernels given via Eisenstein-Kronecker series:

$$F(z,\alpha,\tau) = \frac{1}{\alpha}\sum_{n\geq 0} g^{(n)}(z,\tau)\alpha^n = \frac{\theta_1'(0,\tau)\theta_1(z+\alpha,\tau)}{\theta_1(z,\tau)\theta_1(\alpha,\tau)}$$

Using $\quad (x,y) = (\kappa(z), c_4\kappa'(z)) \quad$ and $\quad z_x = \frac{c_4}{\omega_1}\int_{a_1}^{x}\frac{dx}{y}$

$z_*$ image of $-\infty$

$$dx\,\psi_1(c,x,\vec{a}) = dz\left[g^{(1)}(z-z_c,\tau) + g^{(1)}(z+z_c,\tau) - g^{(1)}(z-z_*,\tau) - g^{(1)}(z+z_*,\tau)\right]$$

$$dx\,\psi_{-1}(c,x,\vec{a}) = dz\left[g^{(1)}(z-z_c,\tau) - g^{(1)}(z+z_c,\tau) + g^{(1)}(z_c-z_*,\tau) + g^{(1)}(z_c+z_*,\tau)\right]$$

$$dx\,\psi_1(\infty,x,\vec{a}) = dz\left[-g^{(1)}(z-z_*,\tau) - g^{(1)}(z+z_*,\tau)\right]$$

$$dx\,\psi_{-1}(\infty,x,\vec{a}) = \frac{a_1\,\omega_1\,dz}{c_4} + dz\left[g^{(1)}(z-z_*,\tau) - g^{(1)}(z+z_*,\tau) + 2g^{(1)}(z_*,\tau)\right]$$

Also: $\quad Z_4(x,\vec{a}) = -\frac{1}{\omega_1}\left[g^{(1)}(z_x-z_*,\tau) + g^{(1)}(z_x+z_*,\tau)\right]$

# Relation between $\mathrm{E}_4$ and $\tilde{\Gamma}$

$\tilde{\Gamma}$ kernels given via Eisenstein-Kronecker series:

$$F(z,\alpha,\tau) = \frac{1}{\alpha}\sum_{n\geq 0} g^{(n)}(z,\tau)\alpha^n = \frac{\theta_1'(0,\tau)\theta_1(z+\alpha,\tau)}{\theta_1(z,\tau)\theta_1(\alpha,\tau)}$$

Using $(x,y) = (\kappa(z), c_4\kappa'(z))$ and $z_x = \frac{c_4}{\omega_1}\int_{a_1}^{x}\frac{dx}{y}$

$$dx\,\psi_1(c,x,\vec{a}) = dz\left[g^{(1)}(z-z_c,\tau) + g^{(1)}(z+z_c,\tau) - g^{(1)}(z-z_*,\tau) - g^{(1)}(z+z_*,\tau)\right]$$

$$dx\,\psi_{-1}(c,x,\vec{a}) = dz\left[g^{(1)}(z-z_c,\tau) - g^{(1)}(z+z_c,\tau) + g^{(1)}(z_c-z_*,\tau) + g^{(1)}(z_c+z_*,\tau)\right]$$

And similarly for all kernels — $\mathrm{E}_4$ and $\tilde{\Gamma}$ span the same class of functions

— Broedel, Duhr, Dulat, Tancredi '17 —

# Relation between $\mathrm{E}_4$ and $\widetilde{\Gamma}$

$$\left( Z_4(x,\vec{a}) = -\frac{1}{\omega_1}\Big[g^{(1)}(z_x - z_*,\tau) + g^{(1)}(z_x + z_*,\tau)\Big] \right)$$

Ex:

$$\mathrm{E}_4\left({}^{-1}_{\ c};x,\vec{a}\right) = \widetilde{\Gamma}\left({}^{\ 1}_{z_c};z_x,\tau\right) - \widetilde{\Gamma}\left({}^{\ \ 1}_{-z_c};z_x,\tau\right) + \Big[g^{(1)}(z_c - z_*,\tau) + g^{(1)}(z_c + z_*,\tau)\Big]\,\widetilde{\Gamma}\left({}^{0}_{0};z_x,\tau\right)$$

$$= \widetilde{\Gamma}\left({}^{\ 1}_{z_c};z_x,\tau\right) - \widetilde{\Gamma}\left({}^{\ \ 1}_{-z_c};z_x,\tau\right) - \omega_1\, Z_4(c,\vec{a})\,\widetilde{\Gamma}\left({}^{0}_{0};z_x,\tau\right)$$

↓

simple pole at $c = \infty$

- $\widetilde{\Gamma}$ have only log singularities in all variables
- $\mathrm{E}_4$ have also poles when seen as a function of many variables ⚠

Not pure!

# To summarise:

We define a basis of eMPLs on the elliptic curve such that

1. They form a basis for all eMPLs
2. They are pure
3. They have definite parity
4. They manifestly contain ordinary MPLs

# Meet the pure eMPLs on the elliptic curve:

$$\mathcal{E}_4\left(\begin{smallmatrix} n_1 & \dots & n_k \\ c_1 & \dots & c_k \end{smallmatrix}; x, \vec{a}\right) = \int_0^x dt\, \Psi_{n_1}(c_1, t, \vec{a})\, \mathcal{E}_4\left(\begin{smallmatrix} n_2 & \dots & n_k \\ c_2 & \dots & c_k \end{smallmatrix}; t, \vec{a}\right)$$

$$n_i \in \mathbb{Z}$$

$$\begin{aligned} dx\, \Psi_{\pm n}(c, x, \vec{a}) &= dz_x \Big[ g^{(n)}(z_x - z_c, \tau) \pm g^{(n)}(z_x + z_c, \tau) \\ &\quad - \delta_{\pm n, 1} \left( g^{(1)}(z_x - z_*, \tau) + g^{(1)}(z_x + z_*, \tau) \right) \Big] \end{aligned}$$

1. They form a basis for all eMPLs ✔

(one-to-one correspondence with basis of $\tilde{\Gamma}$ )

# Meet the pure eMPLs on the elliptic curve:

$$\mathcal{E}_4\left(\begin{smallmatrix} n_1 & \dots & n_k \\ c_1 & \dots & c_k \end{smallmatrix}; x, \vec{a}\right) = \int_0^x dt\, \Psi_{n_1}(c_1, t, \vec{a})\, \mathcal{E}_4\left(\begin{smallmatrix} n_2 & \dots & n_k \\ c_2 & \dots & c_k \end{smallmatrix}; t, \vec{a}\right)$$

$$n_i \in \mathbb{Z}$$

$$\begin{aligned} dx\, \Psi_{\pm n}(c, x, \vec{a}) &= dz_x \Big[ g^{(n)}(z_x - z_c, \tau) \pm g^{(n)}(z_x + z_c, \tau) \\ &\quad - \delta_{\pm n, 1} \left( g^{(1)}(z_x - z_*, \tau) + g^{(1)}(z_x + z_*, \tau) \right) \Big] \end{aligned}$$

2. They are pure ✔

(Linear combination of $\tilde{\Gamma}$ with numeric coefficients)

# Meet the pure eMPLs on the elliptic curve:

$$\mathcal{E}_4\left(\begin{smallmatrix} n_1 & \dots & n_k \\ c_1 & \dots & c_k \end{smallmatrix}; x, \vec{a}\right) = \int_0^x dt\, \Psi_{n_1}(c_1, t, \vec{a})\, \mathcal{E}_4\left(\begin{smallmatrix} n_2 & \dots & n_k \\ c_2 & \dots & c_k \end{smallmatrix}; t, \vec{a}\right)$$

$$n_i \in \mathbb{Z}$$

$$dx\, \Psi_{\pm n}(c, x, \vec{a}) = dz_x \Big[ g^{(n)}(z_x - z_c, \tau) \pm g^{(n)}(z_x + z_c, \tau) - \delta_{\pm n, 1} \Big( g^{(1)}(z_x - z_*, \tau) + g^{(1)}(z_x + z_*, \tau) \Big) \Big]$$

3. They have definite parity ✔

(Recall $g^{(n)}(-z, \tau) = (-1)^n g^{(n)}(z, \tau)$)

# Meet the pure eMPLs on the elliptic curve:

$$\mathcal{E}_4\left(\begin{smallmatrix} n_1 & \dots & n_k \\ c_1 & \dots & c_k \end{smallmatrix}; x, \vec{a}\right) = \int_0^x dt\, \Psi_{n_1}(c_1, t, \vec{a})\, \mathcal{E}_4\left(\begin{smallmatrix} n_2 & \dots & n_k \\ c_2 & \dots & c_k \end{smallmatrix}; t, \vec{a}\right)$$

$$n_i \in \mathbb{Z}$$

$$dx\, \Psi_{\pm n}(c, x, \vec{a}) = dz_x \Big[ g^{(n)}(z_x - z_c, \tau) \pm g^{(n)}(z_x + z_c, \tau) - \delta_{\pm n,1} \left( g^{(1)}(z_x - z_*, \tau) + g^{(1)}(z_x + z_*, \tau) \right) \Big]$$

$$dx\, \Psi_1(c, x, \vec{a}) = \frac{dx}{x - c}, \qquad c \neq \infty$$

4. They manifestly contain ordinary MPLs

# Making it explicit

$$\Psi_0(0,x,\vec{a}) = \frac{1}{\omega_1}\,\psi_0(0,x,\vec{a}) = \frac{c_4}{\omega_1\, y}$$

$$\Psi_1(c,x,\vec{a}) \;= \psi_1(c,x,\vec{a}) = \frac{1}{x-c}\,,$$

$$\Psi_{-1}(c,x,\vec{a}) \;= \psi_{-1}(c,x,\vec{a}) + Z_4(c,\vec{a})\,\psi_0(0,x,\vec{a}) = \frac{y_c}{y(x-c)} + Z_4(c,\vec{a})\,\frac{c_4}{y}\,,$$

$$\Psi_1(\infty,x,\vec{a}) \;= -\psi_1(\infty,x,\vec{a}) = -Z_4(x,\vec{a})\,\frac{c_4}{y}\,,$$

$$\Psi_{-1}(\infty,x,\vec{a}) \;= \psi_{-1}(\infty,x,\vec{a}) - \left[\frac{a_1}{c_4} + 2G_*(\vec{a})\right]\,\psi_0(0,x,\vec{a}) = \frac{x}{y} - \frac{1}{y}\left[a_1 + 2c_4\,G_*(\vec{a})\right]$$

Nothing comes without a price — explicit dependence on

$$G_*(\vec{a}) \equiv \frac{1}{\omega_1}\,g^{(1)}(z_*,\tau) \qquad \text{and} \qquad Z_4(c,\vec{a})$$

Image of $-\infty$ on the torus

In general transcendental, but simplify in specific applications

## Ex 1: the image of $-\infty$ on the torus

$$z_* = \frac{c_4}{\omega_1} \int_{a_1}^{-\infty} \frac{dx}{y} = \frac{1}{2} - \frac{\mathrm{F}(\sqrt{\alpha}|\lambda)}{2\,\mathrm{K}(\lambda)}$$

$$\alpha = \frac{a_1 - a_3}{a_1 - a_4}$$

$$\lambda = \frac{(a_1 - a_4)(a_2 - a_3)}{(a_1 - a_3)(a_2 - a_4)}$$

$$a_1 < a_2 < a_3 < a_4 \in \mathbb{R}$$

$$\mathrm{F}(\sqrt{\alpha}|\lambda) = \int_0^{\sqrt{\alpha}} dt \frac{1}{\sqrt{(1-t^2)(1-\lambda t^2)}}$$

Incomplete elliptic integral of the first kind

$$\mathrm{K}(\lambda) = \int_0^1 dt \frac{1}{\sqrt{(1-t^2)(1-\lambda t^2)}}$$

Complete elliptic integral of the first kind

## Ex 1: the image of $-\infty$ on the torus

$$z_* = \frac{c_4}{\omega_1} \int_{a_1}^{-\infty} \frac{dx}{y} = \frac{1}{2} - \frac{\mathrm{F}(\sqrt{\alpha}|\lambda)}{2\,\mathrm{K}(\lambda)}$$

$$\alpha = \frac{a_1 - a_3}{a_1 - a_4}$$

$$\lambda = \frac{(a_1 - a_4)(a_2 - a_3)}{(a_1 - a_3)(a_2 - a_4)}$$

$$a_1 < a_2 < a_3 < a_4 \in \mathbb{R}$$

$$G_*(\vec{a}) = \frac{1}{\omega_1} g^{(1)}(z_*, \tau) = \left(\frac{2\eta_1}{\omega_1} - \frac{\lambda}{3} + \frac{2}{3}\right) \mathrm{F}(\sqrt{\alpha}|\lambda) - \mathrm{E}(\sqrt{\alpha}|\lambda) + \sqrt{\frac{\alpha(\alpha\lambda - 1)}{\alpha - 1}}$$

$$\mathrm{E}(\sqrt{\alpha}|\lambda) = \int_0^{\sqrt{\alpha}} dt \frac{1 - \lambda t^2}{\sqrt{(1 - t^2)(1 - \lambda t^2)}}$$

Incomplete elliptic
integral of the second kind

$G_*(\vec{a})$ and $Z_4(c, \vec{a})$ typically become *algebraic* for specific examples

# Length and weight

For MPLs, notion of weight and length are straightforward

Length = weight = # of integrations (except for $i\pi$)

For eMPLs, they are not the same!

Roughly speaking: $\omega_1, \eta_1, i\pi$

$$\begin{pmatrix} \omega_1 & \omega_2 \\ \eta_1 & \eta_2 \end{pmatrix} = \begin{pmatrix} \omega_1 & 0 \\ \eta_1 & -i\pi/\omega_1 \end{pmatrix} \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}$$

semi-simple    unipotent

Semi-simple periods have length 0

Unipotent periods have length = # of integrations

Note: $\tau$ has length 1 — $\Delta(\tau) = \tau \otimes 1 + 1 \otimes [d\tau]$

$$\tau = \frac{\log q}{2\pi i} = \widetilde{\Gamma}\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}; \tau, \tau\right) = I\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}; \tau\right), \qquad q = e^{2\pi i \tau}$$

## Weight:

Empirically, by requiring relations between uniform weight functions, we postulate

| | | | |
|---|---|---|---|
| $\omega_1 = 2\mathrm{K}(\lambda)$<br>$\omega_2 = 2i\,\mathrm{K}(1-\lambda)$ | $\longrightarrow$ | $1$ | Recall: $\lim_{x\to 0}\mathrm{K}(x) = \frac{\pi}{2}$ |
| $\tau = \frac{\omega_2}{\omega_1}$ | $\longrightarrow$ | $0$ | |
| $\widetilde{\Gamma}\left(\begin{smallmatrix} n_1 & \dots & n_k \\ z_1 & \dots & z_k \end{smallmatrix}; z, \tau\right)$ | | $\sum_i n_i$ | |
| $I\left(\begin{smallmatrix} n_1 & N_1 \\ r_1 & s_1 \end{smallmatrix}\middle|\dots\middle|\begin{smallmatrix} n_k & N_k \\ r_k & s_k \end{smallmatrix}; \tau\right)$ | $\longrightarrow$ | $\sum_i n_i$ | |
| $\mathcal{E}_4\left(\begin{smallmatrix} n_1 & \dots & n_k \\ c_1 & \dots & c_k \end{smallmatrix}; x, \vec{a}\right)$ | | $\sum_i \lvert n_i \rvert$ | |

We'll see in applications that using these definitions, results are of uniform weight

# Other properties that work the same way as for MPLs:

- Shuffle

$$\mathcal{E}_4(A_1 \cdots A_k; x, \vec{a})\, \mathcal{E}_4(A_{k+1} \cdots A_{k+l}; x, \vec{a}) = \sum_{\sigma \in \Sigma(k,l)} \mathcal{E}_4(A_{\sigma(1)} \cdots A_{\sigma(k+l)}; x, \vec{a})$$

- Unipotent — Symbol

- Shuffle preserving regularisation of $\mathcal{E}_4(\cdots {\pm 1 \atop 0}, x, \vec{a})$

$$\mathcal{E}_4\left({n_1 \atop c_1}\, {\cdots \atop \cdots}\, {n_k \atop c_k}; x, \vec{a}\right) = \int_0^x dt\, \Psi_{n_1}(c_1, t, \vec{a})\, \mathcal{E}_4\left({n_2 \atop c_2}\, {\cdots \atop \cdots}\, {n_k \atop c_k}; t, \vec{a}\right)$$

Analogue of $G(\underbrace{0, \ldots, 0}_{n \text{ times}}; z) = \frac{1}{n!} \log^n z$

# The complete list

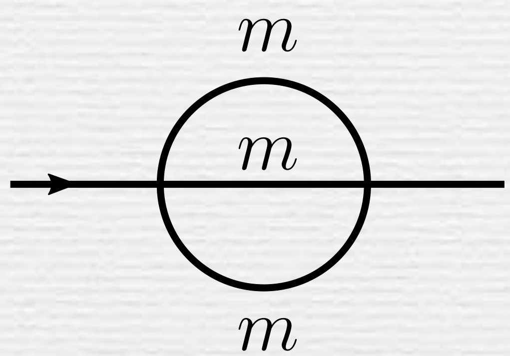
| Name | Unipotent | Length | Weight |
|---|---|---|---|
| Rational Functions | No | 0 | 0 |
| Algebraic Functions | No | 0 | 0 |
| $i\pi$ | No | 0 | 1 |
| $\zeta_{2n}$ | No | 0 | $2n$ |
| $\zeta_{2n+1}$ | Yes | 0 | $2n+1$ |
| $\log x$ | Yes | 1 | 1 |
| $\mathrm{Li}_n(x)$ | Yes | $n$ | $n$ |
| $G(c_1, \ldots, c_k; x)$ | Yes | $k$ | $k$ |
| $\omega_1$ | No | 0 | 1 |
| $\eta_1$ | No | 0 | 1 |
| $\tau$ | Yes | 1 | 0 |
| $g^{(n)}(z, \tau)$ | No | 0 | $n$ |
| $h^{(n)}_{N,r,s}(\tau)$ | No | 0 | $n$ |
| $Z_4(c, \vec{a})$ | No | 0 | 0 |
| $G_*(\vec{a})$ | No | 0 | 0 |
| $\mathcal{E}_4(\begin{smallmatrix} n_1 & \ldots & n_k \\ c_1 & \ldots & c_k \end{smallmatrix}; x, \vec{a})$ | Yes | $k$ | $\sum_i \|n_i\|$ |
| $\widetilde{\Gamma}(\begin{smallmatrix} n_1 & \ldots & n_k \\ z_1 & \ldots & z_k \end{smallmatrix}; z, \tau)$ | Yes | $k$ | $\sum_i n_i$ |
| $I\left(\begin{smallmatrix} n_1 & N_1 \\ r_1 & s_1 \end{smallmatrix} \middle\| \ldots \middle\| \begin{smallmatrix} n_k & N_k \\ r_k & s_k \end{smallmatrix}; \tau\right)$ | Yes | $k$ | $\sum_i n_i$ |

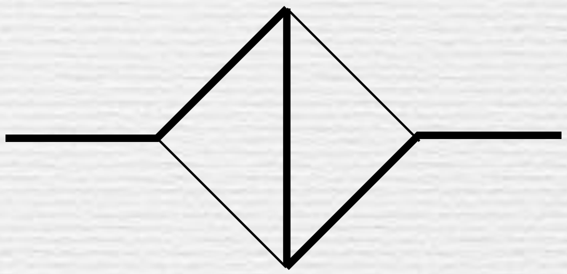
# Applications

— Broedel, Duhr, Dulat, Penante, Tancredi (to appear) —

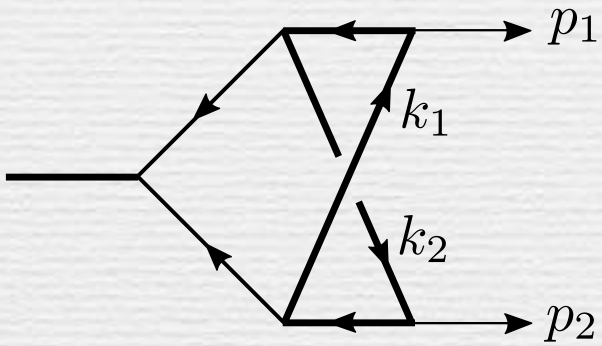
- 2-point functions

  [Weinzierl talk]

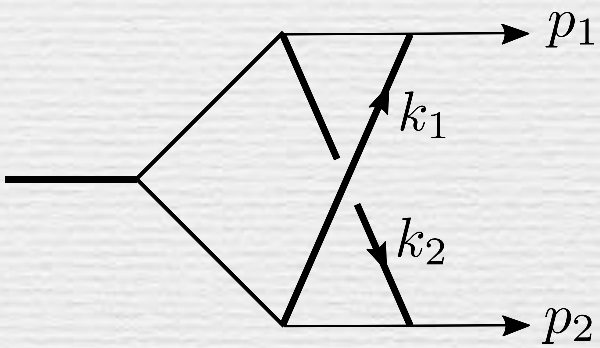


- 3-point functions

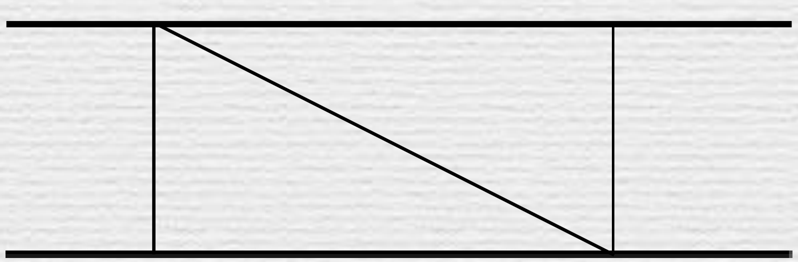



— Tancredi, von Manteuffel '17 —




— Aglietti, Bonciani '07 —

- 4-point functions



— Henn, Smirnov '13 —

# Applications

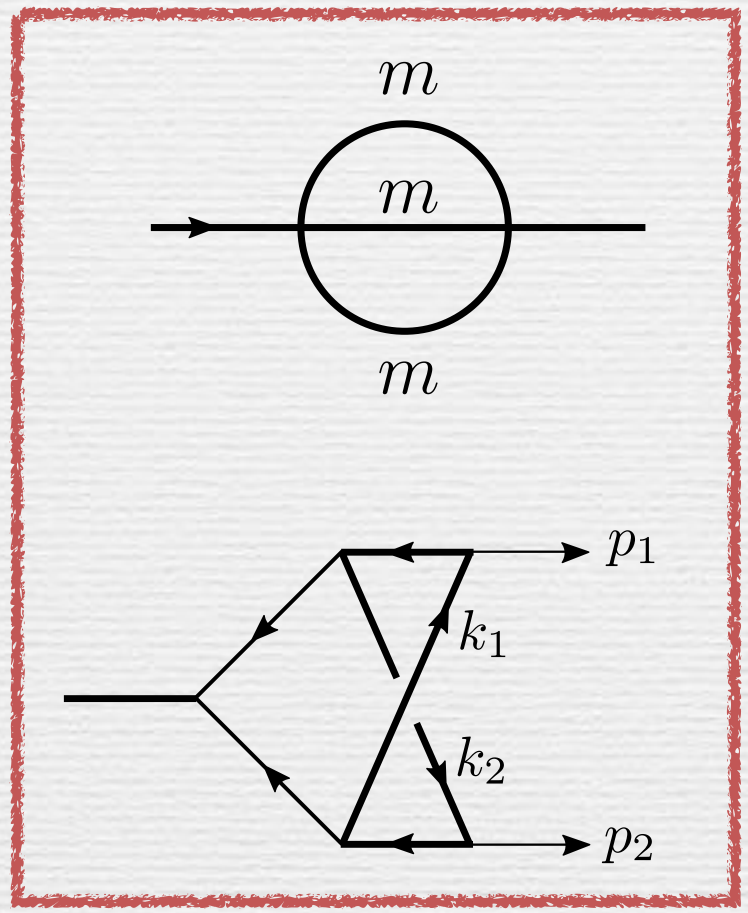
— Broedel, Duhr, Dulat, Penante, Tancredi (to appear) —

- 2-point functions

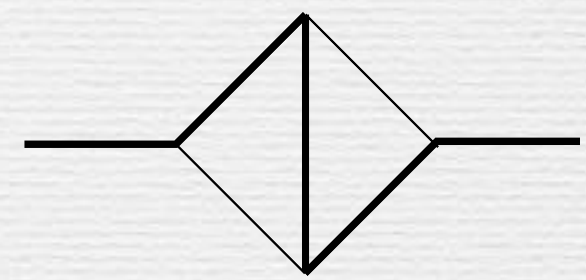
  [Weinzierl talk]

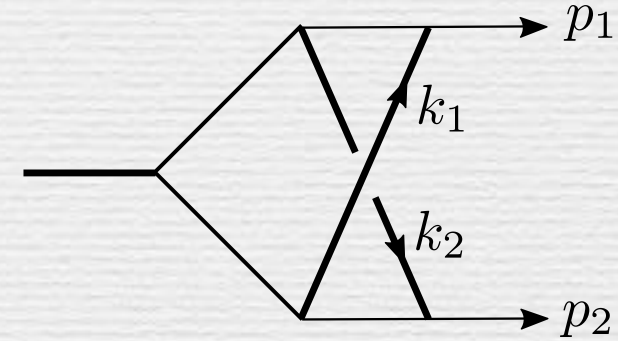
- 3-point functions

- 4-point functions

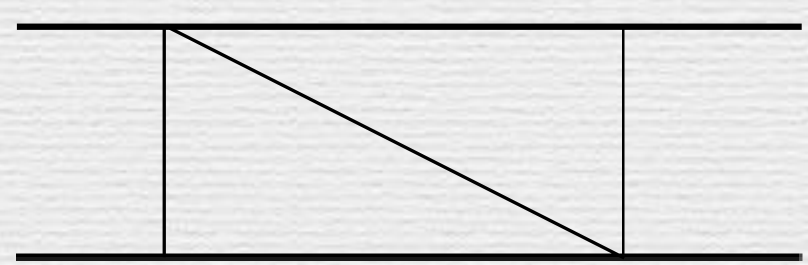



— Tancredi, von Manteuffel ’17 —



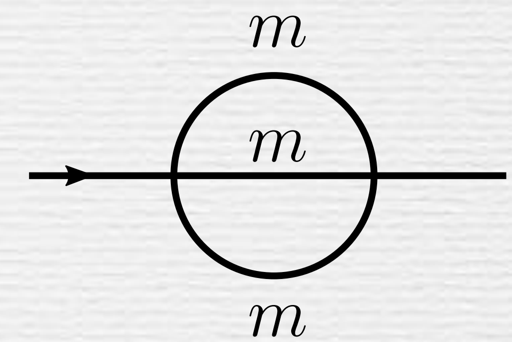



— Aglietti, Bonciani ’07 —



— Henn, Smirnov ’13 —

# Ex 1: Sunrise




Recall first master from Lorenzo's talk:

$$S_1(p^2,m^2) = \frac{1}{(m^2+p^2)c_4}\left[\frac{1}{c_4}\mathrm{E}_4\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix};1,\vec{a}\right) - 2\mathrm{E}_4\left(\begin{smallmatrix}0&-1\\0&\infty\end{smallmatrix};1,\vec{a}\right) - \mathrm{E}_4\left(\begin{smallmatrix}0&-1\\0&0\end{smallmatrix};1,\vec{a}\right)\right.$$
$$\left. - \mathrm{E}_4\left(\begin{smallmatrix}0&-1\\0&1\end{smallmatrix};1,\vec{a}\right) - \mathrm{E}_4\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix};1,\vec{a}\right)\right] + \mathcal{O}(\epsilon)$$

Semi-simple ↖ ↗ Unipotent

After purification: $$S_1(p^2,m^2) = -\frac{\omega_1}{(p^2+m^2)\,c_4}\,T_1(p^2,m^2)$$

$$\mathrm{Cut}[S_1(p^2,m^2)_{|D=2}] = -\frac{\omega_1}{(p^2+m^2)\,c_4}$$

$$T_1(p^2,m^2) = \left(\frac{m^2}{-p^2}\right)^{-2\epsilon}\left[T_1^{(0)} + \epsilon\,T_1^{(1)} + \mathcal{O}(\epsilon^2)\right]$$

$$T_1(p^2, m^2) = \left(\frac{m^2}{-p^2}\right)^{-2\epsilon} \left[T_1^{(0)} + \epsilon\, T_1^{(1)} + \mathcal{O}(\epsilon^2)\right]$$

$$T_1^{(0)} = 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 \\ 0 & \infty \end{smallmatrix}; 1, \vec{a}\right) + \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}; 1, \vec{a}\right) + \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 \\ 0 & 1 \end{smallmatrix}; 1, \vec{a}\right)$$

$$\begin{aligned}
T_1^{(1)} = &-4\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_3 & \infty \end{smallmatrix}; 1, \vec{a}\right) - 4\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_1 & \infty \end{smallmatrix}; 1, \vec{a}\right) - 4\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_4 & \infty \end{smallmatrix}; 1, \vec{a}\right) - 4\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_2 & \infty \end{smallmatrix}; 1, \vec{a}\right) \\
&- 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_3 & 0 \end{smallmatrix}; 1, \vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_3 & 1 \end{smallmatrix}; 1, \vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_1 & 0 \end{smallmatrix}; 1, \vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_1 & 1 \end{smallmatrix}; 1, \vec{a}\right) \\
&- 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_4 & 0 \end{smallmatrix}; 1, \vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_4 & 1 \end{smallmatrix}; 1, \vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_2 & 0 \end{smallmatrix}; 1, \vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_2 & 1 \end{smallmatrix}; 1, \vec{a}\right) \\
&+ 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 \\ 0 & \infty & 0 \end{smallmatrix}; 1, \vec{a}\right) + 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 \\ 0 & \infty & 1 \end{smallmatrix}; 1, \vec{a}\right) + 6\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & 0 & \infty \end{smallmatrix}; 1, \vec{a}\right) + 6\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & 1 & \infty \end{smallmatrix}; 1, \vec{a}\right) \\
&- 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \end{smallmatrix}; 1, \vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \end{smallmatrix}; 1, \vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \end{smallmatrix}; 1, \vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 \\ 0 & 1 & 1 \end{smallmatrix}; 1, \vec{a}\right) \\
&+ 6i\pi\mathcal{E}_4\left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}; 1, \vec{a}\right) + 6i\pi\mathcal{E}_4\left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{smallmatrix}; 1, \vec{a}\right) + 3\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \end{smallmatrix}; 1, \vec{a}\right) + 3\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \end{smallmatrix}; 1, \vec{a}\right) \\
&+ 3\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \end{smallmatrix}; 1, \vec{a}\right) + 3\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \end{smallmatrix}; 1, \vec{a}\right) + \zeta_2\, \mathcal{E}_4\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}; 1, \vec{a}\right) .
\end{aligned}$$

$$T_1(p^2, m^2) = \left(\frac{m^2}{-p^2}\right)^{-2\epsilon} \left[T_1^{(0)} + \epsilon\, T_1^{(1)} + \mathcal{O}(\epsilon^2)\right]$$

$$T_1^{(0)} = 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 \\ 0 & \infty \end{smallmatrix}; 1, \vec{a}\right) + \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}; 1, \vec{a}\right) + \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 \\ 0 & 1 \end{smallmatrix}; 1, \vec{a}\right)$$

$$\begin{aligned}
T_1^{(1)} = &-4\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_3 & \infty \end{smallmatrix}; 1, \vec{a}\right) - 4\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_1 & \infty \end{smallmatrix}; 1, \vec{a}\right) - 4\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_4 & \infty \end{smallmatrix}; 1, \vec{a}\right) - 4\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_2 & \infty \end{smallmatrix}; 1, \vec{a}\right) \\
&- 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_3 & 0 \end{smallmatrix}; 1, \vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_3 & 1 \end{smallmatrix}; 1, \vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_1 & 0 \end{smallmatrix}; 1, \vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_1 & 1 \end{smallmatrix}; 1, \vec{a}\right) \\
&- 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_4 & 0 \end{smallmatrix}; 1, \vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_4 & 1 \end{smallmatrix}; 1, \vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_{\text{[illegible]}} & 0 \end{smallmatrix}; 1, \vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_{\text{[illegible]}} & 1 \end{smallmatrix}; 1, \vec{a}\right) \\
&+ 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 \\ 0 & \infty & 0 \end{smallmatrix}; 1, \vec{a}\right) + 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 \\ 0 & \infty & 1 \end{smallmatrix}; 1, \vec{a}\right) \\
&- 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \end{smallmatrix}; 1, \vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \end{smallmatrix}; 1, \vec{a}\right) \\
&+ 6i\pi\mathcal{E}_4\left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}; 1, \vec{a}\right) + 6i\pi\mathcal{E}_4\left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{smallmatrix}; 1, \vec{a}\right) \\
&+ 3\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \end{smallmatrix}; 1, \vec{a}\right) + 3\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \end{smallmatrix}; 1, \vec{a}\right)
\end{aligned}$$

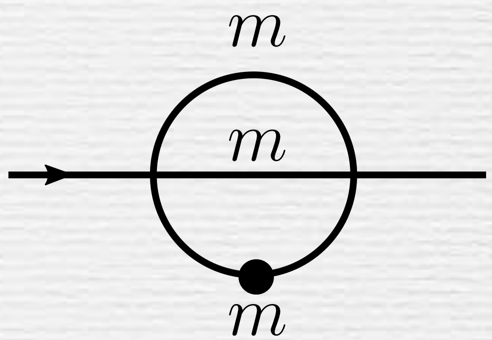
**Recall weights:**

$$\mathcal{E}_4\left(\begin{smallmatrix} n_1 & \cdots & n_k \\ c_1 & \cdots & c_k \end{smallmatrix}; x\right) \;\to\; \sum_i |n_i|$$

$$\omega_1, \pi \;\to\; 1$$

Manifestly pure of weight 2! 🎉

# The second master




Semi-simple

Unipotent

$$\begin{pmatrix} S_1(p^2,m^2) \\ S_2(p^2,m^2) \end{pmatrix} = \begin{pmatrix} \Omega_1 & 0 \\ H_1 & -\frac{2}{m^2(p^2+m^2)(p^2+9m^2)\,\Omega_1} \end{pmatrix} \begin{pmatrix} T_1(p^2,m^2) \\ T_2(p^2,m^2) \end{pmatrix}$$

$$\begin{aligned} \Omega_1 &= -\frac{\omega_1}{c_4\left(m^2+p^2\right)}\,, \\ H_1 &= -\frac{4c_4\eta_1}{m^2\left(9m^2+p^2\right)} - \frac{\omega_1\left(15m^4+12m^2p^2+p^4\right)}{6c_4m^2\left(m^2+p^2\right)^2\left(9m^2+p^2\right)} \end{aligned}$$

$\mathrm{Cut}[S_2(p^2,m^2)_{|D=2}]$

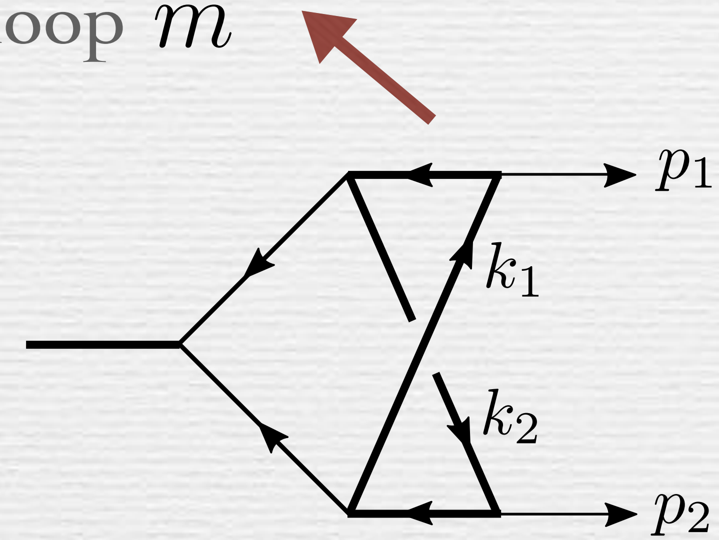
$$T_2(p^2,m^2) = \left(\frac{m^2}{-p^2}\right)^{-2\epsilon}\left[T_2^{(0)} + \epsilon\, T_2^{(1)} + \mathcal{O}(\epsilon^2)\right]$$

$$T_2^{(0)} = 2\mathcal{E}_4\left({\textstyle{-2 \atop \infty}};1,\vec{a}\right) + \mathcal{E}_4\left({\textstyle{-2 \atop 0}};1,\vec{a}\right) + \mathcal{E}_4\left({\textstyle{-2 \atop 1}};1,\vec{a}\right)$$

Uniform weight 2!

# Ex 2: $t\bar{t}$ production — Tancredi, von Manteuffel '17 —

Massive loop $m$




$$I = \int \frac{d^d k_1 d^d k_2}{(i\pi)^4} \frac{1}{\prod_{i=1}^{6} D_i}$$

$$p_1^2 = p_2^2 = 0$$

$$S = -2(p_1 \cdot p_2)$$

$$D_1 = k_1^2 - m^2, \quad D_3 = (k_1 - p_1)^2 - m^2, \quad D_5 = (k_1 - k_2 - p_1)^2,$$

$$D_2 = k_2^2 - m^2, \quad D_4 = (k_2 - p_2)^2 - m^2, \quad D_6 = (k_2 - k_1 - p_2)^2$$

Lorenzo showed the result in terms of $\mathrm{E}_4$

$$r_- = \frac{1}{2} - \frac{1}{2}\sqrt{1-4a}$$

$$r_+ = \frac{1}{2} - \frac{1}{2}\sqrt{1+4a}$$

$$a = \frac{m^2}{S}$$

$$\begin{aligned}
I = \Bigg[ & 5\mathrm{E}_4\left(\begin{smallmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & r_+ \end{smallmatrix}; 1\right) + 5\mathrm{E}_4\left(\begin{smallmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1-r_+ \end{smallmatrix}; 1\right) + 5\mathrm{E}_4\left(\begin{smallmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & r_+ \end{smallmatrix}; 1\right) + 5\mathrm{E}_4\left(\begin{smallmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1-r_+ \end{smallmatrix}; 1\right) \\
& - 3\mathrm{E}_4\left(\begin{smallmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & r_- & 0 \end{smallmatrix}; 1\right) - 3\mathrm{E}_4\left(\begin{smallmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & r_- & 1 \end{smallmatrix}; 1\right) - 3\mathrm{E}_4\left(\begin{smallmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1-r_- & 0 \end{smallmatrix}; 1\right) - 3\mathrm{E}_4\left(\begin{smallmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1-r_- & 1 \end{smallmatrix}; 1\right) \\
& + 3\log a\left(\mathrm{E}_4\left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & r_- \end{smallmatrix}; 1\right) + \mathrm{E}_4\left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 1-r_- \end{smallmatrix}; 1\right)\right)\Bigg] \\
& - \frac{4a^2}{c_4}\Bigg[5\mathrm{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 0 & r_+ \end{smallmatrix}; 1\right) + 5\mathrm{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 0 & 1-r_+ \end{smallmatrix}; 1\right) + 5\mathrm{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 1 & r_+ \end{smallmatrix}; 1\right) + 5\mathrm{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 1 & 1-r_+ \end{smallmatrix}; 1\right) \\
& - 3\mathrm{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & r_- & 0 \end{smallmatrix}; 1\right) - 3\mathrm{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & r_- & 1 \end{smallmatrix}; 1\right) - 3\mathrm{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 1-r_- & 0 \end{smallmatrix}; 1\right) - 3\mathrm{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 1-r_- & 1 \end{smallmatrix}; 1\right) \\
& + 3\log a\left(\mathrm{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 \\ 0 & \infty & r_- \end{smallmatrix}; 1\right) + \mathrm{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 \\ 0 & \infty & 1-r_- \end{smallmatrix}; 1\right)\right)\Bigg]
\end{aligned}$$

Non pure

Changing basis to pure eMPLs $\mathcal{E}_4$ , the result becomes:

$$I = \frac{32\omega_1}{q^2(1+\sqrt{1-16a})}\left[T_0(a) + 3T_-(a) + 5T_+(a) + \mathcal{O}(\epsilon)\right]$$

$$\begin{aligned} T_a = &-\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 0 & 0 \end{smallmatrix};1\right) - \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 0 & 1 \end{smallmatrix};1\right) - \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 1 & 0 \end{smallmatrix};1\right) - \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 1 & 1 \end{smallmatrix};1\right) + \\ &\log(a)\left[\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 \\ 0 & \infty & 0 \end{smallmatrix};1\right) + \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 \\ 0 & \infty & 1 \end{smallmatrix};1\right)\right] + \frac{1}{2}\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 \\ 0 & \infty \end{smallmatrix};1\right)\left(\zeta_2 - \log^2(a)\right) \end{aligned}$$

$$\begin{aligned} T_- = &-\frac{3}{2}\zeta_2\,\mathcal{E}_4\left(\begin{smallmatrix} -1 \\ \infty \end{smallmatrix};r_-\right) + \zeta_2\mathcal{E}_4\left(\begin{smallmatrix} -1 & 0 \\ \infty & 0 \end{smallmatrix};r_-\right) - 2\mathcal{E}_4\left(\begin{smallmatrix} -1 & -1 \\ \infty & \infty \end{smallmatrix};r_-\right)\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 \\ 0 & \infty \end{smallmatrix};1\right) \\ &+ \mathcal{E}_4\left(\begin{smallmatrix} -1 & 0 & 1 & 1 \\ \infty & 0 & 0,0 & \end{smallmatrix};r_-\right) + \mathcal{E}_4\left(\begin{smallmatrix} -1 & 0 & 1 & 1 \\ \infty & 0 & 0 & 1 \end{smallmatrix};r_-\right) - \mathcal{E}_4\left(\begin{smallmatrix} -1 & 0 & 1 & 1 \\ \infty & 0 & 1 & 0 \end{smallmatrix};r_-\right) - \mathcal{E}_4\left(\begin{smallmatrix} -1 & 0 & 1 & 1 \\ \infty & 0 & 1 & 1 \end{smallmatrix};r_-\right) \\ &+ \mathcal{E}_4\left(\begin{smallmatrix} -1 & 1 & 0 & 1 \\ \infty & 0 & 0 & 1 \end{smallmatrix};r_-\right) - \mathcal{E}_4\left(\begin{smallmatrix} -1 & 1 & 0 & 1 \\ \infty & 1 & 0 & 0 \end{smallmatrix};r_-\right) + \mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & 0 & 1 \\ 0 & \infty & 0 & 1 \end{smallmatrix};r_-\right) - \mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & 0 & 1 \\ 1 & \infty & 0 & 0 \end{smallmatrix};r_-\right) \\ &- \mathcal{E}_4\left(\begin{smallmatrix} -1 & 0 & 1 \\ \infty & 0 & 1 \end{smallmatrix};r_-\right)\log(r_-) + \mathcal{E}_4\left(\begin{smallmatrix} -1 & 0 & 1 \\ \infty & 0 & 0 \end{smallmatrix};r_-\right)\log(1-r_-) \end{aligned}$$

$$\begin{aligned} T_+ = &\frac{i\pi}{4}\left(\mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 \\ 0 & \infty \end{smallmatrix};r_+\right) + \mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 \\ 1 & \infty \end{smallmatrix};r_+\right) - 4\left(\mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & 0 \\ 0 & \infty & 0 \end{smallmatrix};r_+\right) + \mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & 0 \\ 1 & \infty & 0 \end{smallmatrix};r_+\right)\right)\right) \\ &- \mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & 0 & 1 \\ 0 & \infty & 1 & 0 \end{smallmatrix};r_+\right) + \mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & 0 & 1 \\ 0 & \infty & 0 & 1 \end{smallmatrix};r_+\right) - \mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & 0 & 1 \\ 1 & \infty & 1 & 0 \end{smallmatrix};r_+\right) + \mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & 0 & 1 \\ 1 & \infty & 0 & 1 \end{smallmatrix};r_+\right) \end{aligned}$$

# Changing basis to pure eMPLs $\mathcal{E}_4$, the result becomes:

$$I = \frac{32\omega_1}{q^2(1+\sqrt{1-16a})}\left[T_0(a) + 3T_-(a) + 5T_+(a) + \mathcal{O}(\epsilon)\right]$$

$$\begin{aligned}
T_a = & -\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 0 & 0 \end{smallmatrix};1\right) - \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 0 & 1 \end{smallmatrix};1\right) - \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 1 & 0 \end{smallmatrix};1\right) - \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 1 & 1 \end{smallmatrix};1\right) + \\
& \log(a)\left[\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 \\ 0 & \infty & 0 \end{smallmatrix};1\right) + \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 \\ 0 & \infty & 1 \end{smallmatrix};1\right)\right] + \frac{1}{2}\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 \\ 0 & \infty \end{smallmatrix};1\right)\left(\zeta_2 - \log^2(a)\right)
\end{aligned}$$

$$\begin{aligned}
T_- = & -\frac{3}{2}\zeta_2\,\mathcal{E}_4\left(\begin{smallmatrix} -1 \\ \infty \end{smallmatrix};r_-\right) + \zeta_2\mathcal{E}_4\left(\begin{smallmatrix} -1 & 0 \\ \infty & 0 \end{smallmatrix};r_-\right) - 2\mathcal{E}_4\left(\begin{smallmatrix} -1 & -1 \\ \infty & \infty \end{smallmatrix};r_-\right)\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 \\ 0 & \infty \end{smallmatrix};1\right) \\
& + \mathcal{E}_4\left(\begin{smallmatrix} -1 & 0 & 1 & 1 \\ \infty & 0 & 0{,}0 & \end{smallmatrix};r_-\right) + \mathcal{E}_4\left(\begin{smallmatrix} -1 & 0 & 1 & 1 \\ \infty & 0 & 0 & 1 \end{smallmatrix};r_-\right) - \mathcal{E}_4\left(\begin{smallmatrix} -1 & 0 & 1 & 1 \\ \infty & 0 & 1 & 0 \end{smallmatrix};r_-\right) - \mathcal{E}_4\left(\begin{smallmatrix} -1 & 0 & 1 & 1 \\ \infty & 0 & 1 & 1 \end{smallmatrix};r_-\right) \\
& + \mathcal{E}_4\left(\begin{smallmatrix} -1 & 1 & 0 & 1 \\ \infty & 0 & 0 & 1 \end{smallmatrix};r_-\right) - \mathcal{E}_4\left(\begin{smallmatrix} -1 & 1 & 0 & 1 \\ \infty & 1 & 0 & 0 \end{smallmatrix};r_-\right) + \mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & 0 & 1 \\ 0 & \infty & 0 & 1 \end{smallmatrix};r_-\right) - \mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & 0 & 1 \\ 1 & \infty & 0 & 0 \end{smallmatrix};r_-\right) \\
& - \mathcal{E}_4\left(\begin{smallmatrix} -1 & 0 & 1 \\ \infty & 0 & 1 \end{smallmatrix};r_-\right)\log(r_-) + \mathcal{E}_4\left(\begin{smallmatrix} -1 & 0 & 1 \\ \infty & 0 & 0 \end{smallmatrix};r_-\right)\log(1-r_-)
\end{aligned}$$

$$\begin{aligned}
T_+ = & \frac{i\pi}{4}\left(\mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 \\ 0 & \infty \end{smallmatrix};r_+\right) + \mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 \\ 1 & \infty \end{smallmatrix};r_+\right) - 4\left(\mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & 0 \\ 0 & \infty & 0 \end{smallmatrix};r_+\right) + \mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & 0 \\ 1 & \infty & 0 \end{smallmatrix};r_+\right)\right)\right) \\
& - \mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & 0 & 1 \\ 0 & \infty & 1 & 0 \end{smallmatrix};r_+\right) + \mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & 0 & 1 \\ 0 & \infty & 0 & 1 \end{smallmatrix};r_+\right) - \mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & 0 & 1 \\ 1 & \infty & 1 & 0 \end{smallmatrix};r_+\right) + \mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & 0 & 1 \\ 1 & \infty & 0 & 1 \end{smallmatrix};r_+\right)
\end{aligned}$$

Uniform weight 4!

# Back to the real world

## The elliptic double box of N=4 SYM

— Caron-Huot, Larsen '12 / Nandan, Paulos, Spradlin, Volovich '14 / Bourjaily, McLeod, Spradlin, von Hippel, Wilhelm '17 —

$$I_{\text{db}}^{\text{ell}} \sim \int \frac{d\alpha}{\sqrt{Q(\alpha)}}\, \mathcal{G}_3(\alpha)$$

Quartic polynomial

Pure combination of MPLs (with quadradic sqrts)

Using

$$\Psi_0(0,x,\vec{a}) = \frac{1}{\omega_1}\,\psi_0(0,x,\vec{a}) = \frac{c_4}{\omega_1\, y}$$

Weight 1

Weight 3

$$I_{\text{db}}^{\text{ell}} \sim \frac{\omega_1}{c_4}\, T_{\text{db}}^{\text{ell}} \qquad T_{\text{db}}^{\text{ell}} = \int d\alpha\, \Psi_0(0,\alpha)\, \mathcal{G}_3(\alpha)$$

Uniform weight 4, as expected!

# Conclusions

- First step into defining a concept of purity and uniform weight in the elliptic case, worked out several examples
- Both conceptual and practical relevance — in the end we are interested in computing amplitudes and obtaining reliable analytical expressions
- Purity is of great relevance in the MPL case (differential equations), hopefully soon we will have a similar understanding for elliptic Feynman integrals too
- Not the end, integrals with multiple elliptic curves, more complicated geometries, etc.

  [Weinzierl & Chaubey's talks]
  [Bourjaily' s talk]

- Lots to do still, but we are definitely moving forward!