on Freudenthal Duality

Alessio MARRANI

Enrico Fermi Center, Roma

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see also (other applications)

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Summary

Maxwell-Einstein-Scalar Gravity Theories

Extremal Black Holes and Attractor Mechanism

U-Duality Orbits and **Attractor Moduli Spaces**

Freudenthal Duality

Groups of type E_7

Geometry of **U-Orbits** and **PVS**

Hints for the Future…

Maxwell-Einstein-Scalar Theories

$$\mathcal{L} = -\frac{R}{2} + \frac{1}{2} g_{ij}(\varphi)\, \partial_\mu \varphi^i \partial^\mu \varphi^j + \frac{1}{4} I_{\Lambda\Sigma}(\varphi)\, F^\Lambda_{\mu\nu} F^{\Sigma|\mu\nu} + \frac{1}{8\sqrt{-G}} R_{\Lambda\Sigma}(\varphi)\, \epsilon^{\mu\nu\rho\sigma} F^\Lambda_{\mu\nu} F^\Sigma_{\rho\sigma}$$

$$H := \left(F^\Lambda, G_\Lambda\right)^T ;$$

$$^*G_{\Lambda|\mu\nu} := 2 \frac{\delta \mathcal{L}}{\delta F^{\Lambda|\mu\nu}} .$$

D=4 Maxwell-Einstein-scalar system (with no potential)

[may be the bosonic sector of D=4 (ungauged) sugra]

Abelian 2-form field strengths

static, spherically symmetric, asympt. flat, **extremal BH**

$$ds^2 = -e^{2U(\tau)} dt^2 + e^{-2U(\tau)} \left[\frac{d\tau^2}{\tau^4} + \frac{1}{\tau^2} \left(d\theta^2 + \sin\theta d\psi^2 \right) \right]$$

$$\tau := -1/r$$

$$\mathcal{Q} \; : \; = \int_{S^2_\infty} H = \left(p^\Lambda, q_\Lambda\right)^T ;$$

$$p^\Lambda \; : \; = \frac{1}{4\pi} \int_{S^2_\infty} F^\Lambda, \;\; q_\Lambda = \frac{1}{4\pi} \int_{S^2_\infty} G_\Lambda .$$

dyonic vector of e.m. fluxes

(**BH charges**)

$$S_{D=1} = \int [(U')^2 + g_{ij}\varphi'^i \varphi'^j + e^{2U} V_{BH}(\varphi(\tau), \mathcal{Q})] d\tau$$

$$' \equiv \frac{d}{d\tau}$$

reduction D=4 →D=1 :effective 1-dimensional (radial) Lagrangian

$$V_{BH}(\varphi, \mathcal{Q}) := -\frac{1}{2}\mathcal{Q}^T \mathcal{M}(\varphi)\mathcal{Q},$$

BH effective potential

Ferrara,Gibbons,Kallosh

eoms

$$\begin{cases} \frac{d^2U}{d\tau^2} = e^{2U} V_{BH}; \\ \frac{d^2\varphi^i}{d\tau^2} = g^{ij} e^{2U} \frac{\partial V_{BH}}{\partial \varphi^j}. \end{cases}$$

in N=2 ungauged sugra, hyper mults. decouple, and we thus disregard them : scalar fields belong to vector mults.

Attractor Mechanism : $\partial_\varphi V_{BH} = 0 \Leftrightarrow lim_{\tau \to -\infty} \varphi^a(\tau) = \varphi^a_H(\mathcal{Q})$

conformally flat geometry $AdS_2 \times S^2$ near the horizon

$$ds^2_{\text{B-R}} = \frac{r^2}{M^2_{\text{B-R}}} dt^2 - \frac{M^2_{\text{B-R}}}{r^2}(dr^2 + r^2 d\Omega)$$

near the horizon, the scalar fields are **stabilized** purely in terms of charges

$$S = \frac{A_H}{4} = \pi\, V_{BH}|_{\partial_\varphi V_{BH}=0} = -\frac{\pi}{2}\mathcal{Q}^T \mathcal{M}_H \mathcal{Q}$$

Bekenstein-Hawking entropy-area formula for extremal dyonic BH

Example : Symmetric Scalar Manifolds

Let's specialize the discussion to theories with scalar manifolds which are **symmetric cosets G/H**

[**N>2** : general, **N=2** : particular, **N=1** : special cases]

H = isotropy group = *linearly* realized; scalar fields sit in an **H**-repr.

G = (global) electric-magnetic duality group
[in string theory : **U-duality**]

In general :

G is an *on-shell* symmetry of the Lagrangian

The 2-form field strengths (F,G) vector and the BH e.m. charges sit in a **G**-repr. **R** which is **symplectic** :

$$\exists ! \mathbb{C}_{[MN]} \equiv 1 \in \mathbf{R} \times_a \mathbf{R};$$

$$\langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle \equiv \mathcal{Q}_1^M \mathcal{Q}_2^N \mathbb{C}_{MN} = - \langle \mathcal{Q}_2, \mathcal{Q}_1 \rangle$$

symplectic product

$$\mathbb{C} = \begin{pmatrix} \mathbf{0}_n & \mathbb{I}_n \\ -\mathbb{I}_n & \mathbf{0}_n \end{pmatrix}$$

$$\begin{array}{ccc} G & \subset & Sp(2n, \mathbb{R}); \\ \mathbf{R} & = & \mathbf{2n} \end{array}$$

Gaillard-Zumino embedding
(generally maximal, but not symmetric)

Dynkin, Gaillard-Zumino

❖ **symmetric** scalar manifolds of **N=2, D=4 sugra** [all but T^3 model]

all special Kaehler of projective type	$\frac{G_V}{H_V}$	r	$dim_{\mathbb{C}} \equiv n_V$
quadratic sequence $n \in \mathbb{N}$	$\frac{SU(1,n)}{U(1) \otimes SU(n)}$	1	n
$\mathbb{R} \oplus \mathbf{\Gamma}_n,\ n \in \mathbb{N}$	$\frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,n)}{SO(2) \otimes SO(n)}$	$2\ (n=1)$ $3\ (n \geqslant 2)$	$n+1$
$J_3^{\mathbb{O}}$	$\frac{E_{7(-25)}}{E_{6(-78)} \otimes U(1)}$	3	27
$J_3^{\mathbb{H}}$	$\frac{SO^*(12)}{U(6)}$	3	15
$J_3^{\mathbb{C}}$	$\frac{SU(3,3)}{S(U(3) \otimes U(3))} = \frac{SU(3,3)}{SU(3) \otimes SU(3) \otimes U(1)}$	3	9
$J_3^{\mathbb{R}}$	$\frac{Sp(6,\mathbb{R})}{U(3)}$	3	6

$$R_{i\bar{j}k\bar{l}} = -g_{i\bar{j}}g_{k\bar{l}} - g_{i\bar{l}}g_{k\bar{j}} + C_{ikm}\overline{C}_{\overline{jlp}}g^{m\bar{p}}$$

symmetric scalar manifolds **G/H** (including symm. SKGs of N=2, D=4 sugra) :

The **G**-representation space **R** of the BH em charges gets **stratified**, under the action of **G**, in **U-orbits** (*non-symmetric* cosets **G/$\mathcal{H}$**). Ferrara, Gunaydin

$\mathcal{H}$ is the **stabilizer** (isotropy) group of the **U-orbit** = symmetry of the charge configs., it relates equivalent BH charge configs

each **U-orbit** supports a class of crit. pts. of V_{BH} , corresponding to specific ***SUSY-preserving properties*** of the near-horizon geometry

When $\mathcal{H}$ is **non-compact**, there is a residual compact symmetry linearly acting on scalars, such that the scalars belonging to the ***"moduli space"* $\mathcal{H}$/mcs($\mathcal{H}$)** (symmetric **submanifold** of **G/H**) are **not** stabilized in terms of BH charges at the event horizon of the extremal BH

Ferrara, AM

The Attractor Mechanism is **inactive** on these **unstabilized** scalar fields, which are ***flat directions*** of V_{BH} at its critical points.

symmetric scalar manifolds **G/H** (cont'd) :

The **absence** of flat directions at **N=2 ½-BPS attractors** can thus be explained by the fact that the stabilizer of the ½-BPS orbit is **compact** : **$\mathcal{H}$=H/U(1)**, where H is the stabilizer of the scalar manifold **G/H** itself

The **massless Hessian modes**, ubiquitous at non-BPS crit pts of V_{BH}, are actually **all *flat directions*** of V_{BH} itself at the considered class of crit. pts.

BH Entropy is Independent on All Unstabilized Scalars

Thus, the ***flat directions*** of V_{BH} at its critical points span various "***moduli spaces"***, related to the solutions of the ***Attractor Eqs*****.**

❖ *"large"* **U-Orbits** of symmetric **N=2, D=4 sugras** [all but T^3 model]

	$\frac{1}{2}$-BPS orbits $\mathcal{O}_{\frac{1}{2}-BPS} = \frac{G}{H_0}$	non-BPS, $Z \neq 0$ orbits $\mathcal{O}_{non-BPS,Z\neq 0} = \frac{G}{\hat{H}}$	non-BPS, $Z = 0$ orbits $\mathcal{O}_{non-BPS,Z=0} = \frac{G}{\tilde{H}}$
Quadratic Sequence $(n = n_V \in \mathbb{N})$	$\frac{SU(1,n)}{SU(n)}$	$-$	$\frac{SU(1,n)}{SU(1,n-1)}$
$\mathbb{R} \oplus \Gamma_n$ $(n = n_V - 1 \in \mathbb{N})$	$\frac{SU(1,1)\otimes SO(2,n)}{SO(2)\otimes SO(n)}$	$\frac{SU(1,1)\otimes SO(2,n)}{SO(1,1)\otimes SO(1,n-1)}$	$\frac{SU(1,1)\otimes SO(2,n)}{SO(2)\otimes SO(2,n-2)}$
$J_3^{\mathbb{O}}$	$\frac{E_{7(-25)}}{E_6}$	$\frac{E_{7(-25)}}{E_{6(-26)}}$	$\frac{E_{7(-25)}}{E_{6(-14)}}$
$J_3^{\mathbb{H}}$	$\frac{SO^*(12)}{SU(6)}$	$\frac{SO^*(12)}{SU^*(6)}$	$\frac{SO^*(12)}{SU(4,2)}$
$J_3^{\mathbb{C}}$	$\frac{SU(3,3)}{SU(3)\otimes SU(3)}$	$\frac{SU(3,3)}{SL(3,\mathbb{C})}$	$\frac{SU(3,3)}{SU(2,1)\otimes SU(1,2)}$
$J_3^{\mathbb{R}}$	$\frac{Sp(6,\mathbb{R})}{SU(3)}$	$\frac{Sp(6,\mathbb{R})}{SL(3,\mathbb{R})}$	$\frac{Sp(6,\mathbb{R})}{SU(2,1)}$

Bellucci, Ferrara, Gunaydin, AM

in N=2 : $\{Q^A_\alpha, Q^B_\beta\} = \epsilon_{\alpha\beta} Z^{[AB]} = \epsilon_{\alpha\beta}\epsilon^{AB} Z$

❖ non-BPS Z<>0 **attractor** ***"moduli spaces"*** of symmetric **N=2, D=4 sugras**

Ferrara,AM

$\hat{h}$ =mcs $\hat{H}$

	$\frac{\hat{H}}{\hat{h}}$	r	$dim_{\mathbb{R}}$
$\mathbb{R} \oplus \Gamma_n$ $(n = n_V - 1 \in \mathbb{N})$	$SO(1,1) \otimes \frac{SO(1,n-1)}{SO(n-1)}$	$1 (n = 1)$ $2 (n \geqslant 2)$	n
$J_3^{\mathbb{O}}$	$\frac{E_{6(-26)}}{F_{4(-52)}}$	2	6
$J_3^{\mathbb{H}}$	$\frac{SU^*(6)}{USp(6)}$	2	14
$J_3^{\mathbb{C}}$	$\frac{SL(3,C)}{SU(3)}$	2	8
$J_3^{\mathbb{R}}$	$\frac{SL(3,\mathbb{R})}{SO(3)}$	2	5

They are nothing but the ***real special*** scalar manifolds
of symmetric **N=2, D=5 sugras**

let's reconsider the starting **Maxwell-Einstein-scalar Lagrangian density**

$$\mathcal{L} = -\frac{R}{2} + \frac{1}{2} g_{ij}(\varphi)\,\partial_\mu \varphi^i \partial^\mu \varphi^j + \frac{1}{4} I_{\Lambda\Sigma}(\varphi)\, F^\Lambda_{\mu\nu} F^{\Sigma|\mu\nu} + \frac{1}{8\sqrt{-G}} R_{\Lambda\Sigma}(\varphi)\, \epsilon^{\mu\nu\rho\sigma} F^\Lambda_{\mu\nu} F^\Sigma_{\rho\sigma}$$

…and introduce the following real 2n x 2n matrix :

$$\mathcal{M} = \begin{pmatrix} \mathbb{I} & -R \\ 0 & \mathbb{I} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{I} & 0 \\ -R & \mathbb{I} \end{pmatrix} = \begin{pmatrix} I + R I^{-1} R & -R I^{-1} \\ -I^{-1} R & I^{-1} \end{pmatrix}$$

$$\mathcal{M} = \mathcal{M}(R, I) = \mathcal{M}(\mathrm{Re}(\mathcal{N}), \mathrm{Im}(\mathcal{N})).$$

$$\mathcal{M}^T = \mathcal{M} \qquad \mathcal{M}\mathbb{C}\mathcal{M} = \mathbb{C}$$

$$\mathcal{M} = -\left(\mathbf{L}\mathbf{L}^T\right)^{-1} = -\mathbf{L}^{-T}\mathbf{L}^{-1},$$

L = element of the **Sp(2n,R)**-bundle over the scalar manifold
(= *coset representative* for homogeneous spaces **G/H**)

…by virtue of this matrix, one can introduce a (scalar-dependent) **anti-involution in *any* Maxwell-Einstein-scalar gravity theory** with **symplectic structure** :

$$\mathfrak{F}(\varphi) := -\mathbb{C}\mathcal{M}(\varphi)$$

$$\mathfrak{F}^2(\varphi) = \mathbb{C}\mathcal{M}(\varphi)\mathbb{C}\mathcal{M}(\varphi) = \mathbb{C}^2 = -Id$$

Ferrara,AM,Yeranyan; Borsten,Duff, Ferrara,AM

This allows to define an **anti-involution** on the dyonic charge vector Q, named (**scalar-dependent**) **Freudenthal duality**

By recalling $$V_{BH}(\varphi, \mathcal{Q}) := -\frac{1}{2}\mathcal{Q}^T\mathcal{M}(\varphi)\mathcal{Q},$$

Freudenthal duality can be related to the **effective BH potential** :

$$\mathfrak{F} : \mathcal{Q} \to \mathfrak{F}(\mathcal{Q}) := \mathbb{C}\frac{\partial V_{BH}}{\partial \mathcal{Q}}.$$

All this enjoys a remarkable physical interpretation when evaluated **at the horizon** :

Attractor Mechanism $\partial_\varphi V_{BH} = 0 \Leftrightarrow lim_{\tau\to-\infty}\varphi^a(\tau) = \varphi^a_H(\mathcal{Q})$

Bekenstein-Hawking entropy $S = \frac{A_H}{4} = \pi V_{BH}|_{\partial_\varphi V_{BH}=0} = -\frac{\pi}{2}\mathcal{Q}^T\mathcal{M}_H\mathcal{Q}$

…by evaluating the matrix M at the horizon $\lim_{\tau\to-\infty}\mathcal{M}(\varphi(\tau)) = \mathcal{M}_H(\mathcal{Q})$

one can define the **horizon Freudenthal duality** as:

$$\lim_{\tau\to-\infty}\mathfrak{F}(\mathcal{Q}) =: \mathfrak{F}_H(\mathcal{Q}) = -\mathbb{C}\mathcal{M}_H\mathcal{Q} = \frac{1}{\pi}\mathbb{C}\frac{\partial S_{BH}}{\partial \mathcal{Q}} =: \tilde{\mathcal{Q}},$$

$$\mathfrak{F}^2_H(\mathcal{Q}) = \mathfrak{F}_H(\tilde{\mathcal{Q}}) = -\mathcal{Q}$$

non-linear (scalar-independent) anti-involutive map on Q (hom of degree one)

Bekenstein – Hawking entropy is **invariant** under its **non-linear symplectic gradient** :

$$S(\mathcal{Q}) = S(\mathfrak{F}_H(\mathcal{Q})) = S\left(\frac{1}{\pi}\mathbb{C}\frac{\partial S}{\partial \mathcal{Q}}\right) = S(\tilde{\mathcal{Q}})$$

This can be extended to include *at least* **all quantum corrections** with **homogeneity 2** or **0** in the BH charges Q

Ferrara, AM, Yeranyan (and late Raymond Stora)

Lie groups of type E_7 : (G,**R**)

Brown (1967);
Garibaldi; Krutelevich;
Borsten,Duff *et al.*
Ferrara,Kallosh,AM;
AM,Orazi,Riccioni

❖ the (ir)repr. **R** is **symplectic** :

$$\exists ! \mathbb{C}_{[MN]} \equiv \mathbf{1} \in \mathbf{R} \times_a \mathbf{R};$$

$$\langle Q_1, Q_2 \rangle \equiv Q_1^M Q_2^N \mathbb{C}_{MN} = - \langle Q_2, Q_1 \rangle ;$$

symplectic product

❖ the (ir)repr. admits a unique completely symmetric **invariant rank-4** tensor

$$\exists ! \ K_{MNPQ} = K_{(MNPQ)} \equiv \mathbf{1} \in [\mathbf{R} \times \mathbf{R} \times \mathbf{R} \times \mathbf{R}]_s$$

(K-tensor)

↓ G-invariant quartic polynomial

$$I_4 := K_{MNPQ} Q^M Q^N Q^P Q^Q =: \epsilon \, |I_4| \, ,$$

→

$$S_{BH} = \pi \sqrt{|I_4|}$$

❖ defining a triple map in **R** as

$$T : \mathbf{R} \times \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$$

$$\langle T(Q_1, Q_2, Q_3), Q_4 \rangle \equiv K_{MNPQ} Q_1^M Q_2^N Q_3^P Q_4^Q$$

it holds

$$\langle T(Q_1, Q_1, Q_2), T(Q_2, Q_2, Q_2) \rangle = \langle Q_1, Q_2 \rangle \, K_{MNPQ} Q_1^M Q_2^N Q_2^P Q_2^Q$$

this third property makes a **group of type E_7** amenable to be defined as the **automorphism group** of a **Freudenthal triple systems**

All electric-magnetic **(U-)duality** groups of D=4 sugras with **symmetric** scalar manifolds and *at least* **8** supersymmetries are **of type E_7**

$N = 2$

G	R
$U(1,n)$	$(\mathbf{1}+\mathbf{n})$
$SL(2,\mathbb{R}) \times SO(2,n)$	$(\mathbf{2},\mathbf{2}+\mathbf{n})$
$SL(2,\mathbb{R})$	$\mathbf{4}$
$Sp(6,\mathbb{R})$	$\mathbf{14}'$
$SU(3,3)$	$\mathbf{20}$
$SO^*(12)$	$\mathbf{32}$
$E_{7(-25)}$	$\mathbf{56}$

N	G	R
3	$U(3,n)$	$(\mathbf{3}+\mathbf{n})$
4	$SL(2,\mathbb{R}) \times SO(6,n)$	$(\mathbf{2},\mathbf{6}+\mathbf{n})$
5	$SU(1,5)$	$\mathbf{20}$
8	$E_{7(7)}$	$\mathbf{56}$

(E_7, 912 – embedding tensor) satisfies the first two axioms, *but* ***not*** *the third one!*

"degenerate" groups of type E_7

$$I_4(p,q) = \left(I_2(p,q)\right)^2$$

$$S_{BH} = \pi\sqrt{|I_4(p,q)|} = \pi\,|I_2(p,q)|\,.$$

In sugras with electric-magnetic duality group **of type E_7**,
the **G**-invariant **K-tensor** determining the
extremal BH Bekenstein-Hawking entropy

$$S_{BH} = \pi\sqrt{|I_4|}$$

$$I_4 := K_{MNPQ}\mathcal{Q}^M\mathcal{Q}^N\mathcal{Q}^P\mathcal{Q}^Q =: \epsilon\,|I_4|\,,$$

can generally be expressed as adjoint-trace of the product of **G**-generators
(dim **R** = 2n, and dim **Adj** = d)

$$K_{MNPQ} = -\frac{n\,(2n+1)}{6d}\left[t^{\alpha}_{MN}t_{\alpha|PQ} - \frac{d}{n(2n+1)}\mathbb{C}_{M(P}\mathbb{C}_{Q)N}\right]$$

The **horizon Freudenthal duality** can be expressed in terms of the **K-tensor**

$$\mathfrak{F}_H(\mathcal{Q})_M = \tilde{\mathcal{Q}}_M = \frac{\partial\sqrt{|I_4(\mathcal{Q})|}}{\partial\mathcal{Q}^M} = \epsilon\frac{2}{\sqrt{|I_4(\mathcal{Q})|}}K_{MNPQ}\mathcal{Q}^N\mathcal{Q}^P\mathcal{Q}^Q$$

Borsten,Dahanayake,Duff,Rubens

the **invariance** of the BH entropy under **horizon Freudenthal duality** reads as

$$I_4\,(\mathcal{Q}) = I_4(\mathbb{C}\tilde{\mathcal{Q}}) = I_4\left(\mathbb{C}\frac{\partial\sqrt{|I_4(\mathcal{Q})|}}{\partial\mathcal{Q}}\right)$$

Metric structure on the U-orbits with non-vanishing I_4 :

$$M^H_{-|MN} = -\partial_M \partial_N \sqrt{|I_4(\mathcal{Q})|} = -\frac{1}{\pi} \partial_M \partial_N S_{BH}$$

Ferrara, AM Orazi, Trigiante

(opposite of the) **Hessian of the BH entropy**

$$\left(M^H_-(\mathcal{Q})\right)^T \mathbb{C} M^H_-(\mathcal{Q}) = \epsilon \mathbb{C} \qquad \epsilon := I_4(\mathcal{Q}) / |I_4(\mathcal{Q})|$$

$$\left(M^H_-(\mathcal{Q})\right)^T = M^H_-(\mathcal{Q}) \qquad \mathcal{Q}^T M^H_-(\mathcal{Q})\, \mathcal{Q} = -2\sqrt{|I_4(\mathcal{Q})|}$$

$$\mathfrak{F}_H\left(M^H_-(\mathcal{Q})\right) = \epsilon M^H_-$$

This matrix is the (opposite of the) **pseudo-Riemannian metric** of a non-compact, real form of a **pre-homogeneous vector space** (**PVS**), associated to a certain orbit of the electric-magnetic duality group

Example : *"large" supersymmetric* orbit in maximal supergravity

$$N=8, D=4: scalar\ manifold\ \mathrm{M}_{N=8}=\frac{E_{7(7)}}{SU(8)},\ dim_{\mathbb{R}}=70,\ rank=7$$

$$I_4>0: \frac{1}{8}-BPS\ E_{7(7)}-orbit\ in\ \mathbf{56}\ repr.space: \mathcal{O}_{I_4>0}=\frac{E_{7(7)}}{E_{6(2)}}$$

$$(quaternionic)\ moduli\ space\ \mathcal{M}_{I_4>0}=\frac{E_{6(2)}}{SU(6)\times SU(2)}\left(\subset\frac{E_{7(7)}}{SU(8)}\right),\ dim_{\mathbb{R}}=40,\ rank=4$$

$$M_{-}^{H}=-\partial^2\sqrt{I_4}: metric\ of\ \mathcal{O}_{I_4>0}\times\mathbb{R}^+=\frac{E_{7(7)}}{E_{6(2)}}\times\mathbb{R}^+;\ (n_+,n_-)=(30,26)$$

As mentioned , $\frac{E_{7(7)}}{E_{6(2)}} \times \mathbb{R}^+$ is a non-compact, real form of $\frac{E_7}{E_6} \times GL(1)$

Regular **Pre-Homogeneous Vector Space** (**PVS**) of type (29) in the classification by Sato and Kimura ('77) :

(29) $(GL(1) \times E_7,\ \square \otimes \Lambda_6,\ V(1) \otimes V(56))$.

(i) $H \sim E_6$, (ii) $\deg f = 4$, (iii) $f(X) = T(x^\sharp, y^\sharp) - \xi N(x) - \eta N(y) - \frac{1}{4}(T(x,y) - \xi\eta)^2$ (see (1.16), or Proposition 52 in § 5).

A **PVS** is a finite-dimensional vector space ***V*** together with a subgroup ***G*** of **GL(*V*)** such that ***G*** has an **open**, **dense orbit** in ***V*** [Sato,Kimura; Knapp]

PVS are subdivided into two types, according to whether there exists a *homogeneous* polynomial **f** on ***V*** which is **invariant** under the semisimple part of ***G***.

In this case : **V** = **56** (fundamental irrep. of **G=E$_7$**), **f** = **quartic** invariant polynomial **I$_4$**
H= isotropy (stabilizer) group = **E$_6$**

Manifestly E$_6$ -invariant expression of the quartic invariant **I$_4$** of the **56** of **E$_7$** :
much before (**'77** = almost contemporary to sugra) the expression introduced by Ferrara,Gunaydin (**'97**) **!**

$$I_4\left(p^0, p^i, q_0, q_i\right) = -\left(p^0 q_0 + p^i q_i\right)^2 + 4\left[q_0 I_3(p) - p^0 I_3(q) + \left\{\frac{\partial I_3(p)}{\partial p}, \frac{\partial I_3(q)}{\partial q}\right\}\right]$$

Simple groups. **“of type E_7” of sugra** almost saturate list of **irr. PVS** with **invariant** deg **4**

G	V	n	Isotropy algebra	Degree	
$SL(2,\mathbb{C})$	$S^3\mathbb{C}^2$	1	0	4	N=2, T^3 model
$SL(6,\mathbb{C})$	$\Lambda^3\mathbb{C}^6$	1	$\mathfrak{sl}(3,\mathbb{C}) \times \mathfrak{sl}(3,\mathbb{C})$	4	N=2 magic on **R**
$SL(7,\mathbb{C})$	$\Lambda^3\mathbb{C}^7$	1	$\mathfrak{g}_2^{\mathbb{C}}$	7	
$SL(8,\mathbb{C})$	$\Lambda^3\mathbb{C}^8$	1	$\mathfrak{sl}(3,\mathbb{C})$	16	
$SL(3,\mathbb{C})$	$S^2\mathbb{C}^3$	2	0	6	
$SL(5,\mathbb{C})$	$\Lambda^2\mathbb{C}^3$	3,4	$\mathfrak{sl}(2,\mathbb{C}), 0$	5,10	
$SL(6,\mathbb{C})$	$\Lambda^2\mathbb{C}^3$	2	$\mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$	6	
$SL(3,\mathbb{C}) \times SL(3,\mathbb{C})$	$\mathbb{C}^3 \otimes \mathbb{C}^3$	2	$\mathfrak{gl}(1,\mathbb{C}) \times \mathfrak{gl}(1,\mathbb{C})$	6	
$Sp(6,\mathbb{C})$	$\Lambda_0^3\mathbb{C}^6$	1	$\mathfrak{sl}(3,\mathbb{C})$	4	N=2 magic on **C**
$Spin(7,\mathbb{C})$	$\mathbb{C}^8$	1,2,3	$\mathfrak{g}_2^{\mathbb{C}}, \mathfrak{sl}(3,\mathbb{C}) \times \mathfrak{so}(2,\mathbb{C}), \mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{so}(3,\mathbb{C})$	2,2,2	
$Spin(9,\mathbb{C})$	$\mathbb{C}^{16}$	1	$\mathfrak{spin}(7,\mathbb{C})$	2	
$Spin(10,\mathbb{C})$	$\mathbb{C}^{16}$	2,3	$\mathfrak{g}_2^{\mathbb{C}} \times \mathfrak{sl}(2,\mathbb{C}), \mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{so}(3,\mathbb{C})$	2,4	3-ctr. Inv. of **N=0** MESGT
$Spin(11,\mathbb{C})$	$\mathbb{C}^{32}$	1	$\mathfrak{sl}(5,\mathbb{C})$	4	**?**
$Spin(12,\mathbb{C})$	$\mathbb{C}^{32}$	1	$\mathfrak{sl}(6,\mathbb{C})$	4	N=2 magic on **H**, N=6
$Spin(14,\mathbb{C})$	$\mathbb{C}^{64}$	1	$\mathfrak{g}_2^{\mathbb{C}} \times \mathfrak{g}_2^{\mathbb{C}}$	8	
$G_2^{\mathbb{C}}$	$\mathbb{C}^7$	1,2	$\mathfrak{sl}(3,\mathbb{C}), \mathfrak{gl}(2,\mathbb{C})$	2,2	
$E_6^{\mathbb{C}}$	$\mathbb{C}^{27}$	1,2	$\mathfrak{f}_4^{\mathbb{C}}, \mathfrak{so}(8,\mathbb{C})$	3,6	
$E_7^{\mathbb{C}}$	$\mathbb{C}^{56}$	1	$\mathfrak{e}_6^{\mathbb{C}}$	4	N=2 magic on **O**, N=8

In sugra, **n** can be associated to the # of centers of the **multi-centered** BH

Here we only consider *irreducible* **PVS**, with **G** simple and **complex** Lie group

→ Classification of groups "**of type E_7**" ? *in progress….*

Some advances in rather recent papers,
e.g. [Garibaldi, Guralnick]

G	V	$\dim V$	$\mathrm{char}\, k$	G	V	$\dim V$	$\mathrm{char}\, k$
B_n	λ_1	$2n+1$	$\neq 2$	A_1	$\lambda_1 + p^i \lambda_1 \ (i \geq 1)$	4	$= p \neq 0$
D_n	λ_1	$2n$	all	A_2	$\lambda_1 + \lambda_2$	7	3
A_1	$2\lambda_1$	3	$\neq 2$	A_3	λ_2	6	all
A_5	λ_3	20	2	B_4	λ_4	16	all
B_3	λ_3	8	all	B_5	λ_5	32	2
C_3	λ_3	8	2	C_3	λ_2	13	3
D_6	half-spin	32	2	G_2	λ_1	7	$\neq 2$
E_7	λ_7	56	2	F_4	λ_4	25	3

p=2 : T^3 model

?

known simple Lie groups "**of type E_7**" occurring in **D=4 (super)gravity theories**

Some Hints for the Future…

- **Freudenthal Duality** for **non-symmetric** PSK manifolds
  [deWit, Van Proeyen; Alekseevsky, Cortes, …]
  and relation to **T-Algebras** [Vinberg, Cecotti]

- **D=5 : Jordan Duality for black holes and black strings**,
  groups **of type E_6, PVS** , and **D=5 Maxwell-Einstein (super)gravity**
  Borsten, Duff *et al.*

- **Freudenthal duality** for intrinsically quantum black holes
  («small» orbits)

- extension to **Multi-Centered (extremal) BH solutions**:
  some progress [Yeranyan; Ferrara,AM,Shcherbakov,Yeranyan]

- into the ***quantum regime*** of gravity [**U-duality over *discrete* fields**]:
  Freudenthal Duality for integer, quantized charges ?
  Borsten, Duff *et al.*,

- …what is the ***«square root»*** of Freudenthal duality?
  [related to establishing the ***«square root»*** of the attractor mechanism…]

Thank You!