Quantum corrections to the dispersion relation in flux-deformed AdS_3/CFT_2

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Outline

1. Dispersion relation in AdS/CFT
2. Classical rigid spinning strings in $\mathbb{R} \times S^3$
3. Quadratic fluctuations
   - S^3 fluctuations
   - AdS_3 and T^4 fluctuations
   - Fermionic fluctuations
4. Algebraic curve
5. Computation of the one-loop correction
6. Conclusions

Outline

Dispersion relation and quantum corrections

- In the context of AdS/CFT correspondence, one of the most successful tests is the comparison between one-loop string correction to the energy with the integrability-based Thermodynamic Bethe Asantz predictions.
- In $AdS_5 \times S^5$ the computation of the one-loop string correction was carried out by Tseytlin, Arutyunov, Frolov, Park, etc. during early 2000 from the perspective of quadratic fluctuations. In [Gromov,Vieira, 2007] the same result was obtained using the algebraic curve.
- This computation was generalized $AdS_3 \times S^3 \times M_4$ by truncating modes, for example, in [Beccaria,Macorini, 2012] and [Beccaria,Levkovich-Maslyuk,Macorini,Tseytlin, 2013].

AdS_3/CFT_2 correspondence and integrability

$AdS_3 \times S^3 \times M_4$ admits an additional NS-NS flux, apart from the usual R-R flux. Here we are going to consider here the $M_4 = T^4$ background with an arbitrary combination of RR and NS-NS fluxes, controlled by a parameter $q \in [0,1]$. This deformation was proven to be integrable and conformally invariant [Cagnazzo, Zarembo, 2012].

The $q = 0$ limit corresponds to the pure R-R flux limit, where the usual $AdS_5 \times S^5$ methods work without any change, up to massless excitations. This is the set-up studied in the references shown on the previous slide.

The $q = 1$ limit corresponds to pure NS-NS flux limit and can be reinterpreted as a WZW model, thus we expect important simplifications therein.

Outline

Spinning string ansatz and Lagrangian for $\mathbb{R} \times S^3$

We are going to focus on classical spinning strings at the center of AdS_3

$$Y_1 + iY_2 = 0 \, , \qquad Y_3 + iY_0 = e^{iw_0\tau} \, ,$$
$$X_1 + iX_2 = r_1(\sigma)\, e^{i[\omega_1\tau + \alpha_1(\sigma)]} \, , \quad X_3 + iX_4 = r_2(\sigma)\, e^{i[\omega_2\tau + \alpha_2(\sigma)]} \, ,$$

with $r_1^2(\sigma) + r_2^2(\sigma) = 1$. This ansatz has to be supplemented with the periodicity conditions

$$r_i(\sigma + 2\pi) = r_i(\sigma) \, , \qquad \alpha_i(\sigma + 2\pi) = \alpha_i(\sigma) + 2\pi m_i \, .$$

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Using the global target space symmetries we can define

$$E = 4\pi h w_0 \ , \qquad J_j = 2h \int_0^{2\pi} d\sigma \left(r_j^2 \omega_j - \sum_{i=1}^{2} q \epsilon_{ij} r_2^2 \alpha_i' \right) \ .$$

The (deformed) Neumann-Rosochatius Lagrangian

The Polyakov action takes the form

$$L_{S^3} = h\left[\sum_{i=1}^{2}\frac{1}{2}\left[(r_i')^2 + r_i^2(\alpha_i')^2 - r_i^2\omega_i^2\right] - \frac{\Lambda}{2}(r_1^2 + r_2^2 - 1)\right] + h\left[q r_2^2\left(\omega_1\alpha_2' - \omega_2\alpha_1'\right)\right] \ ,$$

whereas the Virasoro constrains become

$$\sum_{i=1}^{2}\left(r_i'^2 + r_i^2(\alpha_i'^2 + \omega_i^2)\right) = w_0^2 \ , \qquad \sum_{i=1}^{2} r_i^2\omega_i\alpha_i' = 0 \ .$$

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This Lagrangian is a deformation of the Neumann-Rosochatius Lagrangian used to describe spinning strings in $AdS_5 \times S^5$ [Arutyunov,Russo,Tseytlin, 2003]. Such Lagrangian is integrable, and the flux deformation we are studying does not spoil the integrability.

Integrability and constant radii solutions

Integrability is assured by the existence of the (deformed) Uhlenbeck constant [Hernández,Nieto, 2015]

$$\bar{I}_1 = r_1^2(1-q^2)+\frac{1}{\omega_1-\omega_2}\left[(r_1 r_2' - r_1' r_2)^2 + \frac{(v_1+q\omega_2)^2}{r_1^2} r_2^2 + \frac{v_2^2}{r_2^2} r_1^2\right] .$$

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If we set $r_i = a_i =$const., the Virasoro constrains fix the radii to

$$a_1^2 = \frac{-\omega_2 m_2}{\omega_1 m_1 - \omega_2 m_2} = \frac{J_1 + 4\pi h q m_2}{4\pi h(\omega_1 + q m_2)} , \quad \alpha_1' = \frac{v_1 + q r_2^2 \omega_2}{r_1^2} ,$$

$$a_2^2 = \frac{\omega_1 m_1}{\omega_1 m_1 - \omega_2 m_2} = \frac{J_2}{4\pi h(\omega_2 + q m_1)} , \quad \alpha_2' = \frac{v_2 - q r_2^2 \omega_1}{r_2^2} ,$$

Sadly, a closed dispersion relation cannot be written in general.

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$$\omega_1 = \frac{J}{4\pi h} \; , \qquad \omega_2 = \frac{J}{4\pi h} - (m_1 - m_2) \; ,$$

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- The $\mathfrak{su}(2)$ sector: $m_1 = -m_2$. Here instead

$$\omega_1 = \Upsilon + qm \ , \qquad \omega_2 = \Upsilon - qm \ ,$$

which fixes the dispersion relation to

$$E = \sqrt{J^2 + (1 - q^2) 16\pi^2 h^2 m^2} = 4\pi h \sqrt{\Upsilon^2 + \kappa^2 m^2} \ . \tag{2}$$

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Quadratic fluctuations

The classical solution we are interested in presents a linear relationship between the worldsheet time and target space time $t = w_0\tau$. This allows us to relate the spacetime energy and the worldsheet energy as

$$w_0 E = E_{\text{2-dim.}} \ .$$

Therefore the one-loop correction to the dispersion relation can be obtained by summing the characteristic frequencies of fluctuations around the classical background.

S^3 bosonic fluctuations

The Lagrangian for the quadratic fluctuations can be obtained by substituting

$$r_i \cos\varphi_i \to a_i \cos(\alpha_i + \omega_i \tau) + \tilde{r}_i \cos(\alpha_i + \omega_i \tau) - \rho_i \sin(\alpha_i + \omega_i \tau) \ ,$$
$$r_i \sin\varphi_i \to a_i \sin(\alpha_i + \omega_i \tau) + \tilde{r}_i \sin(\alpha_i + \omega_i \tau) + \rho_i \cos(\alpha_i + \omega_i \tau) \ ,$$

into the Polyakov action with the B-field.

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into the Polyakov action with the B-field. After imposing the orthogonality $a_1 \tilde{r}_1 + a_2 \tilde{r}_2 = 0$, the equations of motion are

$$\begin{aligned}
&-\ddot{\rho}_1 + \rho_1'' + 2\frac{a_2}{a_1}\left[(\omega_1 + q m_2)\dot{\tilde{r}}_2 - (m_1 + q\omega_2)\tilde{r}_2'\right] = 0 \ , \\
&-\ddot{\rho}_2 + \rho_2'' - 2\left[(\omega_2 + q m_1)\dot{\tilde{r}}_2 - (m_2 + q\omega_1)\tilde{r}_2'\right] = 0 \ , \\
&-\frac{\ddot{\tilde{r}}_2}{a_1^2} + \frac{\tilde{r}_2''}{a_1^2} - 2\frac{a_2}{a_1}\left[(\omega_1 + q m_2)\dot{\rho}_1 - (m_1 + q\omega_2)\rho_1'\right] \\
&+ 2\left[(\omega_2 + q m_1)\dot{\rho}_2 - (m_2 + q\omega_1)\rho_2'\right] = 0 \ . \qquad (3)
\end{aligned}$$

Characteristic equation

Expanding in Fourier modes

$$\rho_j = \sum_{n=-\infty}^{\infty} \sum_{k=1}^{6} A_{j,n}^{(k)} e^{i\omega_{k,n}\tau + in\sigma} \ , \quad \tilde{r}_2 = \sum_{n=-\infty}^{\infty} \sum_{k=1}^{6} B_n^{(k)} e^{i\omega_{k,n}\tau + in\sigma} \ ,$$

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The existence of non-trivial solutions requires

$$(\omega_{k,n}^2 - n^2) \left\{ (\omega_{k,n}^2 - n^2)^2 - 4a_1^2 \left[(\omega_1 + qm_2)\omega_{j,n} - (m_1 + q\omega_2)n \right]^2 \right.$$
$$\left. -4a_2^2 \left[(\omega_2 + qm_1)\omega_{k,n} - (m_2 + q\omega_1)n \right]^2 \right\} = 0 \ . \qquad (4)$$

Solving the characteristic equation (1)

In the $\mathfrak{su}(2)$ sector we can solve the equation as a series in Υ

$$\omega_{1,n} = 2\Upsilon - nq + \frac{n^2\kappa^2}{2\Upsilon} + \frac{q\kappa^2 n(n^2-2m^2)}{2\Upsilon^2} + \mathcal{O}\left(\frac{1}{\Upsilon^3}\right) \ , \qquad (5)$$

$$\omega_{2,n} = -2\Upsilon - nq - \frac{n^2\kappa^2}{2\Upsilon} + \frac{q\kappa^2 n(n^2-2m^2)}{2\Upsilon^2} + \mathcal{O}\left(\frac{1}{\Upsilon^3}\right) \ ,$$

$$\omega_{3,n} = nq + \frac{n\kappa^2\sqrt{n^2-4m^2}}{2\Upsilon} - \frac{q\kappa^2 n(n^2-2m^2)}{2\Upsilon^2} + \mathcal{O}\left(\frac{1}{\Upsilon^3}\right) \ ,$$

$$\omega_{4,n} = nq - \frac{n\kappa^2\sqrt{n^2-4m^2}}{2\Upsilon} - \frac{q\kappa^2 n(n^2-2m^2)}{2\Upsilon^2} + \mathcal{O}\left(\frac{1}{\Upsilon^3}\right) \ .$$

Solving the characteristic equation (2)

On the contrary, in the pure NS-NS limit we can derive closed expressions for the roots of the characteristic equation

$$\omega_{1,n} = 2\left(\frac{J}{4\pi h} + m_2\right) - n \ , \qquad \omega_{3,n} = n \ , \tag{6}$$

$$\omega_{2,n} = -2\left(\frac{J}{4\pi h} + m_2\right) - n \ , \qquad \omega_{4,n} = n \ .$$

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In the overlap of both regimes the frequencies can be written as follows

$$\omega_{1,n} = 2\left(\frac{J}{4\pi h} - m\right) - n = 2\Upsilon - n \,, \qquad \omega_{3,n} = n \,, \tag{7}$$
$$\omega_{2,n} = -2\left(\frac{J}{4\pi h} - m\right) - n = 2\Upsilon - n \,, \qquad \omega_{4,n} = n \,.$$

The AdS_3 and T^4 bosonic fluctuations

We proceed analogously for AdS_3 fluctuations using the parameterization

$$\begin{aligned} z_0 \cos t &\to (1+\tilde{z}_0)\cos(\kappa\tau) - \chi_0 \sin(\kappa\tau)\ , \qquad & z_1 \sin\phi &\to \tilde{z}_1\ , \\ z_0 \sin t &\to (1+\tilde{z}_0)\sin(\kappa\tau) + \chi_0 \cos(\kappa\tau)\ , \qquad & z_1 \cos\phi &\to \chi_1\ . \end{aligned}$$

Following similar steps, the characteristic frequencies for the AdS_3 modes are

$$\omega_{k,n} = \pm\sqrt{n^2 \pm 2qw_0 n + w_0^2} \xrightarrow{q\to 1} \pm n \pm w_0\ . \tag{8}$$

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Since we consider no classical dynamics on the torus, we are leaded to a free Lagrangian for the fluctuations. Therefore, the characteristic frequencies are

$$
\omega_{k,n} = \pm n \ . \tag{9}
$$

Fermionic Lagrangian

As our background solution is purely bosonic, the Lagrangian for the fermionic fluctuations reduces to the usual fermionic type IIB Lagrangian computed up to quadratic order in the fermionic fields

$$\tilde{L}_F = i(\eta^{\alpha\beta}\delta_{IJ} - \epsilon^{\alpha\beta}(\sigma_3)_{IJ})\bar{\theta}^I \rho_\alpha (\mathrm{D}_\beta)^J{}_{\dot{K}} \theta^{\dot{K}} \ .$$

Here the covariant derivative is given by

$$(\mathrm{D}_\alpha)^I{}_J = \delta^I{}_J \left(\partial_\alpha - \frac{1}{4}\omega_{\alpha ab}\Gamma^a\Gamma^b \right) + \frac{1}{8}(\sigma_1)^I{}_J e^a_\alpha H_{abc}\Gamma^b\Gamma^c + \frac{1}{48}(\sigma_3)^I{}_J F_{abc}\Gamma^a\Gamma^b\Gamma^c \ .$$

whit fluxes

$$\not{H}_a = 2q\left[\not{E}_a(\Gamma^{012} + \Gamma^{345}) + (\Gamma^{012} + \Gamma^{345})\not{E}_a\right] \ ,$$
$$F_{abc} = 12\kappa(\Gamma^{012} + \Gamma^{345}) \ ,$$

where $\kappa^2 = 1 - q^2$.

Solving the characteristic equation

Expanding in periodic modes in σ, the characteristic frequencies associated to the $\mathfrak{su}(2)$ sector are

$$\omega_{k,n} = \pm n \pm \left(q - \frac{1}{2} \right) w_0 \ , \tag{10}$$

$$\omega_{4+k,n} = \pm \sqrt{n^2 - q^2 w_0^2 + \Upsilon^2} \pm \frac{1}{2} w_0 \ , \tag{11}$$

Note that half of the frequencies cancels the frequencies coming from the T^4.

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Note that half of the frequencies cancels the frequencies coming from the T^4.
If we also take $q = 1$, we obtain

$$\omega_{k,n} = n \pm \frac{1}{2} w_0 \ , \qquad \omega_{k+4,n} = -n \pm \frac{3}{2} w_0 \ . \tag{12}$$

Outline

Presentation of the method

In order to check our results, we are going to derive the characteristic frequencies through a different method: the algebraic curve [Kazakov,Marshakov,Minahan,Zarembo, 2004]. The central object is a collection of quasi-momenta that define a many-sheet Riemann surface. There exists a correspondence between the set of classical solutions and geometrical data [Dorey,Vicedo, 2006].

Features of classical solutions are encoded in cuts on this Riemann surface, hence quantum fluctuations around this classical solutions amounts to add small cuts.

Although [Gromov,Vieira, 2007] used this method to find the one-loop correction in $AdS_5 \times S^5$, it requires some modifications before applying it to $AdS_3 \times S^3 \times T^4$ with flux [Babichenko,Dekel,Ohlsson Sax, 2014].

Steps of the computation

The method [Gromov,Vieira, 2007] is the following:

1. Compute the Lax connection and monodromy matrix associated to our solution. From that, compute the classical quasi-momenta.
2. Perturb them by adding microscopic cuts (which behave like a poles).
3. Reconstruct the perturbation from the known poles and residues, and the asymptotic information (as the Lax connection is chosen to be related to Noether currents).
4. Extract the asymptotic behaviour of the reconstructed quasi-momenta associated to AdS sheets, as they encode the correction to the energy.

Classical quasi-momenta

The quasi-momenta, obtained from the logarithm of the eigenvalues of the monodomy matrix, are [Babichenko,Dekel,Ohlsson Sax, 2014], [Nieto,Ruiz, 2018]

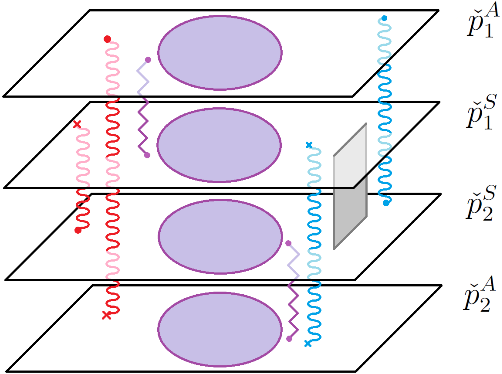
$$\hat{p}_1^A(x) = -\hat{p}_2^A(x) = \check{p}_1^A\left(\frac{1}{x}\right) = -\check{p}_2^A\left(\frac{1}{x}\right) = \frac{2\pi x w_0}{\kappa(x-s)\left(x+\frac{1}{s}\right)} \ ,$$

$$\hat{p}_1^S(x) = -\hat{p}_2^S(x) = \frac{2\pi x K(1/x)}{\kappa(x-s)\left(x+\frac{1}{s}\right)} \ ,$$

$$\check{p}_1^S(x) = -\check{p}_2^S(x) = -\frac{2\pi x K(x)}{\kappa(x+s)\left(x-\frac{1}{s}\right)} + 2\pi m \ ,$$

where $K(x) = \sqrt{m^2 x^2 \kappa^2 + 2q\kappa m^2 x + \Upsilon^2}$.

Graphical representation of the set-up



Final result

$$w_0\delta\Delta = \sum_n \Bigg[\left(n\kappa\hat{x}_n^{AA} - qn - w_0\right)\hat{N}_n^{AA} + \left(\frac{n\kappa}{\check{x}_n^{AA}} - qn\right)\check{N}_n^{AA}$$
$$+ \left(\frac{\kappa(n+2m)}{\check{x}_n^{SS}} - qn\right)\check{N}_n^{SS} + \left(n\kappa\hat{x}_n^{SS} - qn + 2K(0)\right)\hat{N}_n^{SS}$$
$$+ \left(\kappa\frac{n+m}{\check{x}_n^{F}} - qn\right)(\check{N}_n^{AS} + \check{N}_n^{SA})$$
$$+ \left(n\kappa\hat{x}_n^{F} - qn - K(0) - \kappa w_0\right)(\hat{N}_n^{AS} + \hat{N}_n^{SA})\Bigg]$$

where x_n^{XY} solve

$$p_X(x_n^{XY}) - p_Y(x_n^{XY}) = 2\pi n$$

Pure NS-NS frequencies

The method leads to the following frequencies in pure NS-NS regime

$$\begin{aligned} \hat{\Omega}^{AA}_{q\to 1} &= n + 2\Upsilon \ , & \check{\Omega}^{AA}_{q\to 1} &= n - 2\Upsilon \ , \\ \hat{\Omega}^{SS}_{q\to 1} &= n + 2\Upsilon \ , & \check{\Omega}^{SS}_{q\to 1} &= n + 4m - 2\Upsilon \ , \\ \hat{\Omega}^{F}_{q\to 1} &= n + 2\Upsilon \ , & \check{\Omega}^{F}_{q\to 1} &= n + 2m - 2\Upsilon \ . \end{aligned} \tag{13}$$

Outline

Pure NS-NS one-loop correction

In terms of the characteristic frequencies

$$E_{1\text{-loop}} = E_0 + \delta E \ , \qquad \delta E = \frac{1}{2w_0} \sum_{n\in\mathbb{Z}} (\omega_n^B - \omega_n^F) \ . \qquad (14)$$

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Substituting the frequencies obtained from quadratic fluctuations

$$\begin{aligned} \omega_n^B &= 2n + (n + w_0) + (n - w_0) + 4n = 8n \ , \\ \omega_n^F &= 2\left[2\left(n + \frac{w_0}{2}\right) + 2\left(n - \frac{w_0}{2}\right)\right] = 8n \ . \end{aligned} \tag{15}$$

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As their difference vanish, we have

$$\delta E \xrightarrow{q \to 1} 0 \ . \tag{16}$$

General case

The frequencies for general values of q are

$$\omega^{S}_{1,n} = n + (1-q)w_0 + \frac{\kappa^2[\Upsilon^2 - m^2(1+q^2)]}{2n} + \mathcal{O}(n^{-2}) \ ,$$
$$\omega^{S}_{2,n} = n - (1-q)w_0 + \frac{\kappa^2[\Upsilon^2 - m^2(1+q^2)]}{2n} - \mathcal{O}(n^{-2}) \ ,$$
$$\omega^{AdS}_{n} = \sqrt{n^2 + 2qnw_0 + w_0^2} + \sqrt{n^2 - 2qnw_0 + w_0^2}$$
$$= \sqrt{(n+qw_0)^2 + \kappa^2 w_0^2} + \sqrt{(n-qw_0)^2 + \kappa^2 w_0^2} \ ,$$
$$\omega^{T}_{n} = 4n \ ,$$
$$\omega^{F}_{n} = 4n + 4\sqrt{n^2 - q^2 w_0^2 + \Upsilon^2} \ .$$

Finiteness of the contribution

$$\delta E \propto 2 \sum_{n\in\mathbb{Z}^+} \left[2n + \frac{\kappa^2[\Upsilon^2 - m^2(1+q^2)]}{n} + \mathcal{O}(n^{-2}) \right] + I(qw_0, \kappa w_0)$$
$$+ I(-qw_0, \kappa w_0) - 4I\left(0, \sqrt{\Upsilon^2 - q^2 w_0^2}\right) ,$$

where the log divergence cancels due to the identity

$$2\kappa^2[\Upsilon^2 - m^2(1+q^2)] = 4(\Upsilon^2 - q^2 w_0^2) - 2\kappa^2 w_0^2 .$$

This proves that the correction is finite for all values of the mixing parameter q.

Outline

Conclusions

- We have computed the characteristic frequencies and one-loop correction for the rigid spinning string solution on flux-deformed $AdS_3 \times S^3 \times T^4$.
- This correction has been computed through two different methods to check the results.
- The correction is finite for all values of the mixing parameter q, and it vanishes in the pure NS-NS limit.
- This argument, in principle, seems to be generalizable to non-rigid strings under some assumptions. However, an explicit computation is needed.

The importance of this computation

As commented in the introduction, this results is important as it can be compared with the one Bethe string solution of the string Bethe ansatz (Ohlsson Sax, Sfondrini, Stefański, Torrielli, et al.).

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However, it was shown for the pure R-R flux case that massless excitations are an obstruction to this computation [Abbot,Aniceto, 2016]. This is because finite volume correction are exponentially suppressed by the mass of the particle involved in the loop.
We expect that the mixing parameter might be used as a way to control the Lüscher corrections.

The importance of this computation (2)

Although at $q = 1$ a remarkable simplification has to happen, it can be associated not to a control from the q parameter but to the dispersion relation becomes linear and the S-matrix becoming equal for bosonic and fermionic excitations, which imply the cancellation of the Lüscher corrections due to supersymmetry.

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Despite so, we hope that the comparison between the one-loop string correction and the Thermodynamical String Bethe Ansatz for mixed flux can shed some light on the understanding of massless Lüscher corrections.

Thank you for your attention

Representative and Lax connection

Choosing the representative $g_L \oplus g_R = g \oplus 1 \in \mathfrak{psu}(1,1|2)^2$ where, for classical spinning strings,

$$g = \begin{pmatrix} e^{iw_0\tau} & 0 & 0 & 0 \\ 0 & e^{-iw_0\tau} & 0 & 0 \\ 0 & 0 & a_1 e^{i(\omega_1\tau + m_1\sigma)} & a_2 e^{i(\omega_2\tau + m_2\sigma)} \\ 0 & 0 & -a_2 e^{-i(\omega_2\tau + m_2\sigma)} & a_1 e^{-i(\omega_1\tau + m_1\sigma)} \end{pmatrix} ,$$

the Lax connection becomes

$$\mathcal{L}(x) = \begin{pmatrix} \hat{\mathcal{L}}(x) & 0 \\ 0 & \check{\mathcal{L}}(x) \end{pmatrix} = \begin{pmatrix} \hat{\mathcal{L}}(x) & 0 \\ 0 & \hat{\mathcal{L}}\left(\frac{1}{x}\right) \end{pmatrix} ,$$

$$\hat{\mathcal{L}}(x) = \frac{ix}{(x-s)\left(x+\frac{1}{s}\right)} \begin{pmatrix} w_0/\kappa & 0 & 0 & 0 \\ 0 & -w_0/\kappa & 0 & 0 \\ 0 & 0 & -m\left(\frac{q}{\kappa} - x\right) & \frac{\sqrt{\Upsilon^2 - m^2 q^2}}{\epsilon\kappa} \\ 0 & 0 & \frac{\epsilon\sqrt{\Upsilon^2 - m^2 q^2}}{\kappa} & m\left(\frac{q}{\kappa} - x\right) \end{pmatrix} ,$$

with $\epsilon = e^{2im(\sigma + q\tau)}$.

A small trick to compute infinite sums

To perform the infinite sums on the mode number we used the relation [Schafer-Nameki, 2006]

$$2\pi i \sum_{n\in\mathbb{Z}} \omega_n = \oint_{\mathcal{C}} dz\, \pi \cot(\pi z) \omega_z \ . \tag{17}$$

In the semiclassical limit we can approximate the cotangent to 1, and use that

$$\begin{aligned} I(a,b) &= -\int_{a+ib}^{i\Lambda} dz \sqrt{(z-a)^2+b^2} = \int_{b-ia}^{\Lambda} \sqrt{(z+ia)^2-b^2} \\ &= \frac{1}{4}\Big[-2\Lambda^2 - 4ia\Lambda - 2a^2 + b^2(1+\log 4) \\ &\quad -2b^2(\log b - \log\Lambda)\Big] + \mathcal{O}(\Lambda^{-1}) \ . \end{aligned}$$

A non-rigid generalization

A general non-rigid solution is given by

$$r_1^2(\sigma) = c_1 + c_2 \, \text{sn}^2\left(c_3\sigma, \nu\right) \ ,$$

with the same functional form as its $AdS_5 \times S^5$ counterpart.

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The characteristic equation for spinning folded and pulsating strings in $AdS_5 \times S^5$ can be rewritten as the eigenvalue problem of a single-gap Lamé operators

$$\left[\partial_x^2 + 2\bar{\nu}^2 \mathrm{sn}^2(x|\bar{\nu}) + \Omega^2\right] f_\Lambda(x) = \Lambda f_\Lambda(x) \ . \qquad (18)$$

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Since ν vanishes in the limit $q = 1$, all the Lamé equations reduce to a wave equation in such limit as long as the functional form of this eigenvalue problem remains unaltered when we include the NS-NS flux.