Modular graph functions as iterated Eisenstein integrals

Erik Panzer
All Souls College, Oxford

Elliptic integrals in Mathematics and Physics
Congressi Stefano Francscini, ETH Zürich

Ascona, 7th September 2018

Topics

1. MGF: modular graph functions (Green, Vanhove, D'Hoker, Gürdogan)
2. eMPL: elliptic multiple polylogarithms (Brown, Vanhove)
3. single-valued integration (Schnetz)
4. eMZV: elliptic multiple zeta values (Enriquez, Matthes, Zerbini)
5. iEi: iterated Eisenstein integrals (Schlotterer, Brödel)

Modular graph functions

The genus 1 contribution to the graviton amplitude of closed superstrings are integrals over the moduli $\tau = \tau_1 + i\tau_2 \in \mathfrak{M}_{1,1} \cong \mathbb{H}/\mathsf{PSL}_2(\mathbb{Z})$ of

$$\mathcal{B}_4(\{s_{ij}\}\,|\tau) = \left(\prod_{k=1}^{3} \int_{\mathcal{E}_\tau} \frac{\mathrm{d}^2 z_k}{\tau_2}\right) \exp\left(\sum_{1\le i<j\le 4} s_{ij}\mathcal{G}(z_i - z_j|\tau)\right)\Bigg|_{z_4=0}.$$

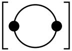
The Green's function on the torus $\mathcal{E}_\tau = \mathbb{C}/\Lambda_\tau$ with $\Lambda_\tau = \mathbb{Z} \oplus \tau\mathbb{Z}$ is

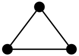
$$\mathcal{G}(z|\tau) = \frac{\tau_2}{\pi} \sum_{\omega \in \Lambda_\tau \setminus \{0\}} \frac{1}{\omega^2} \exp\left[\frac{\pi}{\tau_2}(\omega\bar{z} - \bar{\omega} z)\right].$$

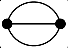
The low energy expansion is indexed by simple graphs G, with coefficients

$$\mathbf{D}\,[G]\,(\tau,\bar{\tau}) := \left(\prod_{v\in V(G)} \int_{\mathcal{E}_\tau} \frac{\mathrm{d}^2 z_k}{\tau_2}\right) \prod_{i\to j\in E(G)} \mathcal{G}(z_i - z_j|\tau).$$

Examples:

$\mathbf{D}\Big[$  $\Big] = \int_{\mathcal{E}_\tau} \frac{\mathrm{d}^2 z_1}{\tau_2}\ \mathcal{G}(z_1|\tau)^2$

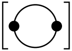
$\mathbf{D}\Big[$  $\Big] = \int_{\mathcal{E}_\tau} \frac{\mathrm{d}^2 z_1}{\tau_2} \int_{\mathcal{E}_\tau} \frac{\mathrm{d}^2 z_2}{\tau_2}\ \mathcal{G}(z_1|\tau)\mathcal{G}(z_2|\tau)\mathcal{G}(z_1 - z_2|\tau)$

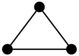
$\mathbf{D}\Big[$  $\Big] = \int_{\mathcal{E}_\tau} \frac{\mathrm{d}^2 z_1}{\tau_2}\ \mathcal{G}(z_1|\tau)^3$

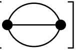
MGFs are modular invariant, real analytic, with MZV coefficients $d_k^{(m,n)}$:

$$\mathbf{D}[G] = \sum_k (\pi\tau_2)^k \sum_{n,m\geq 0} q^m \bar{q}^n d_k^{(m,n)}$$

Examples:

D [] $= \int_{\mathcal{E}_\tau} \frac{\mathrm{d}^2 z_1}{\tau_2}\, \mathcal{G}(z_1|\tau)^2 = E_2$

D [] $= \int_{\mathcal{E}_\tau} \frac{\mathrm{d}^2 z_1}{\tau_2} \int_{\mathcal{E}_\tau} \frac{\mathrm{d}^2 z_2}{\tau_2}\, \mathcal{G}(z_1|\tau)\mathcal{G}(z_2|\tau)\mathcal{G}(z_1 - z_2|\tau) = E_3$

D [] $= \int_{\mathcal{E}_\tau} \frac{\mathrm{d}^2 z_1}{\tau_2}\, \mathcal{G}(z_1|\tau)^3$

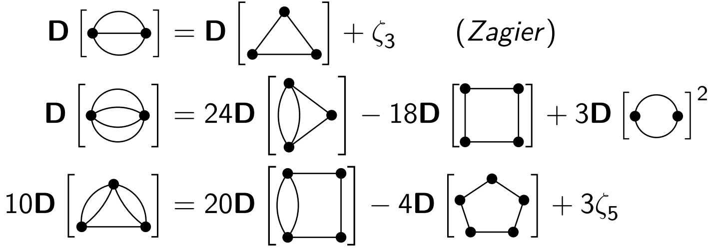
MGFs are modular invariant, real analytic, with MZV coefficients $d_k^{(m,n)}$:

$$\mathbf{D}[G] = \sum_k (\pi\tau_2)^k \sum_{n,m\geq 0} q^m \bar{q}^n d_k^{(m,n)}$$

Real analytic Eisenstein series

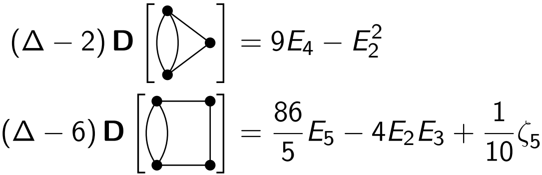
$$E_k(\tau,\bar{\tau}) = \left(\frac{\tau_2}{\pi}\right)^k \sum_{\omega\in\Lambda_\tau\setminus\{0\}} \frac{1}{|\omega|^{2k}}$$

Many identities, for example



Eigenvalue equations with respect to $\Delta = 4\tau_2^2 \partial_\tau \partial_{\bar\tau}$, e.g.

$$(\Delta - k(k-1))\, E_k = 0$$



iterated integrals (Chen 1973)

Take a manifold X and differential forms $\omega_1, \ldots, \omega_n \in \Omega^1(X)$. Integrating these along a path $\gamma \in C^1([0,1], X)$, we can construct functions (on γ):

$$\int_\gamma \omega_1 := \int_0^1 \gamma^*(\omega_1)(t_1)$$

iterated integrals (Chen 1973)

Take a manifold X and differential forms $\omega_1, \dots, \omega_n \in \Omega^1(X)$. Integrating these along a path $\gamma \in C^1([0,1], X)$, we can construct functions (on γ):

$$\int_\gamma \omega_1 := \int_0^1 \gamma^*(\omega_1)(t_1)$$

$$\int_\gamma \omega_2 \omega_1 := \int_0^1 \gamma^*(\omega_2)(t_2) \int_0^{t_2} \gamma^*(\omega_1)(t_1)$$

iterated integrals (Chen 1973)

Take a manifold X and differential forms $\omega_1, \ldots, \omega_n \in \Omega^1(X)$. Integrating these along a path $\gamma \in C^1([0,1], X)$, we can construct functions (on γ):

$$\int_\gamma \omega_r \cdots \omega_1 := \int_0^1 \gamma^*(\omega_r)(t_r) \int_0^{t_r} \cdots \int_0^{t_2} \gamma^*(\omega_1)(t_1)$$

iterated integrals (Chen 1973)

Take a manifold X and differential forms $\omega_1, \ldots, \omega_n \in \Omega^1(X)$. Integrating these along a path $\gamma \in C^1([0,1], X)$, we can construct functions (on γ):

$$\int_\gamma \omega_r \cdots \omega_1 := \int_0^1 \gamma^*(\omega_r)(t_r) \int_0^{t_r} \cdots \int_0^{t_2} \gamma^*(\omega_1)(t_1)$$

1. If $\omega = \mathrm{d}f$ is exact, $\int_\gamma \omega = f(\gamma(1)) - f(\gamma(0))$ is boring.
2. Not all iterated integrals are homotopy invariant.

iterated integrals (Chen 1973)

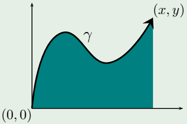
Take a manifold X and differential forms $\omega_1, \dots, \omega_n \in \Omega^1(X)$. Integrating these along a path $\gamma \in C^1([0,1], X)$, we can construct functions (on γ):

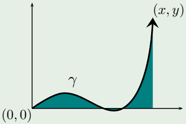
$$\int_\gamma \omega_r \cdots \omega_1 := \int_0^1 \gamma^*(\omega_r)(t_r) \int_0^{t_r} \cdots \int_0^{t_2} \gamma^*(\omega_1)(t_1)$$

1. If $\omega = \mathrm{d}f$ is exact, $\int_\gamma \omega = f(\gamma(1)) - f(\gamma(0))$ is boring.
2. Not all iterated integrals are homotopy invariant.

Example

Take $\omega = y\mathrm{d}x \in \Omega^1(\mathbb{R}^2)$, then $\int_\gamma \omega$ is the area between γ and the x-axis.





$\Rightarrow$ integrability condition (Chen), simplest case:

$$\int_\gamma \omega \quad \text{homotopy invariant} \Leftrightarrow \mathrm{d}\omega = 0$$

Elliptic polylogarithms (Brown & Levin)

Consider $X = \mathcal{E}_\tau = \mathbb{C}/\Lambda_\tau$ where $\Lambda_\tau = \mathbb{Z} \oplus \tau\mathbb{Z}$. The series

$$F(z, \alpha|\tau) = \frac{\vartheta'(0|\tau)\vartheta(z+\alpha|\tau)}{\vartheta(z|\tau)\vartheta(\alpha|\tau)} = \sum_{k\geq 0} \alpha^{k-1} g_k(z|\tau)$$

defines meromorphic functions $g_k(z)$ on $\mathbb{C}$ with

$$g_k(z+1|\tau) = g_k(z|\tau) \qquad g_k(z+\tau|\tau) = g_k(z|\tau) + \sum_{j=1}^{k} g_{k-j}(z|\tau)\frac{(-2i\pi)^j}{j!}.$$

Examples

$$g_0 = 1, \qquad g_1(z) = \frac{\vartheta'(z)}{\vartheta(z)} = \frac{1}{z} + \mathcal{O}(z) \qquad g_2(z) = \frac{\wp(z) - g_1^2(z)}{2}$$

- g_k is smooth for all $k \neq 1$ (on the fundamental domain)
- g_1 has first order poles (with unit residue) on Λ_τ

Fix a finite set $\Sigma \subset \mathbb{C}$ of punctures to define closed forms

$$\omega_{\sigma}^{(n)}(z) = g_n(z-\sigma)\, \mathrm{d}z \quad \in \Omega^1\left(\mathbb{C} \setminus (\sigma + \Lambda_\tau)\right)$$

for each $n \geq 0$ and $\sigma \in \Sigma$. Elliptic MPL are their iterated integrals:

$$\int_0^z \omega_{z_1}^{(n_1)} \cdots \omega_{z_r}^{(n_r)} = \tilde{\Gamma}\left(\begin{matrix} n_1 \cdots n_r \\ z_1 \cdots z_r \end{matrix}; z\right) = \int_0^z \mathrm{d}t\ g_{n_1}(t-z_1)\ \tilde{\Gamma}\left(\begin{matrix} n_2 \cdots n_r \\ z_2 \cdots z_r \end{matrix}; z\right)$$

Fix a finite set $\Sigma \subset \mathbb{C}$ of punctures to define closed forms

$$\omega_\sigma^{(n)}(z) = g_n(z-\sigma)\,\mathrm{d}z \quad \in \Omega^1\left(\mathbb{C} \setminus (\sigma + \Lambda_\tau)\right)$$

for each $n \geq 0$ and $\sigma \in \Sigma$. Elliptic MPL are their iterated integrals:

$$\int_0^z \omega_{z_1}^{(n_1)} \cdots \omega_{z_r}^{(n_r)} = \tilde{\Gamma}\begin{pmatrix} n_1 \cdots n_r \\ z_1 \cdots z_r \end{pmatrix}; z\Big) = \int_0^z \mathrm{d}t\ g_{n_1}(t - z_1)\ \tilde{\Gamma}\begin{pmatrix} n_2 \cdots n_r \\ z_2 \cdots z_r \end{pmatrix}; z\Big)$$

Remarks

1. holomorphic, homotopy invariant
2. not doubly-periodic, not even the forms $\omega_\sigma^{(n)}$
3. functions live on the cover $\mathbb{C} \setminus \bigcup_{\sigma\in\Sigma}(\sigma + \Lambda_\tau)$ of $\mathcal{E} \setminus \Sigma$

Fix a finite set $\Sigma \subset \mathbb{C}$ of punctures to define closed forms

$$\omega_\sigma^{(n)}(z) = g_n(z-\sigma)\,\mathrm{d}z \quad \in \Omega^1\left(\mathbb{C} \setminus (\sigma + \Lambda_\tau)\right)$$

for each $n \geq 0$ and $\sigma \in \Sigma$. Elliptic MPL are their iterated integrals:

$$\int_0^z \omega_{z_1}^{(n_1)} \cdots \omega_{z_r}^{(n_r)} = \tilde{\Gamma}\left(\begin{matrix} n_1 \cdots n_r \\ z_1 \cdots z_r \end{matrix}; z\right) = \int_0^z \mathrm{d}t\; g_{n_1}(t-z_1)\,\tilde{\Gamma}\left(\begin{matrix} n_2 \cdots n_r \\ z_2 \cdots z_r \end{matrix}; z\right)$$

Remarks

1. holomorphic, homotopy invariant
2. not doubly-periodic, not even the forms $\omega_\sigma^{(n)}$
3. functions live on the cover $\mathbb{C} \setminus \bigcup_{\sigma\in\Sigma}(\sigma+\Lambda_\tau)$ of $\mathcal{E} \setminus \Sigma$

$$\mathcal{G}(z|\tau) = \frac{\tau_2}{\pi} \sum_{\omega \in \Lambda_\tau \setminus \{0\}} \frac{1}{\omega^2} \exp\left[\frac{\pi}{\tau_2}(\omega\bar{z} - \bar{\omega} z)\right]$$

Fix a finite set $\Sigma \subset \mathbb{C}$ of punctures to define closed forms

$$\omega_\sigma^{(n)}(z) = g_n(z-\sigma)\,\mathrm{d}z \quad \in \Omega^1\left(\mathbb{C} \setminus (\sigma + \Lambda_\tau)\right)$$

for each $n \geq 0$ and $\sigma \in \Sigma$. Elliptic MPL are their iterated integrals:

$$\int_0^z \omega_{z_1}^{(n_1)} \cdots \omega_{z_r}^{(n_r)} = \tilde{\Gamma}\left(\begin{matrix} n_1 \cdots n_r \\ z_1 \cdots z_r \end{matrix}; z\right) = \int_0^z \mathrm{d}t\; g_{n_1}(t - z_1)\,\tilde{\Gamma}\left(\begin{matrix} n_2 \cdots n_r \\ z_2 \cdots z_r \end{matrix}; z\right)$$

Remarks

1. holomorphic, homotopy invariant
2. not doubly-periodic, not even the forms $\omega_\sigma^{(n)}$
3. functions live on the cover $\mathbb{C} \setminus \bigcup_{\sigma \in \Sigma}(\sigma + \Lambda_\tau)$ of $\mathcal{E} \setminus \Sigma$

$$\mathcal{G}(z|\tau) = -\ln\left|\frac{\vartheta(z|\tau)}{\eta(\tau)}\right|^2 - \frac{\pi}{2\tau_2}(z - \bar{z})^2$$

Fix a finite set $\Sigma \subset \mathbb{C}$ of punctures to define closed forms

$$\omega_\sigma^{(n)}(z) = g_n(z-\sigma)\,\mathrm{d}z \quad \in \Omega^1\left(\mathbb{C} \setminus (\sigma + \Lambda_\tau)\right)$$

for each $n \geq 0$ and $\sigma \in \Sigma$. Elliptic MPL are their iterated integrals:

$$\int_0^z \omega_{z_1}^{(n_1)} \cdots \omega_{z_r}^{(n_r)} = \tilde{\Gamma}\left(\begin{matrix} n_1 \cdots n_r \\ z_1 \cdots z_r \end{matrix}; z\right) = \int_0^z \mathrm{d}t\; g_{n_1}(t-z_1)\,\tilde{\Gamma}\left(\begin{matrix} n_2 \cdots n_r \\ z_2 \cdots z_r \end{matrix}; z\right)$$

Remarks

1. holomorphic, homotopy invariant
2. not doubly-periodic, not even the forms $\omega_\sigma^{(n)}$
3. functions live on the cover $\mathbb{C} \setminus \bigcup_{\sigma\in\Sigma}(\sigma + \Lambda_\tau)$ of $\mathcal{E} \setminus \Sigma$

$$\mathcal{G}(z|\tau) \sim -\ln\left|\frac{\vartheta(z|\tau)}{\vartheta'(0|\tau)}\right|^2 - \frac{\pi}{2\tau_2}(z-\bar{z})^2$$

Fix a finite set $\Sigma \subset \mathbb{C}$ of punctures to define closed forms

$$\omega_\sigma^{(n)}(z) = g_n(z-\sigma)\,\mathrm{d}z \qquad \in \Omega^1\left(\mathbb{C} \setminus (\sigma + \Lambda_\tau)\right)$$

for each $n \geq 0$ and $\sigma \in \Sigma$. Elliptic MPL are their iterated integrals:

$$\int_0^z \omega_{z_1}^{(n_1)} \cdots \omega_{z_r}^{(n_r)} = \tilde{\Gamma}\begin{pmatrix} n_1 \cdots n_r \\ z_1 \cdots z_r \end{pmatrix}; z\Big) = \int_0^z \mathrm{d}t\; g_{n_1}(t-z_1)\,\tilde{\Gamma}\begin{pmatrix} n_2 \cdots n_r \\ z_2 \cdots z_r \end{pmatrix}; z\Big)$$

Remarks

1. holomorphic, homotopy invariant
2. not doubly-periodic, not even the forms $\omega_\sigma^{(n)}$
3. functions live on the cover $\mathbb{C} \setminus \bigcup_{\sigma \in \Sigma}(\sigma + \Lambda_\tau)$ of $\mathcal{E} \setminus \Sigma$

$$\begin{aligned}\mathcal{G}(z|\tau) &\sim -\ln\left|\frac{\vartheta(z|\tau)}{\vartheta'(0|\tau)}\right|^2 - \frac{\pi}{2\tau_2}(z-\bar{z})^2 \\ &= -\tilde{\Gamma}\left(\begin{matrix}1\\0\end{matrix}; z\right) + \text{c.c.} - \frac{\pi}{\tau_2}\left(\tilde{\Gamma}\left(\begin{matrix}0\ 0\\0\ 0\end{matrix}; z\right) + \text{c.c.} - \tilde{\Gamma}\left(\begin{matrix}0\\0\end{matrix}; z\right)\tilde{\Gamma}^*\left(\begin{matrix}0\\0\end{matrix}; z\right)\right)\end{aligned}$$

So, the integrand of MGFs is contained in the algebra $\mathcal{A}_n$ generated by

$$\int_{\sigma_0}^{\sigma_{r+1}} \omega_{z_1}^{(n_1)} \cdots \omega_{z_r}^{(n_r)} \quad \text{and their c.c.}$$

where $n_i \geq 0$ and $\sigma_i \in \Sigma$. This $\mathcal{A}_n$ defines a subsheaf of $\Omega^0(\mathrm{Conf}_n(\mathcal{E}_\tau))$.

Approach

Integrate out each puncture sequentially along the fibrations

$$\mathcal{E}_\tau \setminus \{z_1, \ldots, z_{n-1}\} \hookrightarrow \mathrm{Conf}_n(\mathcal{E}_\tau) \twoheadrightarrow \mathrm{Conf}_{n-1}(\mathcal{E}_\tau)$$

So, the integrand of MGFs is contained in the algebra $\mathcal{A}_n$ generated by

$$\int_{\sigma_0}^{\sigma_{r+1}} \omega_{z_1}^{(n_1)} \cdots \omega_{z_r}^{(n_r)} \quad \text{and their c.c.}$$

where $n_i \geq 0$ and $\sigma_i \in \Sigma$. This $\mathcal{A}_n$ defines a subsheaf of $\Omega^0(\mathrm{Conf}_n(\mathcal{E}_\tau))$.

Approach

Integrate out each puncture sequentially along the fibrations

$$\mathcal{E}_\tau \setminus \{z_1, \ldots, z_{n-1}\} \hookrightarrow \mathrm{Conf}_n(\mathcal{E}_\tau) \twoheadrightarrow \mathrm{Conf}_{n-1}(\mathcal{E}_\tau)$$

Every period $f \in \mathcal{A}_n$ is an iterated integral on the fibre, e.g.

$$f = \sum_{u,v} \int_0^{z_n} u \cdot \left(\int_0^{z_n} v \right)^* \cdot f_{u,v}$$

where $f_{u,v} \in \mathcal{A}_{n-1}$ and u, v are forms independent of z_n.

Example

$$\tilde{\Gamma}\left(\begin{matrix}1\,1\\ z\,0\end{matrix}; z\right) = 2\tilde{\Gamma}\left(\begin{matrix}0\,2\\ 0\,0\end{matrix}; z\right) + \tilde{\Gamma}\left(\begin{matrix}2\,2\\ 0\,0\end{matrix}; z\right) - 2\tilde{\Gamma}\left(\begin{matrix}1\,1\\ 0\,0\end{matrix}; z\right) + \zeta_2$$

Integration

Suppose we have written the integrand in the form

$$f = \left[\sum_{u,v} \int_0^{z_n} u \cdot \left(\int_0^{z_n} v \right)^* \cdot f_{u,v} \right] \cdot \mathrm{d}z_n \wedge \mathrm{d}\bar{z}_n,$$

Then we can easily find a primitive F with $\mathrm{d}F = f$ as

$$F = \left[\sum_{u,v} \int_0^{z_n} \omega_0^{(0)} u \cdot \left(\int_0^{z_n} v \right)^* \cdot f_{u,v} \right] \cdot \mathrm{d}\bar{z}_n.$$

Idea

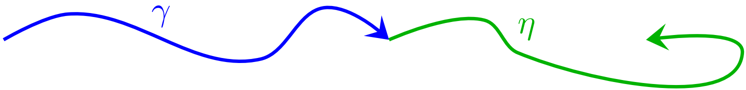
Apply Stokes to the fundamental domain $D = [0,1] \times [0,\tau] \setminus \Sigma$:

$$\int_D f = \int_{\partial D} F.$$

Problem: F does not extend to a smooth function on D°. In other words, F is not single-valued.

Path concatenation

Let $\gamma \star \eta$ denote the concatenation of γ and η at $\gamma(1) = \eta(0) = (\gamma \star \eta)(\frac{1}{2})$:



To decompose

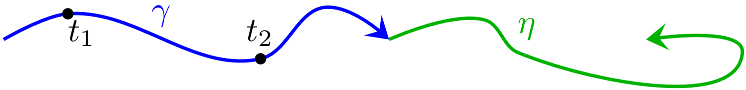
$$\int_{\gamma \star \eta} \omega_2 \omega_1 = \int_{0 \le t_1 \le t_2 \le 1} (\gamma \star \eta)^*(w_2)(t_2)(\gamma \star \eta)^*(w_1)(t_1),$$

split the interval

$$\underbrace{\{t_1 \le t_2\}}_{\int_{\gamma \star \eta} \omega_2 \omega_1} = \{t_1 \le t_2 \le \tfrac{1}{2}\} \cup \{t_1 \le \tfrac{1}{2} \le t_2\} \cup \{\tfrac{1}{2} \le t_1 \le t_2\}$$

Path concatenation

Let $\gamma \star \eta$ denote the concatenation of γ and η at $\gamma(1) = \eta(0) = (\gamma \star \eta)(\frac{1}{2})$:



To decompose

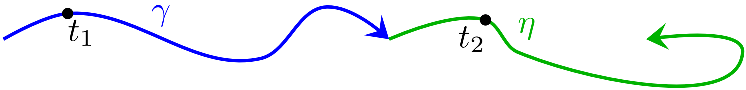
$$\int_{\gamma\star\eta} \omega_2\omega_1 = \int_{0\le t_1\le t_2\le 1} (\gamma \star \eta)^*(w_2)(t_2)(\gamma \star \eta)^*(w_1)(t_1),$$

split the interval

$$\underbrace{\{t_1 \le t_2\}}_{\int_{\gamma\star\eta}\omega_2\omega_1} = \underbrace{\{t_1 \le t_2 \le \tfrac{1}{2}\}}_{\int_{\gamma}\omega_2\omega_1} \cup \{t_1 \le \tfrac{1}{2} \le t_2\} \cup \{\tfrac{1}{2} \le t_1 \le t_2\}$$

Path concatenation

Let $\gamma \star \eta$ denote the concatenation of γ and η at $\gamma(1) = \eta(0) = (\gamma \star \eta)(\frac{1}{2})$:



To decompose

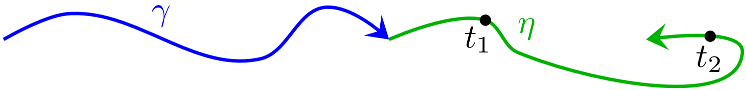
$$\int_{\gamma \star \eta} \omega_2 \omega_1 = \int_{0 \leq t_1 \leq t_2 \leq 1} (\gamma \star \eta)^*(w_2)(t_2)(\gamma \star \eta)^*(w_1)(t_1),$$

split the interval

$$\underbrace{\{t_1 \leq t_2\}}_{\int_{\gamma \star \eta} \omega_2 \omega_1} = \underbrace{\{t_1 \leq t_2 \leq \tfrac{1}{2}\}}_{\int_{\gamma} \omega_2 \omega_1} \cup \underbrace{\{t_1 \leq \tfrac{1}{2} \leq t_2\}}_{\int_{\eta} \omega_2 \int_{\gamma} \omega_1} \cup \{\tfrac{1}{2} \leq t_1 \leq t_2\}$$

Path concatenation

Let $\gamma \star \eta$ denote the concatenation of γ and η at $\gamma(1) = \eta(0) = (\gamma \star \eta)(\frac{1}{2})$:



To decompose

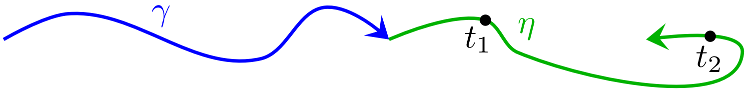
$$\int_{\gamma\star\eta} \omega_2\omega_1 = \int_{0\le t_1\le t_2\le 1} (\gamma\star\eta)^*(w_2)(t_2)(\gamma\star\eta)^*(w_1)(t_1),$$

split the interval

$$\underbrace{\{t_1 \le t_2\}}_{\int_{\gamma\star\eta}\omega_2\omega_1} = \underbrace{\{t_1 \le t_2 \le \tfrac{1}{2}\}}_{\int_\gamma \omega_2\omega_1} \cup \underbrace{\{t_1 \le \tfrac{1}{2} \le t_2\}}_{\int_\eta \omega_2 \int_\gamma \omega_1} \cup \underbrace{\{\tfrac{1}{2} \le t_1 \le t_2\}}_{\int_\eta \omega_2\omega_1}$$

Path concatenation

Let $\gamma \star \eta$ denote the concatenation of γ and η at $\gamma(1) = \eta(0) = (\gamma \star \eta)(\frac{1}{2})$:



To decompose

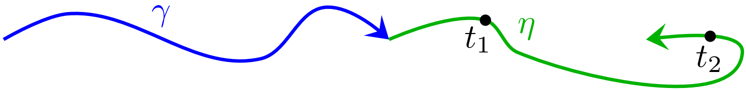
$$\int_{\gamma\star\eta} \omega_2\omega_1 = \int_\gamma \omega_2\omega_1 + \int_\eta \omega_2 \int_\gamma \omega_1 + \int_\eta \omega_2\omega_1,$$

split the interval

$$\underbrace{\{t_1 \le t_2\}}_{\int_{\gamma\star\eta} \omega_2\omega_1} = \underbrace{\{t_1 \le t_2 \le \tfrac{1}{2}\}}_{\int_\gamma \omega_2\omega_1} \cup \underbrace{\{t_1 \le \tfrac{1}{2} \le t_2\}}_{\int_\eta \omega_2 \int_\gamma \omega_1} \cup \underbrace{\{\tfrac{1}{2} \le t_1 \le t_2\}}_{\int_\eta \omega_2\omega_1}$$

Path concatenation

Let $\gamma \star \eta$ denote the concatenation of γ and η at $\gamma(1) = \eta(0) = (\gamma \star \eta)(\frac{1}{2})$:



To decompose

$$\int_{\gamma\star\eta} \omega_2\omega_1 = \int_\gamma \omega_2\omega_1 + \int_\eta \omega_2 \int_\gamma \omega_1 + \int_\eta \omega_2\omega_1,$$

split the interval

$$\underbrace{\{t_1 \leq t_2\}}_{\int_{\gamma\star\eta}\omega_2\omega_1} = \underbrace{\{t_1 \leq t_2 \leq \tfrac{1}{2}\}}_{\int_\gamma \omega_2\omega_1} \cup \underbrace{\{t_1 \leq \tfrac{1}{2} \leq t_2\}}_{\int_\eta \omega_2 \int_\gamma \omega_1} \cup \underbrace{\{\tfrac{1}{2} \leq t_1 \leq t_2\}}_{\int_\eta \omega_2\omega_1}$$

More generally, the path concatenation formula reads

$$\int_{\gamma\star\eta} \omega_r \cdots \omega_1 = \sum_{k=0}^{r} \int_\eta \omega_r \cdots \omega_{k+1} \int_\gamma \omega_k \cdots \omega_1.$$

Monodromy

Analytic continuation $\mathcal{M}_\eta$ along a closed loop η with $\eta(0) = \eta(1) = 0$ is

$$\mathcal{M}_\eta \int_0^z \omega_r \cdots \omega_1 = \sum_{k=0}^{r} \int_0^z \omega_r \cdots \omega_{k+1} \underbrace{\int_\eta \omega_k \cdots \omega_1}_{\in \mathcal{A}_{n-1}} .$$

Monodromy

Analytic continuation $\mathcal{M}_\eta$ along a closed loop η with $\eta(0) = \eta(1) = 0$ is

$$\mathcal{M}_\eta \int_0^z \omega_r \cdots \omega_1 = \sum_{k=0}^{r} \int_0^z \omega_r \cdots \omega_{k+1} \underbrace{\int_\eta \omega_k \cdots \omega_1}_{\in \mathcal{A}_{n-1}}.$$

Monodromy and derivatives commute

$$\partial_z \left(\mathcal{M}_\eta - \mathrm{id}\right) F = \left(\mathcal{M}_\eta - \mathrm{id}\right) \partial_z F = \left(\mathcal{M}_\eta - \mathrm{id}\right) f = 0$$

$\Rightarrow$ the monodromies of F are antiholomorphic:

Monodromy

Analytic continuation $\mathcal{M}_\eta$ along a closed loop η with $\eta(0) = \eta(1) = 0$ is

$$\mathcal{M}_\eta \int_0^z \omega_r \cdots \omega_1 = \sum_{k=0}^{r} \int_0^z \omega_r \cdots \omega_{k+1} \underbrace{\int_\eta \omega_k \cdots \omega_1}_{\in \mathcal{A}_{n-1}} .$$

Monodromy and derivatives commute

$$\partial_z (\mathcal{M}_\eta - \mathrm{id}) F = (\mathcal{M}_\eta - \mathrm{id}) \partial_z F = (\mathcal{M}_\eta - \mathrm{id}) f = 0$$

$\Rightarrow$ the monodromies of F are antiholomorphic:

$$(\mathcal{M}_{\eta_\sigma} - \mathrm{id}) F = \sum_u \left(\int_0^{z_n} u \right)^* F_u^\sigma$$

for any basis $\eta_\sigma \in \pi_1(\mathcal{E}_\tau \setminus \Sigma)$ of loops. We can choose them such that

$$\int_{\eta_\sigma} \omega_z^{(n)} = (2i\pi)\delta_{\sigma,z}\delta_{1,n}$$

Note that the leading length of the monodromy is

$$(\mathcal{M}_\eta - \mathrm{id}) \int_0^z \omega_n \cdots \omega_1 = \int_0^z \omega_n \cdots \omega_2 \int_\eta \omega_1 + \text{lower length}$$

Note that the leading length of the monodromy is

$$(\mathcal{M}_\eta - \mathrm{id}) \int_0^z \omega_n \cdots \omega_1 = \int_0^z \omega_n \cdots \omega_2 \int_\eta \omega_1 + \text{lower length}$$

So there is an antiholomorphic form with the opposite monodromies:

$$(\mathcal{M}_{\eta_\sigma} - \mathrm{id}) \left\{ \sum_{p \in \Sigma} \sum_u \left(\int_0^{z_n} u \omega_p^1 \right) \frac{F_u^\sigma}{2i\pi} \right\} = -\sum_u \left(\int_0^{z_n} u \right)^* F_u^\sigma + \text{lower length}$$

Note that the leading length of the monodromy is

$$(\mathcal{M}_\eta - \mathrm{id}) \int_0^z \omega_n \cdots \omega_1 = \int_0^z \omega_n \cdots \omega_2 \int_\eta \omega_1 + \text{lower length}$$

So there is an antiholomorphic form with the opposite monodromies:

$$(\mathcal{M}_{\eta_\sigma} - \mathrm{id}) \left\{ \sum_{p \in \Sigma} \sum_u \left(\int_0^{z_n} u \omega_p^1 \right) \frac{F_u^\sigma}{2i\pi} \right\} = -(\mathcal{M}_{\eta_\sigma} - \mathrm{id}) F + \text{lower length}$$

Corollary: Existence of single-valued primitives

By adding antiholomorphic functions, we can find a primitive $F \in \mathcal{A}_n$ with

$$\mathrm{d}F = f \quad \text{and} \quad (\mathcal{M}_{\eta_\sigma} - \mathrm{id}) F = 0 \quad \text{for all } \sigma \in \Sigma$$

insert picture of fundamental domain

Stokes' theorem $\int_D f = \int_{\partial D} F$ gets contributions from

1. the punctures $\sigma \in \Sigma$:

$$\lim_{r \to 0} \oint_{|z-\sigma|=r} F = 0$$

2. the sides of D:

$$\int_0^1 F + \int_{1+\tau}^{\tau} F = -\int_0^1 \left(\mathcal{M}_{[0,\tau]} - \mathrm{id}\right) F$$
$$\int_1^{1+\tau} F + \int_{\tau}^{0} F = \int_0^{\tau} \left(\mathcal{M}_{[0,1]} - \mathrm{id}\right) F$$

Stokes' theorem $\int_D f = \int_{\partial D} F$ gets contributions from

1. the punctures $\sigma \in \Sigma$:

$$\lim_{r \to 0} \oint_{|z-\sigma|=r} F = 0$$

2. the sides of D:

$$\int_0^1 F + \int_{1+\tau}^{\tau} F = -\int_0^1 \left(\mathcal{M}_{[0,\tau]} - \mathrm{id}\right) F \in \mathcal{A}_{n-1}$$
$$\int_1^{1+\tau} F + \int_{\tau}^{0} F = \int_0^{\tau} \left(\mathcal{M}_{[0,1]} - \mathrm{id}\right) F \in \mathcal{A}_{n-1}$$

Recall

The monodromies

$$\left(\mathcal{M}_{[0,\tau]} - \mathrm{id}\right) F \quad \text{and} \quad \left(\mathcal{M}_{[0,1]} - \mathrm{id}\right) F$$

are antiholomorphic iterated integrals.

Summary

Given a function $f \in \mathcal{A}_n$ single-valued on $\mathrm{Conf}_n(\mathcal{E}_\tau)$:

1. There is a function $\mathcal{F} \in \mathcal{A}_n$ that is single-valued on D° with $\partial_{z_n} F = f$.
2. We can apply Stokes' theorem to $\mathrm{d}(F\mathrm{d}\bar{z}_n) = f\,\mathrm{d}z_n \wedge \mathrm{d}\bar{z}_n$.
3. All contributions are eMPL on the base $\mathcal{A}_{n-1}$.
4. Due to convergence, the result is necessarily single-valued and descends to $\mathrm{Conf}_{n-1}(\mathcal{E}_\tau)$.

Summary

Given a function $f \in \mathcal{A}_n$ single-valued on $\mathrm{Conf}_n(\mathcal{E}_\tau)$:

1. There is a function $\mathcal{F} \in \mathcal{A}_n$ that is single-valued on D° with $\partial_{z_n} F = f$.
2. We can apply Stokes' theorem to $\mathrm{d}(F\mathrm{d}\bar{z}_n) = f\mathrm{d}z_n \wedge \mathrm{d}\bar{z}_n$.
3. All contributions are eMPL on the base $\mathcal{A}_{n-1}$.
4. Due to convergence, the result is necessarily single-valued and descends to $\mathrm{Conf}_{n-1}(\mathcal{E}_\tau)$.

Corollary

After integrating out all but one puncture, a MGF is thus expressed in terms of iterated integrals on $\mathcal{E}_\tau^\times$, that is, eMZV and their c.c.

$$\omega_A(n_1, \cdots, n_r) = \int_0^1 \omega_0^{(n_1)} \cdots \omega_0^{(n_r)}, \quad \omega_B(n_1, \cdots, n_r) = \int_0^\tau \omega_0^{(n_1)} \cdots \omega_0^{(n_r)}$$

Iterated Eisenstein integrals

Theorem (Enriquez, Matthes, Brödel, Schlotterer, Mafra, Zerbini)

eMZV can be written as iterated Eisenstein integrals.

$$2\pi i \partial_\tau \int_a^b g_{n_1} \cdots g_{n_r} = n_1 g_{1+n_1}(b) \int_a^b g_{n_2} \cdots g_{n_r} - n_r g_{1+n_r}(a) \int_a^b g_{n_1} \cdots g_{n_{r-1}}$$

$$+ \sum_{\mu=1}^{r-1} \sum_{k=0}^{n_\mu + n_{\mu+1}+1} (n_\mu + n_{\mu+1} - k) \left[\binom{k-1}{n_{\mu+1}-1} - \binom{k-1}{k-n_\mu} \right] G_{n_\mu + n_{\mu+1}+1-k}$$

$$\times \int_a^b g_{n_1} \cdots g_{n_{\mu-1}} g_k g_{n_{\mu+2}} \cdots g_{n_r}$$

extras

Shuffle product

The shuffle product of two words

$$w_{n+m} \cdots w_{n+1} \sqcup\!\sqcup\, w_n \cdots w_1 = \sum_{\sigma} w_{\sigma(n+m)} \cdots w_{\sigma(1)}$$

is the sum of all their shuffles σ, i.e. permutations which preserve the relative order of letters in both factors:

$$\sigma^{-1}(1) < \cdots < \sigma^{-1}(n) \quad \text{and} \quad \sigma^{-1}(n+1) < \cdots < \sigma^{-1}(n+m).$$

For arbitrary words u and v, we find that ($\int_\gamma$ is linearly extended)

$$\left(\int_\gamma u\right) \cdot \left(\int_\gamma v\right) = \int_\gamma (u \sqcup\!\sqcup\, v).$$

Shuffle product

The shuffle product of two words

$$w_{n+m} \cdots w_{n+1} \sqcup\!\sqcup\, w_n \cdots w_1 = \sum_{\sigma} w_{\sigma(n+m)} \cdots w_{\sigma(1)}$$

is the sum of all their shuffles σ, i.e. permutations which preserve the relative order of letters in both factors:

$$\sigma^{-1}(1) < \cdots < \sigma^{-1}(n) \quad \text{and} \quad \sigma^{-1}(n+1) < \cdots < \sigma^{-1}(n+m).$$

For arbitrary words u and v, we find that ($\int_\gamma$ is linearly extended)

$$\left(\int_\gamma u\right) \cdot \left(\int_\gamma v\right) = \int_\gamma (u \sqcup\!\sqcup\, v).$$

Example

$$\int_\gamma \omega_3 \cdot \int_\gamma \omega_2\omega_1 = \int_\gamma \omega_3\omega_2\omega_1 + \int_\gamma \omega_2\omega_3\omega_1 + \int_\gamma \omega_2\omega_1\omega_3$$

$$\{t_3\} \times \{t_1 \le t_2\} = \{t_1 \le t_2 \le t_3\} \cup \{t_1 \le t_3 \le t_2\} \cup \{t_3 \le t_1 \le t_2\}$$

Shuffle product

The shuffle product of two words

$$w_{n+m}\cdots w_{n+1} \sqcup\!\sqcup\, w_n \cdots w_1 = \sum_{\sigma} w_{\sigma(n+m)} \cdots w_{\sigma(1)}$$

is the sum of all their shuffles σ, i.e. permutations which preserve the relative order of letters in both factors:

$$\sigma^{-1}(1) < \cdots < \sigma^{-1}(n) \quad \text{and} \quad \sigma^{-1}(n+1) < \cdots < \sigma^{-1}(n+m).$$

For arbitrary words u and v, we find that ($\int_\gamma$ is linearly extended)

$$\left(\int_\gamma u\right) \cdot \left(\int_\gamma v\right) = \int_\gamma (u \sqcup\!\sqcup\, v).$$

Example

$$\int_\gamma \omega_3 \cdot \int_\gamma \omega_2\omega_1 = \int_\gamma (\omega_3\omega_2\omega_1 + \omega_2\omega_3\omega_1 + \omega_2\omega_1\omega_3)$$

$$\{t_3\} \times \{t_1 \leq t_2\} = \{t_1 \leq t_2 \leq t_3\} \cup \{t_1 \leq t_3 \leq t_2\} \cup \{t_3 \leq t_1 \leq t_2\}$$

$$
\begin{aligned}
\mathrm{d}\tilde{\Gamma}\begin{pmatrix} n_1 \cdots n_r \\ z_1 \cdots z_r \end{pmatrix}; z\Big) &= \sum_{p=1}^{k-1} (-1)^{n_p+1} \tilde{\Gamma}\begin{pmatrix} \cdots n_{p-1} & 0 & n_{p+1} \cdots \\ \cdots z_{p-1} & 0 & z_{p+1} \cdots \end{pmatrix} \omega_{p,p+1}^{n_p+n_{p+1}} \\
&+ \sum_{p=1}^{k} \sum_{r=0}^{n_p+1} \Bigg[\binom{n_{p-1}+r-1}{n_{p-1}-1} \tilde{\Gamma}\begin{pmatrix} \cdots & n_{p-1}+r & n_{p+1} & \cdots \\ & z_{p-1} & z_{p+1} & \end{pmatrix} \omega_{p,p-1}^{n_p-r} \\
&- \binom{n_{p+1}+r-1}{n_{p+1}-1} \tilde{\Gamma}\begin{pmatrix} \cdots & n_{p-1} & n_{p+1}+r & \cdots \\ & z_{p-1} & z_{p+1} & \end{pmatrix} \omega_{p,p+1}^{n_p-r} \Bigg]
\end{aligned}
$$

where

$$
\omega_{ij}^{n} = (\mathrm{d}z_j - \mathrm{d}z_i) g_n(z_j - z_i; \tau) + \frac{n\mathrm{d}\tau}{2i\pi} g_{n+1}(z_j - z_i; \tau)
$$