Local β-deformations and Yang-Baxter sigma model

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Introduction

In developments of the AdS/CFT correspondence,
the integrable structure has played an important role.

$\mathrm{AdS}_5/\mathrm{CFT}_4$

$\mathrm{AdS}_5 \times S^5$ superstring ⟺ N=4 super Yang-Mills

Techniques of integrability are very powerful tools to compute various physical observables.

- Energy spectrum of a string / Anomalous dimension of a local op.
- Three point functions

  ……

Question

The techniques are also valid for the systems with *lower supersymmetries* or *no conformal symmetry* ?

Introduction

To find extensions of the well-studied $\mathrm{AdS}_5 \times \mathrm{S}^5$ case,
integrable deformations of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring have been studied.

Yang-Baxter (YB) deformations [Klimcik, 2002,2008]

A systematic way of describing integrable deformations of 2d NLSM

The deformations are characterized by *r*-matrices (sol. of the CYBE)

$\mathrm{AdS}_5 \times S^5$ → YB def.

- gravity duals of NCYM with $[\hat{x}^\mu, \hat{x}^\nu] = i\Theta^{\mu\nu}$
  [Hashimoto-Itzhaki, Maldacena-Russo, 1999]
- solutions of the *generalized SUGRA eq.* with an extra vector $I = I^M \partial_M$
  [Arutyunov-Frolov-Hoare-Roiban-Tseytlin, 1511.05795]
  [Orlando-Reffert-J.S.-Yoshida , 1607.00795]

Introduction

YB deformations can be regarded as duality transformations.

- Non-abelian T-duality [Hoare-Tseytlin, 1609.02550] [Borsato-Wulff, 1609.09834, 1706.10169]

- β-transformation [J.S.-Sakatani-Yoshida, 1703.09213, 1705.07116] [J.S.-Sakatani, 1803.05903]

  (a kind of O(d,d) T-duality transformations)

I want to clarify relations
between YB deformations and β-transformations

Contents

1. YB deformations of the $\mathrm{AdS}_5 \times S^5$ superstring

The $\mathrm{AdS}_5 \times S^5$ superstring

The $\mathrm{AdS}_5 \times S^5$ superstring can be described by using the supercoset

$$\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$$

[Metsaev-Tseytlin, 9805028]

Green-Schwarz (GS) action of the $\mathrm{AdS}_5 \times S^5$ sueperstring

$$S = -\frac{T}{4}\int \mathrm{d}^2\sigma(\gamma^{\alpha\beta}-\epsilon^{\alpha\beta})\mathrm{Str}\left[g^{-1}\partial_\alpha g\, d(g^{-1}\partial_\beta g)\right] \quad g\in SU(2,2|4)$$

$$d = P_1 + 2P_2 - P_3$$

$P_i\,(i=0,1,2,3)$: projections to the $\mathbb{Z}_4$ -grading components $\mathfrak{g}^{(i)}$.

Z_4-grading structure

$$\mathfrak{su}(2,2|4) = \mathfrak{g}^{(0)}\oplus\mathfrak{g}^{(1)}\oplus\mathfrak{g}^{(2)}\oplus\mathfrak{g}^{(3)} \qquad [\,\mathfrak{g}^{(i)},\,\mathfrak{g}^{(j)}\,]\subset\mathfrak{g}^{(i+j)}$$

Blue : bosonic **Red** : fermionic $\mathfrak{g}^{(0)} = \mathfrak{so}(1,4)\times\mathfrak{so}(5)$

➡ existence of a Lax pair (classically integrable)

[Bena-Polchinski-Roiban, hep-th/0305116]

YB deformations of the $AdS_5 \times S^5$ superstring

The action of the YB sigma model for $AdS_5 \times S^5$

$$S_{\rm YB} = -\frac{T}{4}\int d^2\sigma\,(\gamma^{\alpha\beta}-\epsilon^{\alpha\beta})\,{\rm Str}\left[g^{-1}\partial_\alpha g\, d\circ\frac{1}{1-\eta\, R_g\circ d}g^{-1}\partial_\beta g\right]$$

- η is a deformation parameter
- The skew-symmetric linear operator $R_g(x) = g^{-1}R(gxg^{-1})g \quad x\in\mathfrak{su}(2,2|4)$

r-matrix : $r = \frac{1}{2}r^{ij}T_i\wedge T_j$ ⟹ $R(x) = r^{ij}T_i\,{\rm Str}(T_j\,x)$

The classical Yang-Baxter equation

$$[R(x),\,R(y)] - R([R(x),\,y] + [x,\,R(y)]) = 0$$

- The existence of a Lax pair ⟹ Integrable deformations

[Kawaguchi-Matsumoto-Yoshida,1401.4855]

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Outline of the supercoset construction

1. We take a parametrization of a group element

$$g = g_b \cdot g_f \in SU(2,2|4)$$

g_b : the bosonic part $\qquad g_f = \exp(\mathbf{Q}^I \theta_I)\,,$

$\mathbf{Q}^I\theta_I \equiv (\mathbf{Q}^{\check{\alpha}\hat{\alpha}})^I \theta_{I\check{\alpha}\hat{\alpha}}$

$(I = 1,2; \check{\alpha}\,,\hat{\alpha} = 1\,,\dots\,,4)$

θ_I : 10d MW fermions

2. S_{YB} is expanded by θ_I ; $\quad S_{\text{YB}} = S_{(0)} + S_{(2)} + \mathcal{O}(\theta^4)\,.$

3. We compare S_{YB} with the canonical form of the GS action.

The canonical form of the GS action [Cvetic-Lu-Pope-Stelle, 9907202]

$$\begin{aligned} S_{\text{YB}} = -T\int \mathrm{d}^2\sigma\, \Big[& P_-^{\alpha\beta}(g_{mn} + B_{mn})\partial_\alpha X^m \partial_\beta X^n && \mathcal{O}(\Theta^0) \\ & + i\,P_+^{\alpha\beta}\bar{\Theta}_1 e_\alpha{}^a \Gamma_a D_{+\beta}\Theta_1 + i\,P_-^{\alpha\beta}\bar{\Theta}_2 e_\alpha{}^a \Gamma_a D_{-\beta}\Theta_2 && \\ & - i\frac{1}{8}P_+^{\alpha\beta}\bar{\Theta}_1 e_\alpha{}^a \Gamma_a\, e^{\Phi}\boldsymbol{F}\, e_\beta{}^b \Gamma_b \Theta_2 \Big] + \mathcal{O}(\Theta^4) && \mathcal{O}(\Theta^2) \end{aligned}$$

$$P_\pm^{\alpha\beta} = \frac{\gamma^{\alpha\beta} \pm \epsilon^{\alpha\beta}}{2} \qquad D_\pm = \mathrm{d} + \frac{1}{4}\left(\omega^{ab} \pm \frac{1}{2}e_c H^{cab}\right)\Gamma_{ab} \qquad \boldsymbol{F} \equiv \sum_{n=1,3,5,7,9} \frac{1}{n!} F_{a_1\cdots a_n}\Gamma^{a_1\cdots a_n}$$

2. YB deformations as β-transformations

Metric and B-field

Assumption : a r-matrix only contains bosonic generators of su(2,2|4).

The deformed action at $\mathcal{O}(\theta^0)$ is

$$S_{(0)} = -T\int \mathrm{d}^2\sigma\, P_-^{\alpha\beta}\mathrm{Str}\left[g_b^{-1}\partial_\alpha g_b\, P_2\circ\frac{1}{1-2\eta\, R_{g_b}\circ P_2}g_b^{-1}\partial_\beta g_b\right]$$

The left-invariant current can be expanded as

$$g_b^{-1}\mathrm{d}g_b = \left(e_m{}^a\mathbf{P}_a - \frac{1}{2}\omega_m{}^{ab}\mathbf{J}_{ab}\right)\mathrm{d}X^m$$

- $e_m{}^a$: $AdS_5\times S^5$ vielbein
- $\omega_m{}^{ab}$: Spin connection

$$\{\mathbf{J}_{ab}\}\oplus\{\mathbf{P}_a\} = \mathfrak{g}^{(0)}\oplus\mathfrak{g}^{(2)} = \mathfrak{so}(2,4)\times\mathfrak{so}(6)\subset\mathfrak{su}(2,2|4) = \mathfrak{g}^{(0)}\oplus\mathfrak{g}^{(1)}\oplus\mathfrak{g}^{(2)}\oplus\mathfrak{g}^{(3)}$$

In addition, we need to compute

$$\mathrm{Str}(\mathbf{P}^b R_{g_b}(\mathbf{P}_a)) = \lambda_a{}^b$$

Metric and B-field

We can obtain the YB deformed metric and B-field

$$g'_{mn} = e_{(m}{}^{a} e_{n)}{}^{b} k_{+ab}\,, \qquad B'_{mn} = e_{[m}{}^{a} e_{n]}{}^{b} k_{+ab}\,,$$

where $k_{\pm a}{}^{b} \equiv \left[(1 \pm 2\,\eta\,\lambda)^{-1}\right]_{a}{}^{b}\,,$

To understand connections with O(d,d) transformations,
it is useful to introduce the generalized metric H in the DFT.

$$\mathcal{H}_{MN} = \begin{pmatrix} (g - B\,g^{-1}\,B)_{mn} & B_{mk}\,g^{kn} \\ -g^{mk}\,B_{kn} & g^{mn} \end{pmatrix}\,,$$

Metric and B-field

YB deformations can be regarded as O(d,d) T-duality transformations.

$$\mathcal{H}' = e^{\beta^{\mathrm{T}}} \mathcal{H} e^{\beta} \qquad e^{\beta} = \begin{pmatrix} \delta^m_n & \beta^{mn} \\ 0 & \delta_m^n \end{pmatrix} \in O(10,10)$$

where the β-field is

$$\beta^{mn}(x) = 2\eta\, r^{ij} \hat{T}_i^m \hat{T}_j^n$$

($\hat{T}_i$: Killing vectors associated with $T_i \in \mathfrak{so}(2,4) \times \mathfrak{so}(6)$)

➡ The transformations are **(local) $\boldsymbol{\beta}$-transformations** in the context of double field theory.

R-R fields and Dilaton

After lengthy calculations,
we can obtain the canonical GS action at $\mathcal{O}(\Theta^2)$:

$$S_{(2)} = -i\,T \int \mathrm{d}^2\sigma \left[P_+^{\alpha\beta} \bar{\Theta}'_1 e'_\alpha{}^a \Gamma_a D'_{+\beta} \Theta'_1 + P_-^{\alpha\beta} \bar{\Theta}'_2 e'_\alpha{}^a \Gamma_a D'_{-\beta} \Theta'_2 \right.$$

$$\left. - \frac{1}{8} P_+^{\alpha\beta} \bar{\Theta}'_1 e'_\alpha{}^a \Gamma_a \boxed{\hat{\mathcal{F}}_5 \Omega^{-1}} e'_\beta{}^b \Gamma_b \Theta'_2 \right]$$

↑ YB-deformed RR fields and dilaton

$$\hat{\mathcal{F}}_5 = 4\left(\Gamma^{01234} + \Gamma^{56789}\right) ,$$ The undeformed R-R 5-form fields

$$\Omega = (\det k_-)^{\frac{1}{2}} \sum_{p=0}^{5} \frac{(-\eta)^p}{2^p p!} \lambda_{a_1 a_2} \cdots \lambda_{a_{2p-1} a_{2p}} \Gamma^{a_1 \cdots a_{2p}}$$

R-R fields and Dilaton

As a result, YB deformed R-R fields and dilaton are given by

$$e^{\Phi'}\hat{\mathcal{F}}' = \hat{\mathcal{F}}_5\Omega^{-1}\,, \quad e^{\Phi'} = (\det k_{\pm})^{1/2}$$

We can show that the formulas of the RR fields can be rewritten by

$$F' = e^{-B'_2\wedge}e^{-\beta\vee}F_5 \qquad \left(\beta\vee\alpha_p \equiv \frac{1}{2}\beta^{mn}\iota_m\iota_n\alpha_p\right)$$

$$F_5 = 4(\omega_{\mathrm{AdS}_5} + \omega_{\mathrm{S}^5}) \qquad \omega_{\mathrm{AdS}_5}\,,\omega_{\mathrm{S}^5} \;: \mathrm{AdS}_5\,, S^5 \text{ volume forms}$$

$$F' = \sum_{p=1,3,5,7,9} F'_p$$

➡ The transformation is precisely the β-transformation of R-R fields. [Hohm-Kwak-Zwiebach, 1107.0008]

3. An example of β-transformations

An example of β-transformations

Abelian r-matrix : $r = \frac{1}{2} P_1 \wedge P_2 \qquad [P_1, P_2] = 0$ [Matsumoto-Yoshida, 1404.3657]

$\boldsymbol{\beta}$-field : $\beta = \eta\, \hat{P}_1 \wedge \hat{P}_2 = \eta\, \partial_1 \wedge \partial_2$

We take a coordinate system for the AdS_5 part of the original metric as

$$\mathrm{d}s^2 = \frac{\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{d}z^2}{z^2} + \mathrm{d}s^2{}_{\mathrm{S}^5} \qquad (\mu\,, \nu = 0, 1, 2, 3)$$

By performing $\boldsymbol{\beta}$ (or YB) deformations, we obtain

$$\mathrm{d}s^2 = \frac{\mathrm{d}z^2 - (\mathrm{d}x^0)^2 + (\mathrm{d}x^3)^2}{z^2} + \frac{z^2[(\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2]}{z^4 + \eta^2} + \mathrm{d}s^2_{\mathrm{S}^5}\,,$$

$$B_2 = \frac{\eta}{z^4 + \eta^2}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2\,, \qquad \Phi = \frac{1}{2}\,\log\left[\frac{z^4}{z^4 + \eta^2}\right].$$

This b.g. is a gravity dual of NC SYM with $[\hat{x}^1, \hat{x}^2] = i\,\eta$.

[Hashimoto-Itzhaki, Maldacena-Russo, 1999]

An example of β-transformations

original:

$$F_5 = 4\left(\omega_{\mathrm{AdS}_5} + \omega_{\mathrm{S}^5}\right) \qquad \omega_{\mathrm{AdS}_5} = -\frac{\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}z}{z^5},$$

STEP1:

$$\begin{aligned} e^{-\beta\vee} F_5 &= 4\left(\omega_{\mathrm{AdS}_5} + \omega_{\mathrm{S}^5}\right) - 4\,\beta \vee \omega_{\mathrm{AdS}_5} \\ &= 4\left(\omega_{\mathrm{AdS}_5} + \omega_{\mathrm{S}^5}\right) - 4\eta\,\frac{\mathrm{d}x^0 \wedge \mathrm{d}x^3 \wedge \mathrm{d}z}{z^5} \end{aligned}$$

STEP2:

$$\begin{aligned} F' &= e^{-B_2' \wedge} e^{-\beta\vee} F_5 \\ &= -4\,\eta\,\frac{\mathrm{d}x^0 \wedge \mathrm{d}x^3 \wedge \mathrm{d}z}{z^5} + 4\left(\frac{z^4}{z^4+\eta^2}\,\omega_{\mathrm{AdS}_5} + \omega_{\mathrm{S}^5}\right) - 4\,B_2' \wedge \omega_{\mathrm{S}^5} \end{aligned}$$

$$F_1' = 0\,, \qquad F_3' = -4\,\eta\,\frac{\mathrm{d}x^0 \wedge \mathrm{d}x^3 \wedge \mathrm{d}z}{z^5}\,, \qquad F_5' = 4\left(\frac{z^4}{z^4+\eta^2}\omega_{\mathrm{AdS}_5} + \omega_{\mathrm{S}^5}\right),$$

$$F_7' = -4B_2' \wedge \omega_{\mathrm{S}^5}\,, \qquad F_9' = 0\,,$$

The resulting background is a solution of the type IIB SUGRA.

We showed that YB deformations can be regarded as the β-transformations.

In general,

The β-transformations are not a gauge transformation in the SUGRA.

⬇

The β-deformed b.g. may not satisfy the (generalized) SUGRA eq.

However,

The β-transformation with a r-matrix satisfying the CYBE can systematically generate sol. of the (generalized) SUGRA eq.

Summary

- We considered YB-deformations of the $\mathrm{AdS}_5 \times S^5$ superstring.

[Integrable deformations] [T-duality transformations]

YB deformations

$$r = \frac{1}{2} r^{ij} T_i \wedge T_j$$

=

β-deformations

$$\beta^{mn}(x) = 2\eta\, r^{ij} \hat{T}_i^m \hat{T}_j^n$$

- We considered the β-deformations of the $\mathrm{AdS}_3 \times S^3 \times T^4$ with H-flux.

[J.S.-Sakatani, 1803.05903]

Discussions

- YB deformations can also be regarded as the NATD.

[Hoare-Tseytlin, 1609.02550] [Borsato-Wulff, 1609.09834, 1706.10169]

→ Formulations of DFT and the Double sigma models manifesting the symmetry of the NATD ?

- Dual gauge theories → Noncommutative gauge theories ?

[Araujo-Bakhmatov-O Colgain-J.S.-Sheikh Jabbari-Yoshida, 1702.02861, 1705.02063]

Thank you

Appendix

β-deformations of the $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ superstring

In the presence of H-flux,
it is not straightforward to define YB sigma models.

However, we can consider β-deformations of such backgrounds, easily.

As an example, we here consider the $AdS_3 \times S^3 \times T^4$ with H-flux :

$$\mathrm{d}s^2 = \frac{-(\mathrm{d}x^0)^2 + (\mathrm{d}x^1)^2 + \mathrm{d}z^2}{z^2} + \mathrm{d}s^2_{\mathrm{S}^3} + \mathrm{d}s^2_{\mathrm{T}^4}\,,$$

$$B_2 = \frac{\mathrm{d}x^0 \wedge \mathrm{d}x^1}{z^2} + \frac{1}{4}\cos\theta\,\mathrm{d}\phi \wedge \mathrm{d}\psi\,, \qquad \Phi = 0\,,$$

$$\mathrm{d}s^2_{\mathrm{S}^3} \equiv \frac{1}{4}\left[\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2 + \left(\mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi\right)^2\right],$$

β-deformations of the $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ superstring

Abelian : $\quad r = \frac{1}{2} P_0 \wedge P_1 \,.$

We can perform β-deformations by using the r-matrix.

$$ds^2 = \frac{-(\mathrm{d}x^0)^2 + (\mathrm{d}x^1)^2}{z^2 + 2\,\eta} + \frac{\mathrm{d}z^2}{z^2} + \mathrm{d}s^2_{\mathrm{S}^3} + \mathrm{d}s^2_{\mathrm{T}^4}\,,$$

$$B_2 = \frac{\mathrm{d}x^0 \wedge \mathrm{d}x^1}{z^2 + 2\,\eta} + \frac{1}{4}\cos\theta\,\mathrm{d}\phi \wedge \mathrm{d}\psi\,, \qquad e^{-2\Phi} = \frac{z^2 + 2\,\eta}{z^2}\,.$$

The background is a solution of the supergravity.

NOTE

The background can also be reproduced by a TsT transformation.

β-deformations of the $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ superstring

Non-unimordular : $r = \frac{1}{2}\,\bar{c}^{\mu}\, M_{01} \wedge P_{\mu} \qquad c^0 = \pm c^1\,.$

We can perform β-deformations by using the non-unimodular r-matrix.

$$\mathrm{d}s^2 = \frac{\eta_{\mu\nu}\,\mathrm{d}x^{\mu}\,\mathrm{d}x^{\nu}}{z^2 - 2\,c_{\mu}\,x^{\mu}} + \frac{\mathrm{d}z^2}{z^2} + \mathrm{d}s^2_{\mathrm{S}^3} + \mathrm{d}s^2_{\mathrm{T}^4}\,, \qquad \mathrm{e}^{-2\Phi} = \frac{z^2 - 2\,c_{\mu}\,x^{\mu}}{z^2}\,,$$

$$B_2 = \frac{\mathrm{d}x^0 \wedge \mathrm{d}x^1}{z^2 - 2\,c_{\mu}\,x^{\mu}} + \frac{1}{4}\cos\theta\,\mathrm{d}\phi \wedge \mathrm{d}\psi\,.$$

The background is a solution of the generalized supergravity with

$$I = -c^{\mu}\,\hat{P}_{\mu} = -c^{\mu}\,\partial_{\mu}\,,$$

In this case, the Killing vector I satisfies

$$(g+B)_{mn} I^n = 0$$

Then, we can rescale $I^m \to \lambda I^m$ with arbitrary $\lambda \in \mathbb{R}$.

The generalized SUGRA equations (GSE)

[Arutyunov-Frolov-Hoare-Roiban-Tseytlin,1511.05795]

$$R_{MN} - \frac{1}{4} H_{MKL} H_N{}^{KL} + \boxed{D_M X_N + D_N X_M} = T_{MN} \,,$$

$$\frac{1}{2} D^K H_{KMN} + \frac{1}{2} \mathcal{F}^K \mathcal{F}_{KMN} + \frac{1}{12} \mathcal{F}_{MNKLP} \mathcal{F}^{KLP} = \boxed{X^K H_{KMN} + D_M X_N - D_N X_M} \,,$$

$$R - \frac{1}{12} H_3^2 + \boxed{4 D_M X^M - 4 X_M X^M} = 0 \,,$$

$$\mathrm{d} * \mathcal{F}_p - \boxed{Z \wedge * \mathcal{F}_p} + \boxed{*(I \wedge \mathcal{F}_{p-2})} - H_3 \wedge * \mathcal{F}_{p+2} = 0 \,, \quad \mathcal{F}_{n_1 n_2 \dots} = \mathrm{e}^{\Phi} F_{n_1 n_2 \dots}$$

$$\boxed{X_M \equiv I_M + Z_M}$$

➡ The GSE is modified by two extra vectors I and Z.
In addition, there are some constraints for these vectors.

Relations between vectors *X*, *I* and *Z*

The constraints are given by

$$D_M I_N + D_N I_M = 0 \quad (\text{ Killing equation })$$

$$D_M Z_N - D_N Z_M + I^K H_{KMN} = 0 \qquad I^M Z_M = 0$$

Taking a coordinate system such that the *B*-field is isometric $\mathcal{L}_I B = 0$,

$$\Longrightarrow \quad Z_M = \partial_M \Phi - B_{MN} I^N$$

(*generalization of the gradient of dilaton*)

Therefore, setting $I = 0$, we recover the usual SUGRA equations.

The GSE is characterized by a Killing vector $I = I^M \partial_M$