# $D = 6$, $\mathcal{N} = (2, 0)$ and $\mathcal{N} = (4, 0)$ theories


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Geometry, Duality and Strings 2018, May 23 - 26 Murcia, Spain

# Introduction I: The $(4, 0)$ Conjecture

- It was at one time thought that non-trivial conformal quantum field theories exist in at most $D = 4$ spacetime dimensions

- At odds with Nahm's '77 classification, which includes $D = 6$ superconformal algebras

- M-theory: existence of non-trivial $D = 6$ quantum field theories with $\mathcal{N} = (2, 0)$ supersymmetry and $\mathrm{OSp}^{\star}(8|4)$ superconformal symmetry [Gunaydin, Warner '84, Duff, Blencowe '87; Witten '95; Strominger '95; Maldacena '97]

- Consistency of superalgebra does not imply a corresponding non-trivial quantum field theory exists, e.g [Cordova, Intrilligator '16]

- However, taking confidence from the $(2, 0)$ story it is tempting to speculate that the $D = 6, \mathcal{N} = (4, 0)$ multiplet with $\mathrm{OSp}^{\star}(8|8)$ superconformal symmetry, a longstanding and enticing outpost of Nahm's taxonomy, should also correspond to a non-trivial quantum theory

# Introduction I: The $(4,0)$ Conjecture

- Hull argued that a non-trivial "$(4,0)$ theory" may arise in the large $D=5$ Plank length, $l_5$, limit of M-theory compactified on 6-torus
  [Hull '00]
- Warning: highly speculative. For example, we have no M-brane construction supporting its existence
- As emphasised by Hull, the $(4,0)$ theory would constitute the maximally symmetric phase of M-theory
- Moreover, it contains a self-dual "gravi-gerbe" field, suggestive of a $D=6$ chiral theory of conformal gravity.
- Establishing its existence would have profound implications for not only M-theory, but also gravity more broadly understood

# Introduction II: Gravity = Gauge × Gauge

- Here we re-examine the free $(4, 0)$ theory from another, a priori unrelated, but equally provocative, perspective:

$$\text{“gravity} = \text{gauge} \times \text{gauge”}$$

- The scattering amplitudes of (super)gravity are conjectured to be the "double-copy" of (super) Yang-Mills amplitudes to all orders in perturbation theory!

  [Bern, Carrasco, Johansson '08, Bern10ue, Bern:2010yg]

- These fascinating amplitude relations are both computationally expedient and conceptually suggestive, facilitating previously intractable calculations

- Probe profound questions regarding the deep structure of perturbative quantum gravity

# Introduction II: Gravity = Gauge × Gauge

- In this context $D = 5$, $\mathcal{N} = 8$ supergravity, the low energy limit of M-theory on a 6-torus, is the double-copy of $D = 5$, $\mathcal{N} = 4$ super Yang-Mills theory
- M-theory uplift: $(2, 0)$ theory compactified on a circle of radius $R \propto g^2_{YM}$
- Can we formulate $(4, 0) = (2, 0) \times (2, 0)$, morally the M-theory uplift of gravity = gauge × gauge?
  [Chiodaroli, Günaydin, Roiban '11; Anastasiou, LB, Duff, Hughes, Nagy '13]
- Intrinsically non-perturbative nature of the $(2, 0)$ theories makes amplitude relations hard to formulate, although there exist some limited tests
  [Huang, Lipstein '10r, Czech, Rozali '11]
- Avoid this hurdle by appealing to a complementary and independent off-shell field-theoretic realisation of gravity as the "square of Yang-Mills"
  [Anastasiou, LB, Duff, Hughes, Nagy '13; LB '17]

# Plan

I: Review of Gravity = "Gauge $\times$ Gauge" paradigm

II: Review of $\mathcal{N} = (2,0)$ and $\mathcal{N} = (4,0)$ superconformal theories in $D = 6$

III: $(2,0) \times (2,0) = (4,0)$

IV: Conclusions and future directions

# Part I

The "Gravity $=$ Gauge $\times$ Gauge" paradigm

# Gravity and gauge theory

- Gravity as a gauge theory?
  - Gauge theory of Lorentz, (super) Poincaré or de Sitter symmetries
    [Utiyama '56; Kibble '61; MacDowell-Mansouri '77; Chamseddine-West '77; Stelle-West 79]
  - Holographic principle - AdS/CFT correspondence
    ['t Hooft '93; Susskind '94; Maldacena '97]

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- Here, we appeal to a third and (superficially) independent perspective:

$$\text{Gravity} = \text{Gauge} \times \text{Gauge}$$

- The theme of gravity as the "square" of Yang-Mills has appeared in a variety of guises going back to the KLT relations of string theory
  [Kawai-Lewellen-Tye '85]

# Bern-Carrasco-Johansson colour-kinematic duality

- Color-dressed $n$-point tree amplitude of Yang-Mills theory:

$$A_n^{\text{tree}} = \sum_{i \in \text{trivalent graphs}} \frac{c_i n_i}{\prod_{a_i} p_{a_i}^2}$$

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- There is a representation of $A_n^{\text{tree}}$ such that

$$\boxed{c_i + c_j + c_k = 0 \quad \Leftrightarrow \quad n_i + n_j + n_k = 0}$$

(invoking generalised gauge transformations if necessary)

[Bern, Dennen, Huang, Kiermaier '10]

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- Conjectured to hold at loop level [Bern, Carrasco, Johansson '08, '10]

# The BCJ double-copy prescription

- BCJ colour-kinematic duality and the double-copy prescription:

Kinematic factor

Loops

Colour factor

Kinematic factors

$$\underbrace{\sum_i \int \prod^L \frac{d^D p_l}{(2\pi)^D S_i} \frac{c_i n_i}{\prod_{a_i} p_{a_i}^2}}_{\text{gauge theory amplitude}} \longrightarrow \underbrace{\sum_i \int \prod^L \frac{d^D p_l}{(2\pi)^D S_i} \frac{\tilde{n}_i n_i}{\prod_{a_i} p_{a_i}^2}}_{\text{gravity amplitude}}$$

Diagrams

Propagators

$$c_i \longrightarrow \tilde{n}_i$$

[Bern-Carrasco-Johansson '08, '10; Bern-Dennen-Huang-Kiermaier '10]

# The BCJ double-copy: generalisations

- Replace kinematics with colour: $\phi^3$-theory plays a crucial role (more on this later)

  [Hodges '11; Cachazo, He, Yuan '13 '14, Dolan, Goddard '13; Naculich '14 '15, ...]

$$\underbrace{\sum_i \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D S_i} \frac{c_i n_i}{\prod_{a_i} p_{a_i}^2}}_{\text{gauge theory amplitude}} \quad \underbrace{\longrightarrow}_{n_i \to \tilde{c}_i} \quad \underbrace{\sum_i \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D S_i} \frac{c_i \tilde{c}_i}{\prod_{a_i} p_{a_i}^2}}_{\phi^3 \text{ amplitude}},$$

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Inputs: Matter-coupled (super) Yang-Mills, $D = 3$ Chern-Simons-Matter, QCD, Higgsed theories, Z-theory, $(DF)^2$ theories ...

Outputs: Maxwell/scalar/Yang-Mills supergravities, gauged supergravities (Minkowski vacua), NLSM, pure gravity, $\phi^3$-theory, Born-Infeld, conformal gravity, string theories ...

[Cachazo, He, Yuan '13 '14; Chiodaroli et al '14 '15; Johansson, Ochirov '15 '16; Carrasco, Mafra, Schlotterer '16; Johansson, Nohle '17; Azevedo, Chiodaroli, Johansson, Schlotterer '18...]

# The power of BCJ

- Conceptually compelling and computationally powerful: $\mathcal{N} = 8$ supergravity four-point to 5 loops! (finite)
  [Bern, Carrasco, Chen, Edison, Johansson, Parra-Martinez, Roiban, Zeng '18]

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- At 7 loops any would-be cancellations are "not consequences of supersymmetry in any conventional sense" (Bjornsson and Green)

- $D = 4, \mathcal{N} = 5$ supergravity finite to 4 loops, contrary to expectations:

  "Enhanced" cancellations

  No such cancellations seen for $\mathcal{N} = 8$ supergravity at 5 loops: implications unclear
  [Bern-Davies-Dennen '14; Bern et al '18]

# Understanding Amplitudes

- String theory: monodromy $\rightarrow$ BCJ relations, manifest BCJ duality, double, copies, loops, Z-theory ...
  [Bjerrum-Bohr, Damgaard, Vanhove '09; Stieberger '09; Mafra, Schlotterer, Stieberger '11; Broedel, Dixon '12; Broedel, Schlotterer, Stieberger '13; Mafra, Schlotterer '14 '15; Carrasco, Mafra, Schlotterer '17...]
- Twistor theory: Ambitwistor theories $\rightarrow$ scattering equations, loops on the (nodal) Riemann sphere, non-flat backgrounds...
  [Mason, Skinner '13; Adamo, Casali, Skinner '13; Casali, Tourkine '14; Geyer, Mason, Monteiro, Tourkine '15; Adamo, Casali, Mason, Nekovar '17; Geyer, Monteiro '18...]
- Extended supergravity: matter couplings, U-dualities, factorised orbifold projections, gaugings, anomalies...
  [Carrasco, Chiodaroli, Gunaydin, Roiban '12; Chiodaroli, Gunaydin, Johansson, Roiban '14 '15; Carrasco, Kallosh, Roiban, Tseytlin '13; Zvi, Cheung, Chi, Davies, Dixon, Nohle '15 ...]
- Classical understanding: kinematic algebras, Drinfield double, classical solutions ...
  [Monteiro, O'Connell '11 '13; Bjerrum-Bohr, Damgaard, Monteiro, O'Connell '12; Monteiro, O'Connell, White '14; Fu, Krasnov '16; Cardoso, Nagy, Nampuri '16, '17; Luna, Monteiro, Nicholson, Ochirov, O'Connell, Westerberg, White '17...]

# Questions raised by BCJ

## How far?

- Does the BCJ duality hold for all loops?
- Are all supergravity theories "double-copy constructible"?

## How deep?

- Is the double-copy paradigm limited to amplitudes?
- To what extent can one regard gravity as the square of Yang-Mills?

# Field theoretic dictionary

## Taking a step back

- Amplitudes structures revealed by going on-shell - can we now understand their origins?

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## Field theory formulation of "Gauge × Gauge"

- Field theory product
$$A \circ \tilde{A} := A^a \cdot \Phi_{a\tilde{a}} \cdot \tilde{A}^{\tilde{a}},$$
where
$$[f \cdot g](x) = \int d^D y f(y) g(x-y).$$

- Here, $\Phi$ is a "spectator" $G \times \tilde{G}$ bi-adjoint scalar field

[Anastasiou-LB-Duff-Hughes-Nagy '14]

# Basic Properties of Product

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- The convolution reflects the fact that the amplitude relations are multiplicative in momentum space
- Essential for reproducing the local symmetries of (super)gravity from those of the two (super) Yang-Mills factors
- The Killing form accounts for the gauge groups, while the spectator field allows for arbitrary and independent $G$ and $\tilde{G}$
- The appearance of $\Phi$ is quite natural from various perspectives
  [Hodges '11; Cachazo, He, Yuan '13 '14, Monteiro, O'Connell, White '14]

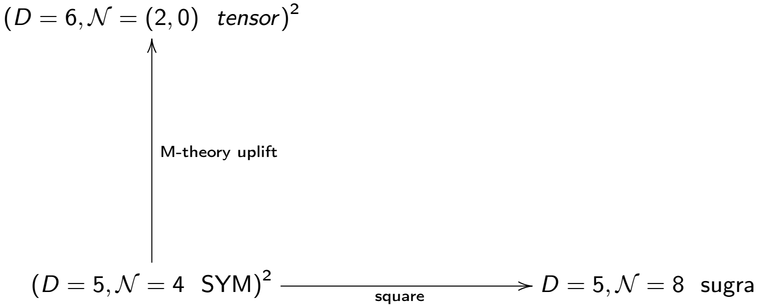
# Classifying Double-Copy Constructible Supergravity Theories

- The field theoretic product maps gauge theory content & symmetries to gravitational content & symmetries
- Generates large class of "factorisable" (super)gravities: all $\mathcal{N} \geq 2$ for $3 \leq D \leq 10$ with homogeneous scalar manifolds, with just two exceptions! [Anastasiou, LB, Duff, Marrani, Nagy, Zoccali '17]
- Agrees with all known BCJ double-copy constructible theories: for example cf. [Chiodaroli, Günaydin, Johansson, Roiban '16]

# Beyond supergravity: speculations on (2,0) squared

$$(D = 5, \mathcal{N} = 4 \;\; \text{SYM})^2 \xrightarrow[\text{square}]{} D = 5, \mathcal{N} = 8 \;\; \text{sugra}$$

# Beyond supergravity: speculations on (2,0) squared



# Part II

$\mathcal{N} = (4, 0)$ superconformal theories in $D = 6$

## The $(4, 0)$ conjecture (be warned - highly speculative)

- Maximally supersymmetric $D = 5, \mathcal{N} = 8$ supergravity has USp(8) R-symmetry and an exceptional non-compact global $E_{6(6)}(\mathbb{R})$
- Its massless fields include 27 one-form Abelian gauge potentials in the fundamental **27** of $E_{6(6)}$
- Hull considered a large $l_5$ limit under the assumption that the $E_{6(6)}$ is preserved and all supersymmetric states are protected

## The $(4, 0)$ conjecture (be warned - highly speculative)

- Decompose $\mathcal{N} = 8$ multiplet under $\mathcal{N} = 4$ subalgebra: five $\mathcal{N} = 4$ Abelian gauge multiplets with coupling constant $g^2 = l_5$

- Each lifts to an Abelian $(2, 0)$ theory as $l_5 \to \infty$, where $g^2 = l_5 = R$

- $E_{6(6)}$ preserved $\Rightarrow$ *all* 27 one-forms lift to two-forms

- Supersymmetries $\Rightarrow$ the entire $\mathcal{N} = 8$ supergravity lifts to a $D = 6$ theory, where $l_5$ is identified with $R$ such that the $l_5 \to \infty$ limit is conformal

- We therefore require a superconformal gravitational theory in $D = 6$ dimensions, consistent with a global $E_{6(6)}$ symmetry, that yields $D = 5, \mathcal{N} = 8$ supergravity when compactified on a circle

- According to Nahm's classification there is a unique candidate satisfying these criteria: the $(4, 0)$ theory

# The free $(4,0)$ theory

- The free $(4,0)$ theory introduced in [Hull '00] consists of:

$$\boxed{8\psi^{A}_{\mu\nu}, \qquad 27B^{[AB]}_{\mu\nu}, \qquad 48\lambda^{[ABC]}, \qquad 42\phi^{[ABCD]}}$$

transforming respectively as the $\mathbf{8}, \mathbf{27}, \mathbf{48}$ and $\mathbf{42}$ of USp(8)

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transforming respectively as the **8**, **27**, **48** and **42** of USp(8)

- Finally, rather than a graviton there is a rank four tensor,

$$G_{\mu\nu\rho\sigma} = G_{[\mu\nu][\rho\sigma]} = G_{[\rho\sigma][\mu\nu]}, \quad G_{[\mu\nu\rho]\sigma} = 0,$$

which might be thought of as a "gravi-gerbe" field
[Mason, Reid-Edwards '11]

- It transforms under generalise gauge transformations,

$$\delta G_{\mu\nu\rho\sigma} = \partial_{[\mu}\xi_{\nu]\rho\sigma} + \partial_{[\rho}\xi_{\sigma]\mu\nu} - 2\partial_{[\mu}\xi_{\nu\rho\sigma]}$$

# The free $(4,0)$ theory

- It has a rank six generalised gauge invariant field strength,

$$R_{\mu\nu\rho\sigma\tau\lambda} = 9\partial_{[\mu}G_{\nu\rho][\sigma\tau,\lambda]} = R_{\sigma\tau\lambda\mu\nu\rho}, \qquad R_{[\mu\nu\rho\sigma]\tau\lambda} = \partial_{[\kappa}R_{\mu\nu\rho]\sigma\tau\lambda} = 0.$$

- The natural free field equation, $R^{\mu}{}_{\nu\rho\mu\tau\lambda} = 0$, describes ten on-shell degrees of freedom in the $(\mathbf{5},\mathbf{1}) + (\mathbf{1},\mathbf{5})$.

- This is reduced to the chiral $(\mathbf{5},\mathbf{1})$ representation by the self-duality relation

$$R = \star R = R\star$$

- On a circle yields a single $D = 5$ graviton: the $(4,0)$ theory is gravitational, but does *not* contain a graviton.

- There exists a a local variational principle, breaking *manifest* covariance, for the complete $(4,0)$ theory
  [Henneaux, Lekeu, Leonard '16 '17]

# Part III

$$(2, 0) \times (2, 0) = (4, 0)$$

# $D = 6$ on-shell multiplet squared and global symmetries

- Product of on-shell $(2,0)$ tensor multiplets yields on-shell $(4,0)$ multiplet

$$[(2,0) \ \text{ on-shell states}] \times [(2,0) \ \text{ on-shell states}] = [(4,0) \ \text{ on-shell states}]$$

with R-symmetry and U-duality, e.g.

$$\phi \in E_{6(6)}/\,\mathrm{USp}(8) \quad \text{and} \quad B_{\mu\nu} \quad \text{in} \quad \mathbf{27}$$

[Strathdee de '86; Chiodaroli, Gunaydin, Roiban '11; Anastasiou, LB, Duff, Hughes, Nagy '13]

# The Free $(2, 0)$ Theory Squared

- In direct analogy with the Yang-Mills case we apply the field theoretic product:

$$\mathcal{G}_{\mu\nu\rho\sigma} := B_{\mu\nu} \circ \tilde{B}_{\rho\sigma}.$$

- Consider BRST variation $\delta$

$$\begin{aligned} \delta\mathcal{G}_{\mu\nu\rho\sigma} &= \delta B_{\mu\nu} \circ \tilde{B}_{\rho\sigma} + B_{\mu\nu} \circ \delta\tilde{B}_{\rho\sigma} \\ &= 2\partial_{[\mu} C^{(\mathbf{10})}_{\nu]\rho\sigma} + 2\partial_{[\rho} C^{(\mathbf{01})}_{\sigma]\mu\nu}, \end{aligned}$$

- Ghost field dictionary

$$C^{(\mathbf{10})}_{\nu\rho\sigma} = C_{\nu} \circ \tilde{B}_{\rho\sigma}, \qquad C^{(\mathbf{01})}_{\sigma\mu\nu} = B_{\mu\nu} \circ \tilde{C}_{\sigma}.$$

# The Free $(2,0)$ Theory Squared

- Repeat variation until everything is annihilated

$$
\begin{aligned}
\delta C^{(10)}_{\nu\rho\sigma} &= \partial_\nu C^{(20)}_{\rho\sigma} - 2\partial_{[\rho} C^{(11)}_{|\nu|\sigma]} \\
\delta C^{(01)}_{\nu\rho\sigma} &= \partial_\nu C^{(02)}_{\rho\sigma} + 2\partial_{[\rho} C^{(11)}_{\sigma]\nu} \\
\delta C^{(11)}_{\rho\sigma} &= \partial_\rho C^{(21)}_{\sigma} - \partial_\sigma C^{(12)}_{\rho} \\
\delta C^{(20)}_{\rho\sigma} &= 2\partial_{[\rho} C^{(21)}_{\sigma]} \\
\delta C^{(02)}_{\rho\sigma} &= 2\partial_{[\rho} C^{(12)}_{\sigma]} \\
\delta C^{(21)}_{\rho} &= \partial_\rho C^{(22)} \\
\delta C^{(12)}_{\rho} &= \partial_\rho C^{(22)}
\end{aligned}
$$

where we have introduced ghost-for-ghosts dictionary

$$
C^{(20)}_{\rho\sigma} = C \circ \tilde{B}_{\rho\sigma}, \quad C^{(11)}_{\rho\sigma} = C_\rho \circ \tilde{C}_\sigma, \quad C^{(02)}_{\rho\sigma} = B_{\rho\sigma} \circ \tilde{C};
$$

$$
C^{(21)}_{\rho} = C \circ \tilde{C}_\rho, \qquad C^{(12)}_{\rho} = C_\rho \circ \tilde{C};
$$

$$
C^{(22)} = C \circ \tilde{C}.
$$

- Removes $125 = (90+90) - (15+15+36) + (6+6) - 1$ components from $\mathcal{G}$, leaving 100 off-shell degrees of freedom as expected

## The Free $(2,0)$ Theory Squared

- Let us now define the irreducible $\mathrm{GL}(6,\mathbb{R})$ representations,

$$\begin{aligned} G_{\mu\nu\rho\sigma} &= \tfrac{1}{2}\left(\mathcal{G}_{\mu\nu\rho\sigma} + \mathcal{G}_{\rho\sigma\mu\nu}\right) - \mathcal{G}_{[\mu\nu\rho\sigma]}, \\ \Phi_{\mu\nu\rho\sigma} &= \mathcal{G}_{[\mu\nu\rho\sigma]}, \\ \mathcal{B}_{\mu\nu\rho\sigma} &= \tfrac{1}{2}\left(\mathcal{G}_{\mu\nu\rho\sigma} - \mathcal{G}_{\rho\sigma\mu\nu}\right), \end{aligned}$$

  transforming as $\mathbf{1}+\mathbf{20}+\mathbf{84}$, $\mathbf{15}$ and $\mathbf{15}+\mathbf{45}+\overline{\mathbf{45}}$ of $\mathrm{Spin}(1,5)$

- Concentrating on $G_{\mu\nu\rho\sigma}$ we find we obtain

$$\delta G_{\mu\nu\rho\sigma} = \partial_{[\mu}\xi_{\nu]\rho\sigma} + \partial_{[\rho}\xi_{\sigma]\mu\nu} - 2\partial_{[\mu}\xi_{\nu\rho\sigma]}, \qquad \xi_{\nu\rho\sigma} := C^{(\mathbf{10})}_{\nu\rho\sigma} + C^{(\mathbf{01})}_{\nu\rho\sigma}.$$

$$\delta\zeta_{\nu\rho\sigma} = \partial_\nu\zeta_{\rho\sigma} + \partial_{[\sigma}\zeta_{\rho]\mu}, \qquad \delta\zeta_{\rho\sigma} = 0,$$

  where $\zeta_{\nu\rho\sigma} = \xi_{\nu\rho\sigma} - \xi_{[\nu\rho\sigma]}$, $\zeta_{\rho\sigma} := 3(C^{(\mathbf{20})}_{\rho\sigma} + C^{(\mathbf{02})}_{\rho\sigma} - 2C^{(\mathbf{11})}_{[\rho\sigma]})/8$

- $\longrightarrow$ generalised gauge transformations of gravi-gerbe

# The Free $(2,0)$ Theory Squared

- Dual scalar $\Phi_{\mu\nu\rho\sigma}$:

$$\delta\Phi_{\mu\nu\rho\sigma} = 4\partial_{[\mu}\Lambda_{\nu\rho\sigma]}, \quad \delta\Lambda_{\nu\rho\sigma} = 3\partial_{[\nu}\Lambda_{\rho\sigma]},$$
$$\delta\Lambda_{\rho\sigma} = 2\partial_{[\rho}\Lambda_{\sigma]}, \quad \delta\Lambda_{\sigma} = \partial_\sigma\Lambda$$

where $\Lambda_{\nu\rho\sigma} = \xi_{[\nu\rho\sigma]}$, $\Lambda_{\rho\sigma} = \xi_{[\rho\sigma]} + 2C^{(\mathbf{11})}_{[\rho\sigma]}$, $\Lambda_\sigma = 3(C^{(\mathbf{21})}_\sigma + C^{(\mathbf{12})}_\sigma)/2$ and $\Lambda = 3C^{(\mathbf{22})}/2$

- Dual two-form $\mathcal{B}_{\mu\nu\rho\sigma}$:

$$\delta\mathcal{B}_{\mu\nu\rho\sigma} = \partial_{[\mu}\alpha_{\nu]\rho\sigma} - \partial_{[\rho}\alpha_{\sigma]\mu\nu},$$
$$\delta\alpha_{\nu\rho\sigma} = \partial_\nu\alpha_{\rho\sigma} - 2\partial_{[\rho}\beta_{\sigma]\nu},$$
$$\delta\alpha_{\rho\sigma} = 2\partial_{[\rho}\alpha_{\sigma]}, \quad \delta\beta_{\sigma\nu} = 2\partial_{(\sigma}\alpha_{\nu)},$$

where $\alpha_{\nu\rho\sigma} := C^{(\mathbf{10})}_{\nu\rho\sigma} - C^{(\mathbf{01})}_{\nu\rho\sigma}$, $\alpha_{\rho\sigma} := C^{(\mathbf{20})}_{\rho\sigma} - C^{(\mathbf{02})}_{\rho\sigma}$, $\alpha_\sigma := C^{(\mathbf{21})}_\sigma - C^{(\mathbf{12})}_\sigma$ and $\beta_{\rho\sigma} := 2C^{(\mathbf{11})}_{(\rho\sigma)}$.

# The Free $(2,0)$ Theory Squared

- Applying global supersymmetries to the factors the rest of the $(4,0)$ multiplet follows

- For example, the eight two-form gravitini

$$\Psi_{\mu\nu} \sim (\chi \circ \tilde{B}_{\mu\nu}, B_{\mu\nu} \circ \tilde{\chi})$$

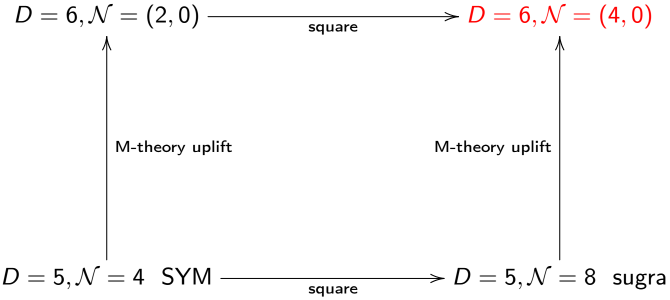
- The super-BRST variation

$$\delta\Psi_{\mu\nu} = 2\partial_{[\mu}\eta_{\nu]}$$

  is generated by the left/right two-form transformations, where the *bosonic* spinor-vector ghosts $\eta_\nu$ are identified with $\chi \circ \tilde{C}_\nu$ and $C_\nu \circ \tilde{\chi}$

- By going first to physical gauge the equations of motion it is simple to verify that all Bianchi identities and self-dualities relations for the free $(4,0)$ theory follow straightforwardly from those of the $(2,0)$ factors

# All (free) roads lead to (4, 0)



## Gravitational S-duality?

- The generalised gauge invariant gravi-gerbe curvature

$$\mathcal{R}_{\mu\nu\rho\sigma\tau\lambda} = 9\partial_{[\mu}\mathcal{G}_{\nu\rho][\tau\lambda,\sigma]} = H_{\mu\nu\rho} \circ \tilde{H}_{\sigma\tau\lambda}.$$

so that $H = \star H, \tilde{H} = \star\tilde{H}, dH = d\tilde{H} = 0$ implies

$$\mathcal{R} = \star\mathcal{R} = \mathcal{R}\star \qquad \partial_{[\mu}\mathcal{R}_{\nu\rho\sigma]\tau\lambda\kappa} = \partial_{[\kappa}\mathcal{R}_{|\mu\nu\rho|\sigma\tau\lambda]} = 0$$

- Recall: a $D = 6$ Abelian two-form with self-dual field strength on $T^2$ yields an $\mathrm{SL}(2,\mathbb{Z})$ doublet:

$$F^i = \star F^j \varepsilon_{jk}\gamma^{ki}$$

- Implies there exists an $\mathrm{SL}(2,\mathbb{Z})$ triplet of $D = 4$ linearised Riemann tensors:

$$\mathcal{R}^{(ij)} \sim F^{(i} \circ \tilde{F}^{j)}$$

obeying the duality constraint $\mathcal{R}^{(ij)} = \star\mathcal{R}^{(kj)}\varepsilon_{jk}\gamma^{ki}$

# Gravitational S-duality?

- The free $(4,0)$ theory compactified on $T^2$ yields linear $\mathcal{N}=8$ supergravity, with an $SL(2,\mathbb{Z})$ symmetry acting on a triplet of duality related gravitational field-strengths
  [Hull '00]

- Here: the "square" of the familiar $SL(2,\mathbb{Z})$ of the Abelian $(2,0)$ multiplet compactified on $T^2$

- Of course, this symmetry is broken by interactions

- Not necessarily an argument against its existence; it simply tells us that it is not a symmetry of classical $\mathcal{N}=8$ supergravity, just as S-duality is not a symmetry of classical $\mathcal{N}=4$ super Yang-Mills theory

- Not contained in the familiar $E_{7(7)}$!

- **Warning: highly speculative**

# Part IV

Conclusions and Future Directions

# Conclusions

- The free $(4,0)$ theory (local/global symmetries, eom, Bianchi identities and self-duality) is generated by the product of two free $(2,0)$ theories:

$$\text{free } (2,0) \times \text{free } (2,0) = \text{free } (4,0)$$

- The hope: this new perspective provides a guide as how to proceed in the non-linear case - hard!

# Conclusions

- A natural setting for such a question is higher gauge theory: promising progress in higher gauge $(2,0)$ models
  [Baez '10; Saemann, Schmidt '17]
- The $(4,0)$ theory will require new structures, gravitational analogs of the $(2,0)$ models - *higher chiral conformal metric on two-forms*
- Here we have an extra input to guide our considerations: the $(4,0)$ higher gauge theory will be *required* to be consistent with the square of the $(2,0)$ theory
- First step: PST action and fake curvature for free gravi-gerbe
  [LB to appear]

# Conclusions

Thank you for listening!