Classical Yang-Baxter from Supergravity

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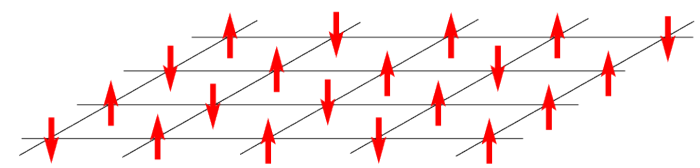
based on work with T. Araujo, I. Bakhmatov (APCTP), Ö. Kelekci (Ankara), J. Sakamoto, K. Yoshida (Kyoto), M. M. Sheikh-Jabbari (IPM Tehran), H. Yavartanoo (KITPC), M. Hong, Y. Kim (Postech)

1702.02861, 1705.02063, 1708.03163, 1710.06784,1801.09567,1803.07498

Yang-Baxter

QYBE is a hallmark of exact solvability (integrability)

- Statistical Physics



2–D Ising Model

- PDEs - KdV equation

$$\frac{\partial u}{\partial t} + au\frac{\partial u}{\partial x} + b\frac{\partial^3 u}{\partial x^3} = 0$$

- Knot theory

- Chern-Simons theory
- Supergravity (CYBE)

Statement

Given any solution to supergravity with an isometry group, there exists a deformation where the equations of motion reduce to the Classical Yang-Baxter Equation.

Surprising?

- Gravity is (usually) dynamical and not algebraic.
- Emergence of CYBE is intriguing. Origin?
- Classifying solutions to CYBE is a math problem.

Classical Yang-Baxter Equation

Consider a Lie-algebra

$$[T_i, T_j] = f_{ij}{}^k T_k, \quad T_i \in \mathfrak{g}$$

with an r-matrix $r = r^{ij} T_i \wedge T_j, \quad r^{ij} = -r^{ji}$

(homogeneous) CYBE takes the form

$$f_{l_1 l_2}{}^i r^{l_1 j} r^{l_2 k} + f_{l_1 l_2}{}^j r^{l_1 k} r^{l_2 i} + f_{l_1 l_2}{}^k r^{l_1 i} r^{l_2 j} = 0$$

Generalized Supergravity

Arutyunov, Hoare, Frolov, Roiban, Tseytlin (2015)

$$R_{MN} - \frac{1}{4}H_{MKL}H_N{}^{KL} - T_{MN} + \nabla_M X_N + \nabla_N X_M = 0$$

$$\frac{1}{2}\nabla^K H_{KMN} + \frac{1}{2}\mathcal{F}^K\mathcal{F}_{KMN} + \frac{1}{12}\mathcal{F}_{MNKLP}\mathcal{F}^{KLP} = X^K H_{KMN} + \nabla_M X_N - \nabla_N X_M$$

$$R - \frac{1}{12}H^2 + 4\nabla_M X^M - 4X_M X^M = 0$$

$$T_{MN} \equiv \frac{1}{2}\mathcal{F}_M\mathcal{F}_N + \frac{1}{4}\mathcal{F}_{MKL}\mathcal{F}_N{}^{KL} + \frac{1}{96}\mathcal{F}_{MPQRS}\mathcal{F}_N{}^{PQRS} - \frac{1}{4}G_{MN}(\mathcal{F}_K\mathcal{F}^K + \frac{1}{6}\mathcal{F}_{PQR}\mathcal{F}^{PQR})$$

$$X = \mathrm{d}\Phi + I + i_I B, \quad \mathcal{F} = e^{\Phi} F$$

RR sector

Page forms

1708.03163

$$Q_1 = F_1, \quad Q_3 = F_3 + B \wedge F_1, \quad Q_5 = F_5 + B \wedge F_3 + \frac{1}{2} B^2 \wedge F_1,$$

$$Q_7 = - * F_3 + B \wedge F_5 + \frac{1}{2} B^2 \wedge F_3 + \frac{1}{3!} B^3 \wedge F_1,$$

$$Q_9 = *F_1 - B \wedge *F_3 + \frac{1}{2} B^2 \wedge F_5 + \frac{1}{3!} B^3 \wedge F_3 + \frac{1}{4!} B^4 \wedge F_1$$

$$\mathrm{d}Q_{2n-1} = i_I Q_{2n+1}, \quad n = 1, 2, 3, 4$$

EOMs can be derived from kappa-symmetry of Green-Schwarz

Tseytlin, Wulff (2016)

“Problem” with NATD

GS EOMs correspond to beta-functions of sigma-models with **rigid scale symmetry** (known from 80s). Friedan; Curtright, Zachos,…

Can apply Buscher procedure to non-Abelian isometries.

de la Ossa, Quevedo (1992)

A Problem with Non–Abelian Duality

M. Gasperini R. Ricci G. Veneziano

(Submitted on 24 Aug 1993 (v1), last revised 7 Sep 1993 (this version, v2))

We investigate duality transformations in a class of backgrounds with non–Abelian isometries, i.e. Bianchi–type (homogeneous) cosmologies in arbitrary dimensions. Simple duality transformations for the metric and the antisymmetric tensor field, generalizing those known from the Abelian isometry (Bianchi I) case, are obtained using either a Lagrangian or a Hamiltonian approach. Applying these prescriptions to a specific conformally invariant $\s$ –model, we show that no dilaton transformation leads to a new conformal background. Some possible ways out of the problem are suggested.

Anomaly

Non-semisimple groups: there exists gauge-gravity anomaly.

Alvarez, Alvarez-Gaumé, Lozano (1994)

NATD sigma-model

$$S = \frac{1}{2\pi}\int \mathrm{d}^2 z \Big(F_{ij}\partial x^i \bar{\partial} x^j + (2\Phi + \ln\det N)\partial\bar{\partial}\sigma$$

$$+ (\partial\lambda_a - \partial x^i F^L_{ia} + \mathrm{tr}T_a\partial\sigma) N^{ab} (\bar{\partial}\lambda_b + F^R_{bj}\bar{\partial}x^j - \mathrm{tr}T_b\bar{\partial}\sigma)\Big),$$

$$N = (E(x) + \lambda_c f^c)^{-1}, \quad (f^c)_{ab} = f^c_{ab}$$

Elitzur, Giveon, Rabinovici, Schwimmer, Veneziano (1994)

Resolution

Anomaly contribution for Bianchi V

$$S = S_0 + S_1$$

$$\pi\delta_\sigma S_0 = \frac{1}{2}[\beta_{G_{ij}} + \beta_{B_{ij}}]\partial x^i \bar{\partial} x^j + \frac{1}{2}\beta_\Phi \partial\bar{\partial}\sigma \quad \text{(one-loop)}$$

$$\pi\delta_\sigma S_1 = -\frac{1}{2}[\beta_{G_{ij}} + \beta_{B_{ij}}]\partial x^i \bar{\partial} x^j$$

Elitzur, Giveon, Rabinovici, Schwimmer, Veneziano (1994)

NATD of Bianchi V is a solution to GS

Fernandez-Melgarejo, Sakamoto, Sakatani, Yoshida (2017)

Point of View

GS initially written down as a NATD sigma-model.

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \sum_{i=1}^{3} a_i(t)^2 \sigma_i^2, \quad \mathrm{d}\sigma_i = \frac{1}{2} f^i_{jk} \sigma_j \wedge \sigma_k$$

Bianchi III, V, VI_h $\qquad I^i = f^j_{ji}$ $\qquad$ **Hong, Kim, Ó C**

$$\delta_\sigma S_1 = -\frac{1}{2\pi} \int \mathrm{d}^2 z \delta\sigma I^i \left(2B_{ji} \left(\frac{1}{2} H^j{}_{kl} - \Gamma^j_{kl} \right) + \partial_l G_{ik} - \partial_k G_{il} - (\partial_l B_{ik} + \partial_k B_{il}) \right) \partial x^k \bar{\partial} x^l$$

Agrees with GS EOMs

$$\nabla_i I_j + \nabla_j I_i = 0$$
$$I^k H_{kij} + \partial_i Z_j - \partial_j Z_i = 0$$
$$X_k = I_k + Z_k$$

Summary I

Generalized supergravity may not be completely new.

Obviously related to NATD w. r. t. non-semisimple groups.

GS resurfaced through integrable sigma models.

(which in turn can be understood as NATD transformations)

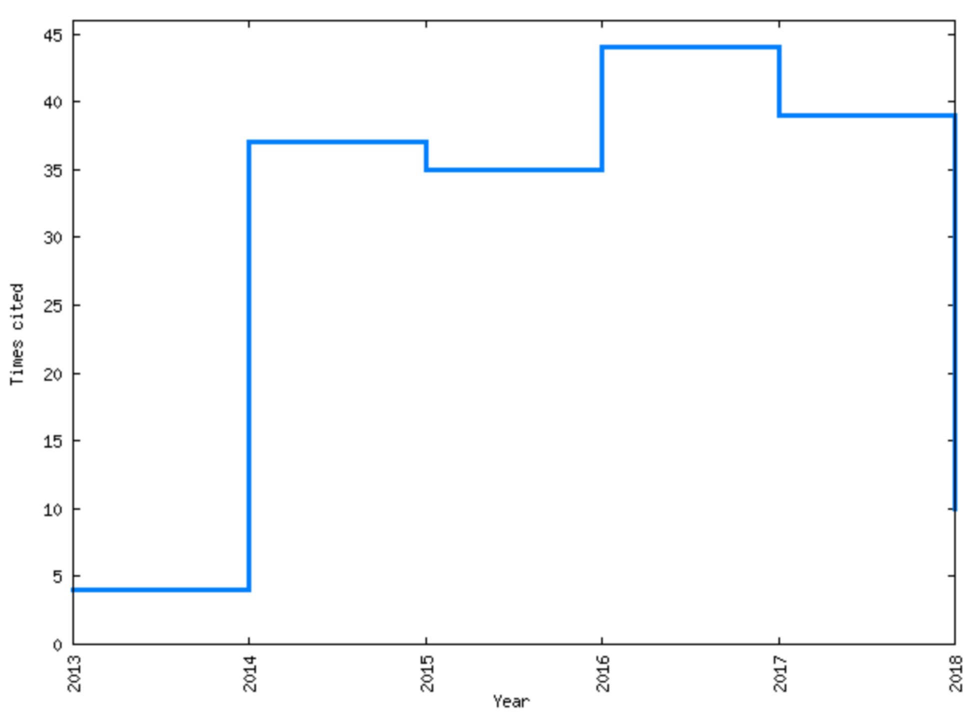
Hoare, Tseytlin (2016); Borsato, Wulff (2016/7)

Yang-Baxter sigma-model

Introduced for principal chiral models. **Klimčík (2002)**

AdS/CFT motivation: less contrived integrable backgrounds

First example (“eta-deformation”) based on modified CYBE.



Delduc, Magro, Vicedo (2013)

Kawaguchi, Matsumoto, Yoshida (2014)

Extended to homogeneous CYBE.

Yang-Baxter sigma-model

A rediscovery of open-closed string map from noncommutativity in string theory.

6. **String theory and noncommutative geometry**
Nathan Seiberg, Edward Witten (Princeton, Inst. Advanced Study). Aug 1999. 99 pp.
Published in **JHEP 9909 (1999) 032**
IASSNS-HEP-99-74
DOI: 10.1088/1126-6708/1999/09/032
e-Print: **hep-th/9908142** | **PDF**
References | BibTeX | LaTeX(US) | LaTeX(EU) | Harvmac | EndNote
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$$(G^{-1} + \Theta) = (g + B)^{-1}$$

Given deformed g, B in literature, NC parameter is always an r-matrix solution to the CYBE, with G undeformed

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Kyoto “empirical” result

$$I^{\mu} = \nabla^{G}_{\nu} \Theta^{\nu\mu}$$

Coset-free recipe

Classical Yang-Baxter Equation from Supergravity

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We promote the open-closed string map, originally formulated by Seiberg & Witten, to a solution generating prescription in generalized supergravity. The approach hinges on a knowledge of an

1. Extract g, B from open-closed string map

2. Dilaton from known T-duality invariant (covers TsT)

3. Killing vector from divergence of “NC parameter”

RR sector

Philosophy: all information about deformation in NC parameter.

Descent using invariant (closed) Page forms.

$$Q_{2(n-p)+1} = \frac{(-1)^p}{p!} \Theta^p \lrcorner \tilde{Q}_{2n+1}$$

This solves RR sector EOMs.

(Now trivial to do TsT by matrix inversion and descent.)

One Unknown

For simple examples, e. g. D=2, can solve for NC parameter.

But assuming it is bi-Killing, nice things happen.

$$\Theta^{\mu\nu} = r^{ij} K_i^\mu K_j^\nu \quad r^{ij} = -r^{ji}$$

$$I^\mu = \frac{1}{2} r^{ij} f_{ij}{}^k K_k^\mu$$

$$\Theta^{[\alpha\rho} \nabla_\rho \Theta^{\beta\gamma]} = K_i^\alpha K_j^\beta K_k^\gamma f_{l_1 l_2}{}^{[i} r^{j l_1} r^{k] l_2} = 0$$

Perturbative Proof

Expand in NC parameter, plug into GS EOMs.

$$g_{\mu\nu} = G_{\mu\nu} + \Theta_{\mu}{}^{\alpha}\Theta_{\alpha\nu} + \mathcal{O}(\Theta^4),$$
$$B_{\mu\nu} = -\Theta_{\mu\nu} - \Theta_{\mu\alpha}\Theta^{\alpha\beta}\Theta_{\beta\nu} + \mathcal{O}(\Theta^5),$$
$$\phi = \Phi + \frac{1}{4}\Theta_{\rho\sigma}\Theta^{\rho\sigma} + \mathcal{O}(\Theta^4)$$

$$K_i^{\alpha}K_k^{\beta}\nabla_{\alpha}K_{\beta m}\left(f_{l_1l_2}{}^{m}r^{il_1}r^{kl_2} + f_{l_1l_2}{}^{k}r^{ml_1}r^{il_2} + f_{l_1l_2}{}^{i}r^{kl_1}r^{ml_2}\right) + \left(\Theta^{\beta\gamma}\Theta^{\alpha\lambda} + \Theta^{\alpha\beta}\Theta^{\gamma\lambda} + \Theta^{\gamma\alpha}\Theta^{\beta\lambda}\right)R_{\beta\gamma\alpha\lambda} = 0.$$

Note, absence of original B-field (open problem).

Example

Method works for all geometries, e .g. Schwarzschild, etc.

$$\mathrm{d}s^2 = \frac{(-\mathrm{d}t^2 + \mathrm{d}z^2)}{z^2} + \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2 + \mathrm{d}s^2(T^6),$$

$$F_5 = (1 + *_{10})\frac{1}{\sqrt{2}z^2}\mathrm{d}t \wedge \mathrm{d}z \wedge (\omega_r - \omega_i)$$

Focus on AdS_2 (here I can solve for deformation in general)

$$K_1 = -t\partial_t - z\partial_z, \quad K_2 = -\partial_t, \quad K_3 = -(t^2 + z^2)\partial_t - 2tz\partial_z$$

Example

$$\begin{aligned}\Theta &= \alpha K_1 \wedge K_2 + \beta K_2 \wedge K_3 + \gamma K_3 \wedge K_1 \\ &= \big(-\alpha z + \beta 2tz + \gamma z(-t^2 + z^2)\big)\, \partial_t \wedge \partial_z \equiv \zeta \partial_t \wedge \partial_z\end{aligned}$$

Deformed solution

$$\mathrm{d}s^2 = \frac{z^2}{(z^4 - \zeta^2)}(-\mathrm{d}t^2 + \mathrm{d}z^2), \quad B = \frac{\zeta}{(z^4 - \zeta^2)}\mathrm{d}t \wedge \mathrm{d}z,$$

$$\Phi = -\frac{1}{2}\log\left[\frac{(z^4 - \zeta^2)}{z^4}\right], \quad I = -\alpha T_1 - 2\beta T_2 + \gamma T_3,$$

$$F_5 = (1 + *_{10})\frac{z^2}{\sqrt{2}(z^4 - \zeta^2)}\mathrm{d}t \wedge \mathrm{d}z \wedge (\omega_r - \omega_i), \quad F_3 = -\frac{\zeta}{\sqrt{2}z^2}(\omega_r - \omega_i)$$

Example

Einstein equation

$$R_{tt} + 2\nabla_t X_t = \frac{z^2(1 - 4\beta^2 + 4\alpha\gamma)(z^4 + \zeta^2)}{(z^4 - \zeta^2)^2}$$

$$T_{tt} = \frac{z^2(z^4 + \zeta^2)}{(z^4 - \zeta^2)^2}$$

$$-\beta^2 + \alpha\gamma = 0$$

Two ways to solve EOMs:

i) impose homogeneous CYBE

ii) absorb constant in dilaton (modified CYBE)

Summary II

Generalized Supergravity can be traced to NATD.

The CYBE emerges from supergravity.

It would be interesting to

- find a proof of the statement
- understand origin of the CYBE
- classify solutions to the CYBE using gravity
- think about the QYB