Path-Integral "Complexity" for Perturbed CFTs

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"*Path-Integral Complexity for Perturbed CFTs*" *arXiv:1804.01999*
with *Pawel Caputa, Sumit R. Das, Nilay Kundu, Masamichi Miyaji andTadashi Takayanagi*

(related paper: ArxIv: 1706.07056 by
Pawel Caputa, Nilay Kundu, Masamichi Miyaji, Kento Watanabe, Tadashi Takayanagi,

and the next talk by Javier Molina-Vilaplana)

Outline

– Motivation

– Optimization of Path Integral

– Local Renormalization Group flow

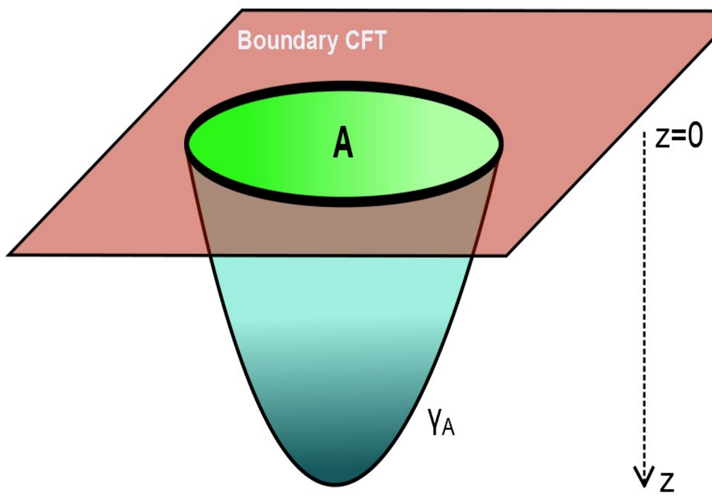
– Comparison with Holography

– Summary and Outlook

Motivation

What is the basic mechanism of AdS/CFT?

There are many paths that currently being pursued:
We will focus on Entanglement for this talk



Ryu-Takayanagi prescription:

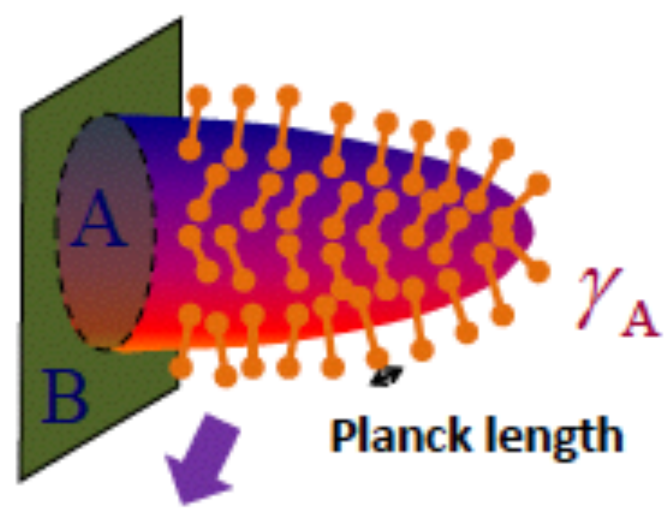
$$S_{EE} = \frac{2\pi \mathrm{Area}(\gamma_{\mathrm{A}})}{\ell_p^{d-1}}$$

(Ryu -Takayanagi, Phys.Rev.Lett.96:181602,2006)

Can we demystify Holography using using *Entanglement* ?

The HEE suggests that

A spacetime in gravity
= Collections of qubits
of quantum entanglement



Network of Quantum Entanglement

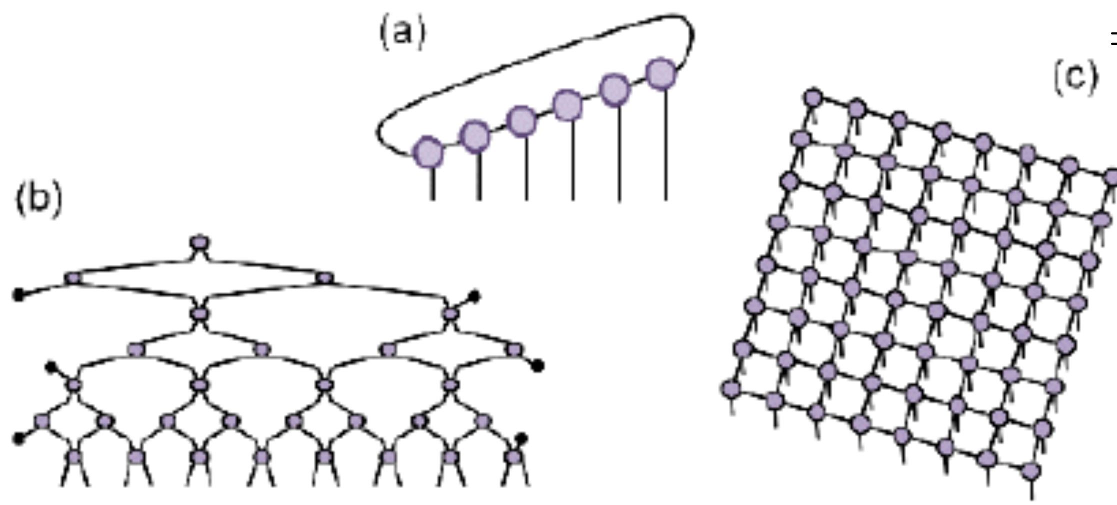
A useful explicit framework for this is the tensor networks.
Tensor networks = A graphical description of wave function.

Tensor Networks :

(representing wavefunctions)

- Well known concept in condensed matter community
- Quantum many body systems: "*curse of dimensionality*"

$$|\psi > = f_{i_1,i_2,\cdots i_N}|i_1,i_2,\cdots i_N >$$
$$= \sum_{\mu_1,\mu_2\cdots\mu_n} T^{I_1}_{i_1\cdots\mu_1} T^{I_2}_{\mu_1\cdots\mu_2}\cdots|i_1,i_2,\cdots i_N >$$



(courtesy Roman Orus : Advances in Tensor network)

figure courtesy Singh and Vidal, Global symmetries in tensor network states: symmetric tensors versus minimal bond dimension

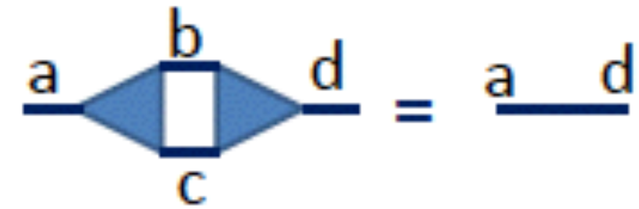
To name a few: MPS, PEPS, MERA : All these exploit entanglement to simplify the many body wavefunctions

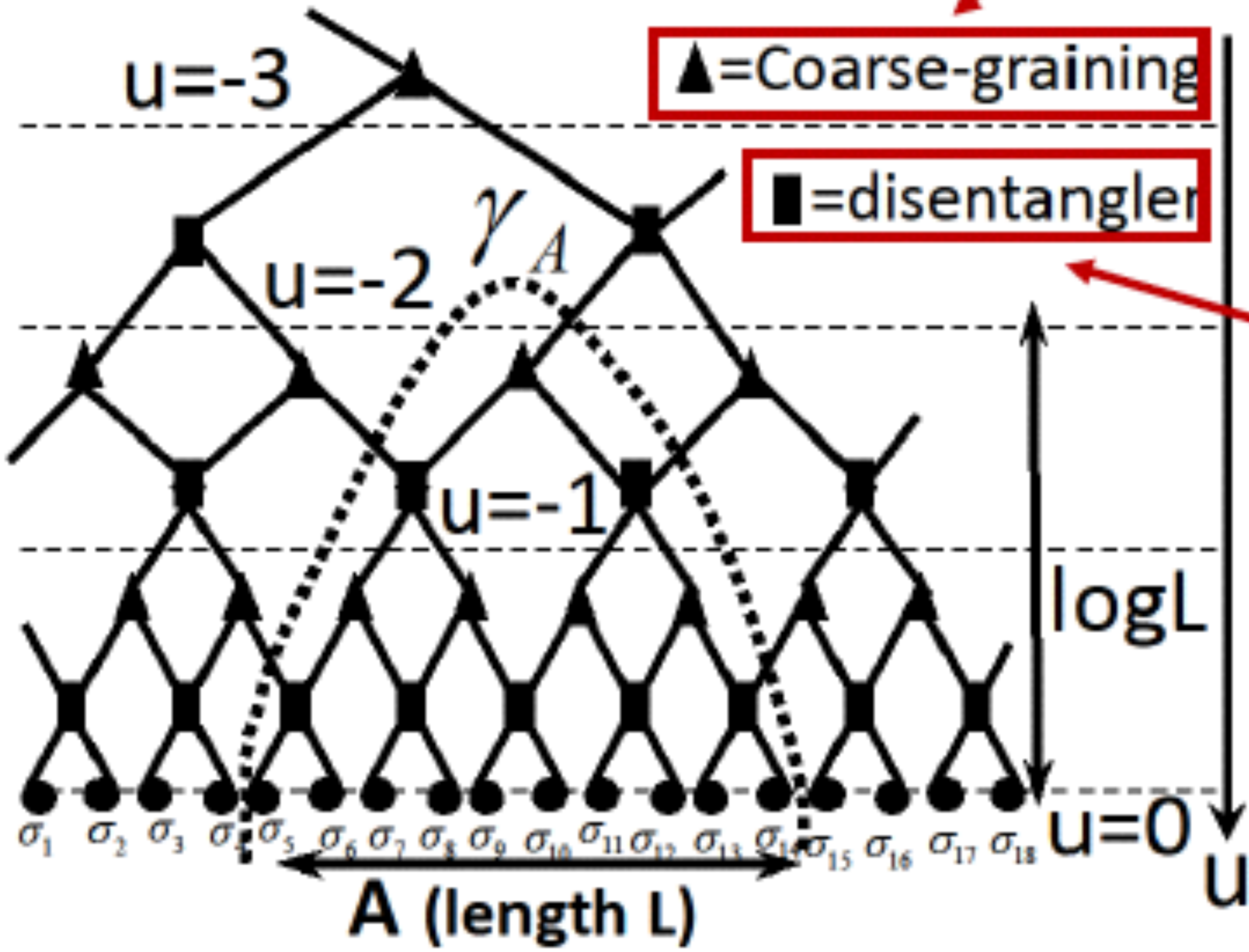
- Numerous application; Classifying gapped systems, topological phases, Correlation functions etc… and also recently in AdS/CFT.

MERA [Vidal 05, …] [TN for AdS/CFT: Swingle 09,…]

Coarse-graining = Isometry

$$[T]^{\dagger}_{abc}[T]_{bcd} = \delta_{ad}$$





Disentangler = Unitary trf.

$S_A \leq \mathrm{Min}[\#\,\mathrm{links}]$
$\propto \log L$
$\Rightarrow$ agrees with results in 2d CFT !

Observation: Tensor Network of MERA=
Describing a time slice of MERA

Further Developments and Motivations

- Models where EE bound is saturated based on Error corrections, Random tensors etc

  (Preskill,Pastawski,Harlow, Yoshida '15, Hayden, Nezami, Qi, Thomas, Walter Yang '16)

- Proposals for computing correlation functions and Bulk/ Boundary reconstruction (HKLL)

  (AB, Hung, Liu, Shen '16, AB, Hung, Li, 17)

- Progress towards understanding Causal structure etc.

  (Benny '11, Czech, Vidal '15, AB and Hung '18)

Some Questions:

- But these are only baby steps , precise reconstruction of Holography dictionary is still eluding for example: role of large central charge, sparse spectrum etc.

- Sub-AdS locality? Role of conformal symmetry on these discrete lattices, How to define a metric? Notion of Stress tensor

- Some of this problems maybe due to the lattice artifacts and to genuinely understand AdS/CFT we want to take somehow a continuous limit.

Also we want to study back reaction through the tensor networks. Perturbing CFT's and studying the back reaction to the geometry is an important aspect of AdS/CFT. But in general it is difficult to study it in terms of tensor network. Lattice discretization makes is generally difficult and hard to control operator growth inside the network.

Motivated by this we want to resort to a different approach based on studying Euclidean path integral, related to the continuum limit of tensor network.

Guiding principle I:
Representation of wave functional (written in terms of path integral) by tensor network = arrangements of these tensor gives a metric

Guiding principle II:

Find an efficient way to represent this network = Optimize w.r.t this metric, gives some sort of dynamical equations.

Efficient for discussing backreactions

By product: Gives a systematic definition of computing "Complexity"in "Quantum Field theory"

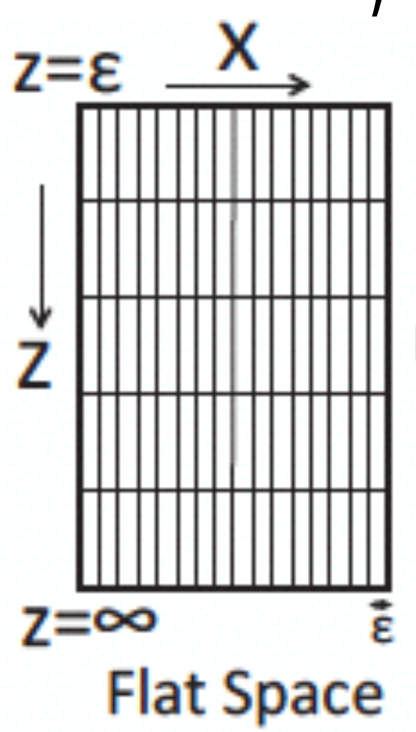
Path-Integral Optimization

We will work in d=2 dimensions

(AB, PC, SD, NK, MM, TT '18
PC, SD, NK, MM, KW, TT'17)

$$\psi_{g_0=\eta_{ab},\lambda_0}[\phi(x)] = \int \prod_x \prod_{\epsilon<z<\infty} [D\phi(x,z)] e^{-S_{g_0=\eta_{ab},\lambda_0}[\phi]} \prod_x \delta(\phi(x,\epsilon) - \phi(x))$$

(similarly for $-\infty < z < \epsilon$)



Area of one unit cell: $ds^2 = \frac{1}{\epsilon^2}(dx^2 + dx^2)$

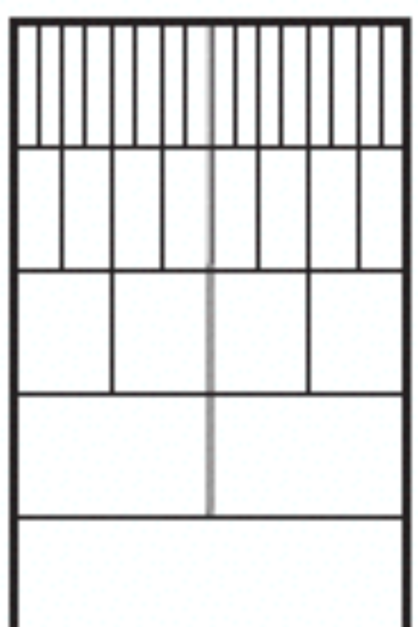
Now we know we can ignore high momentum mode $k > \frac{1}{z}$

Example: Free Massless Scalar

$$\Psi[\phi(x)] = \exp\left(-4\pi \int_0^\infty dz \int_{-\infty}^\infty dk k^2 e^{-2|k|z} \phi(k)\phi(-k)\right)$$

$$= \exp\left(-2\pi \int_{-\infty}^\infty dk\, |k|\, \phi(k)\phi(-k)\right).$$

So we can coarse grain the lattice for large z and fine grain it near small z. That gives the "optimize" path integral.



Now to calculate the area of unit cell we introduce a position dependent cut-off. $\exp(\tilde{\phi}(x,z))$

$$ds^2 = \exp(2\tilde{\phi}(x,z))(dz^2 + dx^2)$$

but with the boundary conditions:

$$g_{zz}(z=\epsilon, x) = \epsilon^{-2}, g_{xx}(z=\epsilon, x) = \epsilon^{-2}, g_{iz} = 0.$$

Basically we have done a Weyl-Rescaling!!!

$$\boxed{\psi_{\eta_{ab}}[\phi(x)] = \mathcal{N}(\tilde{\phi}(x),\lambda_0)\psi_{\eta_{ab}e^{\tilde{\phi}(x)}}[\phi(x)]}$$

Then to make this most optimization most efficient we "minimize" this proportionality factor.

Example I: CFT

$$S^{\eta_{ab}}_{CFT} = S^{\eta_{ab}e^{2\tilde{\phi}(x,z)}}_{CFT}$$

(AB, PC, SD, NK, MM, TT '18
PC, SD, NK, MM, KW, TT'17)

Only the measure changes: $[D\phi]_{\eta_{ab}e^{2\tilde{\phi}(x,z)}} = e^{S_L(\tilde{\phi}(x,z))-S_L(0)}.[D\phi]_{\eta_{ab}}$

(Polyakov `81)

Then, $\mathcal{N}(\tilde{\phi}(x,z),\lambda_0) = e^{S_L(\tilde{\phi}(x,z))}, S_L(\tilde{\phi}(x,z)) = \frac{c}{24\pi}\int_{-\infty}^{\infty}dx\int_{\epsilon}^{\infty}dz\Big[(\partial_x\tilde{\phi})^2+(\partial_z\tilde{\phi})^2+\mu e^{2\tilde{\phi}}\Big]$

Optimizing means finding eom of this Liouville action

$$4\partial_w\partial_{\bar{w}}\tilde{\phi} = e^{2\tilde{\phi}} \qquad e^{2\tilde{\phi}(z=\epsilon,x)} = \frac{1}{\epsilon^2}$$

Solutions: $e^{2\tilde{\phi}} = \frac{4\,A'(w)B'(\bar{w})}{(1-A(w)B(\bar{w}))^2}, \qquad w = z+i\,x, \bar{w} = z-i\,x$

time slice of AdS_{3}

Example II: CFT perturbed by relevant operators

$$S = S_{CFT} + \frac{1}{2}\epsilon^{2-\Delta}\int dxdz\lambda_0\,\mathcal{O}$$

(AB, PC, SD, NK, MM, TT '18)

Then only under the rescaling of the metric:

$$\eta_{ab} \to \eta_{ab}e^{2\,\tilde{\phi}(x,z)}$$

We will have again:

$$[D\phi]_{\eta_{ab}e^{2\tilde{\phi}(z,x)}} = e^{S_L(\tilde{\phi}(x,z))}[D\phi]_{\eta_{ab}}$$

But: $$S_{\eta_{ab}} \not\propto S_{\eta_{ab}e^{2\tilde{\phi}(x,z)}}$$

Hence, $$\psi_{\eta_{ab}}[\phi(x)] \neq \mathcal{N}(\tilde{\phi}(x),\lambda_0)\psi_{\eta_{ab}e^{2\tilde{\phi}(x,z)}}[\phi(x)]$$

To make this work we have to transform the coupling also,

$$\lambda_0 \to \lambda_{\tilde{\phi}}$$

Now one can derive this transformation $\lambda_0 \to \lambda_{\tilde{\phi}}$ **very systematically**

Let us calculate the "beta" function for this theory:

$$\Lambda \frac{d\lambda_{\tilde{\phi}}}{d\Lambda} = \beta(\lambda_{\phi}) \qquad \Lambda = e^{\tilde{\phi}}$$

Also lets do the calculation perturbatively in coupling

$$\Lambda \frac{d\lambda_{\tilde{\phi}}}{d\Lambda} = (\Delta - 2)\lambda_{\tilde{\phi}} + \mathcal{O}(\lambda_0^2)$$

We solve this with the boundary condition,

$$e^{\tilde{\phi}} = \epsilon, \lambda_{\tilde{\phi}} = \lambda_0$$

This gives, $\lambda_{\tilde{\phi}} = e^{\Delta - 2}\lambda_0 + \mathcal{O}(\lambda_0^2)$

Now given this transformation

$$S_{\eta_{ab},\lambda_0} \propto S_{\eta_{ab},\lambda_{\tilde{\phi}}}$$

Hence, $$\psi_{\eta_{ab},\lambda_0}[\phi(x)] = \mathcal{N}(\tilde{\phi}(x),\lambda_0)\psi_{\eta_{ab}e^{2\tilde{\phi}(x,z)}}[\phi(x)]$$

Now this proportionality factor can be evaluated systematically and takes the following form,

$$\mathcal{N}(\tilde{\phi}(x,z),\lambda_0) = e^{S_L+\lambda_0\int dxdz<\mathcal{O}(x,z)>+\lambda_0^2\int dx_1dz_1dx_2dz_2<\mathcal{O}(x_1,z_1)\mathcal{O}(x_2,z_2)>+\mathcal{O}(\lambda_0^3)+\cdots}$$

We can compute this correlators using conformal perturbation on the upper half plane

$$< \mathcal{O}(x,z) >= e^{-\Delta\tilde{\phi}(x,z)}\frac{1}{\left(z^2+e^{-\tilde{\phi}(z,x)}\right)^{\Delta/2}}$$

$$< \mathcal{O}(x_1,z_1)\mathcal{O}(x_2,z_2) >= \frac{e^{-\Delta\tilde{(}x_1,z_1)-\Delta\tilde{\phi}(x_2,z_2)}}{\left(|x_{12}|^2+|z_{12}|^2+e^{-\tilde{\phi}(x_1,z_1)-\tilde{\phi}(x_2,z_2)}\right)^{\Delta}}$$

Keeping only the leading order terms ($\epsilon \to 0$)

$$\ln(\mathcal{N}(\tilde{\phi}(x,z),\lambda_{\tilde{\phi}})) = S_L + \int dxdz\sum_{n=0}^{\infty}N_n(\lambda_0)^{n+2}e^{(2+(\Delta+2)(n+2)\tilde{\phi}(x,z)}$$

Next we try to minimize this:

$$ln(\mathcal{N}(\tilde{\phi}(x,z),\lambda_{\tilde{\phi}}) = \frac{c}{24\,\pi}\int dxdz[(\partial\tilde{\phi})^2 + e^{2\tilde{\phi}} + \lambda_0^2 e^{(2\Delta-2)\tilde{\phi}}] + \cdots$$

We find and solve the eom and we get the following (neglecting some derivatives of coupling),

$$e^{\tilde{\phi}} = \frac{1}{2}\Big(1 - \frac{\lambda_0^2}{2(5-2\Delta)}z^{-2\Delta+4} + \cdots\Big)$$

(AB, PC, SD, NK, MM, TT '18)

Now what about the geometry ? Time slices of AdS geometry ?

Let us compare this with AdS/CFT calculations

Comparison with AdS/CFT

Basic Holographic setup:

$$S = \frac{1}{2\kappa}\int d^3x\sqrt{-g}\Big[R - \Lambda - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\Phi^2\Big]$$

with, $\Delta = 1 + \sqrt{1+m^2}, \phi(z,x) = z^{d-\Delta}\lambda_0(x) + z^{\Delta} < \mathcal{O}(x) >$

To the leading order we identify the scalar as the coupling,

$$\phi(z,x) \approx \lambda_0 e^{(\Delta-2)\tilde{\phi}} + \mathcal{O}(\lambda_0^2)$$

We then look for the perturbative solutions for back reacted metric:

$$ds^2 = \frac{1}{z^2}(dz^2 + f(z)(-dt^2 + dx^2))$$

(*Hung Myers Smolkin '11*)

$$f(z) = 1 - \frac{\lambda_0^2}{4}z^{4-2\Delta} + \sum_{k=1}^{\infty} a_k(\lambda_0 z^{2-\Delta})^{k+2}$$

We consider the time slice:

$$ds^2 = e^{2\tilde{\phi}}(dz^2 + dx^2)$$

with, $$e^{\tilde{\phi}} \approx \frac{1}{z}\Big(1 - \frac{2-\Delta}{5-2\Delta}\lambda_0^2 z^{4-2\Delta}\Big)$$

matches with our result (upto some normalization factor)

Example III: Massive scalar field theory

$$S = \frac{1}{2}\int dxdz(\partial\phi)^2 + \frac{1}{2}\int dxdz m_0^2\phi^2$$

again we need

$$\psi_{\eta_{ab}}[\phi(x)] = \mathcal{N}(\tilde{\phi}(x), m_0)\psi_{\eta_{ab}e^{\tilde{\phi}(x)}}[\phi(x)]$$

for this we need : $$m_{\tilde{\phi}} = m_0 e^{-2\tilde{\phi}}$$

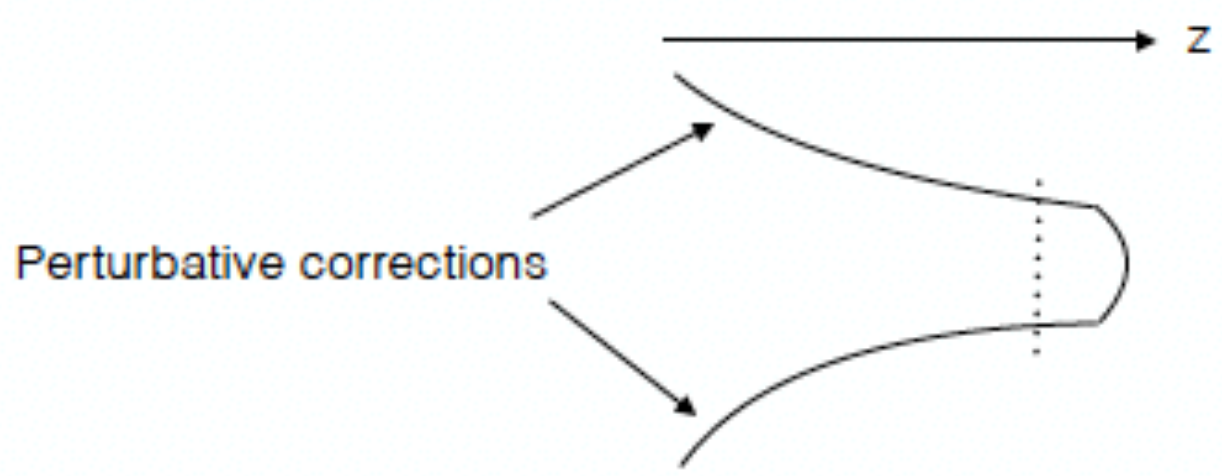
Given this we get, $(\phi = \bar{\phi} + \eta\ )$

$$\mathcal{N}(\tilde{\phi}(x), m_0) = \int \prod_{x,z} [D\eta(x,z)] e^{-\frac{1}{2}\int dx dz [(\partial\eta)^2 + \lambda_0 \eta^2)]}|_{\eta(x,z=0)=0}$$

We can evaluate this using the Heat-Kernel technique

We can do that and find eom from minimizing this and get,

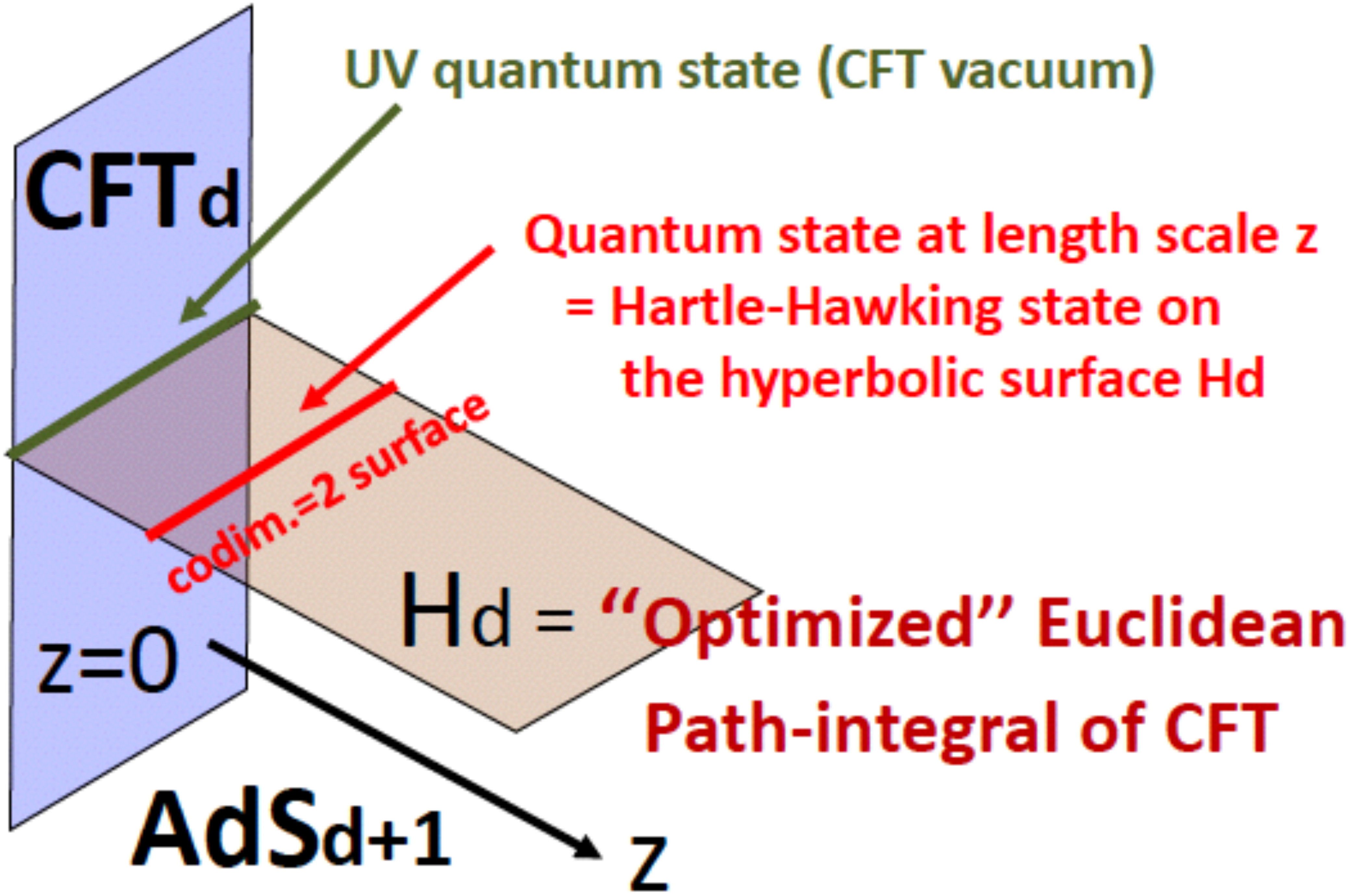
$$e^{2\tilde{\phi}} = \frac{1}{z^2}\Big(1 - \frac{m_0^2}{12\pi} z^2 \log(z/\epsilon) + \mathcal{O}(m_0^4)\Big)$$



(AB, PC, SD, NK, MM, TT '18)

This again matches withe AdS/CFT calculations!!

Brief Summary



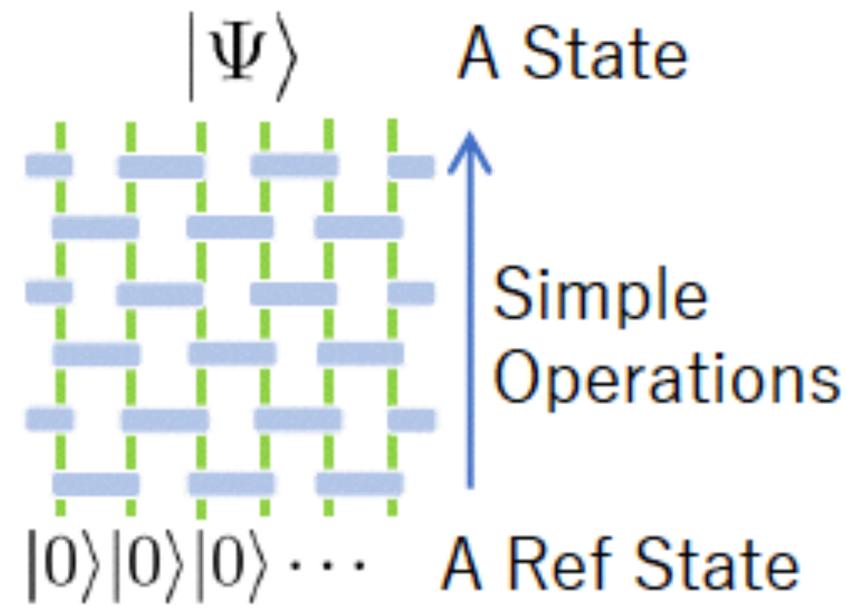
Few words about "Complexity"

(For more details please listen to next talk by Javier Molina-Vilaplana

computational complexity : How difficult is to implement a task ? eg how difficult is to prepare a particular state ?

Quantum circuit model: $|\psi >= U|\psi_0 >$

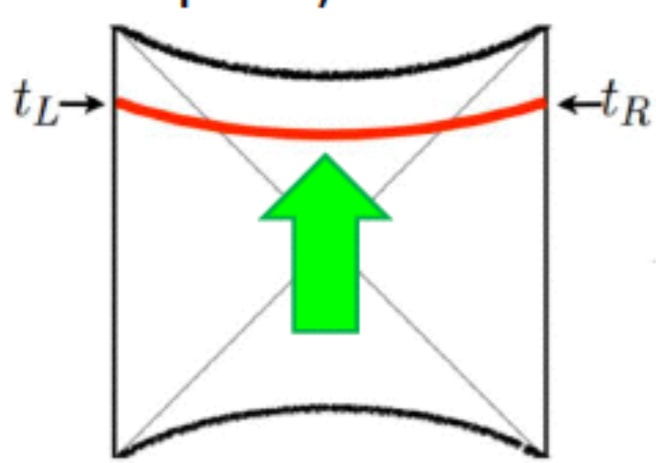
"minimize the number of operations":



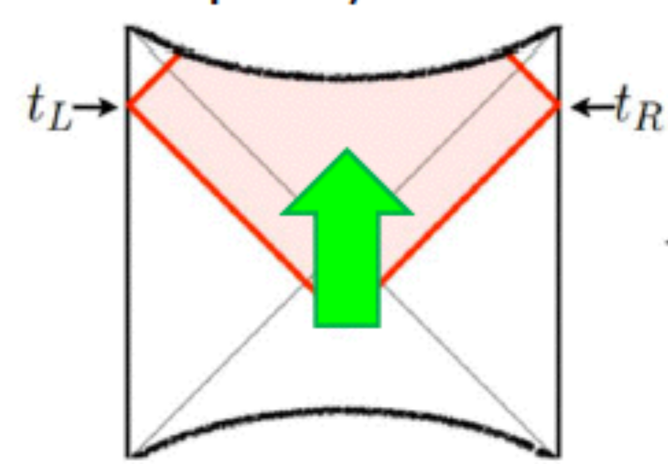
Holography: A tale of two dualities:

(Brown, Roberts, Swingle, Susskind & Zhao)

Complexity = Volume



Complexity = Action



"complexity" dual to these two objects ?

In light of this we can possibly reinterpret our result

$$\psi_{\eta_{ab}e^{2\tilde{\phi}(x,z)},\lambda_{\tilde{\phi}}}[\phi(x)] \quad = \quad \mathcal{N}(\tilde{\phi},\lambda_0)\,\psi_{\eta_{ab},\lambda_0}[\phi(x)]$$

"Target wave functional"

" Reference wave functional"

"counts the number of process which takes from reference to target space "

Minimizing this we are minimizing the complexity and thereby making our circuit "optimal"

Is AdS/CFT the``fastest quantum computer'' ??

For CFT's perturbed by the relevant operators $\mathcal{O}(\lambda_0)$ correction

$$C[\lambda_0] - C[0] = L(\lambda_0)^{\frac{1}{2-\Delta}} - L\,\lambda_0\epsilon^{1-\Delta}$$

(AB, PC, SD, NK, MM, TT '18)

Outlook

– Computations of correlation functions, Entanglement entropy etc?

- Higher dimensions?

- Find a covariant approach ? How to include time in it ?

- Where is the large “N”, sparse spectrum input ?

- Deformation by Stress tensor ?

Many more !!!