Complexity in QFT and Tensor Networks

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Geometry, Duality & Strings, Murcia, 25th May 2018

Talk based on work:

Complexity Functionals and Complexity Growth Limits in Continuous MERA Circuits. [arXiv:1803.02356]

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Holographic Complexity. A Tale of two Dualities.

Complexity of quantum states: useful to investigate the duality between entanglement and spacetime geometries in quantum gravity.

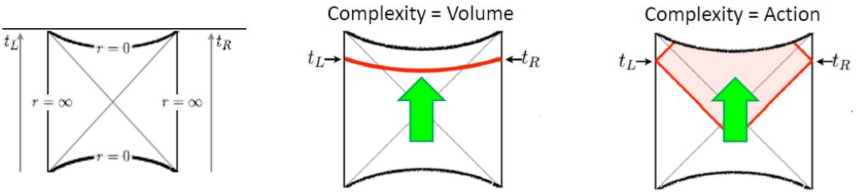
Mainly addressed in the context of the AdS/CFT duality.

"Holographic complexity" as the CFT quantity that encodes the continuous evolution of the interior of the black hole [Susskind 16] .

Two proposals for the quantum complexity of boundary states:

- Complexity=Volume (CV) conjecture. [Susskind 16]
- Complexity= Action (CA) conjecture. [Brown et al 16]

Holographic Complexity. A Tale of two Dualities.



$$\mathcal{C} \sim \frac{V}{G\ell}, \qquad \mathcal{C} = \frac{\mathcal{A}}{\pi\hbar}.$$

The eternal black hole: dual to an entangled state $|\Psi\rangle$ of two CFTs that live on the boundaries.

CV: $\mathcal{C}_\Psi =$ Volume of the maximal spatial slice.

CA: $\mathcal{C}_\Psi =$ Action $\mathcal{A}$ of the Wheeler-DeWitt patch. (roughly corresponds to the black hole interior)

CA and Complexification Rates.

The action of the black hole interior $\mathcal{A}$ increases at a rate such that

$$\frac{d\mathcal{C}_\Psi}{dt} = \frac{2M}{\pi\hbar}$$

i.e, saturates a Margolus–Levitin like bound [Brown et al 16].

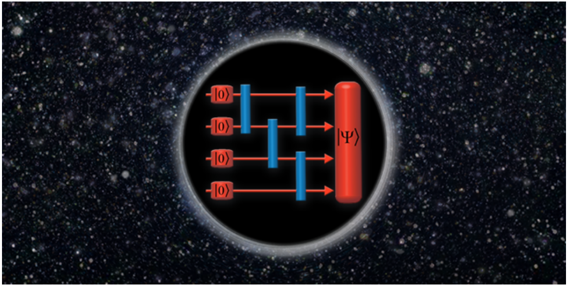
The ML bound is a universal bound on the speed of evolution of any quantum system (Quantum Speed Limit, QSL).

$$\frac{d}{dt}\langle\Psi|\Psi(t)\rangle \leq \frac{2E_\Psi}{\pi\hbar}$$

In Computer Science, ML is a theoretical upper limit on the number of operations that can be performed on unit time.

Black Holes Produce Complexity Fastest.

CA and Quantum Circuits.



Logic gates (blue) in a quantum circuit (red).

Black hole interiors = quantum circuits that produce $|\Psi\rangle$ at the fastest rate allowed by QM

$\mathcal{C}_{|\Psi\rangle} \sim \#\,\text{gates}$ required to produce $|\Psi\rangle$ from a simple reference state $|\Omega\rangle$.

Complexity in QFT: An interesting research program.

1. Complexity measures the cost required to prepare $|\Psi\rangle$ from a specific reference state $|\Omega\rangle$ by applying an optimal unitarity U

$$|\Psi\rangle \approx U\,|\Omega\rangle$$

2. U from iterating generators K taken from some elementary set

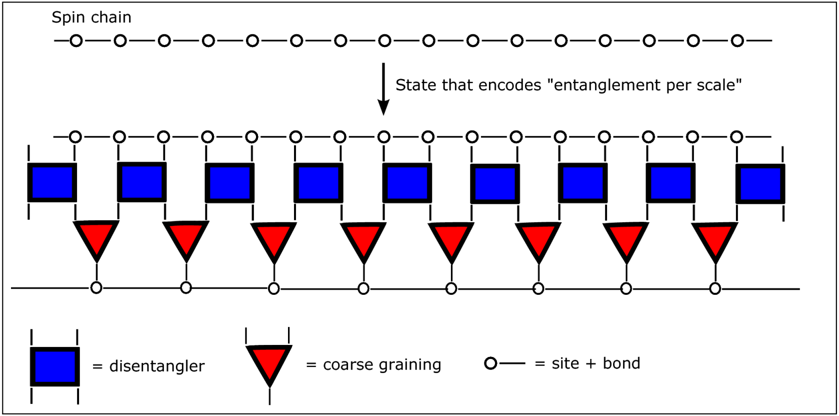
$$U(u) = \mathcal{P}\,\exp\left(\int_{u_I}^{u} K(u')\,du'\right)$$

$$|\Psi\rangle \approx U(u_F)\,|\Omega\rangle$$

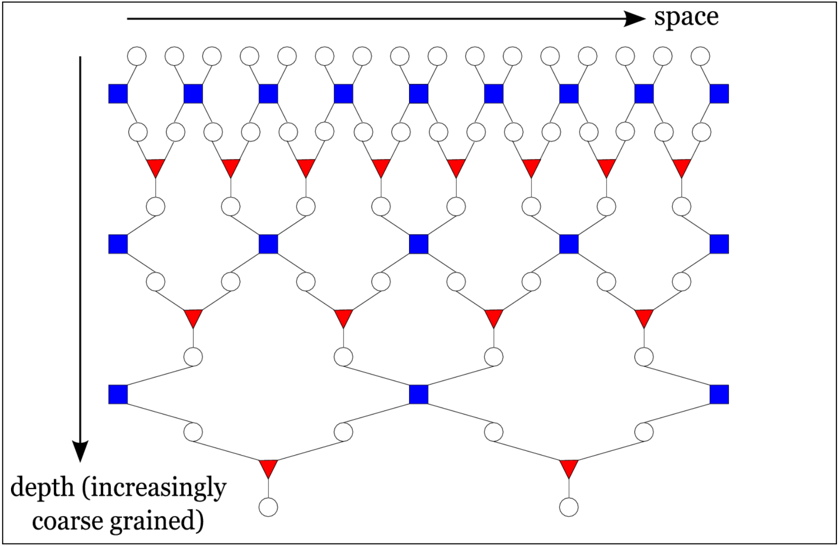
3. A regularization procedure to deal with ultraviolet divergences.
4. A measure of complexity.

Entanglement Renormalization. MERA.

[Vidal 2007]



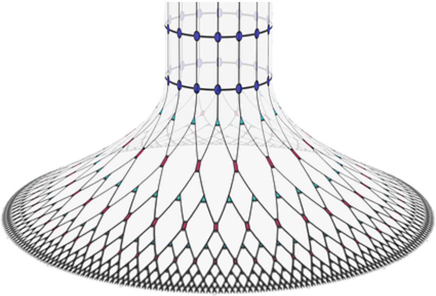
Entanglement Renormalization. MERA



MERA circuits and Holography.

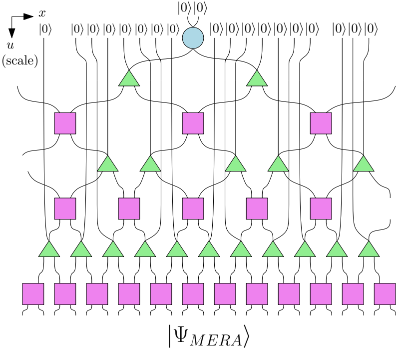
Proposal [Swingle, 2009] connecting the ideas of:

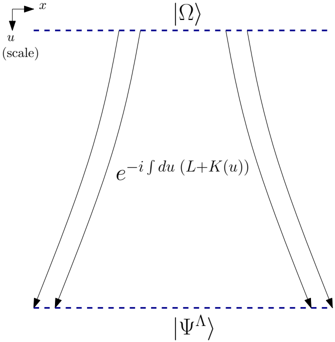
- AdS/CFT [Maldacena, 1998]
- Holographic Entanglement Entropy [Ryu-Takayanagi, 2006]
- Quantum Renormalization Group [Vidal, 2007]



Continuous MERA circuits

cMERA $\equiv$ w.f of the system at each length scale u.





$$|\Psi^{\Lambda}\rangle \equiv \mathcal{P} \exp\left(-i \int_{u_{IR}}^{0} d\hat{u}\, \widetilde{K}(\hat{u})\right) |\Omega\rangle$$

where $\widetilde{K}(\hat{u})$ is the entangler operator.

cMERA Free Boson Theory

For the free boson theory

$$S = \int dt\ dx\ \left((\partial_t \phi)^2 + (\partial_x \phi)^2 - m^2 \phi^2 \right)$$

The entangler operator is given by the Gaussian ansatz

$$\widetilde{K}(u) = \frac{1}{2i} \int dk\ \left(g(k,u)\, a_k^\dagger a_{-k}^\dagger - g^*(k,u)\, a_k a_{-k} \right)$$

with $g(k,u) = g(u)\,\Gamma(|k|e^{-u}/\Lambda)$ where $\Gamma(|k|e^{-u}/\Lambda)$ limits the momentum integral to $|k| \leq \Lambda e^u$

Coherent State Formulation

The state at scale u can be written as a $SU(1,1)$ gaussian squeezed state

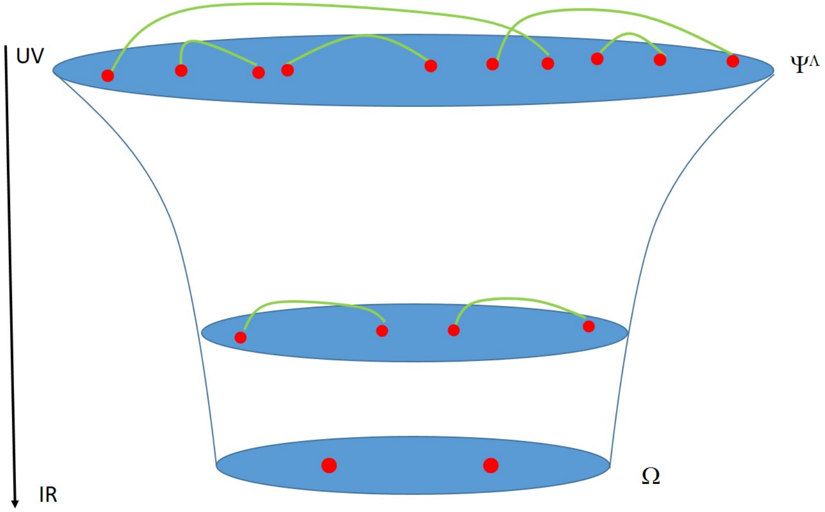
$$|\Psi(u)\rangle = N \exp\left(-\frac{1}{2}\int dk \left[\, \Phi(k,u)\ a^\dagger_k a^\dagger_{-k} - \Phi^*(k,u)\ a_k a_{-k}\right]\right)|\Omega\rangle$$

with

$$\Phi(k,u) = \int_{u_{IR}}^{u} du'\, g_k(k,u')$$

- Scale dependent representations of the state are obtained by adding left-right moving modes with $|k| \leq \Lambda e^u$ to $|\Omega\rangle$

cMERA Entanglement.



cMERA Variational Optimization

The variational parameter $\Phi(k,u)$ of cMERA for the free boson

$$\Phi(k,u) = \left[-\frac{1}{4} \log \frac{k^2+m^2}{\Lambda^2+m^2} \right]_{k=\Lambda\, e^u} = -\frac{1}{4} \log \frac{\Lambda^2 e^{2u}+m^2}{\Lambda^2+m^2}$$

cMERA Coherent State Path Integral

$$\mathcal{Z}_{cMERA} = \langle \Psi^{\Lambda} | \mathcal{P} \exp\left(-i \int_{u_{IR}}^{0} \tilde{K}(\hat{u})\, d\hat{u} \right) |\Omega\rangle$$

as a SU(1,1) coherent state path integral.

$$\mathcal{Z}_{cMERA} = \int \mathcal{D}\Phi \mathcal{D}\Phi^* \exp\left(i\, \mathcal{A}\left[\Phi, \Phi^*\right] \right)$$

$$\mathcal{A}\left[\Phi, \Phi^*\right] = -2\,\mathrm{Vol} \int_{|k|\leq\Lambda} dk \int_{u_{IR}}^{0} du\, \Phi^* \partial \Phi$$

$$= -2\,\mathrm{Vol} \int_{|k|\leq\Lambda e^{u}} dk \int_{u_{IR}}^{0} du\, g(u)\, \Phi(k,u)$$

Our Proposal.

$$\mathcal{C}_A\left[|\Psi^\Lambda\rangle, |\Omega\rangle\right] \equiv \mathcal{A}\left[\Phi, \Phi^*\right]_{\text{on}-\text{shell}}$$

where *on-shell* indicates that the action is evaluated with the parameter $\Phi(k, u)$ obtained from the cMERA variational procedure.

Results.

For the massless case

$$\mathcal{C}_A = \frac{\mathrm{Vol} \cdot \Lambda}{2}$$

we obtain divergencies ≈ CA and CV holographic complexity.

For the massive case

$$\mathcal{C}_A \approx \frac{\mathrm{Vol} \cdot \Lambda}{2} \left[1 - \frac{m}{\Lambda} \left(1 - \log \frac{m}{\Lambda} \right) \right]$$

UV divergences are naturally associated with existence of correlations or entanglement down to arbitrarily small length scales.

Comparison with Circuit Length.

Measures the complexity of a cMERA circuit as a minimal length (in terms of the Fisher information metric) of a path running from $|\Omega\rangle$ to $|\Psi^{\Lambda}\rangle$ [Chapman 18].

$$\mathcal{C}_{\mathrm{cMERA}} = \mathrm{Vol} \int_{\mathrm{u_{IR}}}^{0} \mathrm{du}\, |\, g(u)\, | \int_{|k|\leq \Lambda e^{u}} dk$$

$$= \mathrm{Vol} \int_{|k|\leq \Lambda} dk\, |\, \Phi(k,0)\, |$$

For the massive case

$$\mathcal{C}_{\mathrm{cMERA}} \approx \frac{\mathrm{Vol}\cdot\Lambda}{2}\left[1 - \frac{m}{2\Lambda}\left(\pi - \arctan\frac{m}{\Lambda}\right)\right]$$

Complexodynamics. Complexity Growth Limit.

Is $\mathcal{C}_A$ constrained in any sense?

$$\frac{\Delta \mathcal{C}_A}{\Delta u} = 2\, E_\Psi$$

with

$$E_\Psi = \frac{1}{\Delta u} \int_{u_I}^{u_F} du \, \langle \widetilde{K}(u) \rangle$$

CGL Margolus-Levitin type bound

Equality as opposed to a lower bound reflects the optimality of $\widetilde{K}$ for the generation of complexity along the cMERA circuit.

Complexity from Liouville Field Action.

Similar results can be obtained by considering the functional

$$\mathcal{C}_{\mathcal{A}} = \mathcal{A}_L[\Phi] = \frac{1}{4}\int dx \int_{\epsilon}^{\infty} dz \left[4\left(\partial_z \Phi(z)\right)^2 + \Lambda^2 e^{-4\Phi(z)}\right]$$

$z = \epsilon e^{-u}$ redefines the cMERA RG coordinate.

In the massless case

$$\Phi(z) = \frac{1}{2}\log \Lambda z$$

Complexity from Liouville Field Action.

From

$$\Phi(z) = \frac{1}{2}\log\Lambda - \frac{1}{2}\varphi_L(z)$$

a Liouville Field Theory Action functional can be derived

$$\mathcal{C}_{\mathcal{A}} = \mathcal{A}_L[\varphi_L] = \frac{1}{4}\int dx \int_{\epsilon}^{\infty} dz \left[(\partial_z \varphi_L(z))^2 + e^{2\varphi_L(z)} \right]$$

Complexity from Liouville Field Action.

LFT provides a quantum theory of 2D-gravity that is solved by the metric

$$ds^2 = e^{2\varphi_L(z)} \left(dz^2 + dx^2\right)$$

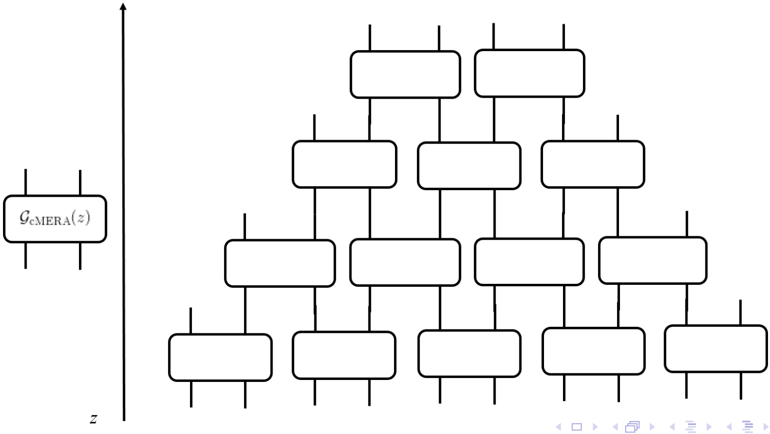
For the massless cMERA $\varphi_L(z) = -\log z$ we obtain the hyperbolic space

$$ds^2 = \frac{1}{z^2} \left(dz^2 + dx^2\right)$$

cMERA Gates and Vertex Operators.

A cMERA gate $\mathcal{G}_{\text{cMERA}}$ is implemented by a Hamiltonian action,

$$\mathcal{G}_{\text{cMERA}}(z) \sim e^{-i\delta\,\widetilde{K}(z)} = e^{-i\delta\,\Phi(z)\,\mathcal{O}} \qquad \mathcal{O} \in \mathfrak{su}(1,1)$$



cMERA Gates and Vertex Operators.

We consider the LFT correlation functions

$$\langle \prod_i e^{\alpha_i \varphi_L(z_i)} \rangle = \int D\varphi_L \, e^{\mathcal{A}_L} \prod_i e^{\alpha_i \varphi_L(z_i)}$$

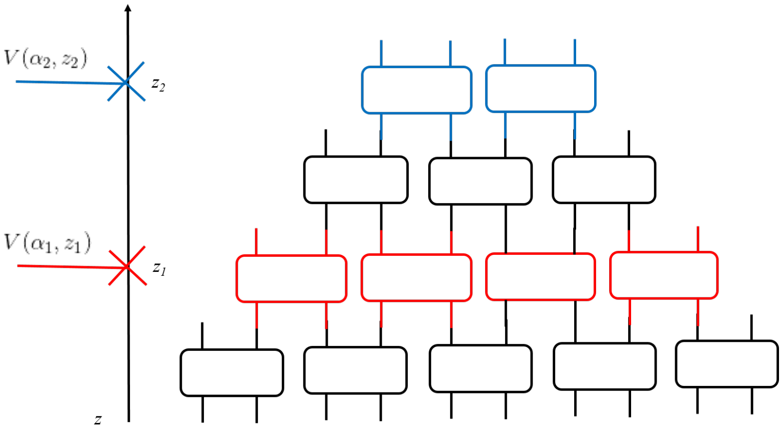
$$V(\alpha_i, z_i) = e^{\alpha_i \varphi_L(z_i)}$$

The functional integral is dominated by configurations

$$\varphi_L(z) = -\log z \qquad |z - z_i| \gg \varepsilon_i$$

$$\varphi_L(z) = -2\alpha_i \log |z - z_i| + \varphi_i + \cdots \qquad |z - z_i| < \varepsilon_i$$

cMERA Gates and Vertex Operators.



cMERA Gates and Vertex Operators.

After proper regularization [Seiberg 90]

$$\mathcal{A}_L(\alpha_i, z) = \mathcal{A}_L(z)_{/\cup\varepsilon_i} + \sum_i \left(\alpha_i\, \varphi_i - 2\alpha_i^2 \log \varepsilon_i^2\right)$$

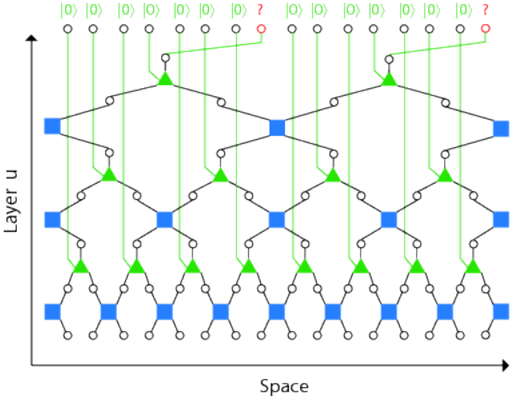
the variation of the cMERA circuit complexity is given by

$$\Delta\, \mathcal{C}_A = \sum_i \left(\alpha_i\, \varphi_i - 2\alpha_i^2 \log \varepsilon_i^2\right)$$

Ground State Entanglement and Liouville Mode.

Entanglement between two neighboring "sites" at scale u_*, is given by $\Phi(u_*)$.

This corresponds to the entanglement of the UV interval $L = \epsilon\, e^{-u_*}$.



Ground State Entanglement and Liouville Mode.

In the massless case of the free boson theory $(c = 1)$

$$S(L) = \frac{c}{6} \log \Lambda \, L$$

$$\frac{6}{c} S(z) = 2\Phi(z) = \log \Lambda - \varphi_L(z)$$

The entanglement structure of the quantum state arises from

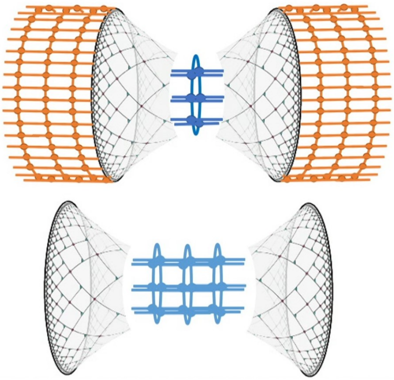
$$\partial \left(\frac{6}{c} S(z) \right) = \Lambda e^{-\frac{6}{c} S(z)}$$

Conclusions.

- Path integral description of “complexity” in cMERA states
- cMERA flow extremize the action functional in the cMERA circuit path integral $\equiv$ Circuit Complexity.
- CA in cMERA saturate Complexity Growth Limits. CGL as new constraints for a cMERA flow. Hints for an holographic description?
- Variational Φ $\equiv$ Liouville mode in LFT $\equiv$ Ground State Entanglement.

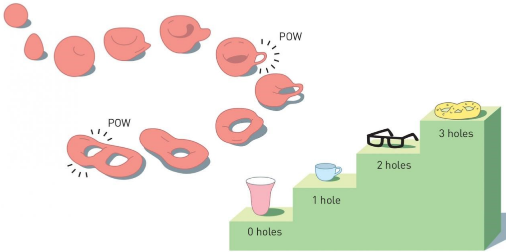
Future.

- Time dependent cMERA circuits.
- FiniteTemperature.



Future.

- Fermions and Bosons. SUSY. Complexity of SUSY states.
- Topological Phases of matter. No adiabatic paths starting from topologically trivial states.



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Thanks.