Linearly reducible elliptic Feynman integrals

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Based on work in collaboration with Martijn Hidding

arXiv:1712.04441 + work in progress soon to appear

Elliptic integrals in high energy physics

- Discovery tool for physics at small scales: compare theory with High energy physics experiments (collider experiments)
- Need to compute scattering amplitudes
- In the energy ranges we probe, all interactions among observed particles are weak. Mathematically this means that the theory depends on a small parameter
- We can apply perturbation theory: we need to compute Feynman diagrams (Feynman integrals)
- Feynman integrals evaluate to multiple polylogarithms and/or their elliptic generalisations (or more complicated functions, still unexplored)

The polylogarithmic case

- Understanding that many Feynman integrals can be expressed in terms of multiple polylogarithms

$$G(a_1, a_2, \ldots, a_n, x) = \int_0^x \frac{1}{t-a_1} G(a_2, \ldots, a_n)$$

Simplification of polylogarithmic expressions via “symbols”, and analytic continuation

[Goncharov] arXiv:1105.2076

- Differential equations method for dimensionally regularised integrals, that naturally captures the polylogarithmic nature of the answer

[Kotikov] Phys.Lett. B254 (1991) 158-164
[Remiddi] arXiv:hep-th/9711188
[Gehrmann,Remiddi] arXiv:hep-ph/9912329

- Canonical basis of integrals: algorithmic solution via differential equations method. Analytic properties of the answer manifest at the level of the differential equations and to all orders, e.g. the symbol

[Henn] arXiv:1304.1806

- These tools are nowadays used as a machinery

Canonical basis —> Differential equations —> Multiple polylogarithms

The elliptic case

- Elliptic multiple polylogarithms (eMPL) form a basis for iterated integrals over an elliptic curve.

  [Brown, Levin] arXiv:1110.6917
  [Broedel,Mafra,Matthes,Schlotterer] arXiv:1412.5535
  [Broedel, Duhr, Dulat, Tancredi] arXiv:1712.07089

- Results are usually obtained performing direct integration + IBP

  [Broedel, Duhr, Dulat, Tancredi] arXiv:1712.07089

- Further exploration of the differential equations method is required when Feynman integrals evaluate to eMPL, i.e. find a “canonical” way to define and solve DE directly in terms of eMPL.

  [Adams, Chaubey, Weinzierl] arXiv:1702.04279
  [Adams, Weinzierl] arXiv:1802.05020

- Analytic continuation done only for single scale Feynman integrals at leading order (i.e. sunrise and kite integrals)

  [Remiddi, Tancredi] arXiv:1602.01481
  [Bogner, Schweitzer, Weinzierl] arXiv:1705.08952

A class of elliptic Feynman integrals

- The aim of this talk is to define a class of elliptic Feynman integrals, dubbed **linearly reducible**.

  [Hidding, FM] arXiv:1712.04441

- <u>We will see that these integrals posses many nice properties.</u>

- We can algorithmically solve them up to arbitrary epsilon order in terms of one-integrals over a polylogarithmic expression

  We can use well established techniques from the polylogarithmic symbol calculus to perform the analytic continuation to the physical region

- Under some assumptions we can use canonical differential equations and IBP reductions to solve linearly reducible elliptic Feynman integrals in a fully algebraic manner in terms of eMPL

Overview

- Feynman parametrisation
- Direct integration and Linearly reducible elliptic Feynman integrals
- Differential equations and Elliptic Polylogarithms
- Analytic continuation
- Application to the unequal masses sunrise and a triangle integral relevant for heavy quark pair production

Parametric representation

$$I_{a_1,\ldots,a_n}(\{s\}) = N \int \left(\prod_{i=1}^{l} d^d k_i \right) \frac{1}{\prod_{i=1}^{n} (m_i^2 - p_i^2)^{a_i}}\,, \qquad N = \left(i\pi^{\frac{d}{2}} \right)^{-l} \frac{\prod_{i=1}^{n} \Gamma(a_i)}{\Gamma(a - \frac{ld}{2})}$$

$$I_{a_1,\ldots,a_n}(\{s\}) = \int_\Delta d^n \vec{\alpha} \left(\prod_{i=1}^{n} \alpha_i^{a_i - 1} \right) U^{a - \frac{d}{2}(l+1)} F^{-a + \frac{ld}{2}}$$

$$\Delta = \left\{ \vec{\alpha} \,\middle|\, \alpha_i \geq 0,\, \sum_{i=1}^{n} \alpha_i = 1 \right\}$$

$$\sum_{i=1}^{n} \alpha_i (m_i^2 - p_i^2) = -\vec{k} \hat{M} \vec{k} + 2 \vec{k} \vec{Q} + J$$

$$U = |\hat{M}|, \quad F = |\hat{M}| (J + \vec{Q} \hat{M}^{-1} \vec{Q})$$

[Bogner, Weinzierl] arXiv:1002.3458

Direct integration

$$I = \sum_{i=1}^{\infty} \epsilon^i \int_\Delta d\alpha_1 \dots d\alpha_n f^{(i)}(\alpha_1, \dots, \alpha_n)$$

$$I = \sum_{i=1}^{\infty} \epsilon^i \int_0^1 d\alpha_p \cdots \underbrace{\int_0^{1-\alpha_4 \cdots -\alpha_p} d\alpha_3 \underbrace{\int_0^{1-\alpha_3 \cdots -\alpha_p} d\alpha_2 f^{(i)}(\alpha_1, \dots, \alpha_p)|_{\alpha_1 = 1-\alpha_2 \cdots -\alpha_p}}_{f_2(\alpha_3, \dots, \alpha_p)}}_{f_3(\alpha_4, \dots, \alpha_p)}$$

$$f_k(\alpha_{k+1}, \dots, \alpha_p) = \int_0^\infty d\alpha_k \, f_{k-1}(\alpha_k, \dots, \alpha_p)$$

1. Express $f_{k-1}(\alpha_k, \dots, \alpha_p)$ as combination of GPL of argument α_k

2. Find a primitive $F_k(\alpha_k, \dots, \alpha_p): \quad \partial_{\alpha_k} F_k(\alpha_k, \dots, \alpha_p) = f_{k-1}(\alpha_k, \dots, \alpha_p)$

3. Compute the Limit $\lim_{\alpha_k \to \infty} F_k(\alpha_k) - \lim_{\alpha_k \to 0} F_k(\alpha_k)$

[Brown] arXiv: 0804.1660
[Panzer] arXiv:1403.3385

Direct integration

1. Express $f_{k-1}(\alpha_k,\ldots,\alpha_p)$ as combination of GPL of argument α_k

2. Find a primitive $F_k(\alpha_k,\ldots,\alpha_p):\quad \partial_{\alpha_k}F_k(\alpha_k,\ldots,\alpha_p)=f_{k-1}(\alpha_k,\ldots,\alpha_p)$

Require that $f_{k-1}(\alpha_k,\ldots,\alpha_p)$ has rational alphabet in α_k

$$\int g(\alpha_k)\; G_w(\alpha_k)\; d\alpha_k$$

$$g(\alpha_k)=\sum_{n>0,a}\frac{C_{a,n}}{(\alpha_k-a)^n}+\sum_{n\geq 0}C_n\alpha_k^n$$

[Brown] arXiv: 0804.1660
[Panzer] arXiv:1403.3385

We can find a primitive by using the definition of GPL and recursively applying the following ibp identities:

$$\int\frac{1}{\alpha_k-a}G_w(\alpha_k)d\alpha_k=G_{a,w}(\alpha_k)$$

$$\int\frac{d\alpha_k\;G_w(\alpha_k)}{(\alpha_k-a)^{n+1}}=-\frac{G_w(\alpha_k)}{n(\alpha_k-a)^n}+\int\frac{d\alpha_k\;\partial_{\alpha_k}G_w(\alpha_k)}{n(\alpha_k-a)^n}$$

$$\int d\alpha_k\;\alpha_k^nG_w(\alpha_k)=\frac{\alpha_k^{n+1}G_w(\alpha_k)}{n+1}-\int\frac{d\alpha_k\;\alpha_k^{n+1}\partial_{\alpha_k}G_w(\alpha_k)}{n+1}$$

Linear reducibility

$$I = \sum_{i=1}^{\infty} \epsilon^i \int_0^1 d\alpha_p \underbrace{\cdots \underbrace{\int_0^{1-\alpha_4 \cdots -\alpha_p} d\alpha_3 \underbrace{\int_0^{1-\alpha_3 \cdots -\alpha_p} d\alpha_2 f^{(i)}(\alpha_1, \ldots, \alpha_p)|_{\alpha_1 = 1-\alpha_2 \cdots -\alpha_p}}_{f_2(\alpha_3, \ldots, \alpha_p)}}_{f_3(\alpha_4, \ldots, \alpha_p)}}_{f_{p-1}(\alpha_p)}$$

- If one of the $f_k(\alpha_{k+1}, \ldots, \alpha_p)$ depends algebraically on the next integration variable, the integral cannot be computed in terms of MPL and the direct integration algorithm stops

- If $f_k(\alpha_{k+1}, \ldots, \alpha_p), \quad k \in \{1, \ldots, p-1\}$ have rational alphabets, then the integral is called **linearly reducible**

[Brown] arXiv: 0804.1660
[Panzer] arXiv:1403.3385

Reducibility

- Integration order
- Cheng-Wu theorem:
  Feynman integrals are projective — invariant under $\alpha_i \to \lambda\alpha_i,\ d\alpha_i \to \lambda d\alpha_i$
  then,

$$I_{a_1,\ldots,a_n}(\{s\}) = \int_{\Delta_S} d^n\vec{\alpha}\,\left(\prod_{i=1}^{n}\alpha_i^{a_i-1}\right) U^{a-\frac{d}{2}(l+1)} F^{-a+\frac{ld}{2}}$$

$$\Delta_S = \left\{\vec{\alpha}\,\middle|\,\alpha_i \geq 0,\ \sum_{i\in S\subseteq\{1,\ldots,n\}}\alpha_i = 1\right\}$$

- Variable change
- Linear reducibility holds to all epsilon orders !

The elliptic case

$$I = \sum_{i=1}^{\infty} \epsilon^i \int_0^1 d\alpha_p \int_0^{1-\alpha_p} d\alpha_{p-1} \cdots \int_0^{1-\alpha_3 \cdots - \alpha_p} d\alpha_2 f^{(i)}(\alpha_1, \ldots, \alpha_p)|_{\alpha_1 = 1 - \alpha_2 \cdots - \alpha_p}$$

An elliptic integral is an integral over an elliptic curve

$$y(x)^2 = (x - a_1)(x - a_2)(x - a_3)(x - a_4)$$

If we are computing a Feynman integral that does not evaluate to MPL it is sensible to expect that the best we can do is to perform the integrations up to the second last parameter included in terms of MPL. We define **linearly reducible elliptic Feynman integrals**

$$I = \sum_{i=0}^{\infty} \epsilon^i \int_0^{\infty} \overbrace{\sum_j R_j^{(i)}(x, y(x)) \mathrm{MPL}_j^{(i)}(x, y(x))}^{\text{Inner Polylogarithmic Part (IPP)}}$$

[Hidding, FM] arXiv:1712.04441

- The outer integration kernels can be reduced to a basis by ibp identities.

$$I = \sum_{i=0}^{\infty} \epsilon^i \int_0^{\infty} \sum_j \phi_j^{(i)}(x, y(x)) \overline{\mathrm{MPL}}_j^{(i)}(x, y(x)), \quad \phi \in \left\{ \frac{1}{x-c}, \frac{1}{y(x)}, \frac{x}{y(x)}, \frac{1}{(x-c)y(x)} \right\}$$

- Expressible to all orders of epsilon in terms of eMPL (by using ibp)

Iterated Integrals

- By using partial fractioning and integration-by-parts identities we can express all iterated integrals over rational functions in terms of Multiple polylogarithms

$$I(x)=\int_{x_0}^{x} d_{t_1} R_1(t_1)\int_{x_0}^{t_1} dt_2 R_2(t_2)...\int_{x_0}^{t_{n-1}} dt_n R_n(t_n) \qquad G(a_1,\ldots,a_n,x)=\int_0^x \frac{1}{t-a_1}G(a_2,\ldots,a_n,t)dt$$

[Goncharov] arXiv:1105.2076

- In the case of rational functions on an elliptic curve we have iterated integrals of the form

$$y(x)^2=(x-a_1)(x-a_2)(x-a_3)(x-a_4)$$

$$I(x)=\int_{x_0}^{x} dt_1 R_1(t_1,y(t_1))\int_{x_0}^{t_1} dt_2 R_2(t_2,y(t_2))...\int_{x_0}^{t_{n-1}} dt_n R_n(t_n,y(t_n))$$

$$\mathrm{E}_4\left(\begin{smallmatrix} n_1 & \ldots & n_k \\ c_1 & \ldots & c_k \end{smallmatrix};x,\vec{a}\right)=\int_0^x dt\,\psi_{n_1}(c_1,t,\vec{a})\,\mathrm{E}_4\left(\begin{smallmatrix} n_2 & \ldots & n_k \\ c_2 & \ldots & c_k \end{smallmatrix};t,\vec{a}\right)$$

[Broedel, Duhr, Dulat, Tancredi] arXiv:1712.07089

- These integrals can be expressed in terms of elliptic Multiple Polylogarithms. eMPL have at most simple poles. Infinite tower of integration kernels that depend on elliptic integrals.

- However, if we relax the simple poles condition, we can express iterated integrals over en elliptic curve in terms of iterated integrals over algebraic integration kernels

$$E(\phi_{i_1},\phi_{i_2},\ldots,\phi_{i_n};x)=\int_0^x dt\phi_{i_1}(t)E(\phi_{i_2},\ldots,\phi_{i_n};t), \quad \phi=\left\{\frac{1}{x-c},\frac{1}{y(x)},\frac{x}{y(x)},\frac{1}{y(x)(x-c)}\right\}$$

Differential equations for linearly reducible elliptic Feynman integrals

- Apply the Feynman trick: $$\frac{1}{D_1^{\nu_1} D_2^{\nu_2}} = \frac{\Gamma(\nu_1+\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_0^\infty \frac{x^{\nu_1-1} dx}{(xD_1+D_2)^{\nu_1+\nu_2}}$$

$$I_{a_1,\ldots,a_n}(\{s\}) = N \int \left(\prod_{i=1}^{l} d^d k_i\right) \frac{1}{\prod_{i=1}^n D_i^{a_i}} = N \frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty dx x^{a_1-1} \tilde{I}_{a_1+a_2,a_3,\ldots,a_n}(s,x)$$

$$\tilde{I}_{a_1+a_2,a_3,\ldots,a_n}(\{s\},x) = N \int \left(\prod_{i=1}^{l} d^d k_i\right) \frac{1}{(xD_1+D_2)^{a_1+a_2} \prod_{i=3}^n D_i^{a_i}}$$

- Since the integral is linearly reducible it is sensible to expect that we can apply the Feynman trick so that the inner integral is **polylogarithmic**

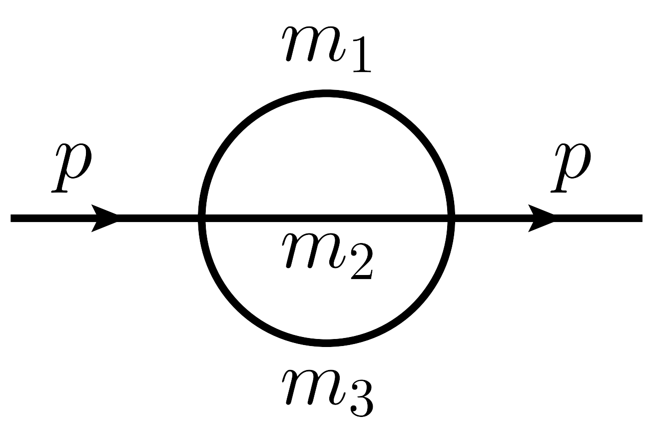
- By completing the set of propagators, we define a new integral topology T, that can be solved with IBP + DE techniques

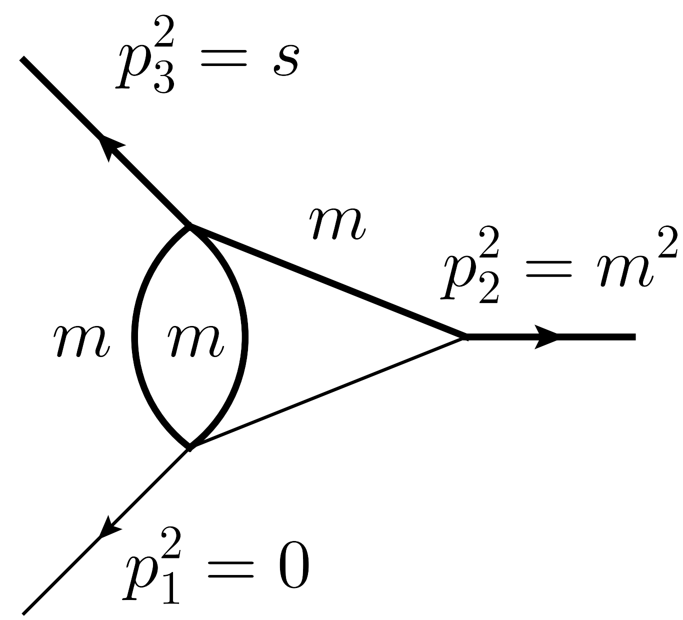
$$I = \sum_{i=0}^{\infty} \epsilon^i \int_0^\infty \sum_j \phi_j^{(i)}(x,y(x)) \mathrm{MPL}_j^{(i)}(x,y(x)), \quad \phi \in \left\{ \frac{1}{x-c}, \frac{1}{y(x)}, \frac{x}{y(x)}, \frac{1}{(x-c)y(x)} \right\}$$

- <u>We will see how we can solve them in terms of eMPL</u>

[Hidding, FM] to appear

Applications

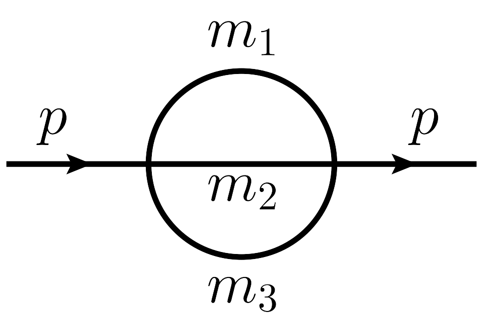




Sunrise

- Four master integrals (three are related by permutations of the masses)

$$S_{1,1,1},\ S_{2,1,1},\ S_{1,2,1},\ S_{1,1,2}$$



$$= S_{1,1,1}(p^2, m_1^2, m_2^2, m_3^2) \quad = \sum_{k=0}^{\infty} \epsilon^k \int_0^\infty d\alpha_3 \int_0^\infty d\alpha_2 \frac{1}{k!} F^{-1} \log\left(\frac{U^3}{F^2}\right)^k |_{\alpha_1=1}$$

$$U = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3,$$
$$F = (\alpha_1\alpha_2 + \alpha_3\alpha_2 + \alpha_1\alpha_3)\left(\alpha_1 m_1^2 + \alpha_2 m_2^2 + \alpha_3 m_3^2\right) - \alpha_1\alpha_2\alpha_3 p^2$$

- The relevant elliptic curve

$$F = m_2^2\,(\alpha_1 + 1)\,(\alpha_2 - R_+)(\alpha_2 - R_-)$$

$$R_\pm(p^2, m_1^2, m_2^2, m_3^2) = \frac{\alpha_3(p^2 - m_1^2 - m_2^2 - m_3^2) - \alpha_3^2 m_3^2 - m_1^2 \pm m_3^2\,y(\alpha_3)}{2m_2^2(\alpha_3 + 1)}$$

$$m_3^2\,y(\alpha_3) = \sqrt{\left(\alpha_3(p^2 - m_1^2 - m_2^2 - m_3^2) - \alpha_3^2 m_1^2 - m_3^2\right)^2 - 4m_2^2\alpha_3\,(\alpha_3 + 1)\,(\alpha_3 m_3^2 + m_1^2)}$$

- At leading epsilon order the solution for the first master integral reads

$$S_{111}^{(0)}(p^2, m_1^2, m_2^2, m_3^2) = \int_0^\infty dx\, \frac{1}{m_3^2\,y(x)} \log\left(\frac{R_-}{R_+}\right)$$

[Hidding, FM] arXiv:1712.04441

Sunrise

$$S_{111}^{(0)}(p^2, m_1^2, m_2^2, m_3^2) = \int_0^\infty d\alpha_3 \, \frac{1}{m_3^2 y(x)} UT^{(1)}$$

- Next to leading order

$$S_{111}^{(1)}(p^2, m_1^2, m_2^2, m_3^2) = \int_0^\infty d\alpha_3 \, \frac{1}{m_3^2 y(x)} UT^{(2)}$$

$$\begin{aligned} UT^{(2)}(x) = & -3G\left(\alpha_3+1, \frac{1}{R_-+1}, 1\right) + 3G\left(\alpha_3+1, \frac{1}{R_++1}, 1\right) - G\left(Q_S, \frac{R_-}{R_--R_+}, 1\right) \\ & - G\left(\frac{R_-}{R_--R_+}, Q_S, 1\right) - G\left(0, \frac{1}{R_-+1}, 1\right) + G\left(0, \frac{1}{R_++1}, 1\right) - 2G\left(\frac{1}{R_-+1}, \frac{1}{R_++1}, 1\right) \\ & + 2G\left(\frac{1}{R_++1}, \frac{1}{R_-+1}, 1\right) \end{aligned}$$

$$Q_S = \frac{(x+1)\, m_2^2 \left(x m_1^2 + m_3^2\right)}{(x+1)\, m_2^2 \left(x m_1^2 + m_3^2\right) - x^2}$$

- If we go a few orders higher we recognise a simple pattern (we will prove this in a moment)

$$S_{111}(p^2, m_1^2, m_2^2, m_3^2) = \sum_{i=0}^{\infty} \epsilon^i \int_0^\infty d\alpha_3 \, \frac{1}{m_3^2 y(x)} UT^{(i+1)}$$

[Hidding, FM] arXiv:1712.04441

Defining DE for the sunrise

- Apply the Feynman trick:

$$S_{\nu_1,\nu_2,\nu_1} = N \int \frac{d^d k_1 d^d k_2}{(i\pi^{d/2})^2} \frac{1}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_1}} = \frac{\Gamma(2\nu_1)\Gamma(\nu_2)}{\Gamma(2\nu_1+\nu_2-d)} \int_0^\infty dx\, x^{\nu_1-1} \int \frac{d^d k_1 d^d k_2}{(i\pi^{d/2})^2} \frac{1}{(D_1+xD_3)^{2\nu_1} D_2^{\nu_2}}$$

$$S_{\nu_1,\nu_2,\nu_1} = \frac{\Gamma(2\nu_1)\Gamma(\nu_2)\Gamma(3-d)}{\Gamma(2\nu_1+\nu_2-d)} \int_0^\infty dx\, x^{\nu_1-1} S^{\mathrm{IPP}}_{2\nu_1,\nu_2,0,0,0}$$

- Complete the set of propagators and define a new, polylogaritmic integral topology

$$S^{\mathrm{IPP}}_{\nu_1,\nu_2,\mu_1,\mu_2,\mu_3} = \frac{1}{\Gamma(3-d)} \int \frac{d^d k_1 d^d k_2}{(i\pi^{d/2})^2} \frac{1}{(D_1+xD_3)^{\nu_1} D_2^{\nu_2} N_1^{\mu_1} N_2^{\mu_2} N_3^{\mu_3}}$$

$$N_1 = -k_1^2, \qquad N_2 = -(k_1+k_2)^2, \qquad N_3 = -(k_1+p)^2$$

- Using IBP define a canonical basis of master integrals and corresponding DE

$$B_1 = 2(m_3^2)^{2\epsilon} x\epsilon S^{\mathrm{IPP}}_{2,0,0,0,0}\,, \quad B_2 = 2(m_3^2)^{2\epsilon}(1+x)\epsilon^2 S^{\mathrm{IPP}}_{1,1,0,0,0}\,, \quad B_3 = (m_3^2)^{2\epsilon+1} y(x)\epsilon S^{\mathrm{IPP}}_{2,1,0,0,0}\,,$$

[Hidding, FM] to appear

DE

$$B_1 = 2(m_3^2)^{2\epsilon} x\epsilon S^{\text{IPP}}_{2,0,0,0,0}\,, \quad B_2 = 2(m_3^2)^{2\epsilon}(1+x)\epsilon^2 S^{\text{IPP}}_{1,1,0,0,0}\,, \quad B_3 = (m_3^2)^{2\epsilon+1} y(x)\epsilon S^{\text{IPP}}_{2,1,0,0,0}\,,$$

- The canonical form of the differential equations reads

$$d\vec{B} = \epsilon dA\vec{B}$$

$$A = \begin{pmatrix} l_1 + 2l_4 - 2l_5 & 0 & 0 \\ \frac{l_7}{2} - \frac{l_3}{2} & -l_1 + 2l_2 - \frac{3l_3}{2} + 2l_4 - \frac{l_7}{2} & -2l_1 - l_3 - l_7 + 2l_{10} \\ l_2 - \frac{l_1}{2} + \frac{l_3}{4} + l_5 + 2l_6 - \frac{3l_7}{4} - 2l_9 + \frac{3l_{10}}{2} - 2l_{11} & \frac{3l_1}{2} + \frac{3l_3}{4} + \frac{3l_7}{4} - \frac{3l_{10}}{2} & 3l_1 + 2l_2 + \frac{l_3}{2} + 6l_4 + \frac{3l_7}{2} - 4l_8 \end{pmatrix}$$

- Where the alphabet is

$$l_1 = \log(x), l_2 = \log(x+1), l_3 = \log\left(m_2^2\right), l_4 = \log\left(m_3^2\right),$$

$$l_5 = \log\left(m_3^2 x + m_1^2\right), l_6 = \log\left(x\left(m_3^2 - s\right) + m_1^2 - m_2^2\right), l_7 = \log(s), l_8 = \log(y),$$

$$l_9 = \log\left(-m_3^2 x^2 + m_1^2(-x) + m_2^2 x + m_3^2 x + m_1^2 - sx - y m_3^2\right),$$

$$l_{10} = \log\left(m_3^2 x^2 + m_1^2 x - m_2^2 x + m_3^2 x + m_1^2 - sx - y m_3^2\right),$$

$$l_{11} = \log\left(m_3^2 x^2 + m_1^2 x - m_2^2 x + m_3^2 x + m_1^2 - 2m_2^2 - sx - y m_3^2\right)$$

Solution

$$d\vec{f}(x,\epsilon) = \epsilon\, dA \vec{f}(x,\epsilon) \qquad \vec{f}(x,\epsilon) = \mathbb{P}\exp\left(\epsilon\int_{x_0}^{x}\frac{\partial A}{\partial x}\,dx\right)\vec{f}(x_0,\epsilon)$$

$$\vec{f}(x,\epsilon) = \vec{f}^{(0)}\left(x_0\right) + \epsilon\left(\int_{x_0}^{x}\frac{\partial A\left(t_1\right)}{\partial t_1}\vec{f}^{(0)}\left(x_0\right)\,dt_1 + \vec{f}^{(1)}\left(x_0\right)\right)$$
$$+\epsilon^2\left(\int_{x_0}^{x}\int_{x_0}^{\mathrm{t1}}\frac{\partial A\left(t_1\right)}{\partial t_1}\frac{\partial A\left(t_2\right)}{\partial t_2}\vec{f}^{(0)}\left(x_0\right)dt_2 dt_1 + \int_{x_0}^{x}\frac{\partial A\left(t_1\right)}{\partial t_1}\vec{f}^{(1)}\left(x_0\right)\,dt_1 + \vec{f}^{(2)}\left(x_0\right)\right) + \cdots$$

- Rational alphabet, iterated integrals translate directly to MPL

$$\frac{\partial A(x)}{\partial x} = \sum_i M_i \frac{1}{x-a_i}$$

- Irrational alphabet, we integrate the symbol up

$$\mathcal{S}(\vec{f}(x,\epsilon)) = \epsilon\left(\otimes A\left(x\right)\vec{f}^{(0)}\left(x_0\right)\right) + \epsilon^2 R\left(A(x)\otimes A\left(x\right)\vec{f}^{(0)}\left(x_0\right)\right) + \cdots$$
$$R(a\otimes b\otimes c) = c\otimes b\otimes a$$

- Result will be pure functions of uniform weight (UT)

$$\vec{B} = \sum_{i=0}^{\infty}\epsilon^i\vec{B}^{(i)}, \quad \vec{B}^{(i)} = \text{pure function of weight } i$$

IBP

$$S_{\nu_1,\nu_2,\nu_1} = \frac{\Gamma(2\nu_1)\Gamma(\nu_2)\Gamma(3-d)}{\Gamma(2\nu_1+\nu_2-d)} \int_0^\infty dx\, x^{\nu_1-1} S^{\text{IPP}}_{2\nu_1,\nu_2,0,0,0}$$

$$S_{1,1,1} = \int_0^\infty S^{\text{IPP}}_{2,1} dx \quad = \frac{(m_3^2)^{-2\epsilon-1}}{\epsilon} \int_0^\infty \frac{B_3}{y} dx \ = (m_3^2)^{-2\epsilon-1} \sum_{i=0}^\infty \epsilon^i \int_0^\infty \frac{B_3^{(i+1)}}{y} dx$$

$$S_{1,2,1} = \frac{(m_3^2)^{-2\epsilon}}{(1+2\epsilon)} \sum_{i=0}^\infty \epsilon^i \int_0^\infty dx \left(\frac{c_1}{x-a_1} B_1^{(i)} + \frac{c_2}{x-a_1} B_2^{(i)} + \frac{c_3}{y(x-a_1)} (4B_3^{(i)} + B_3^{(i+1)}) + \frac{c_4}{y} B_3^{(i)} + \cdots \right)$$

Left-most Integration kernels : $\left\{ \frac{1}{(x-a_i)}, \quad \frac{1}{y}, \quad \frac{1}{y(x-a_i)} \right\}$

- We have proven the all orders structure we observed from direct integration

DE & eMPL

$$x = \frac{x'}{1-x'} \quad \rightarrow \quad S_{\nu_1,\nu_2,\nu_1} = \frac{\Gamma(2\nu_1)\Gamma(\nu_2)\Gamma(3-d)}{\Gamma(2\nu_1+\nu_2-d)} \int_0^1 dx' \, \frac{x'^{\nu_1-1}}{(1-x')^{1+\nu_1}} S^{\mathrm{IPP}}_{2\nu_1,\nu_2,0,0,0}$$

$$d\vec{B} = \epsilon dA\vec{B}$$

$$\frac{\partial \mathbf{A}}{\partial x'} = \left(\begin{array}{c|c|c} -\frac{2(m_3^2-m_1^2)}{m_1^2+x'(m_3^2-m_1^2)} + \frac{1}{x'-1} + \frac{1}{x'} & 0 & 0 \\ \hline 0 & \begin{array}{c} -\frac{1}{x'} - \frac{1}{x'-1} \\ +\frac{2(m_1^2-m_3^2)}{y(p^2-m_2^2)} \end{array} & \frac{2m_1^2}{x'y(m_2^2-p^2)} + \frac{2m_3^2}{(x'-1)y(m_2^2-p^2)} \\ \hline \begin{array}{c} \frac{m_1^2}{2x'y(m_2^2-p^2)} + \frac{m_3^2 m_1^2}{y(m_1^2-m_3^2)(x'm_1^2-m_1^2-x'm_3^2)} \\ +\frac{m_3^2}{2(x'-1)y(m_2^2-p^2)} + \frac{m_1^4+2(p^2-m_2^2-m_3^2)m_1^2+m_3^4}{2y(p^2-m_2^2)(m_1^2-m_3^2)} - \frac{x'}{y} \end{array} & \begin{array}{c} \frac{3m_1^2}{x'y(2p^2-2m_2^2)} + \frac{3(m_1^2-m_3^2)}{2y(m_2^2-p^2)} \\ -\frac{3m_3^2}{2(x'-1)y(m_2^2-p^2)} \end{array} & \begin{array}{c} \frac{3}{x'} - \frac{2}{x'-a_1} - \frac{2}{x'-a_2} - \frac{2}{x'-a_3} - \frac{2}{x'-a_4} \\ +\frac{3}{x'-1} \end{array} \end{array}\right)$$

$$\psi_0(0,x) = \frac{c_4}{y}, \quad \psi_{-1}(\infty,x) = \frac{x}{y}, \quad \psi_{-1}(c,x) = \frac{y_c}{(x-c)y} - \frac{\delta_{c0}}{x}, \quad \psi_1(c,x) = \frac{1}{x-c}$$

[Hidding, FM] to appear

$$\vec{B}(x',p^2,m_1,m_2,m_3) = \mathbb{P}\exp\left(\epsilon \int_{x_0'}^{x'} \frac{\partial A}{\partial x'}\, dx'\right) \vec{B}(x_0',p^2,m_1,m_2,m_3)\,,$$

$$\vec{B}(x',\epsilon) = \vec{B}^{(0)}(x_0') + \epsilon\left(\int_{x_0'}^{x'} \frac{\partial A(t_1)}{\partial t_1}\vec{B}^{(0)}(x_0')\, dt_1 + \vec{B}^{(1)}(x_0')\right) + \epsilon^2\left(\int_{x_0'}^{x'} \frac{\partial A(t_1)}{\partial t_1}\int_{x_0'}^{t1} \frac{\partial A(t_2)}{\partial t_2}\vec{B}^{(0)}(x_0')\, dt_2 dt_1 + \int_{x_0'}^{x'} \frac{\partial A(t_1)}{\partial t_1}\vec{B}^{(1)}(x_0')\, dt_1 + \vec{B}^{(2)}(x_0')\right)$$

$$\mathrm{E}_4\left(\begin{smallmatrix} n_1 & \cdots & n_k \\ c_1 & \cdots & c_k \end{smallmatrix}; x', \vec{a}\right) = \int_0^{x'} dt\, \psi_{n_1}(c_1,t,\vec{a})\, \mathrm{E}_4\left(\begin{smallmatrix} n_2 & \cdots & n_k \\ c_2 & \cdots & c_k \end{smallmatrix}; t, \vec{a}\right)$$

Solution

$$S_{1,1,1} = \int_0^1 \frac{1}{(1-x')^2} S_{2,1,0,0,0} = \frac{(m_3^2)^{-2\epsilon}}{(m_2^2 - s)} \int_0^1 dx' \frac{1}{y} B_3 = \frac{(m_3^2)^{-2\epsilon}}{c_4(m_2^2 - s)} \sum_{i=0}^{\infty} \epsilon^i \int_0^1 dx' \psi_0(0, x') B_3^{(i+1)}$$

$$\frac{1}{y} \longrightarrow \frac{\psi_0(0, x')}{c_4}$$

$$B_3^{(i+1)} \longrightarrow \mathrm{E}_4\left(\begin{smallmatrix} n_1 & \dots & n_k \\ c_1 & \dots & c_k \end{smallmatrix}; x', \vec{a}\right) = \int_0^{x'} dt\, \psi_{n_1}(c_1, t, \vec{a})\, \mathrm{E}_4\left(\begin{smallmatrix} n_2 & \dots & n_k \\ c_2 & \dots & c_k \end{smallmatrix}; t, \vec{a}\right)$$

$$S_{1,2,1} = \frac{(m_3^2)^{-2\epsilon}}{(1+2\epsilon)} \sum_{i=0}^{\infty} \epsilon^i \int_0^1 dx' \; b_1 \; \psi_1(a_1, x') B_1^{(i)} + b_2 \; \psi_1(a_1, x') B_2^{(i)}$$

$$+ \frac{b_3}{y(x' - a_1)} (4B_3^{(i)} + B_3^{(i+1)}) + b_4 \; \psi_0(0, x') B_3^{(i)} + \cdots$$

$$\frac{b_3}{y(x'-a_1)} \longrightarrow E(\phi_{i_1}, \phi_{i_2}, \dots, \phi_{i_n}; x') = \int_0^{x'} dt \phi_{i_1}(t) E(\phi_{i_2}, \dots, \phi_{i_n}; t), \quad \phi = \left\{ \frac{1}{x-c}, \frac{1}{y(x)}, \frac{x}{y(x)}, \frac{1}{y(x)(x-c)} \right\}$$

Up to $\mathcal{O}(\epsilon)$

$$
\begin{aligned}
&c_4\left(m_2^2-p^2\right)\left(m_3^2\right)^{2\epsilon}S_{111}\\
&\quad=-\mathrm{E}_4\left(\begin{smallmatrix}0&-1\\0&0\end{smallmatrix},1\right)-\mathrm{E}_4\left(\begin{smallmatrix}0&-1\\0&1\end{smallmatrix},1\right)-\mathrm{E}_4\left(\begin{smallmatrix}0&-1\\0&\infty\end{smallmatrix},1\right)+\mathrm{E}_4\left(\begin{smallmatrix}0&-1\\0&\frac{m_1^2}{m_1^2-m_3^2}\end{smallmatrix},1\right)+\mathrm{E}_4\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix},1\right)+\mathrm{E}_4\left(\begin{smallmatrix}0\\0\end{smallmatrix},1\right)\log\left(\frac{m_1^2}{m_2^2}\right)\\
&\qquad-\frac{\mathrm{E}_4\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix},1\right)\left(m_1^4+\left(-p^2+m_2^2-2m_3^2\right)m_1^2+m_3^4\right)}{c_4\left(p^2-m_2^2\right)\left(m_1^2-m_3^2\right)}
\end{aligned}
$$

Up to $\mathcal{O}(\epsilon^2)$

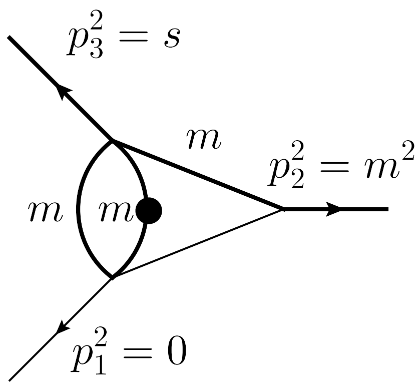
$$
\begin{aligned}
&c_4\left(m_2^2-p^2\right)\left(m_3^2\right)^{2\epsilon}S_{111}\\
&= -\mathrm{E}_4\left(\begin{smallmatrix}0&-1\\0&0\end{smallmatrix},1\right)-\mathrm{E}_4\left(\begin{smallmatrix}0&-1\\0&1\end{smallmatrix},1\right)-\mathrm{E}_4\left(\begin{smallmatrix}0&-1\\0&\infty\end{smallmatrix},1\right)+\mathrm{E}_4\left(\begin{smallmatrix}0&-1\\0&\frac{m_1^2}{m_1^2-m_3^2}\end{smallmatrix},1\right)+\mathrm{E}_4\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix},1\right)+\mathrm{E}_4\left(\begin{smallmatrix}0\\0\end{smallmatrix},1\right)\log\left(\frac{m_1^2}{m_2^2}\right)\\
&\quad-\frac{\mathrm{E}_4\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix},1\right)\left(m_1^4+\left(-p^2+m_2^2-2m_3^2\right)m_1^2+m_3^4\right)}{c_4\left(p^2-m_2^2\right)\left(m_1^2-m_3^2\right)}+\epsilon\Bigg(2\mathrm{E}_4\left(\begin{smallmatrix}0&-1&1\\0&0&1\end{smallmatrix},1\right)-\mathrm{E}_4\left(\begin{smallmatrix}0&-1&1\\0&0&-\frac{m_1^2}{m_3^2-m_1^2}\end{smallmatrix},1\right)+2\mathrm{E}_4\left(\begin{smallmatrix}0&-1&1\\0&1&1\end{smallmatrix},1\right)\\
&\quad-\mathrm{E}_4\left(\begin{smallmatrix}0&-1&1\\0&1&-\frac{m_1^2}{m_3^2-m_1^2}\end{smallmatrix},1\right)-\mathrm{E}_4\left(\begin{smallmatrix}0&-1&1\\0&\infty&1\end{smallmatrix},1\right)+2\mathrm{E}_4\left(\begin{smallmatrix}0&-1&1\\0&\infty&-\frac{m_1^2}{m_3^2-m_1^2}\end{smallmatrix},1\right)+\mathrm{E}_4\left(\begin{smallmatrix}0&-1&1\\0&\frac{m_1^2}{m_1^2-m_3^2}&1\end{smallmatrix},1\right)-2\mathrm{E}_4\left(\begin{smallmatrix}0&-1&1\\0&\frac{m_1^2}{m_1^2-m_3^2}&-\frac{m_1^2}{m_3^2-m_1^2}\end{smallmatrix},1\right)\\
&\quad-5\mathrm{E}_4\left(\begin{smallmatrix}0&1&-1\\0&0&0\end{smallmatrix},1\right)-5\mathrm{E}_4\left(\begin{smallmatrix}0&1&-1\\0&0&1\end{smallmatrix},1\right)-2\mathrm{E}_4\left(\begin{smallmatrix}0&1&-1\\0&0&\infty\end{smallmatrix},1\right)+2\mathrm{E}_4\left(\begin{smallmatrix}0&1&-1\\0&0&\frac{m_1^2}{m_1^2-m_3^2}\end{smallmatrix},1\right)-3\mathrm{E}_4\left(\begin{smallmatrix}0&1&-1\\0&1&0\end{smallmatrix},1\right)-3\mathrm{E}_4\left(\begin{smallmatrix}0&1&-1\\0&1&1\end{smallmatrix},1\right)\\
&\quad-3\mathrm{E}_4\left(\begin{smallmatrix}0&1&-1\\0&1&\infty\end{smallmatrix},1\right)+3\mathrm{E}_4\left(\begin{smallmatrix}0&1&-1\\0&1&\frac{m_1^2}{m_1^2-m_3^2}\end{smallmatrix},1\right)+2\mathrm{E}_4\left(\begin{smallmatrix}0&1&-1\\0&a_1&0\end{smallmatrix},1\right)+2\mathrm{E}_4\left(\begin{smallmatrix}0&1&-1\\0&a_1&1\end{smallmatrix},1\right)+2\mathrm{E}_4\left(\begin{smallmatrix}0&1&-1\\0&a_1&\infty\end{smallmatrix},1\right)-2\mathrm{E}_4\left(\begin{smallmatrix}0&1&-1\\0&a_1&\frac{m_1^2}{m_1^2-m_3^2}\end{smallmatrix},1\right)\\
&\quad+2\mathrm{E}_4\left(\begin{smallmatrix}0&1&-1\\0&a_2&0\end{smallmatrix},1\right)+2\mathrm{E}_4\left(\begin{smallmatrix}0&1&-1\\0&a_2&1\end{smallmatrix},1\right)+2\mathrm{E}_4\left(\begin{smallmatrix}0&1&-1\\0&a_2&\infty\end{smallmatrix},1\right)-2\mathrm{E}_4\left(\begin{smallmatrix}0&1&-1\\0&a_2&\frac{m_1^2}{m_1^2-m_3^2}\end{smallmatrix},1\right)+2\mathrm{E}_4\left(\begin{smallmatrix}0&1&-1\\0&a_3&0\end{smallmatrix},1\right)+2\mathrm{E}_4\left(\begin{smallmatrix}0&1&-1\\0&a_3&1\end{smallmatrix},1\right)\\
&\quad+2\mathrm{E}_4\left(\begin{smallmatrix}0&1&-1\\0&a_3&\infty\end{smallmatrix},1\right)-2\mathrm{E}_4\left(\begin{smallmatrix}0&1&-1\\0&a_3&\frac{m_1^2}{m_1^2-m_3^2}\end{smallmatrix},1\right)+2\mathrm{E}_4\left(\begin{smallmatrix}0&1&-1\\0&a_4&0\end{smallmatrix},1\right)+2\mathrm{E}_4\left(\begin{smallmatrix}0&1&-1\\0&a_4&1\end{smallmatrix},1\right)+2\mathrm{E}_4\left(\begin{smallmatrix}0&1&-1\\0&a_4&\infty\end{smallmatrix},1\right)-2\mathrm{E}_4\left(\begin{smallmatrix}0&1&-1\\0&a_4&\frac{m_1^2}{m_1^2-m_3^2}\end{smallmatrix},1\right)\\
&\quad+5\mathrm{E}_4\left(\begin{smallmatrix}0&1&1\\0&0&1\end{smallmatrix},1\right)-2\mathrm{E}_4\left(\begin{smallmatrix}0&1&1\\0&0&a_1\end{smallmatrix},1\right)-2\mathrm{E}_4\left(\begin{smallmatrix}0&1&1\\0&0&a_2\end{smallmatrix},1\right)-2\mathrm{E}_4\left(\begin{smallmatrix}0&1&1\\0&0&a_3\end{smallmatrix},1\right)-2\mathrm{E}_4\left(\begin{smallmatrix}0&1&1\\0&0&a_4\end{smallmatrix},1\right)-\mathrm{E}_4\left(\begin{smallmatrix}0&1&1\\0&0&-\frac{m_1^2}{m_3^2-m_1^2}\end{smallmatrix},1\right)-2\mathrm{E}_4\left(\begin{smallmatrix}1&0&-1\\0&0&0\end{smallmatrix},1\right)\\
&\quad-2\mathrm{E}_4\left(\begin{smallmatrix}1&0&-1\\0&0&1\end{smallmatrix},1\right)+\mathrm{E}_4\left(\begin{smallmatrix}1&0&-1\\0&0&\infty\end{smallmatrix},1\right)-\mathrm{E}_4\left(\begin{smallmatrix}1&0&-1\\0&0&\frac{m_1^2}{m_1^2-m_3^2}\end{smallmatrix},1\right)+3\mathrm{E}_4\left(\begin{smallmatrix}1&0&1\\0&0&1\end{smallmatrix},1\right)-2\mathrm{E}_4\left(\begin{smallmatrix}1&0&1\\0&0&a_1\end{smallmatrix},1\right)-2\mathrm{E}_4\left(\begin{smallmatrix}1&0&1\\0&0&a_2\end{smallmatrix},1\right)-2\mathrm{E}_4\left(\begin{smallmatrix}1&0&1\\0&0&a_3\end{smallmatrix},1\right)\\
&\quad-2\mathrm{E}_4\left(\begin{smallmatrix}1&0&1\\0&0&a_4\end{smallmatrix},1\right)-\mathrm{E}_4\left(\begin{smallmatrix}1&1&0\\0&0&0\end{smallmatrix},1\right)+6\mathrm{E}_4\left(\begin{smallmatrix}0&1\\0&1\end{smallmatrix},1\right)\log\left(\frac{m_1}{m_2}\right)+\mathrm{E}_4\left(\begin{smallmatrix}0&1\\0&a_1\end{smallmatrix},1\right)\log\left(\frac{m_2^4}{m_1^4}\right)+\mathrm{E}_4\left(\begin{smallmatrix}0&1\\0&a_2\end{smallmatrix},1\right)\log\left(\frac{m_2^4}{m_1^4}\right)\\
&\quad+\mathrm{E}_4\left(\begin{smallmatrix}0&1\\0&a_3\end{smallmatrix},1\right)\log\left(\frac{m_2^4}{m_1^4}\right)+\mathrm{E}_4\left(\begin{smallmatrix}0&1\\0&a_4\end{smallmatrix},1\right)\log\left(\frac{m_2^4}{m_1^4}\right)+\mathrm{E}_4\left(\begin{smallmatrix}0\\0\end{smallmatrix},1\right)\left(-2\log\left(\frac{m_1}{m_2}\right)\log\left(\frac{m_1^3m_2}{m_3^4}\right)-\frac{\pi^2}{6}\right)\\
&\quad+4\mathrm{E}_4\left(\begin{smallmatrix}0&-1\\0&\infty\end{smallmatrix},1\right)\log\left(\frac{m_1}{m_3}\right)-2\mathrm{E}_4\left(\begin{smallmatrix}0&-1\\0&0\end{smallmatrix},1\right)\log\left(\frac{m_1m_3^2}{m_2^3}\right)-2\mathrm{E}_4\left(\begin{smallmatrix}0&-1\\0&1\end{smallmatrix},1\right)\log\left(\frac{m_1m_3^2}{m_2^3}\right)\\
&\quad+\mathrm{E}_4\left(\begin{smallmatrix}0&-1\\0&\frac{m_1^2}{m_1^2-m_3^2}\end{smallmatrix},1\right)\log\left(\frac{m_3^4}{m_1^4}\right)+\frac{2\mathrm{E}_4\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix},1\right)\left(3\log\left(\frac{m_2}{m_1}\right)p^4+\log\left(\frac{m_1m_3^2}{m_2^3}\right)p^4-2\log\left(\frac{m_1}{m_3}\right)m_2^2\left(m_2^2-2p^2\right)\right)}{\left(p^2-m_2^2\right)^2}\\
&\quad+\frac{\mathrm{E}_4\left(\begin{smallmatrix}0&0&1\\0&0&1\end{smallmatrix},1\right)\left(2m_1^4+\left(p^2-m_2^2-4m_3^2\right)m_1^2+2m_3^4\right)}{c_4\left(p^2-m_2^2\right)\left(m_1^2-m_3^2\right)}+\frac{2\mathrm{E}_4\left(\begin{smallmatrix}0&1&0\\0&a_1&0\end{smallmatrix},1\right)\left(m_1^4+\left(-p^2+m_2^2-2m_3^2\right)m_1^2+m_3^4\right)}{c_4\left(p^2-m_2^2\right)\left(m_1^2-m_3^2\right)}\\
&\quad+\frac{2\mathrm{E}_4\left(\begin{smallmatrix}0&1&0\\0&a_2&0\end{smallmatrix},1\right)\left(m_1^4+\left(-p^2+m_2^2-2m_3^2\right)m_1^2+m_3^4\right)}{c_4\left(p^2-m_2^2\right)\left(m_1^2-m_3^2\right)}+\frac{2\mathrm{E}_4\left(\begin{smallmatrix}0&1&0\\0&a_3&0\end{smallmatrix},1\right)\left(m_1^4+\left(-p^2+m_2^2-2m_3^2\right)m_1^2+m_3^4\right)}{c_4\left(p^2-m_2^2\right)\left(m_1^2-m_3^2\right)}\\
&\quad+\frac{2\mathrm{E}_4\left(\begin{smallmatrix}0&1&0\\0&a_4&0\end{smallmatrix},1\right)\left(m_1^4+\left(-p^2+m_2^2-2m_3^2\right)m_1^2+m_3^4\right)}{c_4\left(p^2-m_2^2\right)\left(m_1^2-m_3^2\right)}-\frac{\mathrm{E}_4\left(\begin{smallmatrix}0&0&1\\0&0&-\frac{m_1^2}{m_3^2-m_1^2}\end{smallmatrix},1\right)\left(m_1^4-2\left(-p^2+m_2^2+m_3^2\right)m_1^2+m_3^4\right)}{c_4\left(p^2-m_2^2\right)\left(m_1^2-m_3^2\right)}\\
&\quad+\frac{\mathrm{E}_4\left(\begin{smallmatrix}0&1&0\\0&1&0\end{smallmatrix},1\right)\left(-3m_1^4+3\left(p^2-m_2^2+2m_3^2\right)m_1^2-3m_3^4\right)}{c_4\left(p^2-m_2^2\right)\left(m_1^2-m_3^2\right)}+\frac{\mathrm{E}_4\left(\begin{smallmatrix}1&0&0\\0&0&0\end{smallmatrix},1\right)\left(-2m_1^4+\left(-p^2+m_2^2+4m_3^2\right)m_1^2-2m_3^4\right)}{c_4\left(p^2-m_2^2\right)\left(m_1^2-m_3^2\right)}\\
&\quad+\frac{\mathrm{E}_4\left(\begin{smallmatrix}0&1&0\\0&0&0\end{smallmatrix},1\right)\left(-5m_1^4+2\left(p^2-m_2^2+5m_3^2\right)m_1^2-5m_3^4\right)}{c_4\left(p^2-m_2^2\right)\left(m_1^2-m_3^2\right)}\\
&\quad-\frac{2\mathrm{E}_4\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix},1\right)\left(\log\left(\frac{m_1m_3^2}{m_2^3}\right)m_1^4+2\left(\log\left(\frac{m_1}{m_3}\right)p^2-\log\left(\frac{m_1}{m_3}\right)m_2^2-\log\left(\frac{m_1m_3^2}{m_2^3}\right)m_3^2\right)m_1^2+\log\left(\frac{m_1m_3^2}{m_2^3}\right)m_3^4\right)}{c_4\left(p^2-m_2^2\right)\left(m_1^2-m_3^2\right)}\Bigg)\\
&\quad+\mathcal{O}(\epsilon^2)
\end{aligned}
$$

[Hidding, FM] to appear

Triangle with bubble

- There is one master integral

$(m^2 - s)$



$$= (m^2 - s) \sum_{k=0}^{\infty} \epsilon^k \int_{\Delta} d^4\vec{\alpha}\, \frac{\alpha_2}{k!} (UF)^{-1} \log\left(\frac{U^3}{F^2}\right)^k, \quad \Delta = \{\vec{\alpha} | \alpha_i \geq, \alpha_2 = 1\}$$

- The relevant elliptic curve is the one of the equal mass sunrise

$$m^2 y(x) = \sqrt{\left(sx - m^2\left(x^2 + 3x + 1\right)\right)^2 - 4m^4 x(x+1)^2}$$

- A simple all orders structure holds also for the triangle

$$T_{1211}(s, m^2) = \sum_{i=0}^{\infty} \epsilon^i \int_0^{\infty} dx\, \frac{1}{x} UT^{(i+2)}(x)$$

[Hidding, FM] arXiv:1712.04441

DE & eMPL

$$T_{a_1,a_2,a_3,a_4} = \frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty dx\, x^{a_1-1} T^{\mathrm{IPP}}_{a_1+a_2,a_3,a_4} \qquad T^{\mathrm{IPP}}_{a_1,a_2,a_3} = \frac{m^2(1+t)}{\left(i\pi^{\frac{d}{2}}\right)^2 \Gamma(5-d)} \int \frac{d^dk_1 d^dk_2}{(xD_1+D_2)^{a_1} D_3^{a_2} D_4^{a_3}}$$

$$\vec{B} = \begin{pmatrix} \mathrm{c}^1_{2,2,1} T^{\mathrm{IPP}}_{2,2,1} + \mathrm{c}^1_{3,1,1} T^{\mathrm{IPP}}_{3,1,1} \\ \mathrm{c}^2_{3,1,1} T^{\mathrm{IPP}}_{3,1,1} \\ \mathrm{c}^3_{4,0,1} T^{\mathrm{IPP}}_{4,0,1} \\ \mathrm{c}^4_{2,1,0} T^{\mathrm{IPP}}_{2,1,0} + \mathrm{c}^4_{3,1,0} T^{\mathrm{IPP}}_{3,1,0} + \mathrm{c}^4_{4,0,0} T^{\mathrm{IPP}}_{4,0,0} \\ \mathrm{c}^5_{2,1,0} T^{\mathrm{IPP}}_{2,1,0} + \mathrm{c}^5_{3,1,0} T^{\mathrm{IPP}}_{3,1,0} + \mathrm{c}^5_{4,0,0} T^{\mathrm{IPP}}_{4,0,0} \\ \mathrm{c}^6_{4,0,0} T^{\mathrm{IPP}}_{4,0,0} \end{pmatrix}$$

$$\begin{array}{ll}
\mathrm{c}^1_{2,2,1} = (m^2)^{1+2\epsilon} x(1+t+x)\epsilon^2 & \mathrm{c}^1_{3,1,1} = 2(m^2)^{1+2\epsilon} x^2\epsilon^2 \\
\mathrm{c}^2_{3,1,1} = (m^2)^{1+2\epsilon}(1+t)x\epsilon^2 & \mathrm{c}^3_{4,0,1} = (m^2)^{1+2\epsilon} x^2\epsilon \\
\mathrm{c}^4_{2,1,0} = \frac{(m^2)^{-1+2\epsilon}(1+x)^2(1+x(1+t+x))\epsilon^2(-2+3\epsilon)}{2y} & \mathrm{c}^4_{3,1,0} = -\frac{(m^2)^{2\epsilon}(1+x)\epsilon}{y}(t^2x^2\epsilon + (1+x+x^2)^2\epsilon \\
\mathrm{c}^4_{4,0,0} = \frac{3(m^2)^{2\epsilon}x\epsilon}{y(-1+2\epsilon)}(2x(1+x-tx+x^2)+\epsilon & \quad +2tx(\epsilon + x(2+(-5+x)\epsilon))) \\
\quad +x(-1+t+6tx+(-1+t)x^2+x^3)\epsilon) & \mathrm{c}^5_{2,1,0} = \frac{1}{2}(m^2)^{-1+2\epsilon}(1+x)^2\epsilon^2(-2+3\epsilon) \\
\mathrm{c}^5_{3,1,0} = -(m^2)^{2\epsilon}(1+x)(1+x(1+t+x))\epsilon^2 & \mathrm{c}^5_{4,0,0} = \frac{3(m^2)^{2\epsilon}x(1+x)^2\epsilon^2}{-1+2\epsilon}
\end{array}$$

DE

$$d\vec{B} = \epsilon d A \vec{B} \qquad x = x'/(x'-1)$$

$$\frac{\partial \mathbf{A}}{\partial x'} = \begin{pmatrix} \psi_{1,1} & \psi_{1,2} & \psi_{1,3} & \psi_{1,4} & \psi_{1,5} & \psi_{1,6} \\ \psi_{2,1} & \psi_{2,2} & \psi_{2,3} & \psi_{2,4} & \psi_{2,5} & \psi_{2,6} \\ 0 & 0 & \psi_{3,3} & 0 & 0 & \psi_{3,6} \\ 0 & 0 & 0 & \psi_{4,4} & \psi_{4,5} & \psi_{4,6} \\ 0 & 0 & 0 & \psi_{5,4} & \psi_{5,5} & 0 \\ 0 & 0 & 0 & 0 & 0 & \psi_{6,6} \end{pmatrix},$$

$$\mathrm{E}_4\left(\begin{smallmatrix} n_1 & \dots & n_k \\ c_1 & \dots & c_k \end{smallmatrix}; x, \vec{a}\right) = \int_0^x dt\, \psi_{n_1}(c_1, t, \vec{a})\, \mathrm{E}_4\left(\begin{smallmatrix} n_2 & \dots & n_k \\ c_2 & \dots & c_k \end{smallmatrix}; t, \vec{a}\right)$$

$$T_{1121} = -\frac{2\left(m^2\right)^{-2\epsilon}}{\epsilon^2} \int_0^1 \frac{B_2}{(-1+x')\,x'} dx' = -2\left(m^2\right)^{-2\epsilon} \sum_{i=1}^{\infty} \epsilon^i \int_0^1 (\psi_1(1,x') - \psi_1(0,x')) B_2^{(i+2)}$$

$$
\begin{aligned}
T_{1121} = \frac{\left(m^2\right)^{-2\epsilon}}{c_4} \Big(& 2c_4 \mathrm{E}_4\left(\begin{smallmatrix} -\tilde{1} & -1 & 1 \\ 1 & \infty & 1 \end{smallmatrix}, 1\right) + 2c_4 \mathrm{E}_4\left(\begin{smallmatrix} -\tilde{1} & 1 & -1 \\ 1 & 1 & \infty \end{smallmatrix}, 1\right) + c_4 \mathrm{E}_4\left(\begin{smallmatrix} -\tilde{1} & -1 & 1 \\ 1 & 0 & 1 \end{smallmatrix}, 1\right) + c_4 \mathrm{E}_4\left(\begin{smallmatrix} -\tilde{1} & -1 & 1 \\ 1 & 1 & 1 \end{smallmatrix}, 1\right) \\
& + c_4 \mathrm{E}_4\left(\begin{smallmatrix} -\tilde{1} & 1 & -1 \\ 1 & 1 & 0 \end{smallmatrix}, 1\right) + c_4 \mathrm{E}_4\left(\begin{smallmatrix} -\tilde{1} & 1 & -1 \\ 1 & 1 & 1 \end{smallmatrix}, 1\right) - c_4 \mathrm{E}_4\left(\begin{smallmatrix} -\tilde{1} & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix}, 1\right) - \mathrm{E}_4\left(\begin{smallmatrix} -\tilde{1} & 0 & 1 \\ 1 & 0 & 1 \end{smallmatrix}, 1\right) - \mathrm{E}_4\left(\begin{smallmatrix} -\tilde{1} & 1 & 0 \\ 1 & 1 & 0 \end{smallmatrix}, 1\right) \\
& + 2c_4 \mathrm{E}_4\left(\begin{smallmatrix} -1 & -1 & 1 \\ 0 & \infty & 1 \end{smallmatrix}, 1\right) + 2c_4 \mathrm{E}_4\left(\begin{smallmatrix} -1 & 1 & -1 \\ 0 & 1 & \infty \end{smallmatrix}, 1\right) - 2c_4 \mathrm{E}_4\left(\begin{smallmatrix} -1 & 1 & 1 \\ \infty & 1 & 1 \end{smallmatrix}, 1\right) + 2c_4 \mathrm{E}_4\left(\begin{smallmatrix} 1 & -1 & -1 \\ 0 & 0 & \infty \end{smallmatrix}, 1\right) \\
& + 2c_4 \mathrm{E}_4\left(\begin{smallmatrix} 1 & -1 & -1 \\ 0 & 1 & \infty \end{smallmatrix}, 1\right) + 2c_4 \mathrm{E}_4\left(\begin{smallmatrix} 1 & -1 & 1 \\ 0 & \infty & 1 \end{smallmatrix}, 1\right) + 2c_4 \mathrm{E}_4\left(\begin{smallmatrix} 1 & 1 & -1 \\ 0 & 0 & \infty \end{smallmatrix}, 1\right) + 2c_4 \mathrm{E}_4\left(\begin{smallmatrix} 1 & 1 & -1 \\ 0 & 1 & \infty \end{smallmatrix}, 1\right) + c_4 \mathrm{E}_4\left(\begin{smallmatrix} -1 & -1 & 1 \\ 0 & 0 & 1 \end{smallmatrix}, 1\right) \\
& + c_4 \mathrm{E}_4\left(\begin{smallmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \end{smallmatrix}, 1\right) + c_4 \mathrm{E}_4\left(\begin{smallmatrix} -1 & 1 & -1 \\ 0 & 1 & 0 \end{smallmatrix}, 1\right) + c_4 \mathrm{E}_4\left(\begin{smallmatrix} -1 & 1 & -1 \\ 0 & 1 & 1 \end{smallmatrix}, 1\right) - c_4 \mathrm{E}_4\left(\begin{smallmatrix} -1 & 1 & 1 \\ 0 & 1 & 1 \end{smallmatrix}, 1\right) + c_4 \mathrm{E}_4\left(\begin{smallmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \end{smallmatrix}, 1\right) \\
& + c_4 \mathrm{E}_4\left(\begin{smallmatrix} 1 & -1 & -1 \\ 0 & 0 & 1 \end{smallmatrix}, 1\right) + c_4 \mathrm{E}_4\left(\begin{smallmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \end{smallmatrix}, 1\right) + c_4 \mathrm{E}_4\left(\begin{smallmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \end{smallmatrix}, 1\right) - c_4 \mathrm{E}_4\left(\begin{smallmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \end{smallmatrix}, 1\right) - c_4 \mathrm{E}_4\left(\begin{smallmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \end{smallmatrix}, 1\right) \\
& + c_4 \mathrm{E}_4\left(\begin{smallmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \end{smallmatrix}, 1\right) + c_4 \mathrm{E}_4\left(\begin{smallmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \end{smallmatrix}, 1\right) - c_4 \mathrm{E}_4\left(\begin{smallmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{smallmatrix}, 1\right) - \mathrm{E}_4\left(\begin{smallmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \end{smallmatrix}, 1\right) - \mathrm{E}_4\left(\begin{smallmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \end{smallmatrix}, 1\right) + \mathrm{E}_4\left(\begin{smallmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{smallmatrix}, 1\right) \\
& - \mathrm{E}_4\left(\begin{smallmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{smallmatrix}, 1\right) - \mathrm{E}_4\left(\begin{smallmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \end{smallmatrix}, 1\right) - \mathrm{E}_4\left(\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{smallmatrix}, 1\right) - \mathrm{E}_4\left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{smallmatrix}, 1\right) - \mathrm{E}_4\left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{smallmatrix}, 1\right) + \mathcal{O}(\epsilon) \Big)
\end{aligned}
$$

Analytic continuation

- The analytic continuation to the physical region is defined by the $i\delta$ Feynman prescription. We will assume that

$$I(\{s\}, i\delta) = \sum_{i=0}^{\infty} \epsilon^i \int_0^{\infty} dx\, \phi(x, \{s\}, i\delta) f^{(i+w_0)}(x, \{s\}, i\delta), \quad \phi \in \left\{ \frac{1}{y(x, \{s\}, i\delta)}, \frac{1}{x} \right\},$$

- We have seen that the sunrise(1,1,1) and the triangle(1,2,1,1) have this structure

- However the representation above is not satisfactory (e.g. slowly converging numerical evaluation)

- We will identify a set of regions in the (x,s) space and explicitly perform the limit $\delta \to 0$

Optimised functions

$$I(\{s\}) = \sum_{i=0}^{\infty} \epsilon^i \int_0^{\infty} dx \sum_{j=1}^{n_R} \theta_j(x, \{s\}) \phi_j(x, \{s\}) f_j^{(i+w_0)}(x, \{s\}),$$

$$f_j^{(i)}(x, \{s\}) = \lim_{\delta \to 0} f^{(i)}(x, \{s\}, i\delta),$$

$$\phi_j(x, \{s\}) = \lim_{\delta \to 0} \phi(x, \{s\}, i\delta), \qquad (x, \{s\}) \in R_j.$$

$$\lim_{\delta \to 0} y(x, \{s\}, i\delta) = \begin{cases} y(x, \{s\}, 0) & \text{if} \quad (x, \{s\}) \in R_j : \; y^2(x) > 0 \\ i\sqrt{-y^2(x, \{s\}, 0)} & \text{if} \quad (x, \{s\}) \in R_j : \; y^2(x) < 0. \end{cases}$$

An elementary example

- Suppose the function is

$$f = \log(a - x + i\delta), \quad x, a, \delta > 0$$

$$\text{Alphabet} = \{a - x\}$$

- The relevant regions are identified by requiring that the alphabet is non singular

$$R_1 : a - x > 0, \quad R_2 : a - x < 0$$

- In each region we find a representation of the integrand that is analytic when the regulator is exactly 0

$$f = \theta(a - x)\log(a - x) + \theta(x - a)(\log(x - a) + i\pi)$$

Analytic representations

$$f_j^{(i)}(x,\{s\}) = \lim_{\delta\to 0} f^{(i)}(x,\{s\},i\delta), \qquad (x,\{s\}) \in R_j.$$

- We look for a representation in terms of logarithms and classical polylogarithms (up to weight 3 in our case, but generalises to higher weights)

- We define an ansatz for the limit in terms of analytic functions in the region of interest (we know that the result is a pure function)

$$f_j^{(2)}(x,\{s\}) = \sum_i \rho_i \mathrm{Li}_2(a_i) + \sum_{j,k} \rho_{j,k} \log(b_j)\log(c_k) + \sum_l \sigma_l \log(d_l) + \tau$$

$$\mathrm{Li}_k(a_i) : a_i \notin [1,\infty), \quad \log(b_i) : b_i \notin (-\infty, 0]$$

- We fit the ansatz at the symbol level. Terms in the symbol kernel are fitted against the full expression.

$$\mathcal{S}[f_j^{(i)}(x,\{s\})] \equiv \mathcal{S}[f^{(i)}(x,\{s\},0)]$$

Identifying admissible regions

$$I(\{s\},i\delta)=\sum_{i=0}^{\infty}\epsilon^i\int_0^\infty dx\,\phi(x,\{s\},i\delta)f^{(i+w_0)}(x,\{s\},i\delta),\quad \phi\in\left\{\frac{1}{y(x,\{s\},i\delta)},\frac{1}{x}\right\},$$

- Alphabet of $f^{(w)}(x,\{s\},0)$

$$\alpha_i(x,s)=g(x,s),\quad \beta_j(x,s)=h(x,s)+c\,y(x,s),\quad x,s,c\in\mathbb{R}$$

- Two regions are identified by requiring that the elliptic curve has definite sign

$$\text{Region A}:\ y(x)^2<0,\quad \text{Region B}:\ y(x)^2>0$$

- We further partition Region A and B by requiring that the letters are non-singular

- Note that in Region A, beta letters are non singular

$$\begin{aligned}&\text{Region A}_0:\ \{\alpha_1>0,\alpha_2>0,\dots\}\\&\text{Region A}_1:\ \{\alpha_1<0,\alpha_2>0,\dots\}\\&\qquad\vdots\\&\text{Region A}_{n_A}:\ \{\alpha_1<0,\alpha_2<0,\dots\}\end{aligned}$$

$$\begin{aligned}&\text{Region B}_0:\ \{\alpha_1>0,\alpha_2>0,\dots\ \beta_1>0,\beta_2>0,\dots\}\\&\text{Region B}_1:\ \{\alpha_1<0,\alpha_2>0,\dots\ \beta_1>0,\beta_2>0,\dots\}\\&\qquad\vdots\\&\text{Region B}_{n_B}:\ \{\alpha_1<0,\alpha_2<0,\dots\ \beta_1<0,\beta_2<0,\dots\}\end{aligned}$$

The triangle with bubble

- The alphabet, with $m^2 y(x) = \sqrt{\left(sx - m^2\left(x^2+3x+1\right)\right)^2 - 4m^4 x(x+1)^2}$, reads

$$l_1 = x,\ l_2 = x+1,\ l_3 = m^2,\ l_4 = m^2 - s,\ l_5 = s,\ l_6 = -m^2\left(x^2+x+1\right) + sx + m^2 y(x),$$

$$l_7 = -m^2\left(x^2+3x+1\right) + sx + m^2 y(x),\ \ l_8 = -m^2\left(-x^2-x+1\right) - sx + m^2 y(x)$$

- $y(x)^2 < 0$

$$\text{Region A}_0 :\ \{l_4 < 0\}, \quad \text{Region A}_1 :\ \{l_4 > 0\}$$

- $y(x)^2 > 0$

$$\text{Region B}_0 :\ \{l_4 < 0, l_6 < 0, l_7 < 0, l_8 < 0\} \qquad \text{Region B}_2 :\ \{l_4 < 0, l_6 > 0, l_7 > 0, l_8 < 0\}$$

$$\text{Region B}_1 :\ \{l_4 < 0, l_6 < 0, l_7 < 0, l_8 > 0\} \qquad \text{Region B}_3 :\ \{l_4 > 0, l_6 < 0, l_7 < 0, l_8 > 0\}$$

- A posteriori, we find that only (at most) three regions are required

$$R_1 = A_0 \cup A_1, \quad R_2 = B_0 \cup B_1 \cup B_3, \quad R_3 = B_2$$

Result

$$T^{(0)}_{1,2,1,1} = \int_0^\infty \frac{1}{x} \sum_{i=1}^{3} \theta_{R_i}\; g^{(2)}_{R_i}$$

- E.g., we have the following manifestly real valued expression, $y(x)^2 > 0$

$$\begin{aligned}
g^{(2)}_{R_2} = & -\mathrm{Li}_2\left(\frac{\left(-x^2+x+1\right)m^2 - sx + m^2\sqrt{y(x)^2}}{2m^2(x+1)}\right) + \mathrm{Li}_2\left(\frac{-\left(x^2+x+1\right)m^2 + s(x+2) + m^2\sqrt{y(x)^2}}{2s(x+1)}\right) \\
& +\mathrm{Li}_2\left(-\frac{\left(x^2+x+1\right)m^2 - s(x+2) + m^2\sqrt{y(x)^2}}{2s(x+1)}\right) - \mathrm{Li}_2\left(-\frac{((x-1)x-1)m^2 + sx + m^2\sqrt{y(x)^2}}{2m^2(x+1)}\right) \\
& +\mathrm{Li}_2\left(1-\frac{s}{m^2}\right) + \log^2\left(m^2\left(x^2+x+1\right) - sx - m^2\sqrt{y(x)^2}\right) + \frac{1}{2}\log^2\left(m^2\right) + \log(2)\log\left(m^2\right) \\
& -\log(s)\log\left(m^2\left(x^2+x+1\right) - sx - m^2\sqrt{y(x)^2}\right) - 2\log(x)\log\left(m^2\left(x^2+x+1\right) - sx - m^2\sqrt{y(x)^2}\right) \\
& -2\log(2)\log\left(m^2\left(x^2+x+1\right) - sx - m^2\sqrt{y(x)^2}\right) + 2\log\left(m^2\right)\log(x) - \log\left(m^2\right)\log(x+1) \\
& -\log\left(m^2\right)\log\left(m^2\left(x^2+x+1\right) - sx - m^2\sqrt{y(x)^2}\right) + \log(s)\log(x+1) + \frac{\log^2(s)}{2} + \log(2)\log(s) \\
& +2\log(2)\log(x) + \log^2(2)
\end{aligned}$$

[Hidding, FM] to appear

- Fast numerical evaluation: O(10^3) points in O(1 minute)

R3

$$y(x)^2 > 0$$

$$
\begin{aligned}
g_{R_3}^{(2)} = \; & -\mathrm{Li}_2\left(\frac{\left(-x^2+x+1\right)m^2 - sx + m^2\sqrt{y(x)^2}}{2m^2(x+1)}\right) + \mathrm{Li}_2\left(\frac{-\left(x^2+x+1\right)m^2 + s(x+2) + m^2\sqrt{y(x)^2}}{2s(x+1)}\right) \\
& + \mathrm{Li}_2\left(-\frac{\left(x^2+x+1\right)m^2 - s(x+2) + m^2\sqrt{y(x)^2}}{2s(x+1)}\right) - \mathrm{Li}_2\left(-\frac{((x-1)x-1)m^2 + sx + m^2\sqrt{y(x)^2}}{2m^2(x+1)}\right) \\
& + \mathrm{Li}_2\left(1-\frac{s}{m^2}\right) + \log^2\left(-m^2\left(x^2+x+1\right) + sx + m^2\sqrt{y(x)^2}\right) + 2\log\left(m^2\right)\log(x) + \log(s)\log(x+1) \\
& - \log(s)\log\left(-m^2\left(x^2+x+1\right) + sx + m^2\sqrt{y(x)^2}\right) - 2\log(x)\log\left(-m^2\left(x^2+x+1\right) + sx + m^2\sqrt{y(x)^2}\right) \\
& - 2\log(2)\log\left(-m^2\left(x^2+x+1\right) + sx + m^2\sqrt{y(x)^2}\right) - 2i\pi\log\left(-m^2\left(x^2+x+1\right) + sx + m^2\sqrt{y(x)^2}\right) \\
& - \log\left(m^2\right)\log\left(-m^2\left(x^2+x+1\right) + sx + m^2\sqrt{y(x)^2}\right) - \log\left(m^2\right)\log(x+1) + \log(2)\log\left(m^2\right) \\
& + \frac{1}{2}\log^2\left(m^2\right) + i\pi\log\left(m^2\right) + \frac{\log^2(s)}{2} + \log(2)\log(s) + i\pi\log(s) + 2\log(2)\log(x) + 2i\pi\log(x) \\
& + \log^2(2) + 2i\pi\log(2) - \pi^2
\end{aligned}
$$

- Explicit imaginary parts

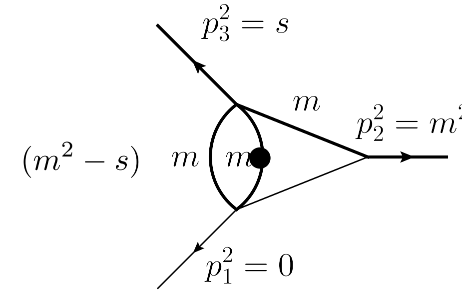
R1

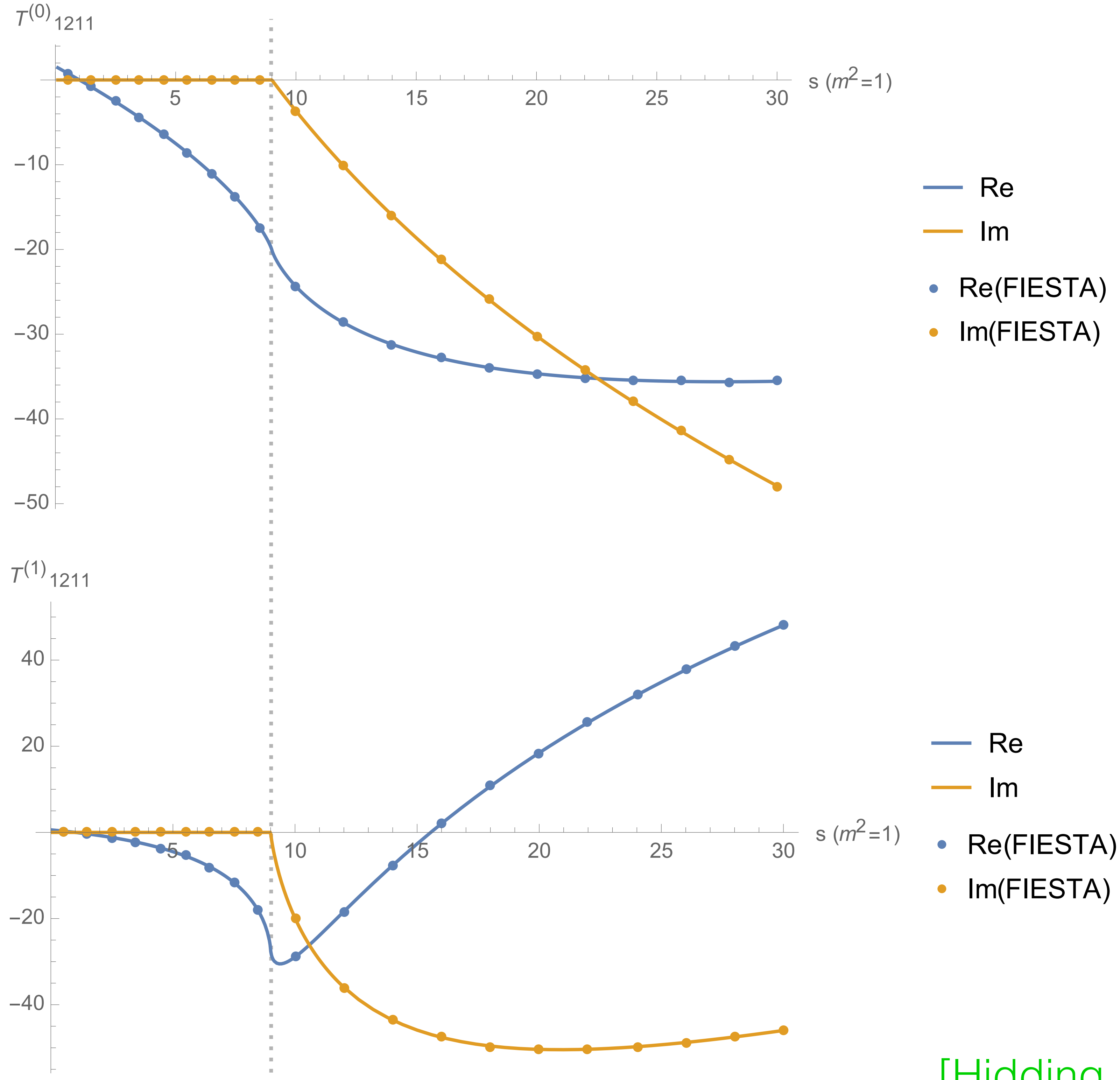
$$y(x)^2 < 0$$

$$
\begin{aligned}
g_{R_1}^{(2)} = \; & -\mathrm{Li}_2\left(\frac{\left(-x^2+x+1\right)m^2 - sx + i\,m^2\sqrt{-y(x)^2}}{2m^2(x+1)}\right) + \mathrm{Li}_2\left(\frac{-\left(x^2+x+1\right)m^2 + s(x+2) + i\,m^2\sqrt{-y(x)^2}}{2s(x+1)}\right) \\
& +\mathrm{Li}_2\left(-\frac{\left(x^2+x+1\right)m^2 - s(x+2) + i\,m^2\sqrt{-y(x)^2}}{2s(x+1)}\right) - \mathrm{Li}_2\left(-\frac{((x-1)x-1)m^2 + sx + i\,m^2\sqrt{-y(x)^2}}{2m^2(x+1)}\right) \\
& +\mathrm{Li}_2\left(1-\frac{s}{m^2}\right) + \log^2\left(-m^2\left(x^2+x+1\right) + sx + i\,m^2\sqrt{-y(x)^2}\right) + 2\log\left(m^2\right)\log(x) + \log(s)\log(x+1) \\
& -\log(s)\log\left(-m^2\left(x^2+x+1\right) + sx + i\,m^2\sqrt{-y(x)^2}\right) - 2\log(x)\log\left(-m^2\left(x^2+x+1\right) + sx + i\,m^2\sqrt{-y(x)^2}\right) \\
& -2\log(2)\log\left(-m^2\left(x^2+x+1\right) + sx + i\,m^2\sqrt{-y(x)^2}\right) - 2i\pi\log\left(-m^2\left(x^2+x+1\right) + sx + i\,m^2\sqrt{-y(x)^2}\right) \\
& -\log\left(m^2\right)\log\left(-m^2\left(x^2+x+1\right) + sx + i\,m^2\sqrt{-y(x)^2}\right) - \log\left(m^2\right)\log(x+1) + \log(2)\log\left(m^2\right) \\
& +\frac{1}{2}\log^2\left(m^2\right) + i\pi\log\left(m^2\right) + \frac{\log^2(s)}{2} + \log(2)\log(s) + i\pi\log(s) + 2\log(2)\log(x) + 2i\pi\log(x) \\
& +\log^2(2) + 2i\pi\log(2) - \pi^2
\end{aligned}
$$

- Analytic expression, complex valued functions

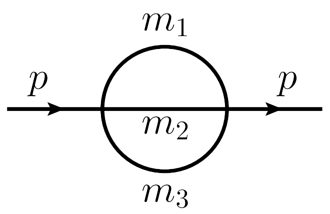
$$T_{1211}(s,m^2) = \sum_{i=0}^{\infty} \epsilon^i \int_0^{\infty} dx \, \frac{1}{x} UT^{(i+2)}(x)$$

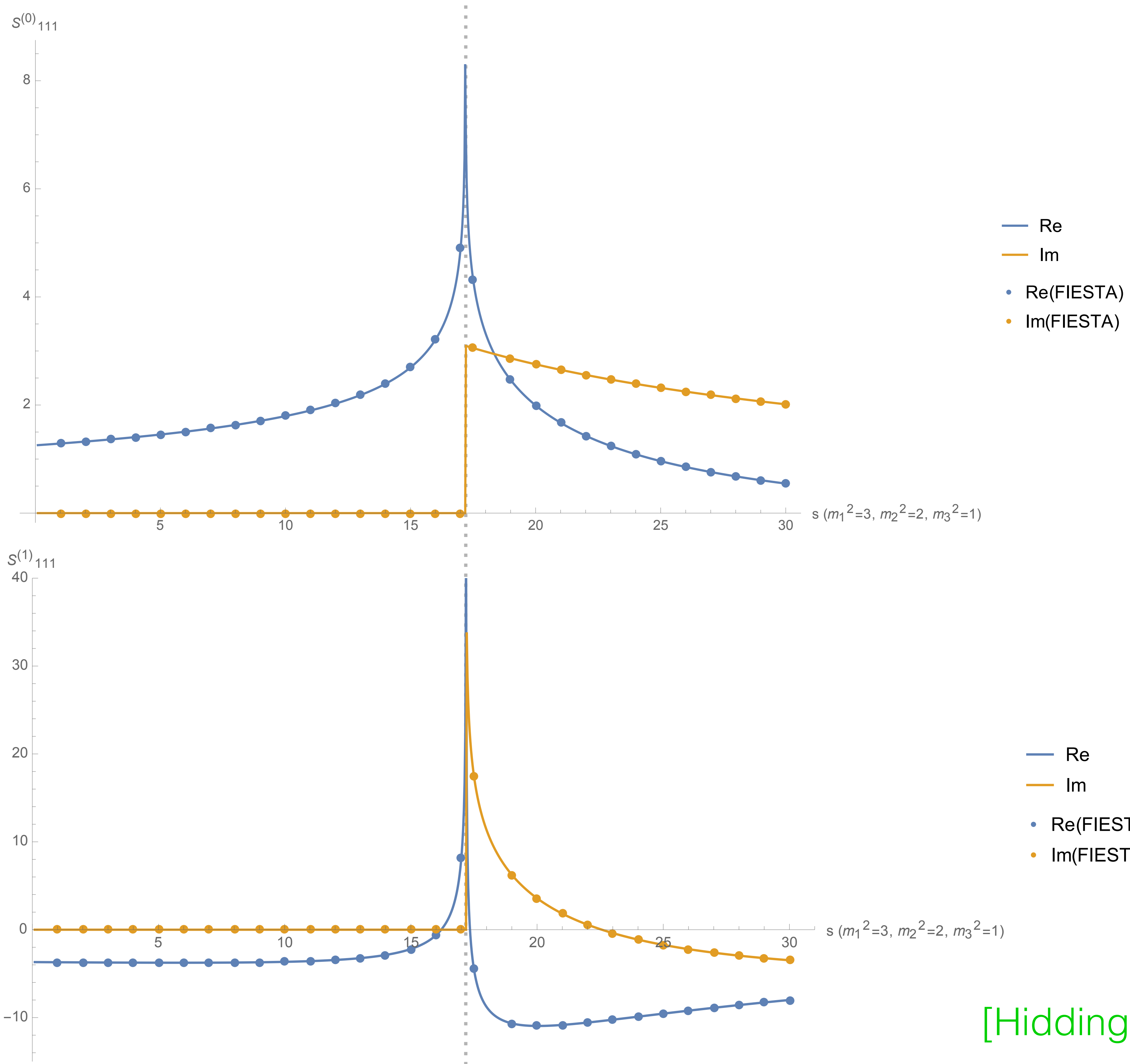




[Hidding, FM] to appear

$$S_{111}(p^2, m_1^2, m_2^2, m_3^2) = \sum_{i=0}^{\infty} \epsilon^i \int_0^\infty d\alpha_3 \, \frac{1}{m_3^2 y(x)} UT^{(i+1)}$$



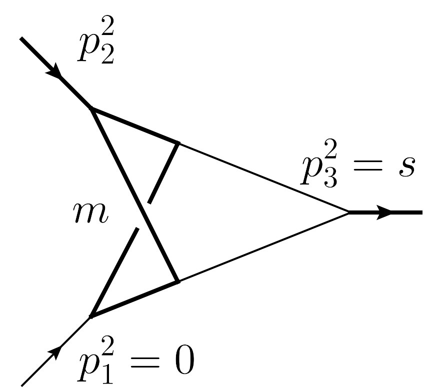


[Hidding, FM] to appear

linear reducibility and eMPL

- Linear reducibility does not imply eMPL. An example from Higgs+Jet:

$(1+2\epsilon)(p_2^2 - s)$



$$= \sum_{i=0}^{\infty} \epsilon^i \int_0^{\infty} dx \frac{1}{y(x)} \mathrm{UT}^{(i+3)}(x, s, p_2^2, m^2)\,,$$

[Hidding, FM] arXiv:1712.04441

- The IPP depends on the elliptic curve and two algebraic functions. It is not clear how/if an eMPL representation exists

$$y(x) = \sqrt{(x+1)^4 m^4 - 2x\,(x+1)^2 m^2\,(xp_2^2 + p_2^2 - 2xs - s) + x^2\,(xp_2^2 + p_2^2 - s)^2},$$

$$\rho_1(x) = \sqrt{(xp_2^2 + p_2^2 - s)^2 - 4\,(x+1)^2 m^2\,(p_2^2 - s)}\,,$$

$$\rho_2(x) = \sqrt{x^2\,(p_2^4 - 4m^2 s) + 2xs\,(p_2^2 - 4m^2) + s\,(s - 4m^2)}$$

Conclusions and outlook

- We have defined **Linearly reducible elliptic Feynman integrals**
- These integrals can be solved to all epsilon orders in terms of one-integrals over an elliptic curve and an inner polylogarithmic part (IPP) by direct parametric integration. Analytic continuation can be performed by using well established techniques from polylogarithmic symbol calculus
- We can use canonical DE + IBP reductions to solve them in a fully algebraic way in terms of eMPL and/or a basis of iterated integrals over higher poles.
- **New results:** computation of a two loop triangle topology relevant for heavy quark pair production in terms of eMPL, analytic continuation thereof and of the unequal masses sunrise.
- **Outlook:** Further exploration is required to understand how these methods apply to integrals depending on multiple algebraic curves.

Thanks !

Pure functions

- Choose an integral basis that evaluates to pure functions (canonical basis)

[Henn] arXiv:1304.1806

- Pure functions: rational linear combinations of polylogarithms of the same weight

$$\frac{1}{2}\mathrm{Li}_3(x) + \pi\,\mathrm{Li}_2\left(\frac{1}{x}\right) + \pi^3$$

- Once this basis has been found the integrals can be algorithmically computed with the differential equations method

$$d\vec{f}(x,\epsilon) = \epsilon\, dA \vec{f}(x,\epsilon) \qquad \vec{f}(x,\epsilon) = \mathbb{P}\exp\left(\epsilon\int_{x_0}^{x}\frac{\partial A}{\partial x}\,dx\right)\vec{f}(x_0,\epsilon)$$

$$\vec{f}(x,\epsilon) = \vec{f}^{(0)}(x_0) + \epsilon\left(\int_{x_0}^{x}\frac{\partial A(t_1)}{\partial t_1}\vec{f}^{(0)}(x_0)\;dt_1 + \vec{f}^{(1)}(x_0)\right) + \epsilon^2\left(\int_{x_0}^{x}\int_{x_0}^{\mathrm{t1}}\frac{\partial A(t_1)}{\partial t_1}\frac{\partial A(t_2)}{\partial t_2}\vec{f}^{(0)}(x_0)\,dt_2 dt_1 + \int_{x_0}^{x}\frac{\partial A(t_1)}{\partial t_1}\vec{f}^{(1)}(x_0)\;dt_1 + \vec{f}^{(2)}(x_0)\right) + \cdots$$

Choosing functions arguments

- The ansatz has to satisfy

$$g(R) = \sum_i \rho_i \mathrm{Li}_2(a_i) + \rho_{j,k} \sum_{j,k} \log(b_i)\log(c_j) + \sum_l \sigma_l \log(d_l) + \tau$$

$$\mathcal{S}(UT^{(k)}(x, y(x), i\delta)) \stackrel{!}{=} \mathcal{S}(g(R))$$

- This requires that candidate functions have the same symbol alphabet as

$$\mathrm{Alphabet}(UT^{(k)}(x, y(x), i\delta)) = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$$

$$\mathcal{S}[\mathrm{Li}_k(a_i)] = -(1-a_i) \otimes a_i \otimes \cdots \otimes a_i, \quad \rightarrow \quad a_i = \prod_{i=1}^n \alpha_i^{m_i}, \quad 1 - \alpha_i = \prod_{i=1}^n \alpha_i^{m_i'}, \quad (m_i, m_i' \in \mathbb{N})$$

$$\mathcal{S}[\log(b_i)] = \otimes b_i \rightarrow \quad b_i = \prod_{i=1}^n \alpha_i^{m_i}, \quad (m_i, \in \mathbb{N})$$

[Duhr, Gangl, Rhodes] arXiv:1110.0458
[Bonciani, Del Duca, Frellesvig, Henn, FM, Smirnov] arXiv:1609.06685
[von Manteuffel, Tancredi] arXiv:1701.05905

- Require that the functions are analytic in R

$$\mathrm{Li}_k(a_i) : a_i \notin [1, \infty), \quad \log(b_i) : b_i \notin (-\infty, 0]$$

Fixing the ansatz

$$g(R) = \sum_i \rho_i \mathrm{Li}_2(a_i) + \rho_{j,k} \sum_{j,k} \log(b_i)\log(c_j) + \sum_l \sigma_l \log(d_l) + \tau$$

- We fix the first part of the ansatz for g(R) by requiring that the symbol matches the symbol of the IPP

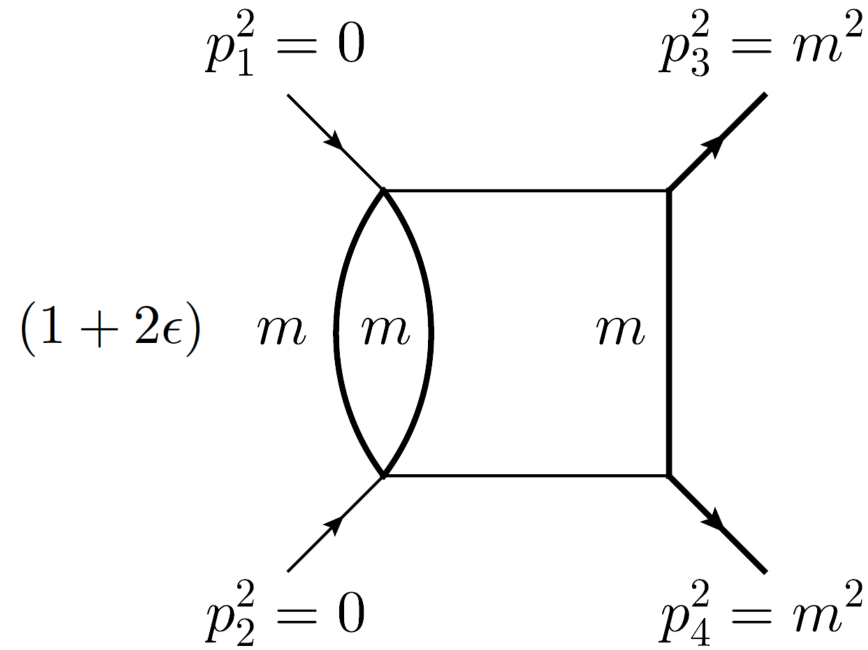
$$\mathcal{S}(UT^{(k)}(x, y(x), i\delta)) = \mathcal{S}(g(R))$$

- Terms in the symbol kernel are fixed by sampling the full result in several phase space points and matching it to the ansatz (no coproduct required)

$$UT^{(k)}(x, y(x), i\delta)) = g(R)$$

- Since by construction g(R) is analytic, we can safely set the Feynman regulator to zero.

Linear reducibility and eMPL



[Hidding, FM] arXiv:1712.04441

- Also this integral is linearly reducible
- However it depends on two elliptic curves, its maximal cut and the sunrise subtopology

Boundary conditions

$$B_1 = 2(m_3^2)^{2\epsilon} x \epsilon S^{\mathrm{IPP}}_{2,0,0,0,0}\,, \quad B_2 = 2(m_3^2)^{2\epsilon}(1+x)\epsilon^2 S^{\mathrm{IPP}}_{1,1,0,0,0}\,, \quad B_3 = (m_3^2)^{2\epsilon+1} y(x) \epsilon S^{\mathrm{IPP}}_{2,1,0,0,0}\,,$$

$$S^{\mathrm{IPP}}_{\nu_1,\nu_2,\mu_1,\mu_2,\mu_3} = \frac{1}{\Gamma(3-d)} \int \frac{d^d k_1 d^d k_2}{(i\pi^{d/2})^2} \frac{1}{(D_1 + x D_3)^{\nu_1} D_2^{\nu_2} N_1^{\mu_1} N_2^{\mu_2} N_3^{\mu_3}}$$

$$\lim_{x\to 0} \left(\vec{B}(x, s, m_1, m_2, m_3) \right) = \lim_{x \to 0} \left(\vec{B}(x, 0, m_1, m_2, m_3) \right)$$

$$B_1(x', 0, m_1, m_2, m_3) = ((1-x')x')^{\epsilon} \left(\frac{m_3^2}{m_1^2(-x') + m_3^2 x' + m_1^2} \right)^{2\epsilon}$$

$$B_2(x', 0, m_1, m_2, m_3) = \epsilon \left(m_3^2\right)^{2\epsilon} (-(x'-1)x')^{\epsilon} \left(m_3^2 x' - m_1^2(x'-1)\right)^{-2\epsilon}$$
$$\left(\frac{\pi^{3/2} 2^{1-2\epsilon} \csc(\pi\epsilon) \left(-\frac{m_2^2(x'-1)x'\left(x'\left(m_2^2(x'-1)+m_3^2\right) - m_1^2(x'-1)\right)}{\left(m_1^2(x'-1)-m_3^2 x'\right)^2} \right)^{-\epsilon}}{\Gamma(1-\epsilon)\Gamma\left(\epsilon + \frac{1}{2}\right)} - \frac{{}_2F_1\left(1, 2\epsilon; \epsilon+1; \frac{(x'-1)x' m_2^2}{(x'-1)m_1^2 - x' m_3^2}\right)}{\epsilon} \right)$$

$$B_3(x', 0, m_1, m_2, m_3) = m_2^2 \epsilon \sqrt{\frac{\left(m_1^2(-x') + x'\left(m_2^2(x'-1) + m_3^2\right) + m_1^2\right)^2}{m_2^4}} \left(m_3^2\right)^{2\epsilon} (-(x'-1)x')^{\epsilon}$$
$$\left(m_3^2 x' - m_1^2(x'-1)\right)^{-2\epsilon-1} \left(\frac{\pi^{3/2} 4^{-\epsilon} \csc(\pi\epsilon) \left(\frac{m_2^2(x'-1)x'}{m_1^2(x'-1) - m_3^2 x'} \right)^{-\epsilon} \left(1 - \frac{m_2^2(x'-1)x'}{m_1^2(x'-1)-m_3^2 x'}\right)^{-\epsilon-1}}{\Gamma(1-\epsilon)\Gamma\left(\epsilon+\frac{1}{2}\right)} \right.$$
$$\left. - \frac{{}_2F_1\left(1, 2\epsilon+1; \epsilon+1; \frac{(x'-1)x' m_2^2}{(x'-1)m_1^2 - x' m_3^2}\right)}{\epsilon} \right)$$

Triangle kernels

$$t = -s/m^2$$

$$y^2 = 1 + x\left(2 + 2t + 3x + t(6+t)x + 2(1+t)x^2 + x^3\right)$$

$$\psi_{1,1} = \psi_1\left(1+\tfrac{1}{t}, x'\right) - \psi_1(0,x')$$
$$\psi_{1,2} = 2\psi_1(0,x') - 4\psi_1(1,x') + 2\psi_1\left(1+\tfrac{1}{t}, x'\right)$$
$$\psi_{1,3} = 6\psi_1(0,x') - 6\psi_1(1,x')$$
$$\psi_{1,4} = -\psi_{-1}(1,x') - \tfrac{1}{2}\psi_{-1}(\infty,x') + \frac{(t-1)\psi_0\left(0,x'\right)}{2c_4(t+1)}$$
$$\tfrac{1}{2}\left(\psi_{-1}(0,x') + \psi_1(0,x')\right)$$
$$\psi_{1,5} = \psi_1(1,x') - \tfrac{1}{2}\psi_1(0,x')$$
$$\psi_{1,6} = \tfrac{1}{2}\psi_1(0,x') - \psi_1(1,x')$$
$$\psi_{2,1} = \psi_1(0,x') - 2\psi_1(1,x') + \psi_1\left(1+\tfrac{1}{t}, x'\right)$$
$$\psi_{2,2} = 2\psi_1\left(1+\tfrac{1}{t}, x'\right) - 2\psi_1(1,x')$$
$$\psi_{2,3} = 6\psi_1(1,x') - 6\psi_1(0,x')$$
$$\psi_{2,4} = \tfrac{1}{2}\left(-\psi_{-1}(0,x') - \psi_1(0,x')\right) - \tfrac{1}{2}\psi_{-1}(1,x')$$
$$\psi_{2,5} = \tfrac{1}{2}\psi_1(0,x') - \tfrac{1}{2}\psi_1(1,x')$$

$$\psi_{2,6} = \tfrac{1}{2}\psi_1(1,x') - \tfrac{1}{2}\psi_1(0,x')$$
$$\psi_{3,3} = 2\psi_1(0,x') - 2\psi_1(1,x')$$
$$\psi_{3,6} = -\tfrac{1}{3}\psi_1(1,x')$$
$$\psi_{4,4} = 3\psi_1(0,x') + 3\psi_1(1,x') - 2\psi_1(a_1,x')$$
$$-2\psi_1(a_2,x') - 2\psi_1(a_3,x') - 2\psi_1(a_4,x')$$
$$\psi_{4,5} = -3\psi_{-1}(1,x') - 3\left(\psi_{-1}(0,x') + \psi_1(0,x')\right)$$
$$\psi_{4,6} = \psi_{-1}(0,x') + \psi_{-1}(1,x') - 4\psi_{-1}(\infty,x')$$
$$\frac{2\psi_0\left(0,x'\right)}{c_4} + \psi_1(0,x')$$
$$\psi_{5,4} = \psi_{-1}(0,x') + \psi_{-1}(1,x') + \psi_1(0,x')$$
$$\psi_{5,5} = -\psi_1(0,x') - \psi_1(1,x')$$
$$\psi_{6,6} = \psi_1(0,x') + \psi_1(1,x')$$

eMPL kernels

$$y(x)^2 = (x-a_1)(x-a_2)(x-a_3)(x-a_4)$$

$$a_{ij} = a_i - a_j$$

$$\psi_{-\tilde{1}}(1,x) = \frac{y(1)}{(x-1)y} - \frac{1}{(x-1)} \qquad c_4 = \frac{1}{2}\sqrt{a_{13}a_{24}}$$

$$\boxed{\begin{aligned}
\psi_0(0,x) &= \frac{c_4}{y}\,, \\
\psi_1(c,x) &= \frac{1}{x-c}\,, \qquad \psi_{-1}(c,x) = \frac{y_c}{y(x-c)}\,, \\
\psi_1(\infty,x) &= \frac{c_4}{y}\,Z_4(x)\,, \qquad \psi_{-1}(\infty,x) = \frac{x}{y}\,, \\
\psi_{-n}(\infty,x) &= \frac{x}{y}\,Z_4^{(n-1)}(x) - \frac{\delta_{n2}}{c_4}\,, \\
\psi_n(c,x) &= \frac{1}{x-c}\,Z_4^{(n-1)}(x) - \delta_{n2}\,\Phi_4(x)\,, \\
\psi_n(\infty,x) &= \frac{c_4}{y}\,Z_4^{(n)}(x)\,, \qquad \psi_{-n}(c,x) = \frac{y_c}{y(x-c)}\,Z_4^{(n-1)}(x)\,,
\end{aligned}}$$

The Higgs+jet case

- High transverse Higgs momentum (p_T) distribution has been shown to be very sensitive to new physics effects in many BSM models
  [Grojean, Salvioni, Schlaffer, Weiler] arXiv:1312.3317
  [Azatov, Paul] arXiv:1309.5273
  [Azatov, Grojean, Paul, Salvioni] arXiv:1608.00977
- So far known analytically at NLO only in the large top mass limit
  [Gehrmann, Jaquier, Glover, Koukoutsakis] arXiv:1112.3554
  [Boughezal, Caola, Melnikov, Petriello, Schulze] arXiv:1504.07922
- The two loop amplitudes required for the NLO are known analytically only in the planar case
  [Bonciani, Del Duca, Frellesvig, Henn, FM, Smirnov] arXiv:1609.06685
- While the large top mass expansion rapidly diverges for scales greater than m_t (173 GeV).
  [Harlander, Neumann, Ozeren, Wiesemann] arXiv:1206.0157
  [Neumann, Williams] arXiv:1609.00367
- E.g. In order to study high p_t distribution we need full top mass dependence
- The NLO full top mass dependence is known only numerically.
  [Jones, Kerner, Luisoni] arXiv:1802.00349