Beyond one elliptic curve

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Elliptic Functions in Mathematics and Physics, Ascona
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Based on [arXiv:1806.04981]

Outline

Introduction

Motivation

Higher order loop corrections inevitable for precision particle physics. Starting from 2-loops multiple polylogarithms (MPLs) not sufficient to describe Feynman Integrals (FIs).

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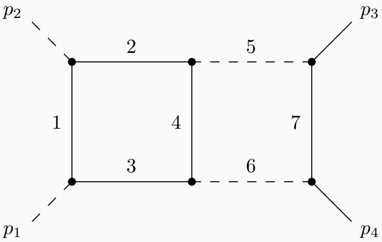
Single scale example: the sunrise.

Multiscale example: NNLO contribution for the process $pp \to t\bar{t}$ involves calculating the planar double box integral with a closed top loop (The Topbox).



Tool at our disposal: Differential Equations (DEs)

Finding the canonical-form

- First we try to find a basis that brings the DEs to the 'ϵ-form' [J. Henn, '13].
  - In cases where rational transformation is sufficient : several algorithms exist; e.g. in massless processes.
  - for algebraic cases (involving roots): not many transformations well known.

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- One such algorithm is enlarging the set of transformations from basis

  from rational functions in kinematic variables

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  rational functions in the kinematic variables, the periods of the elliptic curve and their derivatives

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Linear form of DEs

We may slightly relax the form of DE and consider

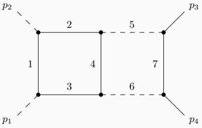
$$d\vec{J} = (A^{(0)} + \epsilon A^{(1)})\vec{J},$$

where $A^{(0)}$ and $A^{(1)}$ are independent of ϵ and $A^{(0)}$ is strictly lower-triangular and $A^{(1)}$ is block triangular.

Preliminaries

The Topbox

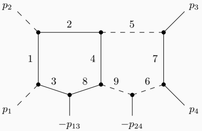
- Solid lines → massive propagators,
  all external momenta → out-going and on-shell.
  $s = (p_1 + p_2)^2$ and $t = (p_2 + p_3)^2$.
- Nine independent scalar products involving the loop momenta.



Auxiliary Topology

- Integral family for the auxiliary topology given by:

$$I_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7\nu_8\nu_9}\left(D, s, t, m^2, \mu^2\right)$$
$$= e^{2\gamma_E \varepsilon}\left(\mu^2\right)^{\nu - D}\int \frac{d^D k_1}{i\pi^{\frac{D}{2}}}\frac{d^D k_2}{i\pi^{\frac{D}{2}}}\prod_{j=1}^{9}\frac{1}{P_j^{\nu_j}},$$



$P_1 = -(k_1 + p_2)^2 + m^2$, $P_2 = -k^2 + m^2$, $P_3 = -(k_1 + p_1 + p_2)^2 + m^2$, $P_4 = -(k_1 + k_2)^2 + m^2$,
$P_5 = -k_2^2$, $P_6 = -(k_2 + p_3 + p_4)^2$, $P_7 = -(k_2 + p_3)^2 + m^2$, $P_8 = -(k_1 + p_2 - p_3)^2 + m^2$,
$P_9 = -(k_2 - p_2 + p_3)^2$.

Sector id:

$$\mathrm{id} \quad = \quad \sum_{j=1}^{9} 2^{j-1}\Theta(\nu_j).$$

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Aim:

Interested in the Laurent expansion of these integrals in ϵ, where $\epsilon = (4-D)/2$ is the dimensional regularisation parameter.

$$I_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7}(4-2\epsilon) = \sum_{j=j_{min}}^{\infty} \epsilon^j I^{(j)}_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7}.$$

Iterated integrals

Chen's definition

For $\lambda \in [0,1]$ the k-fold iterated integral of $\omega_1, ...\omega_k$ along the path γ is defined by

$I_\gamma(\omega_1, ..., \omega_k; \lambda) = \int_0^\lambda d\lambda_1 f_1(\lambda_1) \int_0^{\lambda_1} d\lambda_2 f_2(\lambda_2) ... \int_0^{\lambda_k - 1} d\lambda_k f_k(\lambda_k).$

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Choice of co-ordinate system

1. We set $\mu = m$ and can take $\frac{s}{m^2}$, $\frac{t}{m^2}$ as the two dimensionless ratios on which the FI depends; the (s,t) coordinates.
2. We can also choose the (x,y) coordinates where $\frac{s}{m^2} = -\frac{(1-x)^2}{x}$, $\frac{t}{m^2} = y$ (to rationalise the square root $\sqrt{-s(4m^2 - s)}$).
3. In order to simultaneously rationalise also the square root $\sqrt{-s(-4m^2 - s)}$, we may use the coordinate $\frac{s}{m^2} = -\frac{(1+\tilde{x}^2)^2}{\tilde{x}(1-\tilde{x}^2)}$.

Working bottom-up we choose coordinates suitable to the sector.

Iterated integrals

Multiple Polylogarithms

- For $z_k \neq 0$, defined by

$$G(z_1, ..z_k; y) = \int_0^y \frac{dy_1}{y_1 - z_1} \int_0^{y_1} \frac{dy_2}{y_2 - z_2} ... \int_0^{y_k - 1} \frac{dy_k}{y_k - z_k}.$$

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Iterated integrals of modular forms

- Let $f_1(\tau)$, $f_2(\tau)$,...,$f_k(\tau)$ be modular forms of a congruence subgroup.
- Assuming $f_k(\tau)$ vanishes at the cusp $\tau = i\infty$, we define the k-fold iterated integral by

$$F(f_1, f_2, .., f_k; q) = (2\pi i)^k \int_{i\infty}^{\tau} d\tau_1 f_1(\tau_1) \int_{i\infty}^{\tau_1} d\tau_2 f_2(\tau_2) ... \int_{i\infty}^{\tau_k - 1} d\tau_k f_k(\tau_k),$$
$$q = e^{2\pi i \tau}.$$

Elliptic Curves

Maximal Cut

Baikov representations: the loop by loop approach.

1. We first consider a one-loop sub-graph with a minimal number of propagators and change the integration variables for this sub-graph as:

$$\frac{d^D k}{i\pi^{\frac{D}{2}}} = u \frac{2^{-e}\pi^{-\frac{e}{2}}}{\Gamma(\frac{D-e}{2})} G(p_1, ..., p_e)^{\frac{1+e-D}{2}} G(k, p_1, ..., p_e)^{\frac{D-e-2}{2}} \prod_{j=1}^{e+1} dP_j,$$

then repeat the procedure for the second loop.

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then repeat the procedure for the second loop.

2. For an integral of the form

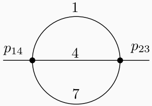
$$I = e^{2\gamma_E \varepsilon} \left(\mu^2\right)^{n-D} \int \frac{d^D k_1}{i\pi^{\frac{D}{2}}} \frac{d^D k_2}{i\pi^{\frac{D}{2}}} N(k_1, k_2) \prod_{j=1}^{n} \frac{1}{P_j},$$

a maximal cut is given by

$$\mathrm{MaxCut}_{\mathcal{C}}\, I = e^{2\gamma_E \varepsilon} \left(\mu^2\right)^{n-D} \int_{\mathcal{C}} \frac{d^D k_1}{i\pi^{\frac{D}{2}}} \frac{d^D k_2}{i\pi^{\frac{D}{2}}} N(k_1, k_2) \prod_{j=1}^{n} \delta(P_j),$$

Coming up: Extraction of all the 3 curves using Maximal Cut

Sector 73: **Elliptic Curve a, $E^{(a)}$**



- Starting with the sub-loop C_1 first we obtain

$$\text{MaxCut}_{\mathcal{C}}\ I_{1001001}\left(2-2\varepsilon\right) =$$
$$\frac{u\mu^2}{\pi^2}\int\limits_{\mathcal{C}} \frac{dP'}{\left(P'-t+2m^2\right)^{\frac{1}{2}}\left(P'-t+6m^2\right)^{\frac{1}{2}}\left(P'^2+6m^2P'-4m^2t+9m^4\right)^{\frac{1}{2}}}+\mathcal{O}\left(\varepsilon\right).$$

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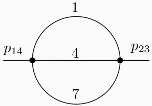

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- We could have equally well started with the sub-loop C_2, where we find

$$\mathrm{MaxCut}_{\mathcal{C}}\ I_{1001001}\left(2-2\varepsilon\right) = \frac{u\mu^2}{\pi^2}\int_{\mathcal{C}} \frac{dP}{\left(P-t\right)^{\frac{1}{2}}\left(P-t+4m^2\right)^{\frac{1}{2}}\left(P^2+2m^2P-4m^2t+m^4\right)^{\frac{1}{2}}} + \mathcal{O}\left(\varepsilon\right).$$

Extraction of the elliptic curve

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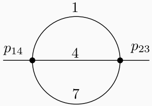

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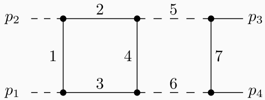
The two representations are related by $P^{'} = P - 2m^2$.

Sector 127: **Elliptic Curve b,** $E^{(b)}$



For the double box integral in 4 space-time dimensions.

$$\text{MaxCut}_{\mathcal{C}}\ I_{1111111}\left(4-2\varepsilon\right) = \frac{u\mu^6}{4\pi^4 s^2}\int_{\mathcal{C}} \frac{dP}{\left(P-t\right)^{\frac{1}{2}}\left(P-t+4m^2\right)^{\frac{1}{2}}\left(P^2+2m^2P-4m^2t+m^4-\frac{4m^2\left(m^2-t\right)^2}{s}\right)^{\frac{1}{2}}} + \mathcal{O}\left(\varepsilon\right).$$

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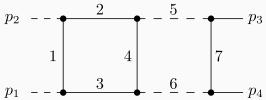
Sector 127: **Elliptic Curve b, $E^{(b)}$**



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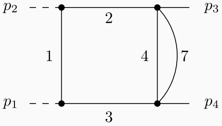
The term $-\frac{4m^2(m^2-t)^2}{s}$ vanishes in the limit $s \to \infty$.

Sector 79: **Elliptic Curve b, $E^{(b)}$**



For the maximal cut in the sector 79

$$\text{MaxCut}_{\mathcal{C}}\ I_{1112001}\,(4-2\varepsilon) = \frac{u\mu^4}{4\pi^3 s}\int_{\mathcal{C}} \frac{dP}{(P-t)^{\frac{1}{2}}\,(P-t+4m^2)^{\frac{1}{2}}\left(P^2+2m^2P-4m^2t+m^4-\frac{4m^2\left(m^2-t\right)^2}{s}\right)^{\frac{1}{2}}} + \mathcal{O}\left(\varepsilon\right).$$

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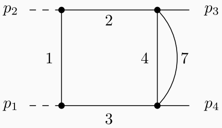

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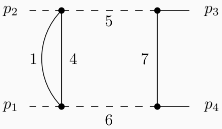
Up to the prefactor, this is the same maximal cut as for the full topology. So sectors 79 and 127 are associated to the same elliptic curve.

Sector 121: **Elliptic Curve c, $E^{(c)}$**



For the sector 121,

$$\mathrm{MaxCut}_{\mathcal{C}}\ I_{2001111}\left(4-2\varepsilon\right) = \frac{u\mu^4}{4\pi^3\left(-s\right)^{\frac{1}{2}}\left(4m^2-s\right)^{\frac{1}{2}}}$$

$$\times \int\limits_{\mathcal{C}} \frac{dP}{\left(P-t\right)^{\frac{1}{2}}\left(P-t+4m^2\right)^{\frac{1}{2}}\left(P^2+2m^2\frac{(s+4t)}{(s-4m^2)}P+m^2(m^2-4t)\frac{s}{s-4m^2}-\frac{4m^2t^2}{s-4m^2}\right)^{\frac{1}{2}}}$$

$$+\mathcal{O}\left(\varepsilon\right).$$

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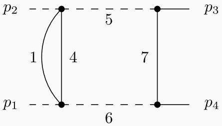

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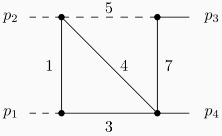
$$\times\int_{\mathcal{C}}\frac{dP}{\left(P-t\right)^{\frac{1}{2}}\left(P-t+4m^2\right)^{\frac{1}{2}}\left(P^2+2m^2\frac{(s+4t)}{(s-4m^2)}P+m^2(m^2-4t)\frac{s}{s-4m^2}-\frac{4m^2t^2}{s-4m^2}\right)^{\frac{1}{2}}}$$

$$+\mathcal{O}\left(\varepsilon\right).$$

This corresponds to an elliptic curve different from the ones found in sectors 79 and 127. In the limit $s\to\infty$ the maximal cut integral reduces again, up to a prefactor, to one of the sunrise.

Sector 93: **Elliptic Curve b,** $E^{(b)}$



For the sector 93,

$$\frac{1}{\varepsilon}\,\mathrm{MaxCut}_{\mathcal{C}}\ I_{1012101}\,(4-2\varepsilon) =$$

$$\frac{u\mu^4}{\pi^2 s}\int_{\mathcal{C}} \frac{dP}{(P-t)^{\frac{1}{2}}\,(P-t+4m^2)^{\frac{1}{2}}\left(P^2+2m^2P-4m^2t+m^4-\frac{4m^2\left(m^2-t\right)^2}{s}\right)^{\frac{1}{2}}}$$

$$+\mathcal{O}\left(\varepsilon\right).$$

Extraction of the elliptic curve

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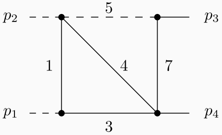

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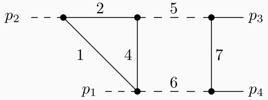
$$+\mathcal{O}\left(\varepsilon\right).$$

This is again the same elliptic curve from the sector 79 and 127.

Extraction of the elliptic curve

Sector 123: **No elliptic curve**



The last maximal cut example is from the sector 123. We find:

$$\mathrm{MaxCut}_{\mathcal{C}}\ I_{1101111}\left(4-2\varepsilon\right) = \frac{u\mu^4}{4\pi^3\left(-s\right)^{\frac{1}{2}}\left(4m^2-s\right)^{\frac{1}{2}}}$$

$$\times \int\limits_{\mathcal{C}} \frac{dP}{\left(P-t\right)\left(P^2+2m^2\frac{(s+4t)}{(s-4m^2)}P+m^2\left(m^2-4t\right)\frac{s}{s-4m^2}-\frac{4m^2t^2}{s-4m^2}\right)^{\frac{1}{2}}}$$

$$+\mathcal{O}\left(\varepsilon\right).$$

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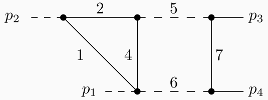

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The denominator may be viewed as a square root of a quartic polynomial, where two roots coincide. This does not involve an elliptic curve and corresponds to genus zero.

Reading an elliptic curve from the maximal cut

After 'getting' the curve:

1. We may read off the elliptic curve (e.g. for the sunrise integral) from the maximal cut:

$$E^a : w^2 - \left(z - \frac{t}{\mu^2}\right)\left(z - \frac{t-4m^2}{\mu^2})(z^2 + \frac{2m^2}{\mu^2}z + \frac{m^4 - 4m^2 t}{\mu^4}\right) = 0.$$

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2. The roots of the quartic polynomial are

$$z_1^{(a)} = \frac{t-4m^2}{\mu^2}, \quad z_2^{(a)} = \frac{-m^2 - 2m\sqrt{t}}{\mu^2}, \quad z_3^{(a)} = \frac{-m^2 + 2m\sqrt{t}}{\mu^2}, \quad z_4^{(a)} = \frac{t}{\mu^2}$$

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$$z_1^{(a)} = \frac{t-4m^2}{\mu^2}, \quad z_2^{(a)} = \frac{-m^2 - 2m\sqrt{t}}{\mu^2}, \quad z_3^{(a)} = \frac{-m^2 + 2m\sqrt{t}}{\mu^2}, \quad z_4^{(a)} = \frac{t}{\mu^2}$$

3. This curve has the j-invariant

$$j(E^{(a)}) = \frac{(3m^2+t)^3(3m^6+75m^4t-15m^2t^2+t^3)}{m^6 t(m^2-t)^6(9m^2-t)^2}$$

Differential Equations (DEs)

The system of DEs

Pre-canonical MIs

- Let the DEs for $\vec{I}$ read

$$d\vec{I} \;=\; A\vec{I}, \qquad A \;=\; A_s \frac{ds}{m^2} + A_t \frac{dt}{m^2}.$$

- Matrix-valued one-form A satisfies the integrability condition

$$dA - A \wedge A \;=\; 0.$$

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Change of Basis

- We can change the basis,

$$\vec{J} \;=\; U\vec{I},$$

to obtain

$$d\vec{J} \;=\; A'\vec{J},$$

where the matrix A' is related to A by

$$A' \;=\; UAU^{-1} - UdU^{-1}.$$

The system of DEs

'Linear-form'

- We choose $\vec{J}$ so that it brings the DEs linear in ϵ,

$$d\vec{J} \quad = \quad \left(A^{(0)} + \varepsilon A^{(1)}\right) \vec{J},$$

- The matrices $A^{(0)}$ and $A^{(1)}$ are independent of ϵ and $A^{(0)}$ is strictly lower-triangular and $A^{(1)}$ is block triangular.

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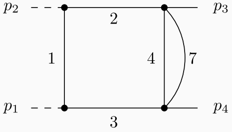
Simple DEs

- The system of DEs simplifies for $t = m^2$ (i.e. for y=1), as well as for $s = \infty$ (i.e. for x=0).
- For y=1 the solution for MIs can be expressed in terms of MPLs.
- For x=0 the MIs are expressed in terms of iterated integrals of modular forms.

Basis for the Linear form of DEs

An example for the basis:



$$J_{24} = \varepsilon^3 \frac{(1-x)^2}{x} \frac{\pi}{\psi_1^{(b)}} I_{1112001},$$

$$J_{25} = \varepsilon^3 (1-2\varepsilon) \frac{(1-x)^2}{x} I_{1111001} - \frac{1}{3}(y-9) \frac{\psi_1^{(b)}}{\pi} J_{24},$$

$$J_{26} = \frac{6}{\varepsilon} \frac{\left(\psi_1^{(b)}\right)^2}{2\pi i W_y^{(b)}} \frac{d}{dy} J_{24} - \frac{1}{4}\left(3y^2 - 10y - 9\right) \left(\frac{\psi_1^{(b)}}{\pi}\right)^2 J_{24}$$
$$- \frac{1}{24}\left(y^2 - 30y - 27\right) \frac{\psi_1^{(b)}}{\pi} \frac{\psi_1^{(a)}}{\pi} J_6,$$

Integration kernels

For our system of DEs we find 107 independent integration kernels.

In case of multiple polylogarithms:

- For the cases with a singular point at $s = 4m^2$, $\Big($i.e. to rationalise the square root $\sqrt{-s(4m^2 - s)}$ $\Big)$ we make the replacements as :

$$\frac{s}{m^2} = -\frac{(1-x)^2}{x}, \quad \frac{ds}{\sqrt{-s(4m^2 - s)}} = \frac{dx}{x}$$

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- In order to simultaneously rationalise the two square roots $\sqrt{-s(4m^2 - s)}$ and $\sqrt{-s(-4m^2 - s)}$, we introduce a variable $\tilde{x}$ through $x = \tilde{x}\frac{(1-\tilde{x})}{1+\tilde{x}}$.

Integration kernels for multiple polylogarithms

Overall we have the following kernels in this case:

$$\begin{aligned}
\omega_0 &= \frac{ds}{s} &&= \frac{2\,(2\tilde{x})\,d\tilde{x}}{\tilde{x}^2+1} - \frac{d\tilde{x}}{\tilde{x}-1} - \frac{d\tilde{x}}{\tilde{x}+1} - \frac{d\tilde{x}}{\tilde{x}},\\
\omega_4 &= \frac{ds}{s-4m^2} &&= \frac{2\,(2\tilde{x}-2)\,d\tilde{x}}{\tilde{x}^2-2\tilde{x}-1} - \frac{d\tilde{x}}{\tilde{x}-1} - \frac{d\tilde{x}}{\tilde{x}+1} - \frac{d\tilde{x}}{\tilde{x}},\\
\omega_{-4} &= \frac{ds}{s+4m^2} &&= \frac{2\,(2\tilde{x}+2)\,d\tilde{x}}{\tilde{x}^2+2\tilde{x}-1} - \frac{d\tilde{x}}{\tilde{x}-1} - \frac{d\tilde{x}}{\tilde{x}+1} - \frac{d\tilde{x}}{\tilde{x}},\\
\omega_{0,4} &= \frac{ds}{\sqrt{-s\,(4m^2-s)}} &&= \frac{d\tilde{x}}{\tilde{x}-1} - \frac{d\tilde{x}}{\tilde{x}+1} + \frac{d\tilde{x}}{\tilde{x}},\\
\omega_{-4,0} &= \frac{ds}{\sqrt{-s\,(-4m^2-s)}} &&= -\frac{d\tilde{x}}{\tilde{x}-1} + \frac{d\tilde{x}}{\tilde{x}+1} + \frac{d\tilde{x}}{\tilde{x}}.
\end{aligned}$$

Integration kernels

Modular form kernels

- For MIs depending only t, integration kernels are of the form $(2\pi i) f(\tau) d\tau_6^{(a)}$ $\left(\tau_6^{(a)} = \frac{1}{6}\frac{\psi_2^{(a)}}{\psi_1^{(a)}}\right)$; f is a modular form of $\Gamma_1(6)$ from the set $\boxed{\{1, f_2, f_3, f_4, g_{2,1}\}}$,

$$f_2 = -\frac{1}{4}\left(3y^2 - 10y - 9\right)\left(\frac{\psi_1^{(a)}}{\pi}\right)^2, \qquad f_3 = -\frac{3}{2}y\,(y-1)\,(y-9)\left(\frac{\psi_1^{(a)}}{\pi}\right)^3,$$

$$f_4 = \frac{1}{16}\,(y+3)^4\left(\frac{\psi_1^{(a)}}{\pi}\right)^4, \qquad g_{2,1} = -\frac{1}{2}y\,(y-9)\left(\frac{\psi_1^{(a)}}{\pi}\right)^2.$$

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The high energy limit

- Let $g_{n,r} = -\frac{1}{2}\frac{y(y-1)(y-9)}{y-r}\left(\frac{\Psi_1^a}{\pi}\right)^n$ and $h_{n,s} = -\frac{1}{2}y(y-1)^{1+s}(y-9)\left(\frac{\Psi_1^{(a)}}{\pi}\right)^n$.
- In the limit $x \to 0$, $E^{(b)}$ and $E^{(c)}$ degenerate to $E^{(a)}$ and we may express all MIs in terms of iterated integrals of modular forms. Corresponding full set is

$$\boxed{\{1, g_{2,0}, g_{2,1}, g_{2,9}, g_{3,1}, h_{3,0}, g_{4,0}, g_{4,1}, g_{4,9}, h_{4,0}, h_{4,1}\}}\,.$$

The full set of Integration kernels

Notations:

- We define 'm-weight' = scaling power + 2.
- The integration kernels appearing in the ϵ^0 part $A^{(0)}$ denoted by $a_{n,j}^{(r)}$, where n gives the m-weight, (r) indicates the periods and j indexes different integration kernels with the same n and (r).
- Integration kernels appearing in the ϵ^1-part $A^{(1)}$ denoted by $\eta_{n,j}^{(r)}$.
- For d-log form we use $d_{2,j}$.

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- Integration kernels appearing in the ϵ^1-part $A^{(1)}$ denoted by $\eta^{(r)}_{n,j}$.
- For d-log form we use $d_{2,j}$.

$$\Big\{\omega_0, \omega_4, \omega_{-4}, \omega_{0,4}, \omega_{-4,0}, f_2, f_3, f_4, g_{2,1}, \eta_0^{(r)}, \eta^{(b)}_{1,1-4}, \eta^{(c)}_{1,1-3}, d_{2,1-5},$$
$$\eta_{2,1-12}, \eta_2^{(\frac{r}{s})}, a^{(b)}_{3,1-4}, a^{(c)}_{3,1-3}, \eta^{(a)}_{3,1-3}, \eta^{(b)}_{3,1-24}, \eta^{(c)}_{3,1-11}, a^{(a,b)}_{4,1},$$
$$a^{(a,c)}_{4,1}, a^{(b,b)}_{4,1-5}, a^{(c,c)}_{4,1}, a^{(b,c)}_{4,1}, \eta^{(a,b)}_{4,1-3}, \eta^{(a,c)}_{4,1}, \eta^{(b,b)}_{4,1-5}, \eta^{(c,c)}_{4,1}, \eta^{(b,c)}_{4,1}\Big\}.$$

Integrating the system of DE

Boundary Conditions (BCs)

- We integrate the system of DE starting from the point $(x, y) = (0, 1)$.
- The BC may be expressed as a linear combination of transcendental constants.
- A basis of these transcendental constants up to weight four is given by

$$
\begin{aligned}
w = 1: &\qquad \ln(2), \\
w = 2: &\qquad \zeta_2, \quad \ln^2(2), \\
w = 3: &\qquad \zeta_3, \quad \zeta_2 \ln(2), \quad \ln^3(2), \\
w = 4: &\qquad \zeta_4, \quad \mathrm{Li}_4\left(\frac{1}{2}\right), \quad \zeta_3 \ln(2), \quad \zeta_2 \ln^2(2), \quad \ln^4(2).
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- For MIs which do not depend on s or t we need to calculate explicitly the BCs . Two such integrals: J_1 (which is also a product of tadpoles) and J_8 (the sunrise at the pseudo threshold).

Integrating the system of DEs

The tadpole integral

$$T_\nu\left(D, m^2, \mu^2\right) \quad = \quad e^{\gamma_E \epsilon} \frac{\Gamma\left(\nu - \frac{D}{2}\right)}{\Gamma(\nu)} \left(\frac{m^2}{\mu^2}\right)^{\frac{D}{2} - \nu}.$$

For $D = 2 - 2\epsilon$, $\mu = m$ and $\nu = 1$ we have

$$T_1(2 - 2\epsilon) \quad = \quad e^{\gamma_E \epsilon} \Gamma(\epsilon) = \frac{1}{\epsilon}\left[1 + \frac{1}{2}\zeta_2 \epsilon^2 - \frac{1}{3}\zeta_3 \epsilon^3 + \frac{9}{16}\zeta_4 \epsilon^4 + \mathcal{O}\left(\epsilon^5\right)\right].$$

Integrating the system of DEs

The tadpole integral

$$T_{\nu}\left(D, m^2, \mu^2\right) \quad = \quad e^{\gamma_E \epsilon} \frac{\Gamma\left(\nu - \frac{D}{2}\right)}{\Gamma\left(\nu\right)} \left(\frac{m^2}{\mu^2}\right)^{\frac{D}{2}-\nu} .$$

For $D = 2 - 2\epsilon$, $\mu = m$ and $\nu = 1$ we have

$$T_1\left(2-2\epsilon\right) \quad = \quad e^{\gamma_E \epsilon}\Gamma\left(\epsilon\right) = \frac{1}{\epsilon}\left[1 + \frac{1}{2}\zeta_2\epsilon^2 - \frac{1}{3}\zeta_3\epsilon^3 + \frac{9}{16}\zeta_4\epsilon^4 + \mathcal{O}\left(\epsilon^5\right)\right].$$

Sunrise at Pseudo-Threshold **[L. Adams, C. Bogner, S. Weinzierl, arxiv: 1302.7004]**

$$J_8 \quad = \quad 6\epsilon^2 e^{2\gamma_E \epsilon}\Gamma\left(1+2\epsilon\right) \int_0^1 dx_2 \int_0^1 dx_4 \left[\frac{1}{x_2-1} - \frac{1}{x_2+1}\right]\left[\frac{1}{x_4+1} - \frac{1}{x_4+x_2}\right]$$
$$\times \left(x_2+1\right)^{\epsilon}\left(x_4+1\right)^{-2\epsilon}\left(x_4+x_2\right)^{-2\epsilon}\left(x_4 + \frac{x_2}{x_2+1}\right)^{\epsilon}.$$

- For all the other MIs we obtain BCs from the behaviour at a specific point, where the MI vanishes or reduces to simpler integrals, here these are (x,y) equal to (0,1), (1,1) & (-1,1).

Solutions of DEs for the Master Integrals (MIs)

A peek at the results

$$J_k \quad = \quad \sum_{j=0}^{\infty} \varepsilon^j J_k^{(j)}.$$

The integrals which do not depend on s nor t

$$\begin{aligned} J_1 \quad &= \quad 1 + \zeta_2 \varepsilon^2 - \frac{2}{3}\zeta_3 \varepsilon^3 + \frac{7}{4}\zeta_4 \varepsilon^4 + \mathcal{O}\left(\varepsilon^5\right), \\ J_8 \quad &= \quad 6\zeta_2 \varepsilon^2 + \varepsilon^3 \left(21\zeta_3 - 36\zeta_2 \ln 2\right) + \varepsilon^4 \left(144\,\mathrm{Li}_4\left(\frac{1}{2}\right) - 78\zeta_4 + 72\zeta_2 \ln^2(2) + 6\ln^4(2)\right) \\ &\quad + \mathcal{O}\left(\varepsilon^5\right). \end{aligned}$$

A peek at the results

One of the MIs which depend only on s

$$
\begin{aligned}
J_2^{(0)} &= 0, \\
J_2^{(1)} &= -G\left(0;x\right), \\
J_2^{(2)} &= 2G\left(-1,0;x\right) - G\left(0,0;x\right) + \zeta_2, \\
J_2^{(3)} &= -4G\left(-1,-1,0;x\right) + 2G\left(-1,0,0;x\right) + 2G\left(0,-1,0;x\right) - G\left(0,0,0;x\right) \\
&\quad -2\zeta_2 G\left(-1;x\right) + 2\zeta_3, \\
J_2^{(4)} &= 8G\left(-1,-1,-1,0;x\right) - 4G\left(-1,-1,0,0;x\right) - 4G\left(-1,0,-1,0;x\right) \\
&\quad -4G\left(0,-1,-1,0;x\right) + 2G\left(-1,0,0,0;x\right) + 2G\left(0,-1,0,0;x\right) \\
&\quad +2G\left(0,0,-1,0;x\right) - G\left(0,0,0,0;x\right) + 4\zeta_2 G\left(-1,-1;x\right) \\
&\quad -2\zeta_2 G\left(0,-1;x\right) - 4\zeta_3 G\left(-1;x\right) + \frac{8}{3}\zeta_3 G\left(0;x\right) + \frac{19}{4}\zeta_4.
\end{aligned}
$$

A peek at the results

One of the MIs which depend only on t

$$
\begin{aligned}
J_6^{(0)} &= 0, \\
J_6^{(1)} &= 0, \\
J_6^{(2)} &= F\left(1, f_3; q_6\right) + 3\zeta_2, \\
J_6^{(3)} &= -F\left(f_2, 1, f_3; q_6\right) - F\left(1, f_2, f_3; q_6\right) + 3\zeta_2 F\left(1; q_6\right) - 3\zeta_2 F\left(f_2; q_6\right) + \frac{21}{2}\zeta_3 \\
&\quad -18\zeta_2 \ln(2) \\
J_6^{(4)} &= F\left(f_2, f_2, 1, f_3; q_6\right) + F\left(f_2, 1, f_2, f_3; q_6\right) + F\left(1, f_2, f_2, f_3; q_6\right) + F\left(1, f_4, 1, f_3; q_6\right) \\
&\quad +3\zeta_2 F\left(f_2, f_2; q_6\right) - 3\zeta_2 F\left(1, f_2; q_6\right) - 3\zeta_2 F\left(f_2, 1; q_6\right) + 3\zeta_2 F\left(1, f_4; q_6\right) \\
&\quad +\zeta_2 F\left(1, f_3; q_6\right) + \left(\frac{21}{2}\zeta_3 - 18\zeta_2 \ln(2)\right)\left(F\left(1; q_6\right) - F\left(f_2; q_6\right)\right) - 39\zeta_4 + 72\mathrm{Li}_4\left(\frac{1}{2}\right) \\
&\quad +36\zeta_2 \ln^2(2) + 3\ln^4(2).
\end{aligned}
$$

One of the MIs which depend on both s and t

$$
\begin{aligned}
J_{24}^{(0)} &= 0, \\
J_{24}^{(1)} &= 0, \\
J_{24}^{(2)} &= 0, \\
J_{24}^{(3)} &= I_\gamma\left(\eta_0^{(b)}, \eta_2^{(\frac{b}{a})}, f_3; \lambda\right) - \frac{3}{2}\, I_\gamma\left(\eta_0^{(b)}, \eta_{3,5}^{(b)}, \omega_{0,4}; \lambda\right) \\
&\quad -3\, I_\gamma\left(\eta_{1,1}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda\right) + I_\gamma\left(\eta_2^{(\frac{a}{b})}, \eta_0^{(a)}, f_3; \lambda\right) \\
&\quad +\frac{9}{2}\, I_\gamma\left(\eta_0^{(b)}, a_{3,2}^{(b)}, \omega_{0,4}, \omega_{0,4}; \lambda\right) + I_\gamma\left(\eta_0^{(b)}, a_{4,1}^{(a,b)}, \eta_0^{(a)}, f_3; \lambda\right) \\
&\quad +\frac{7}{4}\, \zeta_2\, I_\gamma\left(\eta_0^{(b)}; \lambda\right) - 2\, \zeta_2\, I_\gamma\left(\eta_{1,1}^{(b)}; \lambda\right) + 3\, \zeta_2\, I_\gamma\left(\eta_2^{(\frac{a}{b})}; \lambda\right) \\
&\quad +3\, \zeta_2\, I_\gamma\left(\eta_0^{(b)}, a_{3,2}^{(b)}; \lambda\right) + 3\, \zeta_2\, I_\gamma\left(\eta_0^{(b)}, a_{4,1}^{(a,b)}; \lambda\right) - 3\, \ln(2)\, \zeta_2 - \frac{7}{4}\, \zeta_3.
\end{aligned}
$$

Outlook

Summary

1. Analytic results for the planar double box relevant to top-pair production with a closed top loop presented.
2. This system depends on two scales and involves several elliptic sub-sectors.
3. Extraction of the elliptic curves shown.
4. Results expressed in terms of iterated integrals and the occuring integration kernels discussed.

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Thanks!