Instantons in AdS_5 x S^5/Z_k

Dr. Mario Trigiante

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Geometry, Duality and Strings, Murcia 2018

T. Hertog, M.T., T. Van Riet, JHEP **1706**, 067 (2017); D. Ruggeri, M.T., T. Van Riet, JHEP **1803** (2018) 091;
D. Ruggeri, M.T., T. Van Riet, work in progress…

Motivations: AdS/CFT duality

- 20 years ago Maldacena conjectured for the first time a correspondence between superstring theory on an anti de Sitter space-time and a CFT on its boundary

J. M. Maldacena, Adv. Theor. Math. Phys **2** (1998) 231

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$$\text{Type IIB on } AdS_5 \times S^5 \longleftrightarrow \mathcal{N}=4,\ D=4,\ SU(N) \text{ SYM theory}$$

$$g_{YM}^2 = 4\pi\, g_s\ ,\quad g_s = e^{\phi}$$

Near horizon geometry of a stack of overlapping *N* D3 branes

$$L_{AdS}^2 = \sqrt{4\pi g_s\, N}\, \alpha'$$

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- Effective SUGRA description: $L_{AdS}^2 \gg \alpha' \ , \quad g_s \ll 1$

$$g_{YM}^2\, N = 4\pi\, g_s\, N \gg 1 \quad \Longrightarrow \quad \text{Strongly coupled SYM}$$

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- For scalars $m^2\,\ell^2 = \Delta(\Delta - d)$, solutions $\Delta,\ d - \Delta \leq \Delta$

- Correspondence made more precise:

E. Witten, Adv. Theor. Math. Phys. **2** (1998) 253
S. S. Gubser, I. R. Klebanov, A. M. Polyakov, Phys.Lett. B428 (1998) 105-114

$$\Phi(z,\vec{y}) = z^{d-\Delta}\,\Phi_0(\vec{y}) + z^{\Delta}\,\Phi_1(\vec{y}) + \ldots$$

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v.e.v. <O> in the presence of J

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- Type IIB on AdS_5 x S^5 has two moduli scalars: the dilaton and the RR axion

$m^2 = 0\,,\ \Delta = 4$

$$g_s = e^{\phi} = \frac{g_{YM}^2}{4\pi} \quad \leftrightarrow \quad \mathcal{O} \propto \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$$

$$\chi = \frac{\theta_{YM}}{2\pi} \quad \leftrightarrow \quad \mathcal{O} \propto \mathrm{Tr}(F_{\mu\nu}{}^{*}F^{\mu\nu})$$

- D-instantons on EAdS5: non trivial supersymmetric space-time axio-dilaton configurations corresponding to instanton solutions on the CFT side

$$\langle \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})\rangle = \pm\langle \mathrm{Tr}(F_{\mu\nu}{}^{*}F^{\mu\nu})\rangle$$

T.Banks, M.B. Green, 9804170; M. Bianchi, M.B. Green, S. Kovaks, G. Rossi, 9807033; C. S. Chu, P.M.Ho, Y.Y.Wu, 9806103; I.I. Kogan, G. Luzon, 9806197; V. Balasubramanian, P. Kraus, A. Lawrence, S.P. Trivedi, 9808017; N. Dorey, V.V. Khoze, M.P. Mattis, S. Vandoren, 9808157; N. Dorey, T. J. Hollowood, V.V. Khoze, M.P. Mattis, S. Vandoren, 9901128; M.B. Green, S. Kovacs, 0212332 ….

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- More general (non-supersymmetric) instantonic solutions in Type IIB on flat and AdS backgrounds studied in

M.B. Einhorn, L.A. Pando-Zayas, 0003072 ; J. Y. Kim, Y. B. Kim, J. E. Hetrick, 90301191 ; M. Gutperle, W. Sabra, 0206153 ; ; E. Bergshoeff, A. Collinucci, U. Gran, D. Roest, S. Vandoren, 0406038; E. Bergshoeff, A. Collinucci, A. Ploegh, S. Vandoren, T. Van Riet, 0510048 ....

A less SUSY background

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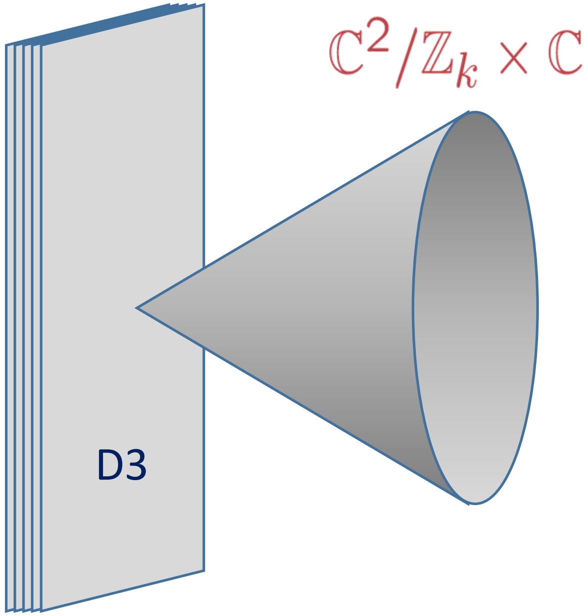
near horizon geometry of a stack of N coinciding D3 branes on the apex of an orbifold $\mathbb{C}^2/\mathbb{Z}_k \times \mathbb{C}$



M.R. Douglas, G.W.Moore 9603167; C.V. Johnson, R.C. Myers, 9610140

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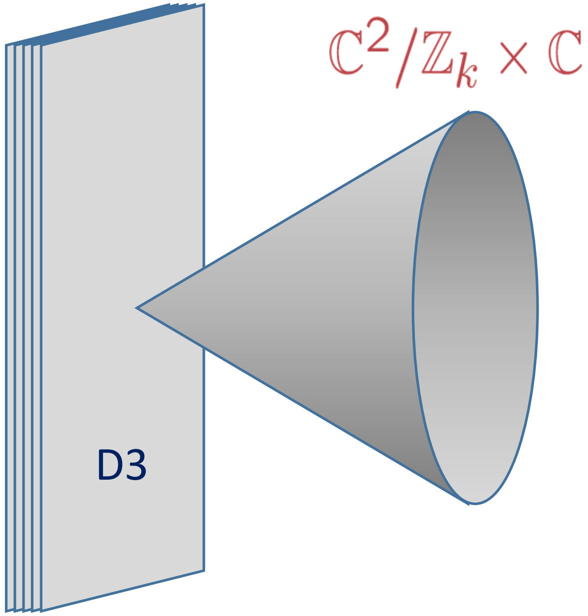

- Background preserves 16 of the original 32 supercharges and the dual field theory is an A_{k-1} quiver gauge theory with gauge group $U(N)^k$ S. Kachru, E. Silverstein, 9802183
- Pure SUGRA analysis: construct instantons on this (Euclideanized) background

- Effective D=5 description: gauged $\mathcal{N} = 4,\ D = 5$ SUGRA

R. Corrado, M. Gunaydin, N.P. Warner, M. Zagermann, 0203057; J. Louis, H.Triendl, M. Zagermann, 1507.01623

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Gauge group is the Z_k-invariant subgroup of the S^5 isometry group

$$SO(6) \supset SU(2)_L \times \underbrace{SU(2)_R \times U(1)}$$

$$\cup$$

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gauge group of the half-maximal theory, R-symmetry group of the dual CFT

- ➢ k=2 gauge group also includes $SU(2)_L$

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- Twisted sector: 2 x (k-1) tensor multiplets (5 scalars each) [S. Gukov, 9806180]

- Scalar potential defines an *N*=4 AdS_5 vacuum:

Untwisted sector

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$$\begin{pmatrix} 1\times g_{\mu\nu} \\ 4\times \psi_\mu \\ (3+1)\times A_\mu \\ 2\times B_{\mu\nu} \\ 4\times \chi \\ 1\times \phi \end{pmatrix} + 2\times \begin{pmatrix} 1\times B_{\mu\nu} \\ 4\times \lambda \\ 5\times \phi \end{pmatrix} + 1\times \begin{pmatrix} 1\times A_\mu \\ 4\times \lambda \\ 5\times \phi \end{pmatrix}$$

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Twisted sector

- Scalar manifold: $$\mathcal{M}_{\text{scal}} = \mathrm{SO}(1,1) \times \frac{SO(5,n)}{SO(5)\times SO(n)}$$

$$n = 2k+1$$ k>2

$$n = 2k+3$$ k=2

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- k complex scalars $\leftrightarrow \tau_\alpha = \frac{\theta_\alpha}{2\pi} + i\,\frac{4\pi}{g_\alpha^2}\,,$ $\alpha = 1,\ldots,k$ complexified coupling constants of the N=2 U(N)^k boundary SYM theory

Instantonic solutions

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$$\Lambda = V_0 = -\frac{12}{\ell^2} \qquad \phi^I \in \mathcal{M}_{\text{moduli}}$$

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- Shift symmetry in the **axion fields** $\chi^i \to \chi^i + \xi^i$

- Corresponding Noether currents and conserved charges:

$$J_{\mu\, i} = \sqrt{|g|}\, G_{ij}\, \partial_\mu \chi^j \;, \quad \partial_\mu J^\mu = 0 \;\Rightarrow\; Q_i = \int_{\Sigma_\tau} n^\mu J_{\mu\, i} = \text{const.}$$

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  S.R. Coleman, K.M. Lee, Nucl.Phys. **B329** (1990) 387-409; N. Arkani-Hamed, J. Orgera, J. Polchinski, 0705.2760; E. Bergshoeff, A. Collinucci, A. Ploegh, S. Vandoren, T. Van Riet, 0510048

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T. Hertog, M.T., T. Van Riet, 0702.04622

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$$\tilde{S} = S_{\text{grav}} + \int_{\text{EAAdS}_5} \frac{1}{2}\left(G_{ab}\, d\varphi^a \wedge {}^* d\varphi^b + G^{ij}\, F_i \wedge {}^* F_j\right) > 0$$

$$G^{ik} G_{kj} = \delta^i_j\ ,\ \ G_{ij},\, G^{ij} > 0$$

- $e^{-\tilde{S}}$ computed on the solution with charges Q_i is the corresponding instanton contribution to the path integral. *It has contributions only from the boundary terms*

Instantons from geodesics

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$$\mathcal{R}_{\mu\nu} = \tfrac{\Lambda}{3}\, g_{\mu\nu} + \tfrac{1}{2}\, G_{IJ}\partial_\mu\phi^I\partial_\nu\phi^J$$

$$\ddot{\phi}^I + \Gamma^I_{JK}\,\dot{\phi}^J\,\dot{\phi}^K = 0$$

Geodesic on $\mathcal{M}_{\text{moduli}}$

$$\dot{\phi}^I \equiv \frac{d\phi^I}{d\tau}$$

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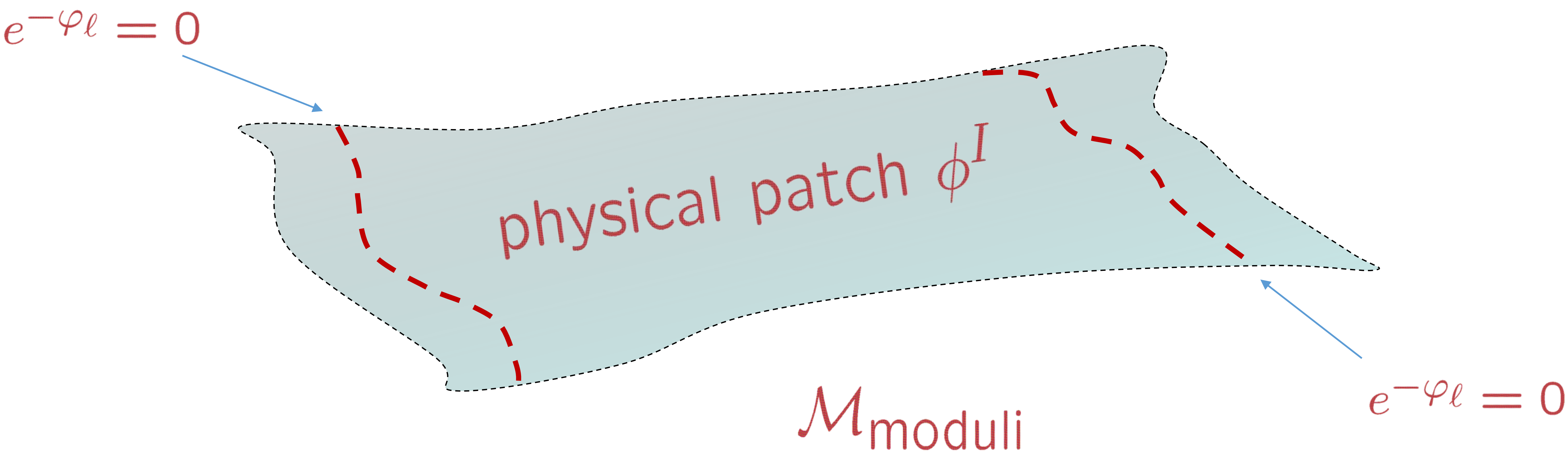
- c > 0, «over-extremal»: ‘spike-like’ singularity in the bulk where the non-extremal D-instanton is located.

In ***AdS_5 x S^5*** : E. Bergshoeff, A. Collinucci, A. Ploegh, S. Vandoren, T. Van Riet, 0510048

Regularity of the «under-extremal» solution:



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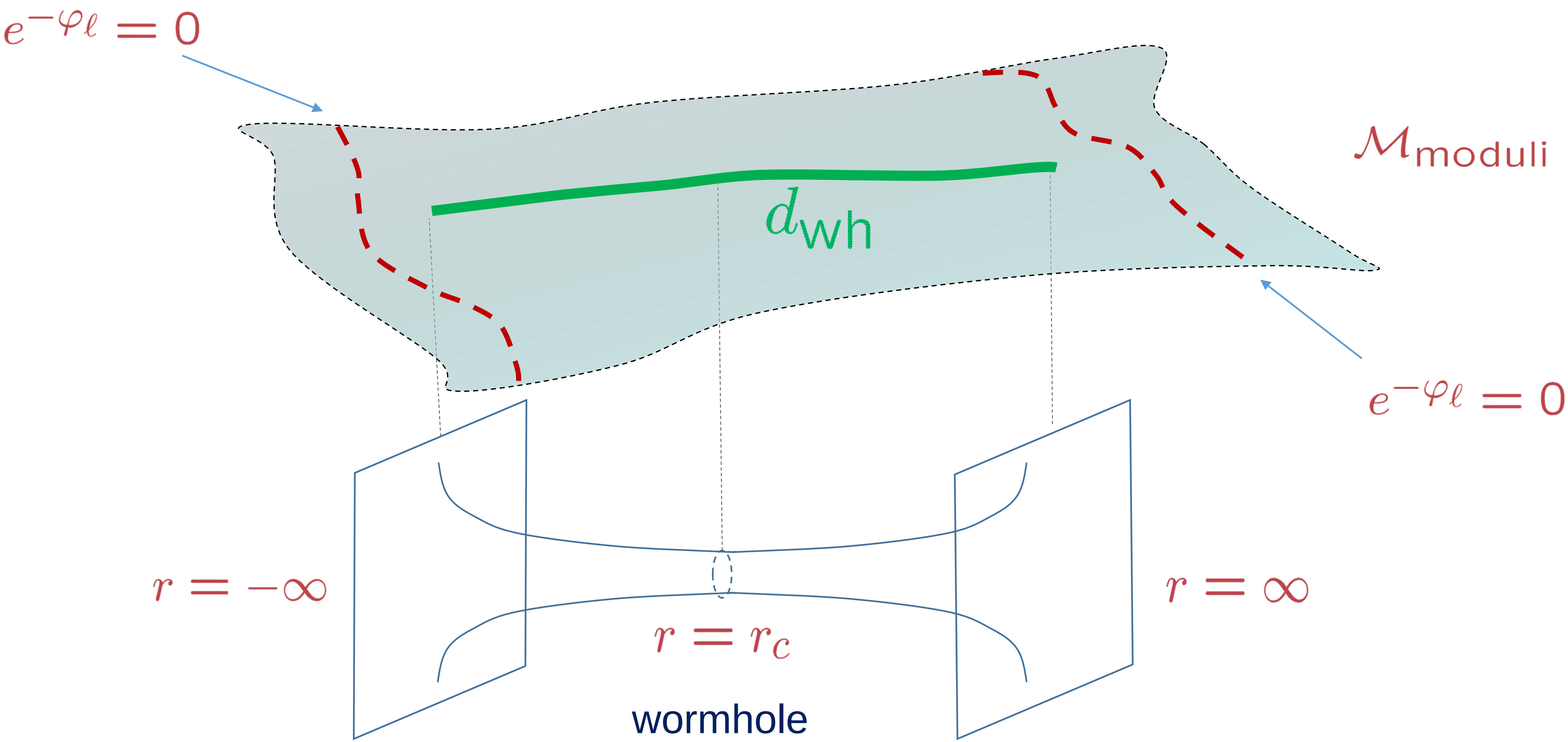

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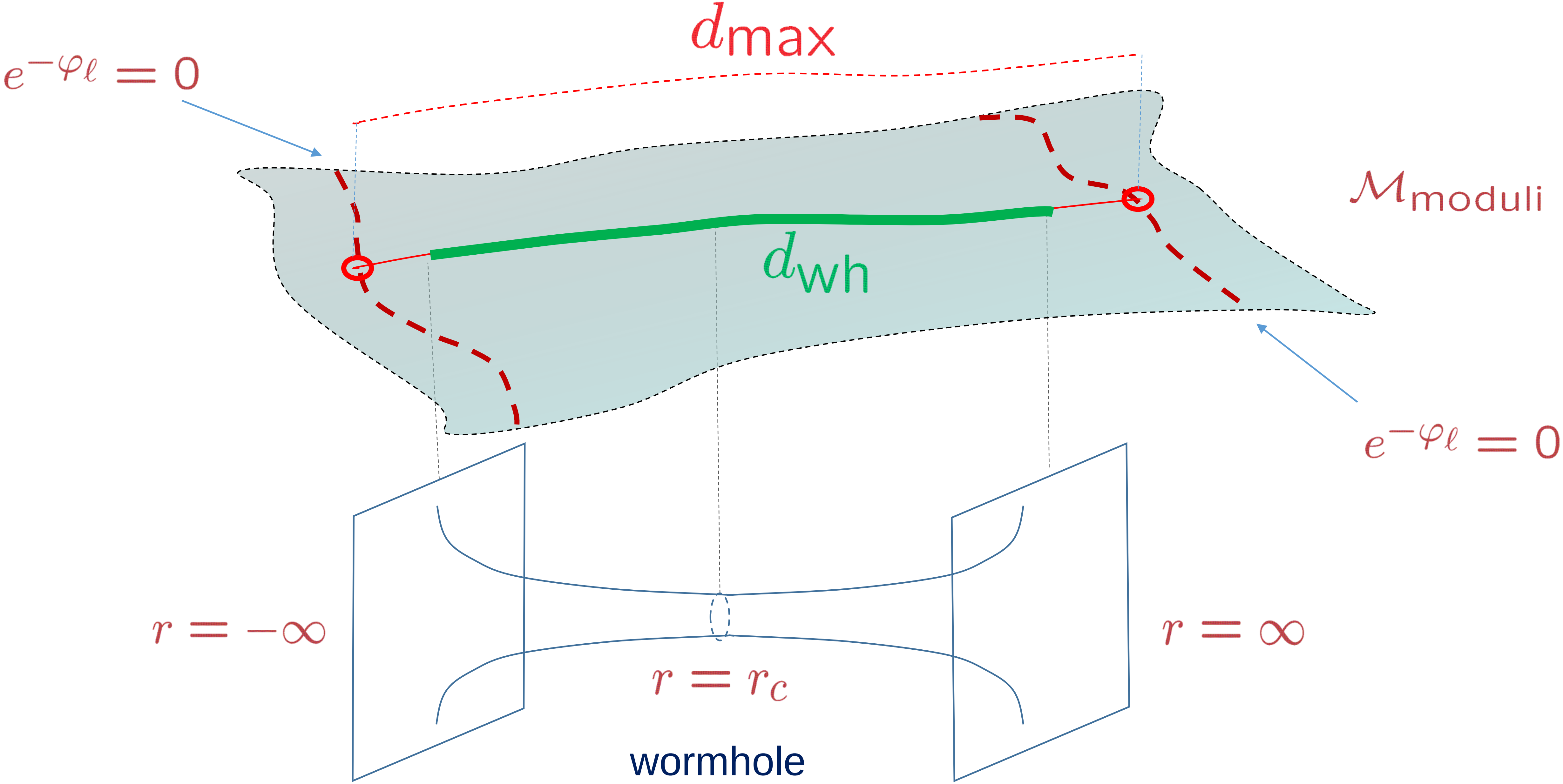

Regularity of the «under-extremal» solution: the solution along the geodesic should not reach the boundaries of the physical patch N. Arkani-Hamed, J. Orgera, J. Polchinski, 0705.2760;

$$d_{\mathrm{max}} > d_{\mathrm{wh}}$$

depends on the target space geometry and on the geodesic

Fixed by the field equations

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Condition not satisfied by wormholes in Euclidean SUGRAS obtained from time-like dimensional reduction of Lorentzian theories

E. Bergshoeff, W. Chemissany, A. Ploegh, M. T., T. Van Riet, 0806.2310

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Condition satisfied by certain geodesics in our model: There exist regular wormhole solutions in Euclidean $AdS_5 \times S^5/Z_k$

T. Hertog, M.T., T. Van Riet, 0702.04622

The explicit solutions

D. Ruggeri, M.T., T. Van Riet, arXiv:1712.06081

Physical patch defined by local solvable parametrization of the moduli space

$$\mathcal{M}_{\text{moduli}} = \frac{\mathsf{SL}(k+1)}{\mathsf{GL}(k)} \sim e^{\mathsf{Solv}} = \mathsf{O}(1,1) \ltimes e^{\mathsf{Heis}}$$

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$$\text{Solv} = \phi^I T_I = U\, T_0 + \underbrace{\zeta^s T_s + \tilde{\zeta}_s T^s + a\, T_\bullet}_{\text{Heis}}$$

$s = 1, \ldots, k-1$

$$[T_0, T_s] = \tfrac{1}{2} T_s\,,\ [T_0, T^s] = \tfrac{1}{2} T^s$$
$$[T_0, T_\bullet] = T_\bullet\ ,\ \ [T_s, T^r] = \delta_s^r\, T_\bullet$$

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Coset represenatative: $\mathbb{L}(\phi^I) = e^{-a T_\bullet}\, e^{\sqrt{2}\,(\zeta^s\, T_s + \tilde{\zeta}_s\, T^s)}\, e^{2U\, T_0} \in e^{\mathrm{Solv}}$

Axions are defined by maximal abelian ideal of Solv

L. Andrianopoli, R. D'Auria, S. Ferrara, P. Frè, M.T. , 9611014

$$\chi^i = \{\tilde{\zeta}_s, \tilde{a}\}\ ,\ \ \tilde{a} \equiv a - \zeta^s\, \tilde{\zeta}_s$$

Define involution

$$\mathfrak{sl}(k+1) = \mathfrak{gl}(k) \oplus \mathfrak{p}$$

$$\sigma : \qquad +1 \qquad -1$$

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On a geodesic $\phi^I(\tau)$ we define the following constant Noether charge matrix (the 'initial velocity')

$$\mathbb{Q} = \frac{1}{2} M(\phi(\tau))^{-1} \frac{d}{d\tau} M(\phi(\tau)) \in \mathfrak{sl}(k+1) \quad \Rightarrow \quad M(\phi(\tau)) = M(\phi(\tau=0))\, e^{2\,\mathbb{Q}\,\tau}$$

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Geodesic	$\mathbb{L}_0 = \mathbb{L}(\phi_0)$ →	Geodesic
$\phi^I(\tau = 0) = 0,\ \mathbb{Q}_O$		$\phi^I(\tau = 0) = \phi^I_0,\ \mathbb{Q}$

$$\mathbb{Q} = \sigma[\mathbb{L}_0]\, \mathbb{Q}_O\, \sigma[\mathbb{L}_0^{-1}]$$

Generating geodesics through the origin:

$$\mathbb{Q}_O = \left(\begin{array}{c|cccc} 0 & m_1 & \cdots & m_{k-1} & m_0 \\ \hline p_1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ p_{k-1} & 0 & \cdots & 0 & 0 \\ p_0 & 0 & \cdots & 0 & 0 \end{array}\right)$$

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Noether charge matrix defined by two vectors

$$\vec{m} = (m_i)\ ,\ \ \vec{p} = (p_i)$$
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Two orbits:

- $\vec{p}$ and $\vec{m} \neq \vec{0} \ \Rightarrow \ \mathbb{Q}^3 = \mathbf{0}$
- $\vec{p} = \vec{0}$ or $\vec{m} = \vec{0} \ \Rightarrow \ \mathbb{Q}^2 = \mathbf{0}$

Checked SUSY by embedding the extremal solutions in the D=5 gauged SUGRA

orbit $\mathbb{Q}^2 = \mathbf{0}$: $\frac{1}{2}$ -BPS

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Checked SUSY by embedding the extremal solutions in the D=5 gauged SUGRA

new equation

orbit $\mathbb{Q}^2 = \mathbf{0}$: $\frac{1}{2}$-BPS

orbit $\mathbb{Q}^3 = \mathbf{0}$: non-BPS

New

The generating solution

$$
\begin{aligned}
U &= \frac{1}{2}\log\left[\frac{1}{(1+\tau|p_0|)\,(1+\tau|m_0|)}\right] \\
\zeta^s &= -\tau\left(\frac{p_s}{\sqrt{2}\,(1+\tau|p_0|)} + \frac{m_s}{\sqrt{2}\,(1+\tau|m_0|)}\right) \\
\tilde{\zeta}_s &= -\tau\left(\frac{p_s}{\sqrt{2}\,(1+\tau|p_0|)} - \frac{m_s}{\sqrt{2}\,(1+\tau|m_0|)}\right) \\
a &= -\frac{1}{(1+\tau|p_0|)} + \frac{1}{(1+\tau|m_0|)}
\end{aligned}
$$

Non-extremal solutions

$$\mu \equiv \sqrt{|\vec{m} \cdot \vec{p}|}$$

c > 0

$$\begin{aligned}
U &= \frac{1}{2} \log\left[\frac{\mu^2}{\left(m_0 \sinh(\mu\tau) - \mu \cosh(\mu\tau)\right)\left(p_0 \sinh(\mu\tau) - \mu \cosh(\mu\tau)\right)}\right] \\
\zeta^s &= \frac{1}{\sqrt{2}}\left[\frac{m_s}{m_0 - \mu \coth(\mu\tau)} + \frac{p_s}{p_0 - \mu \coth(\mu\tau)}\right] \\
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\end{aligned}$$

- Geodesics through a generic point $\phi^I(\tau=0)=\{U(0),\, a(0),\, \zeta^s(0),\, \tilde{\zeta}_s(0)\}$ obtained by acting on the generating one by means of the solvable group of isometries

$$\begin{aligned} U &\to U + U(0) \\ \tilde{\zeta} &\to \tilde{\zeta}e^{U(0)} + \tilde{\zeta}(0) \\ \zeta &\to \zeta e^{U(0)} + \zeta(0) \\ a &\to a e^{2U(0)} + \zeta\tilde{\zeta}(0)e^{U(0)} - \tilde{\zeta}\zeta(0)e^{U(0)} + a(0) \end{aligned}$$

- Agreement for k=1 with the solutions found in

E. Bergshoeff, A. Collinucci, A. Ploegh, S. Vandoren, T. Van Riet, 0510048

Conclusions

- If the effective D=5 gauged SUGRA is a consistent truncation of Type IIB on $AdS_5 \times S^5/Z_k$, we constructed and classified the instantonic solutions on this background

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Conclusions

- If the effective D=5 gauged SUGRA is a consistent truncation of Type IIB on $AdS_5 \times S^5/Z_k$, we constructed and classified the instantonic solutions on this background
- By suitably choosing the parameters of the geodesics, we recover the known solutions

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- Found regular wormhole solutions
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Perspectives…

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- Understand string interpretation of these new solutions: Embedding the D=5 theory in D=10 by D=5 extension of '*DFT at SL(2) angles*' (F. Ciceri, G. Dibitetto,J.J.Fernandez-Melgarejo, A. Guarino, G. Inverso, 1612.05230)

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- Find correspondence with instanton solutions in the dual quiver SYM at the boundary (for a study of instantons in the A_{k-1} quiver theory see e.g. T.J. Hollowood,V.V. Khoze 9908035; R. Argurio, D. Forcella, A. Mariotti, D. Musso, C. Petersson, 1211.1884)

Thank You!

On-shell actions for the generating solution

$$S^{(\text{real})}_{\text{on-shell}} = \frac{\text{Vol}(S^4)}{2\,\kappa_5^2}\left[|(m_0+p_0)|\left(1+\frac{1}{2}\left[\sum_{i=1}^{k-1}\frac{m_i^2}{m_0^2}+\sum_{i=1}^{k-1}\frac{p_i^2}{p_0^2}\right]\right)\right]$$

c = 0

$$S^{\text{real}}_{\text{on-shell}} = \frac{Vol(S^4)}{2\,\kappa_5^2}\frac{1}{\hat{m}_0^2\hat{p}_0^2}\text{Abs}\left[\frac{(\hat{m}_0+\hat{p}_0)}{2}\sum_{i=1}^{k-1}(\hat{m}_0\,p_i-\hat{p}_0\,m_i)^2-\mu\,\hat{m}_0\hat{p}_0(m_0-p_0)\right],$$

$$\hat{m}_0 = m_0-\mu\,,\quad \hat{p}_0=p_0-\mu\,,\quad \mu=\sqrt{\vec{m}\cdot\vec{p}}$$

c > 0

The harmonic function for c=0

$$ds^2 = \frac{\ell^2}{z^2}\left(dz^2 + |d\vec{y}|^2\right) = \frac{dr^2}{1+\frac{r^2}{\ell^2}} + r^2\, d^2\Omega(S_4)$$

$$r(z,\vec{y}) = \frac{\sqrt{[(\ell-z)^2+|\vec{y}|^2][(\ell+z)^2+|\vec{y}|^2]}}{2z}$$

$$\tau(z,\vec{y}) = \frac{1}{3}\, r^{-3}\left(\left(1-\frac{2r^2}{\ell^2}\right)\sqrt{1+\frac{r^2}{\ell^2}}\right) + \frac{2}{3\ell^3}$$

$$\sqrt{|g_5|}\, g^{rr}\, \partial_r \tau = -\sqrt{|g(S_4)|}$$