Ascona Workshop: Elliptic functions

in Mathematics and Physics

String amplitudes, elliptic multiple zeta values and single-valued maps

Oliver Schlotterer (AEI Potsdam & Perimeter Institute)

based on arXiv:1803.00527 with J. Brödel & F. Zerbini

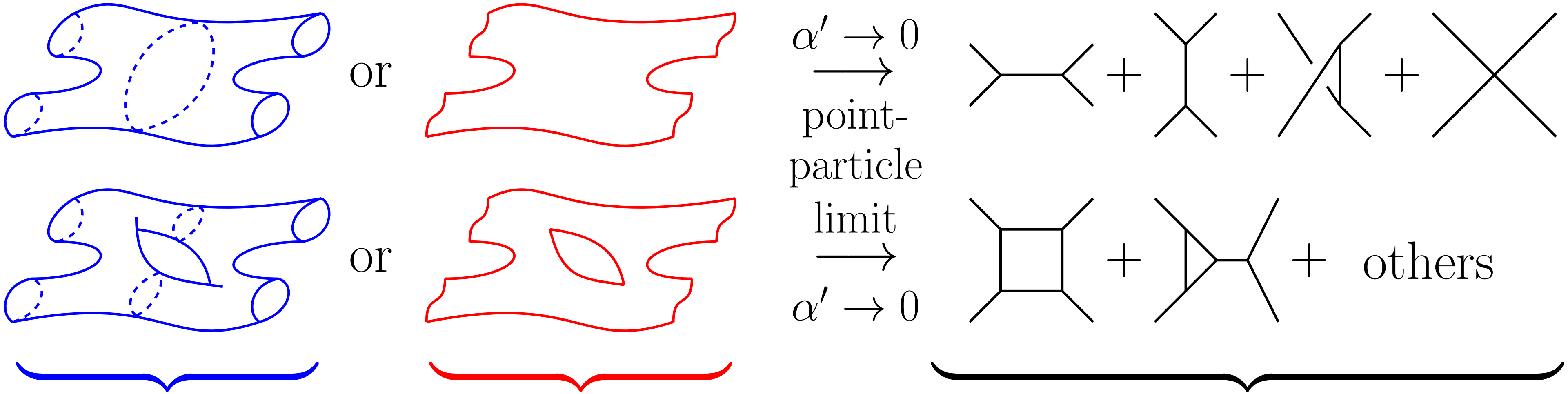
arXiv:18xy.abcde with J. Gerken & A. Kleinschmidt

07.09.2018

Intro I – string perturbation theory

String amplitudes $\longleftrightarrow$ Riemann surfaces as "fattened" Feynman diag's

loop order in perturbation theory = genus of the Riemann surface



closed-string states:
external gravitons

open-string states:
non-abelian gauge bosons

convenient organization of loop integrand
"gravity = (gauge theory)^2" (BCJ)

This talk: Study corrections to field theory ∼ inverse string tension α'

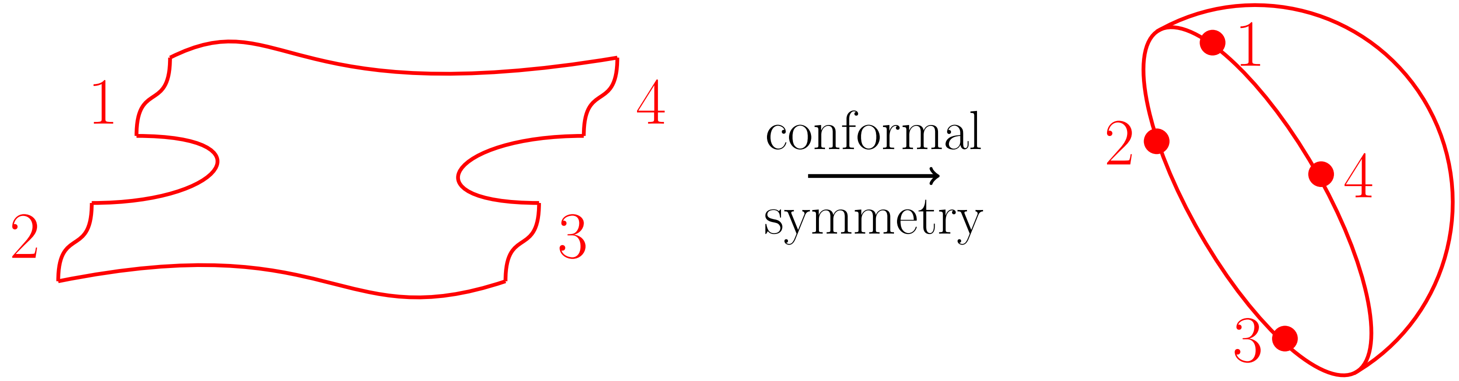
$\Longrightarrow$ rewarding laboratory for multiple zeta values,

elliptic generalizations & their single-valued projection

Intro I – string perturbation theory

Map external states to punctures • on the Riemann surface, e.g.

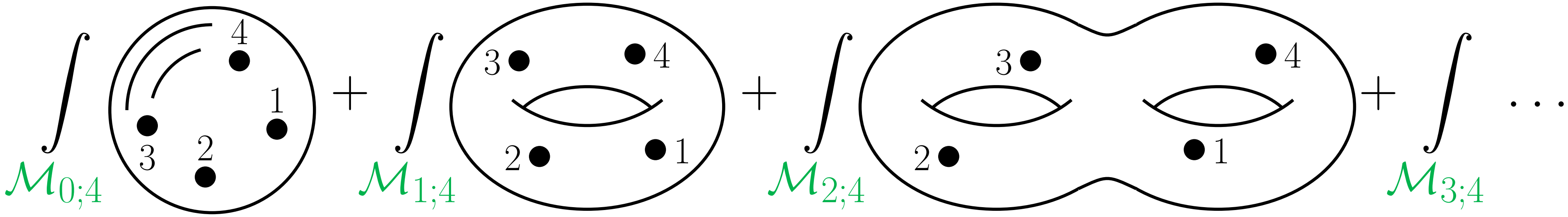
open strings
at tree level:



$\xrightarrow[\text{symmetry}]{\text{conformal}}$

String amplitudes (n points, g loop) ↔ integrals over moduli spaces $\mathcal{M}_{g;n}$

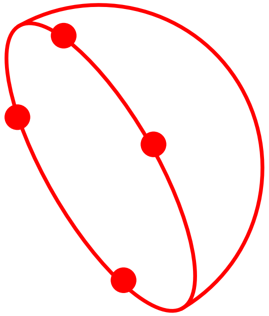
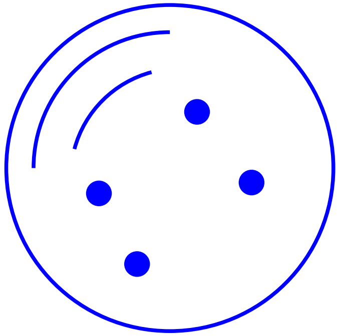
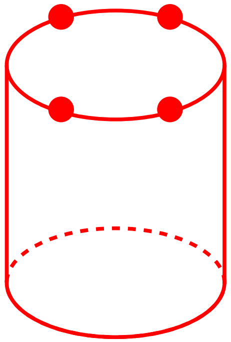
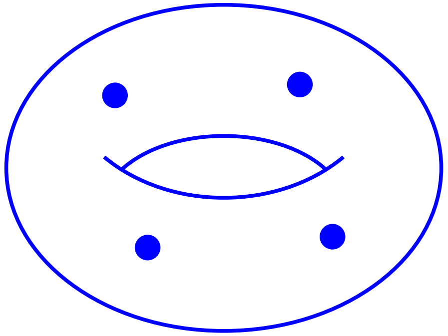
of n-punctured Riemann surfaces of genus g,



$$\int_{\mathcal{M}_{0;4}} + \int_{\mathcal{M}_{1;4}} + \int_{\mathcal{M}_{2;4}} + \int_{\mathcal{M}_{3;4}} \ldots$$

α'-expansions ↔ generating series for (large classes of) periods of $\mathcal{M}_{g;n}$.

Intro II – periods of moduli spaces at genus 0 & 1

	open strings	closed strings
tree level	disk 	sphere 
one loop	cylinder Möbius strip 	torus 

Intro II – periods of moduli spaces at genus 0 & 1

	open strings	closed strings
tree level	disk ⇒ multiple zeta values (MZVs) = polylog's at $z = 1$	sphere ⇒ single-valued MZVs = single-valued polylog's at $z = 1$
one loop	cylinder / Möbius strip ⇒ elliptic MZVs	torus $\frac{\mathbb{C}}{\mathbb{Z}+\tau\mathbb{Z}}$ ⇒ modular graph forms (modular covariant fct's of τ)

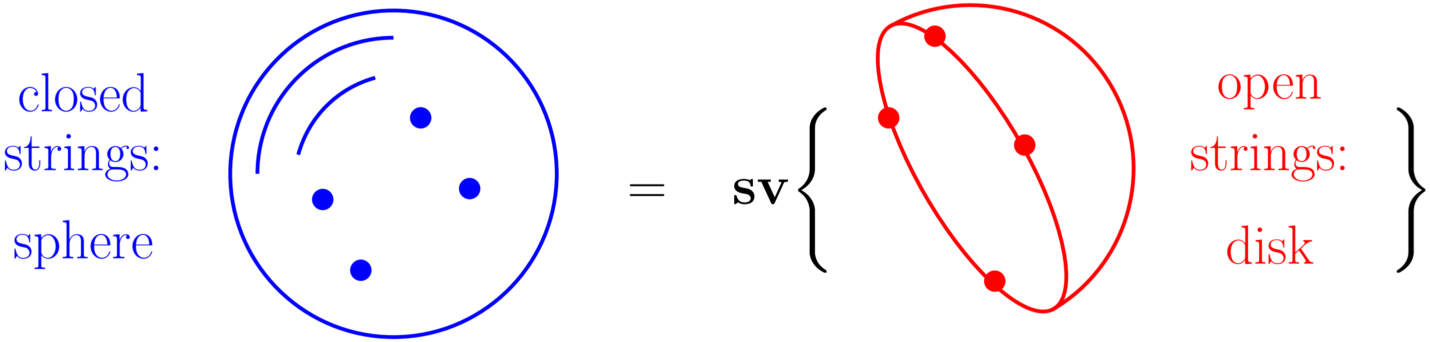
[Brödel, Mafra, Matthes, Richter, OS 2014 - 2017]

[D'Hoker, Green, Vanhove et al. 1999 - 2017]

Intro II – periods of moduli spaces at genus 0 & 1

At tree level, can obtain closed-string α'-expansions

from single-valued projection "**sv**" of open-string α'-expansions



closed strings: sphere $=$ **sv** $\Big\{$ open strings: disk $\Big\}$

[Brown, Dupont, OS, Schnetz, Stieberger, Taylor]

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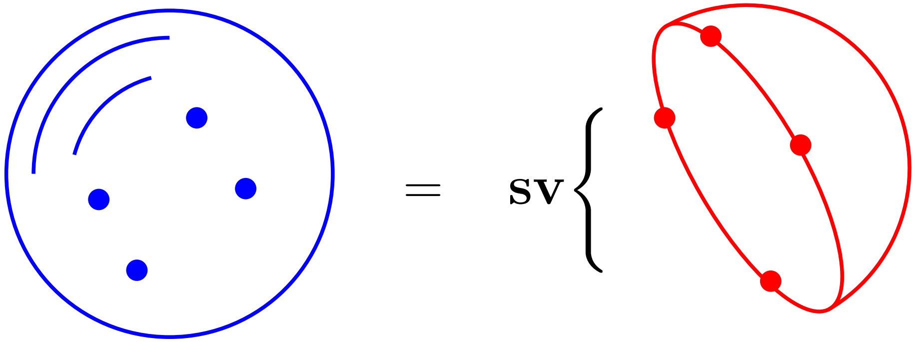
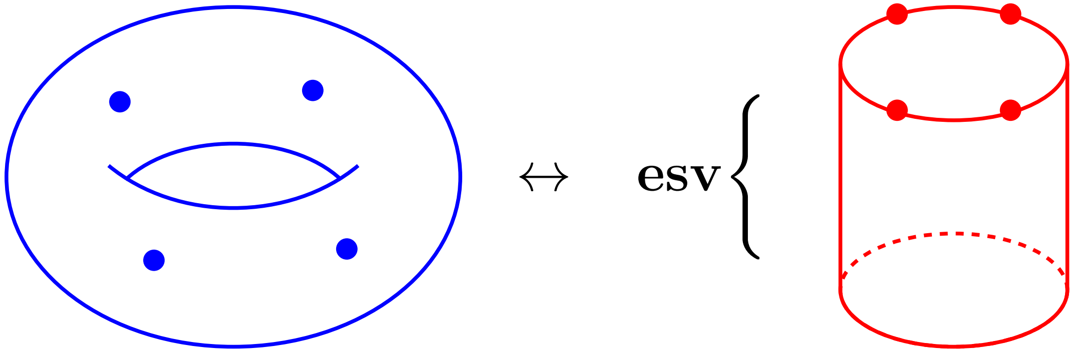


closed strings: sphere $=$ **sv** $\Big\{$ open strings: disk $\Big\}$

[Brown, Dupont, OS, Schnetz, Stieberger, Taylor]

This talk: Empirically obtain elliptic single-valued projection "**esv**"

by comparing closed-string α'-expansions ↔ open-string α'-expansions



closed strings: torus ↔ **esv** $\Big\{$ open strings: cylinder $\Big\}$

[Brödel, Gerken, Kleinschmidt, OS, Zerbini]

Outline

I. The single-valued map at tree level

[Brown, Dupont, OS, Schnetz, Stieberger, Taylor]

II. Elliptic MZVs and open strings at one loop

[Brödel, Mafra, Matthes, OS 1412.5535 & Brödel, Matthes, OS 1507.02254]

III. From closed strings to an elliptic single-valued map

[Brödel, OS, Zerbini 1803.00527]

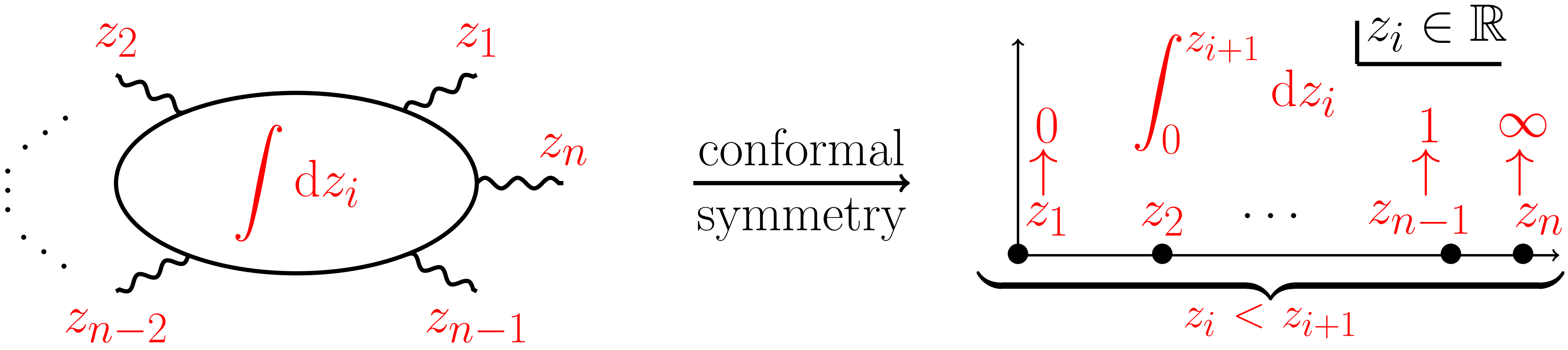
IV. The elliptic single-valued map and heterotic strings

[Gerken, Kleinschmidt, OS: in progress]

V. Conclusions & Outlook

I. The single-valued map at tree level

I. 1 Four open strings on the disk



Veneziano amplitude 1968 (4pt tree level, massless open-string states)

involving dim'less Mandestam invariants $s_{ij} := 2\alpha' k_i \cdot k_j$

$$Z_{\text{4-pt}} = \int_{0=z_1}^{1=z_3} \frac{\mathrm{d}z_2}{z_2} \, z_2^{s_{12}} \, (1-z_2)^{s_{23}} = \frac{\Gamma(s_{12})\,\Gamma(1+s_{23})}{\Gamma(1+s_{12}+s_{23})}$$

$$= \frac{1}{s_{12}} \exp\left(\sum_{n=2}^{\infty} \frac{\zeta_n}{n} (-1)^n \left[s_{12}^n + s_{23}^n - (s_{12}+s_{23})^n \right] \right)$$

$$= \frac{1}{s_{12}} - \zeta_2 \, s_{23} + \zeta_3 \, s_{23} \, (s_{12}+s_{23}) + \dots$$

Expansion in α' or s_{ij} ⇒ all Riemann zeta values $\zeta_n = \sum_{k=1}^{\infty} k^{-n}$.

I. 2 Four closed strings on the sphere

Again fix $(z_1, z_3, z_4) \rightarrow (0, 1, \infty)$, integrate $z = z_2$

and use dim'less Mandestam invariants $s_{ij} := 2\alpha' k_i \cdot k_j$

$$J_{\text{4-pt}} = \frac{1}{\pi} \int_{\mathbb{C}\setminus\{0,1,\infty\}} \mathrm{d}^2 z \, \frac{|z|^{2s_{12}} \, |1-z|^{2s_{23}}}{z \, \bar{z} \, (1-\bar{z})} = \frac{1}{s_{12}} \prod_{i<j}^{3} \frac{\Gamma(1+s_{ij})}{\Gamma(1-s_{ij})}$$

$$= \frac{1}{s_{12}} \exp\left(-2 \sum_{k=1}^{\infty} \frac{\zeta_{2k+1}}{2k+1} \left[s_{12}^{2k+1} + s_{23}^{2k+1} + s_{13}^{2k+1} \right] \right)$$

Only ζ_{2k+1} at odd argument (no ζ_{2k} from open-string case)

$$Z_{\text{4-pt}} = \frac{1}{s_{12}} \exp\left(\sum_{n=2}^{\infty} \frac{\zeta_n}{n} (-1)^n \left[s_{12}^n + s_{23}^n - (s_{12}+s_{23})^n \right] \right)$$

Formally, at the level of α'-expansions, relate closed & open strings via

$$J_{\text{4-pt}} = Z_{\text{4-pt}} \Big|_{\substack{\zeta_{2k+1} \rightarrow 2\,\zeta_{2k+1} \\ \zeta_{2k} \rightarrow 0}}$$

I. 3 Single-valued MZVs

α'-expansion of n-point tree amplitudes involves multiple zeta values (MZVs)

$$\zeta_{n_1,n_2,\ldots,n_r} \equiv \sum_{0<k_1<k_2<\ldots<k_r}^{\infty} k_1^{-n_1} k_2^{-n_2} \ldots k_r^{-n_r} \ , \qquad n_r \geq 2$$

Define single-valued projection **sv** of MZVs via their polylogarithm origin

[Schnetz 1302.6445 & Brown 1309.5309]

$$\begin{tikzcd}
\text{MZVs} \arrow[rr, dashed, "\textbf{sv}"'] & & \text{single-valued MZVs} \\
\begin{array}{c}\text{polylogarithms}\\ G(\{\text{word in } 0,1\};z)\end{array} \arrow[u, "z\to1"] \arrow[rr, "\begin{array}{c}\text{Francis Brown 2004:}\\ \exists\text{ unique procedure to}\end{array}", "\text{cancel monodromies}"'] & & \begin{array}{c}\text{single-valued polylog's}\\ G_{\rm SV}(\{\text{word in } 0,1\};z)\end{array} \arrow[u, "z\to1"']
\end{tikzcd}$$

e.g. $G(1;z) \ = \ \log(1-z) \ \longrightarrow \ G_{\rm SV}(1;z) = \log|1-z|^2$

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\end{tikzcd}$$

$$\mathbf{sv}(\zeta_{2k}) \ = \ 0 \, , \qquad \mathbf{sv}(\zeta_{2k+1}) \ = \ 2\,\zeta_{2k+1} \, , \qquad \mathbf{sv}(\zeta_{3,5}) \ = \ -10\,\zeta_3\,\zeta_5 \, , \qquad \text{etc.}$$

I. 4 Closed-string trees = sv(open-string trees)

At four points, have observed formal relation

$$\mathbf{sv}\, Z_{\text{4-pt}} \; = \; J_{\text{4-pt}}$$

$$\underbrace{\begin{array}{c}\text{disk ordering}\\ -\infty < z_1 < z_2 < z_3 < z_4 < \infty\end{array} \qquad \begin{array}{c}\text{antiholomorphic "Parke–Taylor" factor}\\ (\bar z_{12}\bar z_{23}\bar z_{34}\bar z_{41})^{-1} \text{ with } \bar z_{ij} = \bar z_i - \bar z_j\end{array}}_{\text{modulo SL}_2 \text{ fixing } (z_1,z_3,z_4) \to (0,1,\infty)}$$

$$\mathbf{sv}(\zeta_{2k}) \; = \; 0\,, \qquad \mathbf{sv}(\zeta_{2k+1}) \; = \; 2\,\zeta_{2k+1}\,, \qquad \text{etc.}$$

I. 4 Closed-string trees = sv(open-string trees)

Extends to n-point disk & sphere integrals

$$\mathbf{sv}\ Z_{n\text{-pt}} \ = \ J_{n\text{-pt}}$$

$$\underbrace{\begin{matrix}\text{disk ordering} \\ -\infty<z_1<z_2<\ldots<z_{n-1}<z_n<\infty\end{matrix} \qquad \begin{matrix}\text{antiholomorphic “Parke–Taylor” factor} \\ (\bar{z}_{12}\bar{z}_{23}\ldots\bar{z}_{n-1,n}\bar{z}_{n,1})^{-1} \text{ with } \bar{z}_{ij}=\bar{z}_i-\bar{z}_j\end{matrix}}_{\text{modulo SL}_2 \text{ fixing } (z_1,z_{n-1},z_n)\,\to\,(0,1,\infty)}$$

For an arbitrary rational function $Q(z) = Q(z_2, z_3, \ldots, z_{n-2})$:

$$\begin{tikzcd}[column sep=small] Z(1,2,\ldots,n\,|\,Q(z)) \arrow[d, "\mathbf{sv}"'] & = & \displaystyle\int_{0<z_2<z_3<\ldots<z_{n-2}<1} \mathrm{d}z_2 \ldots \mathrm{d}z_{n-2} \prod_{i<j}^{n-1} |z_{ij}|^{s_{ij}}\, Q(z) \\ J(1,2,\ldots,n\,|\,Q(z)) & = & \displaystyle\frac{1}{\pi^{n-3}} \int_{\mathbb{C}^{n-3}\setminus\{z_i=z_j\}} \frac{\mathrm{d}^2z_2 \ldots \mathrm{d}^2z_{n-2} \prod_{i<j}^{n-1} |z_{ij}|^{2s_{ij}}\, Q(z)}{\bar{z}_{12}\bar{z}_{23}\cdots\bar{z}_{n-3,n-2}\bar{z}_{n-2,n-1}} \end{tikzcd}$$

I. 4 Closed-string trees = sv(open-string trees)

$$\begin{tikzcd}
Z(1,2,\ldots,n\,|\,Q(z)) \arrow[d, "\mathbf{sv}"'] & = \displaystyle\int_{0<z_2<z_3<\ldots<z_{n-2}<1} \mathrm{d}z_2 \ldots \mathrm{d}z_{n-2} \prod_{i<j}^{n-1} |z_{ij}|^{s_{ij}} \, Q(z) \\
J(1,2,\ldots,n\,|\,Q(z)) & = \displaystyle\frac{1}{\pi^{n-3}} \int_{\mathbb{C}^{n-3}\setminus\{z_i=z_j\}} \frac{\mathrm{d}^2 z_2 \ldots \mathrm{d}^2 z_{n-2} \prod_{i<j}^{n-1} |z_{ij}|^{2s_{ij}} \, Q(z)}{\bar{z}_{12}\bar{z}_{23} \cdots \bar{z}_{n-3,n-2}\bar{z}_{n-2,n-1}}
\end{tikzcd}$$

- conjectured after order-by-order inspection of α'-expansion

[OS, Stieberger 1205.1516; Stieberger 1310.3259; Stieberger, Taylor 1401.1218]

- announced as a theorem

[Brown: talk at String Math 2018 (Sendai, Japan); Brown, Dupont: to appear]

- physicist's proof (assuming e.g. standard transcendentality conjectures)

[OS, Schnetz 1808.00713]

II. Elliptic MZVs and open strings at one loop

II. 1 Four open strings on a cylinder

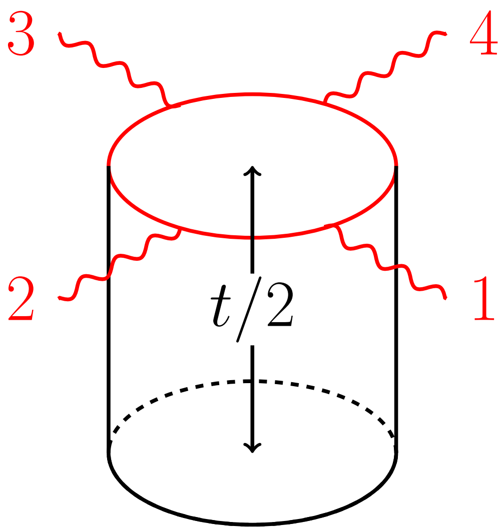
Cylinder contribution to planar one-loop amplitude $\sim \mathrm{Tr}(t^1 t^2 t^3 t^4)$

$$A^{\text{1-loop}}_{\text{string}}(1,2,3,4) = s_{12} s_{23} A^{\text{tree}}_{\text{YM}}(1,2,3,4) \int_0^\infty \mathrm{d}t \; I_{1234}(s_{ij}, \tau = it)$$

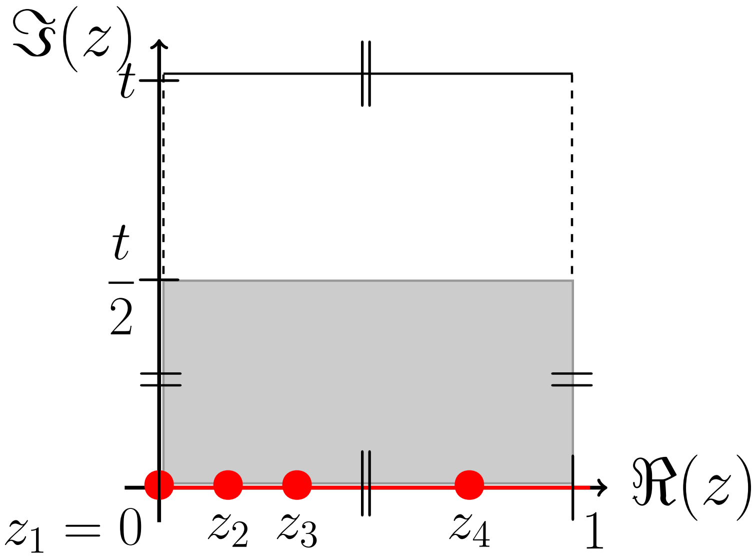
$$I_{1234}(s_{ij},\tau) = \int_{0=z_1<z_2<z_3<z_4<1} \mathrm{d}z_2 \, \mathrm{d}z_3 \, \mathrm{d}z_4 \; \exp\Big(\sum_{i<j}^{4} s_{ij} P(z_i - z_j, \tau) \Big)$$

[Brink, Green, Schwarz 1982]

with $\sum_{i<j}^4 s_{ij} = 0$ and Green function $\partial_z P(z,\tau) = \partial_z \log\theta(z,\tau) + 2\pi i \frac{\mathrm{Im}\, z}{\mathrm{Im}\, \tau}$.



$$\xrightarrow[\text{“half a torus”}]{\text{parametrized as}}$$



II. 1 Four open strings on a cylinder

Main interest in this talk on the integral over the punctures z_2, z_3, z_4

$$I_{1234}(s_{ij},\tau) = \int\limits_{0=z_1<z_2<z_3<z_4<1} \mathrm{d}z_2\,\mathrm{d}z_3\,\mathrm{d}z_4 \ \exp\Bigg(\sum_{i<j}^{4} s_{ij}\underbrace{P(z_i-z_j,\tau)}_{P_{ij}}\Bigg)$$

- Taylor expand $\exp(s_{ij}P_{ij}) = \sum_{n=0}^{\infty}\frac{1}{n!}(s_{ij}P_{ij})^n$ for each pair $1\leq i<j\leq 4$

- integrating $\prod_{i<j}(P_{ij})^{n_{ij}}$ over cyl. boundary ⇒ elliptic MZVs (eMZVs)

[Brödel, Mafra, Matthes, OS 1412.5535; see talk of Nils Matthes]

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- eMZVs are proper subset of iterated τ-integrals $\mathcal{E}_0(k_1, k_2, \ldots, k_r;\tau)$

  over holomorphic Eisenstein series $\mathrm{G}_k(\tau)$ with $\mathbb{Q}[\mathrm{MZV}, \frac{1}{2\pi i}]$ coeff's

  [Enriquez 1301.3042 & Brödel, Matthes, OS 1507.02254]

- cylinder ↔ specialize $\tau = it$ with $t \in \mathbb{R}_+$, Moebius strip has $\tau = \frac{1}{2} + it$

II. 2 Iterated Eisenstein integrals

Holomorphic Eisenstein series ($k \geq 4$ even, $q = e^{2\pi i\tau}$) & $\mathrm{G}_0 = -1$

$$\mathrm{G}_k(\tau) \ = \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{(m\tau{+}n)^k} \ = \ 2\,\zeta_k \ + \ \frac{2\,(2\pi i)^k}{(k{-}1)!} \sum_{m,n=1}^{\infty} m^{k-1} q^{mn}$$

II. 2 Iterated Eisenstein integrals

Holomorphic Eisenstein series ($k \geq 4$ even, $q = e^{2\pi i \tau}$) & $\mathrm{G}_0 = -1 = \mathrm{G}_0^0$

$$\mathrm{G}_k(\tau) = \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{1}{(m\tau+n)^k} = 2\,\zeta_k + \underbrace{\frac{2\,(2\pi i)^k}{(k-1)!} \sum_{m,n=1}^{\infty} m^{k-1} q^{mn}}_{\mathrm{G}_k^0(\tau)}$$

Define iterated Eisenstein integrals recursively by $\mathcal{E}_0(\,;\tau) = 1$ and

$$\mathcal{E}_0(k_1,k_2,\ldots,k_r;\tau) = (2\pi i)^{1-k_r} \int_\tau^{i\infty} \mathrm{d}\tau' \, \mathrm{G}^0_{k_r}(\tau') \, \mathcal{E}_0(k_1,k_2,\ldots,k_{r-1};\tau')$$

$k_1 \geq 4 \;\Rightarrow$ convergent integrals by zero-mode subtraction $\mathrm{G}^0_{k_1} = \mathrm{G}_{k_1} - 2\zeta_{k_1}$

$$\text{e.g.} \quad \mathcal{E}_0(k,\underbrace{0,0,\ldots,0}_{p-1};\tau) = \frac{-2}{(k-1)!} \sum_{m,n=1}^{\infty} \frac{m^{k-1}}{(mn)^p} q^{mn}$$

q-expansion straightforwardly inherited from above $\mathrm{G}_k^0(\tau)$.

II. 2 Iterated Eisenstein integrals

Back to open-string integral

$$I_{1234}(s_{ij},\tau) = \int_{0=z_1<z_2<z_3<z_4<1} \mathrm{d}z_2\,\mathrm{d}z_3\,\mathrm{d}z_4\ \exp\Big(\sum_{i<j}^{4} s_{ij}\,\underbrace{P(z_i-z_j,\tau)}_{P_{ij}}\Big)$$

with Mandelstam relations $s_{34}=s_{12},\ \ s_{14}=s_{23}$ & $s_{13}=s_{24}=-s_{12}-s_{23}$

$$\begin{aligned} I_{1234}(s_{ij},\tau) \ &= \ \frac{1}{6} + \frac{3s_{13}}{2\pi^2}\big[\zeta_3 - 6\,\mathcal{E}_0(4,0,0;\tau)\big] \\ &\quad + \frac{60}{\pi^2}(s_{13}^2+2s_{12}s_{23})\big[\mathcal{E}_0(6,0,0,0;\tau) - \tfrac{\zeta_4}{120}\big] \\ &\quad - 2\,(s_{12}^2+s_{12}s_{23}+s_{23}^2)\big[\mathcal{E}_0(4,0;\tau) - \tfrac{\zeta_2}{12}\big] + \mathcal{O}(\alpha'^3) \end{aligned}$$

Order by order in s_{ij}, get eMZVs and therefore iterated Eisenstein integrals

$$\mathcal{E}_0(k_1,k_2,\ldots,k_r;\tau) = (2\pi i)^{1-k_r}\int_\tau^{i\infty}\mathrm{d}\tau'\,\mathrm{G}^0_{k_r}(\tau')\,\mathcal{E}_0(k_1,k_2,\ldots,k_{r-1};\tau')$$

II. 3 Symmetrized open-string integral

To connect with closed strings, combine permutations of $\int_{0<z_2<z_3<z_4<1}$

$$
\begin{aligned}
I_{\substack{\text{symm}\\\text{open}}}(s_{ij},\tau) \; &= \; \sum_{\rho\in S_3} I_{1\rho(234)}(s_{ij},\tau) \\
&= \; 1 \; + \; (s_{12}^2+s_{12}s_{23}+s_{23}^2)\,\big[\zeta_2 - 12\,\mathcal{E}_0(4,0;\tau)\big] \\
&\quad + \; s_{12}s_{13}s_{23}\Big[12\,\mathcal{E}_0(4,0,0;\tau)+300\,\mathcal{E}_0(6,0,0;\tau)-\frac{5}{2}\zeta_3\Big] \; + \; \mathcal{O}(\alpha'^4)
\end{aligned}
$$

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$$= 1 + (s_{12}^2 + s_{12}s_{23} + s_{23}^2)\left[\zeta_2 - 12\,\mathcal{E}_0(4,0;\tau)\right]$$

$$+ s_{12}s_{13}s_{23}\left[12\,\mathcal{E}_0(4,0,0;\tau) + 300\,\mathcal{E}_0(6,0,0;\tau) - \frac{5}{2}\zeta_3\right] + \mathcal{O}(\alpha'^4)$$

Modular S-transformation $\tau \rightarrow -\frac{1}{\tau}$ follows from $\mathrm{G}_k(-\frac{1}{\tau}) = \tau^k \mathrm{G}_k(\tau)$

$$I_{\substack{\text{symm}\\\text{open}}}(s_{ij},-\tfrac{1}{\tau}) = 1 - (s_{12}^2{+}s_{12}s_{23}{+}s_{23}^2)\left[\frac{T^2}{90}-\frac{\pi^2}{9}-\frac{\pi^4}{30T^2}-\frac{2i\zeta_3}{T}+12\,\mathcal{E}_0(4,0;\tau)+\frac{12i}{T}\mathcal{E}_0(4,0,0;\tau)\right]$$

$$+ s_{12}s_{13}s_{23}\left[-\frac{iT^3}{756}+\frac{i\pi^2T}{45}-\frac{\zeta_3}{2}-\frac{7i\pi^4}{72T}-\frac{2\pi^2\zeta_3}{T^2}+\frac{15\zeta_5}{2T^2}+\frac{17i\pi^6}{1890T^3}+\frac{12\pi^2}{T^2}\mathcal{E}_0(4,0,0;\tau)\right.$$

$$\left.+ 300\,\mathcal{E}_0(6,0,0;\tau) + \frac{900i}{T}\mathcal{E}_0(6,0,0,0;\tau) - \frac{900}{T^2}\mathcal{E}_0(6,0,0,0,0;\tau)\right] + \mathcal{O}(\alpha'^4)$$

$\Longrightarrow$ coeff's of $q^{0,1,2,\ldots}$ = Laurent polynomials in $T{:=}\pi\tau$ along with MZVs.

III. From closed strings to an elliptic single-valued map

III. 1 Four closed strings on a torus

Four-point closed-string amplitude at one loop (gravitons in type IIA/B)

$$M^{\text{1-loop}}_{\text{string}}(1,2,3,4) = \left|s_{12}s_{23}A^{\text{tree}}_{\text{YM}}(1,2,3,4)\right|^2 \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{(\operatorname{Im}\tau)^2}\, J_{\text{closed}}(s_{ij},\tau)$$

$$\underbrace{J_{\text{closed}}(s_{ij},\tau)}_{\text{mod. invariant}} = \left(\prod_{j=2}^{4}\int_{\mathcal{T}(\tau)}\frac{\mathrm{d}^2 z_j}{\operatorname{Im}\tau}\right)\exp\left(\sum_{i<j}^{4} s_{ij}\, g(z_i-z_j,\tau)\right)\bigg|_{z_1=0}$$

[Brink, Green, Schwarz 1982]

- fund. domain $\mathcal{F}$ of modular group $\mathrm{SL}_2(\mathbb{Z})$ and torus $\mathcal{T}(\tau) = \frac{\mathbb{C}}{\mathbb{Z}+\tau\mathbb{Z}}$
- Fourier expansion of the Green function with $z = r+\tau s$ and $r,s\in\mathbb{R}$

$$g(z,\tau) = \frac{\operatorname{Im}\tau}{\pi}\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}}\frac{e^{2\pi i(nr-ms)}}{|m+\tau n|^2} \qquad \text{modular invariant!}$$

- α'-expansion of $J_{\text{closed}}(s_{ij},\tau)$ & generalizations has long history

[D'Hoker, Green, Vanhove et al. 1999 - 2017]

III. 1 Four closed strings on a torus

Main interest in this talk on the integral over the punctures z_2, z_3, z_4

$$J_{\text{closed}}(s_{ij},\tau) \ = \ \left(\prod_{j=2}^{4} \int_{\mathcal{T}(\tau)} \frac{\mathrm{d}^2 z_j}{\operatorname{Im}\tau} \right) \ \exp\left(\sum_{i<j}^{4} s_{ij}\, g(z_i - z_j,\tau) \right)\Big|_{z_1=0}$$

Again, Taylor expand the exponentials in s_{ij}

$\Longrightarrow$ need to evaluate $\int_{\mathcal{T}(\tau)} \mathrm{d}^2 z_j$ over $\prod_{i<j} (g(z_{ij},\tau))^{n_{ij}}$

$\Longrightarrow$ modular invariance order by order in s_{ij}

III. 1 Four closed strings on a torus

Main interest in this talk on the integral over the punctures z_2, z_3, z_4

$$J_{\text{closed}}(s_{ij},\tau) \ = \left(\prod_{j=2}^{4} \int_{\mathcal{T}(\tau)} \frac{\mathrm{d}^2 z_j}{\operatorname{Im}\tau} \right) \ \exp\left(\sum_{i<j}^{4} s_{ij}\, g(z_i - z_j,\tau) \right) \Big|_{z_1=0}$$

Integrating monomials of Green fct's @ $z = r + \tau s$ over $r, s \in (0,1)$ ...

$$g(z,\tau) \ = \frac{\operatorname{Im}\tau}{\pi} \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{e^{2\pi i(nr-ms)}}{|m+\tau n|^2} \qquad \text{modular invariant!}$$

... naturally lands on non-holomorphic Eisenstein series ...

$$\mathrm{E}_k(\tau) \ := \ \left(\frac{\operatorname{Im}\tau}{\pi}\right)^k \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{1}{|m+\tau n|^{2k}} \, , \qquad k \geq 2$$

... and generalizations to nested lattice sums "*modular graph functions*".

[D'Hoker, Green, Gürdogan, Vanhove 1512.06779; see talk of Erik Panzer]

III. 1 Four closed strings on a torus

Main interest in this talk on the integral over the punctures z_2, z_3, z_4

$$J_{\rm closed}(s_{ij},\tau) \ = \left(\prod_{j=2}^4 \int_{\mathcal{T}(\tau)} \frac{{\rm d}^2 z_j}{{\rm Im}\, \tau} \right) \ \exp\Big(\sum_{i<j}^4 s_{ij}\, g(z_i-z_j,\tau) \Big) \Big|_{z_1=0}$$

E_k and generalizations $\longrightarrow$ (real parts of) iterated Eisenstein integrals

$$\mathcal{E}_0(k_1,k_2,\ldots,k_r;\tau) = (2\pi i)^{1-k_r} \int_\tau^{i\infty} {\rm d}\tau' \, {\rm G}^0_{k_r}(\tau')\, \mathcal{E}_0(k_1,k_2,\ldots,k_{r-1};\tau')$$

[Gangl, Zagier 2000; D'Hoker, Green 1603.00839;

Brödel, OS, Zerbini 1803.00527; see talk of Erik Panzer]

III. 1 Four closed strings on a torus

Main interest in this talk on the integral over the punctures z_2, z_3, z_4

$$J_{\rm closed}(s_{ij},\tau) \ = \left(\prod_{j=2}^4 \int_{\mathcal{T}(\tau)} \frac{{\rm d}^2 z_j}{{\rm Im}\, \tau} \right) \ \exp \left(\sum_{i<j}^4 s_{ij} \, g(z_i-z_j,\tau) \right) \Big|_{z_1=0}$$

E_k and generalizations $\longrightarrow$ (real parts of) iterated Eisenstein integrals

$$\mathcal{E}_0(k_1,k_2,\ldots,k_r;\tau) = (2\pi i)^{1-k_r} \int_\tau^{i\infty} {\rm d}\tau' \, {\rm G}^0_{k_r}(\tau') \, \mathcal{E}_0(k_1,k_2,\ldots,k_{r-1};\tau')$$

$\longrightarrow$ series in $q^m \bar q^n$, coefficients are Laurent polynomials in $y := \pi \, {\rm Im}\, \tau$

$$\begin{aligned} J_{\rm closed}(s_{ij},\tau) = 1 + (s_{12}^2+s_{12}s_{23}+s_{23}^2) \Big[\frac{2y^2}{45} + \frac{2\zeta_3}{y} - 24\,{\rm Re}\,\mathcal{E}_0(4,0;\tau) - \frac{12}{y}\,{\rm Re}\,\mathcal{E}_0(4,0,0;\tau) \Big] \\ + s_{12}s_{13}s_{23} \Big[-\frac{2y^3}{189} - \zeta_3 - \frac{15\zeta_5}{4y^2} + 600\,{\rm Re}\,\mathcal{E}_0(6,0,0;\tau) \\ + \frac{900}{y}\,{\rm Re}\,\mathcal{E}_0(6,0,0,0;\tau) + \frac{450}{y^2}\,{\rm Re}\,\mathcal{E}_0(6,0,0,0,0;\tau) \Big] \ + \mathcal{O}(\alpha'^4) \end{aligned}$$

$\longrightarrow$ structure familiar from open strings & $I^{\rm symm}_{\rm open}(s_{ij},-\frac{1}{\tau})$ ☺

III. 2 An elliptic single-valued projection?

Compare open- and closed-string α'-expansion

$$I_{\substack{\text{symm}\\\text{open}}}(s_{ij},-\tfrac{1}{\tau}) \quad \longleftrightarrow \quad J_{\text{closed}}(s_{ij},\tau)$$

- leading orders $1+\mathcal{O}(\alpha'^2)$ on both sides
- series in q^m or $q^m\bar{q}^n$ & Laurent polynomials in $T:=\pi\tau$ or $y:=\pi\,\mathrm{Im}\,\tau$
- compare the first non-trivial order $(\alpha')^2$: very similar coefficients!

$$I_{\substack{\text{symm}\\\text{open}}}(s_{ij},-\tfrac{1}{\tau})\,\Big|_{(\alpha')^2} \sim -\frac{T^2}{90}+\frac{\pi^2}{9}+\frac{\pi^4}{30T^2}+\frac{2i\zeta_3}{T}-12\,\mathcal{E}_0(4,0;\tau)-\frac{12i}{T}\mathcal{E}_0(4,0,0;\tau)$$

$$J_{\text{closed}}(s_{ij},\tau)\,\Big|_{(\alpha')^2} \sim \frac{2y^2}{45}+\frac{2\zeta_3}{y}-24\,\mathrm{Re}\,\mathcal{E}_0(4,0;\tau)-\frac{12}{y}\mathrm{Re}\,\mathcal{E}_0(4,0,0;\tau)$$

$\rightarrow$ no closed-string analogue of $\pi^{2k}\sim\zeta_{2k}$, only **sv**(MZV) ζ_3 survives

$\rightarrow$ closed string involves real parts $\mathrm{Re}\,\mathcal{E}_0(\ldots)$ of iterated Eisenstein int's

III. 2 An elliptic single-valued projection?

How to map open-string data to closed-string data?

$$\begin{aligned} I_{\substack{\text{symm}\\ \text{open}}}(s_{ij},-\tfrac{1}{\tau})\Big|_{(\alpha')^2} &\sim -\frac{T^2}{90}+\frac{\pi^2}{9}+\frac{\pi^4}{30T^2}+\frac{2i\zeta_3}{T}-12\,\mathcal{E}_0(4,0;\tau)-\frac{12i}{T}\,\mathcal{E}_0(4,0,0;\tau) \\ J_{\text{closed}}(s_{ij},\tau)\Big|_{(\alpha')^2} &\sim \frac{2y^2}{45}+\frac{2\zeta_3}{y}-24\,\mathrm{Re}\,\mathcal{E}_0(4,0;\tau)-\frac{12}{y}\,\mathrm{Re}\,\mathcal{E}_0(4,0,0;\tau) \end{aligned}$$

III. 2 An elliptic single-valued projection?

How to map open-string data to closed-string data?

$$I^{\rm symm}_{\rm open}(s_{ij},-\tfrac{1}{\tau})\Big|_{(\alpha')^2} \sim -\frac{T^2}{90}+\frac{\pi^2}{9}+\frac{\pi^4}{30T^2}+\frac{2i\zeta_3}{T}-12\,\mathcal{E}_0(4,0;\tau)-\frac{12i}{T}\mathcal{E}_0(4,0,0;\tau)$$

$$J_{\rm closed}(s_{ij},\tau)\Big|_{(\alpha')^2} \sim \frac{2y^2}{45}+\frac{2\zeta_3}{y}-24\,{\rm Re}\,\mathcal{E}_0(4,0;\tau)-\frac{12}{y}{\rm Re}\,\mathcal{E}_0(4,0,0;\tau)$$

Engineer *elliptic single-valued projection* **esv**

$$\textbf{esv} \ : \ \begin{cases} ({\rm i}) : & \zeta_{n_1,n_2,\ldots} \ \rightarrow \ \textbf{sv}(\zeta_{n_1,n_2,\ldots}) \\ ({\rm ii}) : & T \ \rightarrow \ 2iy \ \ {\rm i.e.} \ \ \tau \ \rightarrow \ 2i\,{\rm Im}\,\tau \\ ({\rm iii}) : & \mathcal{E}_0(k_1,\ldots;\tau) \ \rightarrow \ 2\,{\rm Re}\,\mathcal{E}_0(k_1,\ldots;\tau) \end{cases}$$

to match above expressions @ $(\alpha')^2$ and in fact complete $(\alpha')^{\leq 6}$ orders!

$$\boxed{\textbf{esv}\ I^{\rm symm}_{\rm open}(s_{ij},-\tfrac{1}{\tau}) \ = \ J_{\rm closed}(s_{ij},\tau)}$$

[Brödel, OS, Zerbini 1803.00527]

III. 2 An elliptic single-valued projection?

Conjectural *elliptic single-valued projection* **esv** (works to order α'^6)

$$\textbf{esv} \ : \ \begin{cases} \text{(i)} : & \zeta_{n_1,n_2,\ldots} \ \rightarrow \ \textbf{sv}(\zeta_{n_1,n_2,\ldots}) \\ \text{(ii)} : & T \ \rightarrow \ 2iy \ \ \text{i.e.} \ \ \tau \ \rightarrow \ 2i \, \mathrm{Im}\, \tau \\ \text{(iii)} : & \mathcal{E}_0(k_1,\ldots;\tau) \ \rightarrow \ 2 \, \mathrm{Re}\, \mathcal{E}_0(k_1,\ldots;\tau) \end{cases}$$

$$\boxed{\textbf{esv} \, I^{\text{symm}}_{\text{open}}(s_{ij}, -\tfrac{1}{\tau}) \ = \ J_{\text{closed}}(s_{ij},\tau)}$$

[Brödel, OS, Zerbini 1803.00527]

- so far requires ad-hoc convention how to use shuffle multiplication of $\mathcal{E}_0$

$$\Big(\textbf{esv}\big[\mathcal{E}_0(4,0,0;\tau)\big]\Big)^2 \ \neq \ 2\,\textbf{esv}\big[\mathcal{E}_0(4,0,0,4,0,0;\tau)\big]$$

$$+ \, 6\,\textbf{esv}\big[\mathcal{E}_0(4,0,4,0,0,0;\tau)\big] + 12\,\textbf{esv}\big[\mathcal{E}_0(4,4,0,0,0,0;\tau)\big]$$

- **esv** should relate to equivariant iterated Eisenstein integrals of Brown

[Brown 1407.5167, 1707.01230, 1708.03354]

IV. The elliptic single-valued map & heterotic strings

IV. 1 The esv map on non-symmetrized integration cycles?

So far, studied **esv** on symmetrized integration cycles $I_{\substack{\text{symm}\\ \text{open}}} = \sum_{\rho \in S_3} I_{1\rho(234)}$

$$I_{1234}(s_{ij},\tau) = \int_{0=z_1<z_2<z_3<z_4<1} \mathrm{d}z_2\, \mathrm{d}z_3\, \mathrm{d}z_4\ \exp\Big(\sum_{i<j}^{4} s_{ij}\, \underbrace{P(z_i{-}z_j,\tau)}_{P_{ij}} \Big) = I_{4321}(s_{ij},\tau)$$

What is the **esv**-image of the non-symmetric components?

$$Z_{1234}(s_{ij},\tau) := \frac{1}{3}\left[2\, I_{1234}(s_{ij},\tau) - I_{1342}(s_{ij},\tau) - I_{1423}(s_{ij},\tau) \right]$$

- by definition cycle to zero

$$Z_{1234}(s_{ij},\tau) + Z_{1342}(s_{ij},\tau) + Z_{1423}(s_{ij},\tau) = 0\,.$$

- from earlier results on $I_{1234}(s_{ij},\tau)$-expansion:

$$Z_{1234}(s_{ij},\tau)\ =\ \frac{3s_{13}}{2\pi^2}\left[6\,\mathcal{E}_0(4,0,0){-}\zeta_3 \right] + \frac{s_{13}^2+2s_{12}s_{23}}{2\pi^2}\left[120\,\mathcal{E}_0(6,0,0,0){-}\zeta_4 \right] + \mathcal{O}(\alpha'^3)$$

IV. 1 The esv map on non-symmetrized integration cycles?

What is the **esv**-image of the non-symmetric components?

$$Z_{1234}(s_{ij},\tau) := \frac{1}{3}\left[2\, I_{1234}(s_{ij},\tau) - I_{1342}(s_{ij},\tau) - I_{1423}(s_{ij},\tau)\right]$$

step 1: modular transformation (recalling $T := \pi\tau$)

$$Z_{1234}(s_{ij},-\tfrac{1}{\tau}) = s_{13}\Big(\frac{iT}{60} - \frac{3\zeta_3}{2T^2} - \frac{i\pi^2}{12T} + \frac{i\pi^4}{60T^3} + \frac{9}{T^2}\mathcal{E}_0(4,0,0)\Big)$$
$$+ (s_{13}^2+2s_{12}s_{23})\Big(\frac{T^2}{3780} - \frac{i\zeta_5}{T^3} - \frac{\pi^2}{216} + \frac{\pi^4}{360T^2} - \frac{\pi^6}{756T^4} + \frac{60}{T^2}\mathcal{E}_0(6,0,0,0) + \frac{120i}{T^3}\mathcal{E}_0(6,0,0,0,0)\Big) + \mathcal{O}(\alpha'^3)$$

step 2: apply **esv**-rules to T, ζ_k and $\mathcal{E}_0$ ...

$$\textbf{esv}\, Z_{1234}(s_{ij},-\tfrac{1}{\tau}) = s_{13}\Big(-\frac{y}{30} + \frac{3\zeta_3}{4y^2} - \frac{9}{2y^2}\,\mathrm{Re}\,\mathcal{E}_0(4,0,0)\Big)$$
$$+ (s_{13}^2 + 2s_{12}s_{23})\Big(-\frac{y^2}{945} + \frac{\zeta_5}{4y^3} - \frac{30}{y^2}\,\mathrm{Re}\,\mathcal{E}_0(6,0,0,0) - \frac{30}{y^3}\,\mathrm{Re}\,\mathcal{E}_0(6,0,0,0,0)\Big) + \mathcal{O}(\alpha'^3)$$

next section: compare to closed-string quantities.

IV. 2 esv(Z) from the heterotic string

- Heterotic string theories: also gauge bosons are closed-string excitations

- Motivation: type–II integral $J_{\rm closed}$ constrained by maximal SUSY

  ⇒ but generalities of **esv** should not depend on SUSY

  ⇒ non-SUSY current-algebra sector of heterotic string theories

  is more likely to give a complete picture of **esv**

IV. 2 esv(Z) from the heterotic string

- Heterotic string theories: also gauge bosons are closed-string excitations
- 1-loop 4-gluon amplitude @ $\mathrm{Tr}(t^1t^2t^3t^4) \ni$ integral over torus $\mathcal{T}(\tau)$

$$J^{\mathrm{het}}_{1234}(s_{ij},\tau) \ = \ \left(\prod_{j=2}^{4} \int_{\mathcal{T}(\tau)} \frac{\mathrm{d}^2 z_j}{\mathrm{Im}\,\tau} \right) V_2(1,2,3,4|\tau) \ \exp\left(\sum_{i<j}^{4} s_{ij}\, g(z_i - z_j,\tau) \right) \Big|_{z_1=0}$$

[Dolan, Goddard 0710.3743]

- Current algebra of heterotic strings $\Rightarrow$ elliptic function $V_2 \in$ integrand

$$V_2(1,2,3,4|\tau) \ = \ F(z_{12},\alpha,\tau)F(z_{23},\alpha,\tau)F(z_{34},\alpha,\tau)F(z_{41},\alpha,\tau)\Big|_{\alpha^{-2}}$$

$$F(z,\alpha,\tau) \ = \ \frac{\theta_1'(0,\tau)\theta_1(z+\alpha,\tau)}{\theta_1(z,\tau)\theta_1(\alpha,\tau)} \quad \text{Kronecker Eisenstein series}$$

$\Longrightarrow$ modular weight $(2,0)$ & simple Fourier expansion in $z_j = r_j + \tau s_j$

- α'-expansion of $\mathcal{J}^{\mathrm{het}}_{1234} \rightarrow$ *modular graph forms* **[D'Hoker, Green 1603.00839]**

[Gerken, Kleinschmidt, OS: in progress]

IV. 2 esv(Z) from the heterotic string

Integrate order by order & simplify modular graph forms

$$J^{\text{het}}_{1234}(s_{ij},\tau) \;=\; \Bigg(\prod_{j=2}^{4}\int_{\mathcal{T}(\tau)}\frac{\mathrm{d}^2 z_j}{\mathrm{Im}\,\tau}\Bigg)\, V_2(1,2,3,4|\tau)\,\exp\Bigg(\sum_{i<j}^{4} s_{ij}\,g(z_i-z_j,\tau)\Bigg)\Bigg|_{z_1=0}$$

$$= 2\pi i\Bigg\{3s_{13}\,\frac{\partial \mathrm{E}_2}{\partial\tau} \;+\; \frac{2}{3}\,(s_{12}^2+4s_{12}s_{23}+s_{23}^2)\,\frac{\partial \mathrm{E}_3}{\partial\tau} \;+\; \mathcal{O}(\alpha'^3)\Bigg\}$$

Higher orders in α' also involve

$$\text{e.g.}\quad \mathrm{E}_m\frac{\partial \mathrm{E}_n}{\partial\tau} \quad \text{or} \quad \frac{\partial}{\partial\tau}\left(\begin{matrix}\text{more challenging mo-}\\ \text{dular graph functions}\end{matrix}\right)$$

$\longrightarrow$ can all be expressed in terms of iterated Eisenstein integrals $\mathcal{E}_0$

IV. 2 esv(Z) from the heterotic string

Integrate order by order & simplify modular graph forms

$$J^{\rm het}_{1234}(s_{ij},\tau) \; = \; \bigg(\prod_{j=2}^4 \int_{\mathcal{T}(\tau)} \frac{{\rm d}^2 z_j}{{\rm Im}\, \tau} \bigg) \, V_2(1,2,3,4|\tau) \, \exp\bigg(\sum_{i<j}^4 s_{ij} \, g(z_i-z_j,\tau) \bigg) \Big|_{z_1=0}$$

$$= 2\pi i \Big\{ 3 s_{13} \, \frac{\partial {\rm E}_2}{\partial \tau} \; + \; \frac{2}{3} \, (s_{12}^2 + 4 s_{12} s_{23} + s_{23}^2) \, \frac{\partial {\rm E}_3}{\partial \tau} \; + \; \mathcal{O}(\alpha'^3) \Big\}$$

After importing $\mathcal{E}_0(\ldots)$-representation of ${\rm E}_k$ ...

$$J^{\rm het}_{1234}(s_{ij},\tau) = \pi^2 s_{13} \Big(\frac{2y}{15} - \frac{3\zeta_3}{y^2} + \frac{18}{y^2} {\rm Re}\, \mathcal{E}_0(4,0,0) + 72 \mathcal{E}_0(4) + \frac{36}{y} \mathcal{E}_0(4,0) \Big) + \mathcal{O}(\alpha'^2)$$

,

... most terms match open-string target up to few exceptions $\sim y^m \mathcal{E}_0(\ldots)$

$$(2\pi i)^2 \, \textbf{esv} \, Z_{1234}(s_{ij}, -\tfrac{1}{\tau}) \; = \; \pi^2 s_{13} \Big(\frac{2y}{15} - \frac{3\zeta_3}{y^2} + \frac{18}{y^2} {\rm Re}\, \mathcal{E}_0(4,0,0) \Big) + \mathcal{O}(\alpha'^2)$$

IV. 2 esv(Z) from the heterotic string

Integrate order by order & simplify modular graph forms

$$J^{\rm het}_{1234}(s_{ij},\tau) \;=\; \left(\prod_{j=2}^{4}\int_{\mathcal{T}(\tau)}\frac{\mathrm{d}^2 z_j}{\mathrm{Im}\,\tau}\right) V_2(1,2,3,4|\tau)\;\exp\left(\sum_{i<j}^{4} s_{ij}\,g(z_i-z_j,\tau)\right)\Big|_{z_1=0}$$

$$= 2\pi i\left\{3s_{13}\,\frac{\partial \mathrm{E}_2}{\partial\tau} \;+\; \frac{2}{3}\,(s_{12}^2+4s_{12}s_{23}+s_{23}^2)\,\frac{\partial \mathrm{E}_3}{\partial\tau} \;+\; \mathcal{O}(\alpha'^3)\right\}$$

After importing $\mathcal{E}_0(\ldots)$-representation of E_k ...

$$J^{\rm het}_{1234}(s_{ij},\tau) = \pi^2 s_{13}\Big(\frac{2y}{15}-\frac{3\zeta_3}{y^2}+\frac{18}{y^2}\mathrm{Re}\,\mathcal{E}_0(4,0,0)+72\mathcal{E}_0(4)+\frac{36}{y}\mathcal{E}_0(4,0)\Big)$$
$$+\,\pi^2(s_{12}^2+4s_{12}s_{23}+s_{23}^2)\Big(\frac{4y^2}{945}-\frac{\zeta_5}{y^3}+\frac{120}{y^2}\mathrm{Re}\,\mathcal{E}_0(6,0,0,0)+\frac{120}{y^3}\mathrm{Re}\,\mathcal{E}_0(6,0,0,0,0)$$
$$+160\mathcal{E}_0(6,0)+\frac{240}{y}\mathcal{E}_0(6,0,0)+\frac{120}{y^2}\mathcal{E}_0(6,0,0,0)\Big)+\mathcal{O}(\alpha'^3)\;,$$

... most terms match open-string target up to few exceptions $\sim y^m\mathcal{E}_0(\ldots)$

$$(2\pi i)^2\,\mathbf{esv}\,Z_{1234}(s_{ij},-\tfrac{1}{\tau}) \;=\; \pi^2 s_{13}\Big(\frac{2y}{15}-\frac{3\zeta_3}{y^2}+\frac{18}{y^2}\mathrm{Re}\,\mathcal{E}_0(4,0,0)\Big)$$
$$+\,\pi^2(s_{12}^2+4s_{12}s_{23}+s_{23}^2)\Big(\frac{4y^2}{945}-\frac{\zeta_5}{y^3}+\frac{120}{y^2}\mathrm{Re}\,\mathcal{E}_0(6,0,0,0)+\frac{120}{y^3}\mathrm{Re}\,\mathcal{E}_0(6,0,0,0,0)\Big)+\mathcal{O}(\alpha'^3)$$

IV. 2 esv(Z) from the heterotic string

Up to and including α'^3, have verified the following conjecture:

$$\boxed{J^{\text{het}}_{1234}(s_{ij},\tau) \ = \ (2\pi i)^2\, \mathbf{esv}\, Z_{1234}(s_{ij},-\tfrac{1}{\tau}) \text{ modulo } y^m\mathcal{E}_0(\ldots)}$$

open string predicts
coefficient of $y^m q^0 \bar{q}^n$,
i.e. without factors of q

no control over terms
with factors of q or $\mathcal{E}_0(\ldots)$

[Gerken, Kleinschmidt, OS: in progress]

Does modular weight $(2,0)$ of $J^{\text{het}}_{1234}(s_{ij},\tau)$ fix the missing $y^m\mathcal{E}_0(\ldots)$??

→ at order α'^3, above **esv** conjecture uniquely fixes the coeff's in an ansatz

$$J^{\text{het}}_{1234}(s_{ij},\tau)\Big|_{\alpha'^3} \ \sim \ \frac{4}{5}\frac{\partial \mathrm{E}_4}{\partial\tau} + 6\,\mathrm{E}_2\frac{\partial \mathrm{E}_2}{\partial\tau} + 12\,\frac{\partial \mathrm{E}_{2,2}}{\partial\tau}$$

modular graph function of “depth 2”

IV. 2 esv(Z) from the heterotic string

Up to and including α'^3, have verified the following conjecture:

$$\boxed{J^{\rm het}_{1234}(s_{ij},\tau) \ = \ (2\pi i)^2 \, \textbf{esv} \, Z_{1234}(s_{ij},-\tfrac{1}{\tau}) \text{ modulo } y^m \mathcal{E}_0(\ldots)}$$

- tree level correspondence: closed-string integrands ↔ integration cycles

$$\left. \begin{array}{c} \text{“Parke–Taylor” factor} \\ (z_{12}z_{23}\ldots z_{n-1,n}z_{n,1})^{-1} \text{ with } z_{ij}=z_i{-}z_j \end{array} \right\} \ \leftrightarrow \ \left\{ \begin{array}{c} \text{disk ordering} \\ -\infty{<}z_1{<}z_2{<}\ldots{<}z_{n-1}{<}z_n{<}\infty \end{array} \right.$$

- have now seen two conjectural generalizations to genus-one

$$1 \ \leftrightarrow \ \bigcup_{\rho\in S_{n-1}} \{0<z_{\rho(2)}<z_{\rho(3)}<\ldots<z_{\rho(n)}<1\}$$

$$V_2(1,2,3,4) \ \leftrightarrow \ \{0<z_2<z_3<z_4<1\} - \frac{1}{6}\bigcup_{\rho\in S_3}\{0<z_{\rho(2)}<z_{\rho(3)}<z_{\rho(4)}<1\}$$

- **expect:** combination of elliptic $V_w(1,2,\ldots,n)$ ↔ ∀ cylinder ordering

  modular weight w depends on the (anti-)symmetrization of the cycles

V. Conclusion & Outlook

- α'-expansion of string amplitudes $\longleftrightarrow$ periods of moduli spaces $\mathcal{M}_{g;n}$

	open strings	closed strings
tree level	disk $\Rightarrow$ multiple zeta values (MZVs) $=$ polylog's at $z=1$	sphere $\Rightarrow$ single-valued MZVs $=$ single-valued polylog's at $z=1$
one loop	cylinder / Möbius strip $\Rightarrow$ elliptic MZVs	torus $\frac{\mathbb{C}}{\mathbb{Z}+\tau\mathbb{Z}}$ $\Rightarrow$ modular graph forms (modular covariant fct's of τ)

V. Conclusion & Outlook

- α'-expansion of string amplitudes $\longleftrightarrow$ periods of moduli spaces $\mathcal{M}_{g;n}$

- conjectural elliptic single-valued projection from one-loop α'-expansions:

$$\textbf{esv}: \ \text{eMZVs (open string)} \ \rightarrow \left\{ \begin{array}{c} \text{modular graph functions (type II)} \\ \text{up to ambiguities with } ⧢\text{-product} \\ \\ \text{modular graph forms (heterotic)} \\ \text{up to mismatch} \sim q \text{ or } \mathcal{E}_0 \end{array} \right.$$

- should underpin by explicitly constructing single-valued elliptic polylog's

- should relate higher-genus modular graph functions with open strings

[D'Hoker, Green, Pioline 1712.06135, 1806.02691]

Thank you for your attention !