Gravitational duality and deformations of action principles for generalized gauge fields

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Outlook:

1. Review of basic notions of EM duality

2. Generalized gauge fields

3. Discussion of work in progress

1. Review of basic aspects of EM duality

i) Twisted self-duality

Maxwell equations in vacuum

$$\partial_\mu F^{\mu\nu} = 0$$
$$\partial_\mu {}^*F^{\mu\nu} = 0$$

are invariant under $SO(2)$ duality rotations

$$\begin{aligned} F &\rightarrow F' = \cos\alpha F - \sin\alpha {}^*F \\ {}^*F &\rightarrow {}^*F' = \sin\alpha F + \cos\alpha {}^*F \end{aligned}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Symmetry between electric and magnetic degrees of freedom.

In the previous analysis of Maxwell equations the form F has been prioritized over its Hodge dual ${}^*F$, for its equation has implicitly been solved in terms of a potential. Its equation of motion is an identity.

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\ dF &\equiv 0 \end{aligned}$$

But the duality symmetry is telling that there is no necessity of prioritizing any of these forms.

We would like to reformulate Maxwell equations in such a way that F and ${}^*F$ are treated on an equal footing.

This is achieved by solving for F and ${}^*F$, considered as independent fields

$$F = dA, \quad H \equiv {}^*F = dB$$

and then replacing the second-order equations by the first-order twisted self-duality condition

$$\begin{pmatrix} F \\ H \end{pmatrix} = \mathcal{S} \begin{pmatrix} {}^*F \\ {}^*H \end{pmatrix}$$

with

$$\mathcal{S} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Redundancy in the covariant formulation: one equation implies the other one by Hodge duality. It can be overcome by a 3+1 space-time splitting: by selecting the purely spatial components one gets a non-redundant set of equations that imply the full set of Maxwell equations (Bunster-Henneaux, 2011).

ii) Duality as an off-shell symmetry

Is duality a symmetry of the action?

The form of the Maxwell action

$$S = \int d^4x(\mathbf{E}^2 - \mathbf{B}^2)$$

has been used to argue that duality does not hold off-shell. This is a misconception: the dynamical variables are the components of the vector potential A_μ, not the fields **E** and **B**.

Duality is an off-shell symmetry of Maxwell theory in its Hamiltonian formulation (Deser-Teitelboim, 76).

Duality transformations:

$$\begin{aligned} \delta A_0 &= 0 \\ \delta A_i &= \beta \epsilon_{ipq} \Delta^{-1}(\partial^p E^q) \end{aligned}$$

reproduce on-shell the infinitesimal form of the duality rotations with parameter β:

$$\begin{aligned} \delta E^k &= -\beta B^k + \beta \epsilon^{kpq} \Delta^{-1}(\partial^\mu \partial_p F_{\mu q}) \\ \delta B^k &= \beta E^k - \beta \Delta^{-1}(\partial^k \partial_m E^m) \end{aligned}$$

The previous transformation leaves Maxwell action invariant (up to total derivatives), but the symmetry is not manifest at this stage.

One can introduce a second potential by solving the Gauss law (the constraint in the Hamiltonian formulation) and write down a **manifestly duality-invariant action principle.**

The action principle in its Hamiltonian form is

$$S[A_i, \pi^j, A_0] = \int dt\, d^3x [\pi^i \dot{A}_i - \mathcal{H} - A_0 C]$$

with the Hamiltonian density

$$\mathcal{H} = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)$$

and the constraint

$$C \equiv \partial^i E_i$$

A_0 is a Lagrange multiplier (its gauge variation involves a time derivative).

Solving for the constraint and substituting in the action principle:

$$S[A^a_i] = \int d^4x(\epsilon_{ab}B^a\dot{A}^b - \delta_{ab}B^aB^b) \quad a, b = 1, 2$$

where

$$B^a = \nabla \times A^a$$

Gauge symmetries:

$$\delta A^a_i = \partial_i v^a$$

This action principle is manifestly invariant under $SO(2)$ rotations

$$\begin{aligned} A^1 &\rightarrow A'^1 = \cos\alpha A^1 + \sin\alpha A^2 \\ A^2 &\rightarrow A'^2 = -\sin\alpha A^1 + \cos\alpha A^2 \end{aligned}$$

(both ϵ_{ab} and δ_{ab} are $SO(2)$-invariant tensors).

Duality is a "hidden symmetry" of Maxwell action.

Lorentz invariance is not manifest in the duality-symmetric action principle. Its (now hidden) presence may be verified by the fulfillment of the Dirac-Schwinger commutation relations on the energy-momentum tensor.

It might be that EM duality is a more fundamental symmetry than Poincaré invariance. The condition $\nabla \mathbf{B} = 0$ and the Poisson brackets $[\mathbf{B}^{a\,i}(x), \mathbf{B}^{b\,j}(x')] = \epsilon^{ijk}\epsilon^{ab}\delta_{,k}(x - x')$ imply the Dirac-Schwinger relations. Duality invariance implies Poincaré invariance (but not vice versa) (Bunster-Henneaux, 2011).

In higher dimensions, the dual of the vector field is a $D - 3$ form. One cannot rotate the prepotentials into each other. Nevertheless, twisted self-duality survives.

iii) What about gravity?

Indications of the presence of duality in gravitational theories: Ehlers group, Geroch group.

Ehlers phenomenon in gravitational theories: the emergence of hidden symmetries in gravitational theories upon dimensional reduction and suitable dualisations of certain Kaluza-Klein fields

Duality in linearized gravity (Henneaux-Teitelboim):

Hamiltonian form of the Pauli-Fierz action

$$S[h_{ij}, \pi^{ij}, n, n_i] = \int dt \left(\int d^3x \, \pi^{ij} \dot{h}_{ij} - H - \int d^3x (n\mathcal{C} + n_i \mathcal{C}^i) \right)$$

with H is the Hamiltonian

$$H = \int d^3 \left[\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2 + \frac{1}{4}\partial^m h^{ij}\partial_m h_{ij} - \frac{1}{2}\partial_m h^{mn}\partial_r h^r{}_n + \frac{1}{2}\partial^m h \partial^n h_{mn} - \frac{1}{4}\partial^m h \partial_m h \right]$$

The constraints

$$\begin{aligned} \mathcal{C} &\equiv \partial^i\partial^j h_{ij} - \Delta h = 0 \\ \mathcal{C}^i &\equiv -2\partial_j \pi^{ij} = 0 \end{aligned}$$

generate the gauge transformations of the dynamical variables.

The resolution of the constraints yields

$$\pi_{ij} = \epsilon^{imn}\epsilon^{jkl}\partial_m\partial_k P_{nl}$$
$$h_{ij} = \epsilon_{imn}\partial^m\phi_j{}^n + \epsilon_{jmn}\partial^m\phi_i{}^n + \partial_i u_j + \partial_j u_i$$

where ϕ_{ij} and P_{ij} are two symmetric potentials and u_i is a vector prepotential that can be gauged away (Henneaux-Teitelboim, 2005).

The gauge transformations acting on the potentials are

$$\delta Z^a_{ij} = \partial_i\eta^a_j + \partial_j\eta^a_i + \delta_{ij}\eta^a$$
$$(Z_\alpha{}^{ij}) = (P^{ij}, \phi^{ij}) \quad \alpha = 1, 2$$

These transformations have the form of the symmetries of conformal gravity.

Then the action can be written as (Bunster-Henneaux-Hörtner)

$$S[Z_\alpha{}^{ij}] = \int dt \left(-2\int d^3x \epsilon^{\alpha\beta} D^{ij}_\alpha \dot{Z}_{\beta\ ij} - \int d^3x (4R^\alpha_{ij}R^{\beta\ ij} - \frac{3}{2}R^\alpha R^\beta)\delta_{\alpha\beta} \right)$$

$D^{ij}[Z^a] = \epsilon^{iab}C_{ab}{}^j$ and $R_{ij}[Z^a]$ are respectively the dual of the Cotton tensor and the Ricci tensor constructed out of the prepotentials.

This action is manifestly invariant under the $SO(2)$ rotations.

Observation: one can define a dual metric in terms of the potential P_{ij} as follows

$$f_{ij}[P] = \epsilon_{irs}\partial^r P^s{}_j + \epsilon_{jrs}\partial^r P^s{}_i + \partial_i v_j + \partial_j v_i$$

This relation may be inverted:

$$P_{ij} \quad = \quad -\frac{1}{4}\left[\epsilon_{irs}\Delta^{-1}(\partial^r f^s_j) + \epsilon_{jrs}\Delta^{-1}(\partial^r f^s_i)\right]$$

(same for $\phi_{ij}[h]$) and then substitute in the two-prepotential action. One gets an action principle that accommodates two metrics (Bunster-Henneaux-Hörtner):

$$S[h^a_{mn}, n^a] = K[h^a_{mn}] - \int dt\, H[h^a_{mn}] - \int dt d^3x \delta_{ab} n^a R^b$$

Doubling of metric and (linearized) diffeomorphisms. Doubling of (not compactified) coordinates?

Twisted self-duality can be discussed along the same lines (Bunster-Henneaux-Hörtner): a subset of the covariant twisted-self duality condition (containing at most one time derivative) is equivalent to the full set.

2. Generalized gauge fields

The study of gauge theories of mixed symmetry tensor fields ("generalized gauge fields") was motivated by the emergence in string field theory of massive higher spin excitations transforming in arbitrary representations of the Lorentz group (mid-80's), including mixed-symmetry representations.

Mixed-symmetry tensors also appear in the study of electric-magnetic duality of linearized gravity and higher spin fields.

Dual formulation in higher dimensions ($D > 4$): the following irreducible representations of the massless little group $SO(D-2)$ are equivalent:

$$\ydiagram{2} = D-3\left\{\ydiagram{2,1,1}\right. = D-3\left\{\ydiagram{2,2,2}\right.$$

This can be seen by dualizing the corresponding fields in the physical gauge:

$$h_{ij} = \epsilon_{ik_1..k_{D-3}} t_j{}^{k_1..k_{D-3}} = \epsilon_{ik_1..k_{D-3}} \epsilon_{jl_1..l_{D-3}} t'^{k_1..k_{D-3} l_1..l_{D-3}}$$

and bearing in mind the trace conditions. When considered as representations of $GL(D, R)$, they are not equivalent. This originates the covariant dual action principles based on tensors of different Young symmetry type (with their corresponding gauge symmetries).

The Curtright action is the action principle for a free massless tensor field of mixed symmetry (2,1) (the simplest one). It is constructed by solely relying on the principle of gauge symmetry:

i) one postulates the most general form of the gauge symmetries

$$\delta T_{\alpha_1\alpha_2\beta} = 2\partial_{[\alpha_1}\sigma_{\alpha_2]\beta} + 2\partial_{[\alpha_1}\alpha_{\alpha_2]\beta} - 2\partial_\beta \alpha_{\alpha_1\alpha_2} \quad ; \sigma_{\mu\nu} = \sigma_{\nu\mu}, \alpha_{\mu\nu} = \alpha_{\nu\mu}$$

ii) then one constructs an invariant Lagrangian

$$S[T_{\alpha_1\alpha_1\beta}] = -\frac{1}{6}\int d^5x \left[F_{\alpha_1\alpha_2\alpha_3\beta}F^{\alpha_1\alpha_2\alpha_3\beta} - 3F_{\alpha_1\alpha_2\beta}{}^{\beta}F^{\alpha_1\alpha_2\gamma}{}_{\gamma}\right]$$

The field strength

$$F_{\alpha_1\alpha_2\alpha_3\beta} = 3\partial_{[\alpha_1}T_{\alpha_2\alpha_3]\beta} = \partial_{\alpha_1}T_{\alpha_2\alpha_3\beta} + \partial_{\alpha_2}T_{\alpha_3\alpha_1\beta} + \partial_{\alpha_3}T_{\alpha_1\alpha_2\beta}$$

is only invariant under the $\sigma_{\mu\nu}$ gauge transformations:

$$\delta F_{\alpha_1\alpha_2\alpha_3\beta} = -6\partial_\beta\partial_{[\alpha_1}\alpha_{\alpha_2\alpha_3]}$$

This is the dual formulation of linearized gravity in five dimensions.

One needs at least two derivatives to construct a fully gauge invariant object (generalized Riemann tensor):

$$E_{\alpha_1\alpha_2\alpha_3\beta_1\beta_2} = 2F_{\alpha_1\alpha_2\alpha_3[\beta_1,\beta_2]} = 6\partial_{[\beta_2}\partial_{[\alpha_1}T_{\alpha_2\alpha_3]\beta_1]}$$

The action is

$$S = \int d^5x T^{\mu\nu\rho}G_{\mu\nu\rho}$$

and the equation of motion

$$G_{\alpha_1\alpha_2\beta} = 0$$

with

$$G_{\alpha_1\alpha_2\beta} = E_{\alpha_1\alpha_2\beta} + \frac{1}{2}(\eta_{\alpha_1\beta}E_{\alpha_2} - \eta_{\alpha_2\beta}E_{\alpha_1})$$

the analogue of the linearized Einstein tensor in the dual theory.

Twisted self-duality form of the equations of motion (Bunster-Henneaux-Hörtner): The set containing at most first-order time derivatives is equivalent to the full set (it implies both the dynamical and constraint equations, making use of the gauge freedom of the theory).

Two-potential formulation of five dimensional linearized gravity (Bunster-Henenaux-Hortner)

It requires the resolution of the constraints in the Hamiltonian formalism (either in the Pauli-Fierz or the Curtright pictures) and the subsequent substitution in the action principle.

$$S = \int dt\, d^4x \Big[2\epsilon^{imab}\epsilon^{jncd}\epsilon_{ilxy}\partial_m\partial_n P_{abcd}\partial^l\dot{\phi}^{xy}{}_j$$
$$-\Big(f^{-4}(R_{ij}[P]R^{ij}[P] - \frac{7}{27}R^2[P]) + f^2(2E_{ijk}[\phi]E^{ijk}[\phi] - \frac{3}{2}E_i[\phi]E^i[\phi])\Big)\Big]$$

The dynamical variables are now a (2,1) potential ϕ_{ijk} and a (2,2) potential P_{ijkl} with gauge transformations:

$$\begin{aligned}
\delta_1\phi_{rsm} &= B_{[r}\delta_{s]m} \\
\delta_2\phi_{mrs} &= \partial_r S_{sm} - \partial_s S_{rm} + \partial_r A_{sm} - \partial_s A_{rm} + 2\partial_m A_{sr} \\
\delta_1 P_{ijkl} &= \chi_{kl[i,j]} + \chi_{ij[k,l]}, \quad \chi = (2,1) \\
\delta_2 P_{ijkl} &= \frac{1}{4}[\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}]\xi_0
\end{aligned}$$

The equations of motion derived from the variational principle coincide with the twisted self-duality equations. An action principle involving the graviton and its dual was also derived, with a spatially-non-local kinetic term.

Difficulties in the construction of action principles for generalized gauge fields.

No-go result: no consistent deformations of the action for a free (2,1) tensor field under the hypotheses of locality and manifest space-time covariance (Bekaert-Boulanger-Henneaux, 2003).

The analysis is based on BRST cohomological methods.

Lack of a notion of diffeomorphism covariance for mixed-symmetry tensors.

The problem is relevant in the context of the E_{10} and E_{11} conjectures.

Deformation of the Curtright action (Hörtner, 2017)

Consider the linearized Einstein-Hilbert action in the ADM formalism

$$S = \int d^D x \ \left[\pi^{ij}\dot{g}_{ij} + Ng^{1/2}(R - 2\Lambda) + Ng^{-1/2}(\frac{1}{D-2}\pi^2 - g_{ik}g_{jl}\pi^{ij}\pi^{kl}) + 2N_i\pi^{ij}{}_{|j}\right]$$

($g = \det(g_{ij})$, $N_i = g_{0i}$, $N = (-g^{00})^{-1/2}$, $\pi = g_{ij}\pi^{ij}$) around a de Sitter background (use of planar coordinates for convenience):

$$g_{ij} = \bar{g}_{ij} + h_{ij}, \quad \pi^{ij} = \bar{\pi}^{ij} + p^{ij}$$
$$N = 1 + n, \quad N_i = n_i$$

with

$$\bar{g}_{ij} = f^2(t)\delta_{ij}$$
$$\bar{\pi}^{ij} = \sqrt{\bar{g}}(\bar{g}^{ij}\bar{K} - \bar{K}^{ij}) = -(D-2)kf^{D-3}\delta^{ij}$$

After linearization, the ADM action takes the form

$$S = \int d^D x \left[p^{ij}\dot{h}_{ij} - \mathcal{H} - nC - n_i C^i \right]$$

with the Hamiltonian

$$\begin{aligned}
\mathcal{H} &= f^{-D+5} p_{ij}p^{ij} - \frac{f^{-D+5}}{D-2}p^2 - 2(D-3)kp_{ij}h^{ij} + khp \\
&+ f^{D-7}\left[\frac{1}{4}\partial^i h^{jk}\partial_i h_{jk} - \frac{1}{4}\partial_i h \partial^i h + \frac{1}{2}\partial^i h \partial^j h_{ij} - \frac{1}{2}\partial_i h^{ij}\partial^k h_{kj}\right] \\
&- k^2 f^{D-5}\frac{(D-2)(-2D+6)}{4}h_{ij}h^{ij}
\end{aligned}$$

The linearized constraints

$$\begin{aligned}
C &= f^{D-5}(\Delta h - \partial_i\partial_j h^{ij}) + 2kpf^2 + f^{D-3}k^2 h(D-2)(D-3) = 0 \\
C^i &= -2\partial_j p^{ij} + (D-2)f^{D-5}k(2\partial_k h^{ik} - \partial^i h) = 0
\end{aligned}$$

generate the gauge transformation of the canonical variables.

In order to solve the constraints it is useful to perform the canonical transformation

$$
\begin{aligned}
h_{ij} &\mapsto \hat{h}_{ij} = h_{ij} \\
p^{ij} &\mapsto \hat{p}^{ij} = p^{ij} - \frac{D-2}{2} k f^{D-5} (2h^{ij} - \delta^{ij} h).
\end{aligned}
$$

derived from the generating functional

$$
F[h_{ij}, \hat{p}^{ij}] = \int d^{D-1}x \left[\hat{p}^{ij} h_{ij} + \frac{(D-2)}{2} k f^{D-5} (h_{ij} h^{ij} - \frac{1}{2} h^2) \right]
$$

as follows:

$$
\begin{aligned}
\frac{\delta F}{\delta h^{ij}} &\equiv p_{ij} = \hat{p}_{ij} + \frac{D-2}{2} k f^{D-5} (2h_{ij} - \delta_{ij} h) \\
\frac{\delta F}{\delta \hat{p}^{ij}} &\equiv \hat{h}_{ij} = h_{ij}
\end{aligned}
$$

The action principle reduces then to

$$S[\hat{p}^{ij}, h_{ij}, n, n_i] = \int d^D x \left[\hat{p}^{ij}\dot{h}_{ij} - H - nC - n_i C^i\right], \qquad (0.\text{-}66)$$

where

$$H = f^{-D+5}\hat{p}_{ij}\hat{p}^{ij} - \frac{f^{-D+5}}{D-2}\hat{p}^2 + 2k\hat{p}_{ij}h^{ij} + f^{D-7}\left[\frac{1}{4}\partial^i h^{jk}\partial_i h_{jk} - \frac{1}{4}\partial_i h \partial^i h + \frac{1}{2}\partial^i h \partial^j h_{ij} - \frac{1}{2}\partial_i h^{ij}\partial^k h_{kj}\right]$$

and

$$C = f^{D-5}(-\partial^i\partial^j h_{ij} + \Delta h) + 2kf^2\hat{p}$$
$$C^i = -2\partial_j \hat{p}^{ij}.$$

The new canonical variable $\hat{p}^{ij}$ transforms as

$$\delta\hat{p}^{ij} = f^{D-5}(-\partial^i\partial^j\xi + \delta^{ij}\Delta\xi)$$

Focusing on five dimensions, the momentum constraint is solved as in the flat case:

$$\hat{p}^{ij} = \epsilon^{iklm}\epsilon^{jnpq}\partial_k\partial_n P_{lmpq}.$$

with the ambiguities

$$\begin{aligned}\delta P_{abcd} &= 2\chi_{cd[b,a]} + 2\chi_{ab[d,c]} + \frac{1}{4}[\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}]\xi, \\ & \chi_{abc} = -\chi_{bac}, \chi_{[abc]} = 0\end{aligned}$$

ξ induces the gauge transformation on $\hat{p}^{ij}$, whereas χ_{abc} defines an internal invariance.

Substitution of the trace in the scalar constraint produces

$$\Delta h - \partial_i\partial_j h^{ij} + 4f^2 k \Delta P^{ab}{}_{ab} - 8f^2 k \partial_i\partial_j P^{imj}{}_m = 0.$$

We shall decompose the potential P_{ijkl} as follows:

$$P_{abcd} = Q_{abcd} + \frac{1}{12}\left[\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}\right] P_{mn}{}^{mn}$$

with Q_{ijkl} a (2,2) tensor whose double trace vanishes. The final expression for h_{ij} reads

$$\begin{aligned}h_{ij} &= \partial^k \epsilon_{ikab}\phi^{ab}{}_j + \partial^k \epsilon_{jkab}\phi^{ab}{}_i + \partial_i u_j + \partial_j u_i \\ & -8kf^2 P_{ikj}{}^k + \frac{4}{3}kf^2\delta_{ij}P_{mn}{}^{mn}. \qquad (0.\text{-}76)\end{aligned}$$

The ambiguities in the choice of the prepotential are

$$\delta\phi_{abc} = \partial_a S_{bc} - \partial_b S_{ac} + \partial_a A_{bc} - \partial_b A_{ac} + 2\partial_c A_{ba}$$
$$+B_{[a}\delta_{b]c} - 16kf^2(\tilde{\chi}_{cab} + \tilde{\chi}_{abc})$$
$$\delta u_i = \xi_i + 16kf^2\epsilon_{ibxy}\tilde{\chi}^{bxy} - 2\partial^l\epsilon_{il}{}^{ab}A_{ba}$$

We can now write the action principle in terms of the potentials

$$S[\phi_{ijk}, P_{abcd}] = \int dt\, d^4x \Big[2\epsilon^{imab}\epsilon^{jncd}\epsilon_{ilxy}\partial_m\partial_n P_{abcd}\partial^l\dot{\phi}^{xy}{}_j$$

$$+\frac{32}{3}k\dot{P}\partial_a\partial_b P^{ab} - 8k\dot{P}_{ij}\partial_a\partial_b P^{iajb} + 8kf^2\epsilon^{jlab}\partial^i\partial_l\phi_{ab}{}^k\partial_i P_{jk} - 8kf^2\epsilon_{jlab}\partial_i\partial^l\phi^{abi}\partial_k P^{kj}$$

$$+\frac{72}{9}k^2f^2\partial_j P\partial^j P + 32k^2f^2\partial^i P_{ik}\partial_j P^{jk} - 16f^2k^2\partial_i P_{jk}\partial^i P^{jk} - \frac{64}{3}k^2f^2\partial_i P\partial_j P^{ij}$$

$$-\left(f^{-4}(R_{ij}[P]R^{ij}[P] - \frac{7}{27}R^2[P]) + f^2(2E_{ijk}[\phi]E^{ijk}[\phi] - \frac{3}{2}E_i[\phi]E^i[\phi])\right)\Big]$$

$$R_{ijklmn} = 18\partial_{[i}P_{jk][lm,n]}$$

and

$$E_{ijkmn} = 6\partial_{[n}\partial_{[i}T_{jk]m]}$$

Although the constraints can be solved without prior fixing of the gauge, in order to construct the dual theory it is useful to use the gauge choice

$$\begin{aligned} \hat{p}^{ij} &= a^{ij} + \delta\hat{p}^{ij} = \hat{a}^{ij} - \partial^i\partial^j\xi + \delta^{ij}\Delta\xi \\ h_{ij} &= b_{ij} - 2kf^2\xi \end{aligned}$$

$a = 0$: achieved through the gauge choice $\xi = \frac{1}{3}\Delta^{-1}\hat{p}$.

The constraints take the same form as in the flat case:

$$\begin{aligned} \partial_j a^{ij} &= 0 \\ \Delta b - \partial^i\partial^j b_{ij} &= 0 \end{aligned}$$

so they are solved

$$\begin{aligned} a^{ij} &= f^{-2}\partial^k\partial^l\epsilon_{ikab}\epsilon_{jlcd}P^{abcd} \\ b_{ij} &= f^2(\partial^l\epsilon_{ilab}\phi^{ab}_{\ \ j} + \partial^l\epsilon_{jlab}\phi^{ab}_{\ \ i}) + \partial_i u_j + \partial_j u_i \end{aligned} \tag{0.-88}$$

After substituting in the action, the terms proportional to k and k^2 are no longer present:

$$S[P_{ijkl}, \phi_{abc}] = \int dt\ d^4x\ \Big[2\partial_m \partial_k \epsilon^{imnp} \epsilon^{jkst} P_{npst} \partial^l \epsilon_{ilab} \dot{\phi}^{ab}_{j}$$

$$- \Big(f^{-4} (R_{ij}[P] R^{ij}[P] - \frac{7}{27} R^2[P])$$

$$+\ f^2 (2E^{ijk}[\phi] E_{ijk}[\phi] - \frac{3}{2} E_i[\phi] E^i[\phi])\Big)\Big]$$

In order to construct the dual theory one defines the canonical pair of dual variables as in the flat case:

$$
\begin{aligned}
\hat{t}^{ijk} &= -\frac{2}{3} f^{-2} \partial_l \left[2\epsilon^{klab} P_{ab}{}^{ij} + \epsilon^{ilab} P_{ab}{}^{kj} - \epsilon^{jlab} P_{ab}{}^{ki} \right] \\
\hat{\pi}_{ijk} &= f^2 \epsilon_{ijmn} \epsilon_{krst} \partial^m \partial^r \phi^{stn}.
\end{aligned}
$$

The action

$$
S[\hat{t}_{ijk}, \hat{\pi}_{ijk}, m_j, m_{ij}] = \int d^5x \left[\hat{\pi}^{ijk} \dot{\hat{t}}_{ijk} - \mathcal{H} - m_j \Gamma^j - m_{jk} \Gamma^{jk} \right]
$$

reproduces the form of the prepotential action, with the Hamiltonian density

$$
\begin{aligned}
\mathcal{H} &= -2k \hat{\pi}^{ijk} \hat{t}_{ijk} + \frac{1}{2} \partial_i \hat{t}_{jkl} \partial^i \hat{t}^{jkl} + \partial_i \hat{t}_{jkl} \partial^j \hat{t}^{kil} \\
&\quad - \frac{1}{2} \partial^k \hat{t}_{ijk} \partial_l \hat{t}^{ijl} + \frac{1}{2} \hat{\pi}_{ijk} \hat{\pi}^{ijk} - \frac{1}{2} \hat{\pi}_i{}^{ji} \hat{\pi}^k{}_{jk}
\end{aligned}
$$

and the constraints

$$
\begin{aligned}
\Gamma^j &= \partial_i \partial_k \hat{t}^{ijk} \\
\Gamma^{ij} &= -2 \partial_k (\hat{\pi}^{ijk} + \hat{\pi}^{kji}).
\end{aligned}
$$

This is regarded as the dual of the standard action written in terms of the 'new' variables (b_{ij}, a^{ij}).

In order to get the action principle dual to Pauli-Fierz in the untransformed variables (h_{ij}, π^{ij}) one has to undo the canonical transformation in the dual picture. Express the generating functional in terms of the relevant dual variables.

Since the canonical transformation leaves h_{ij} invariant, it is natural to expect in the dual theory the action of the canonical transformation on $\hat{\pi}_{ijk}$ to be the identity map: we set $\hat{\pi}^{ijk} = \pi^{ijk}$.

We introduce the inversion formulas (valid in the flat case and also in our gauge choice)

$$\phi_{ijk}[\hat{\pi}] = -\frac{1}{2}\Delta^{-1}\hat{\pi}_{ijk}$$

$$P_{abcd}[\hat{t}] = \frac{1}{8}\left[\epsilon_{abij}\partial^i\Delta^{-1}\hat{t}_{cd}{}^j + \epsilon_{cdij}\partial^i\Delta^{-1}\hat{t}_{ab}{}^j\right] - \frac{1}{24}\left[\epsilon_{abij}\partial^i\Delta^{-1}\hat{t}_{cd}{}^j + \epsilon_{cdij}\partial^i\Delta^{-1}\hat{t}_{ab}{}^j\right.$$
$$\left.+\epsilon_{caij}\partial^i\Delta^{-1}\hat{t}_{bd}{}^j + \epsilon_{adij}\partial^i\Delta^{-1}\hat{t}_{bc}{}^j + \epsilon_{bcij}\partial^i\Delta^{-1}\hat{t}_{ad}{}^j + \epsilon_{bdij}\partial^i\Delta^{-1}\hat{t}_{ca}{}^j\right]$$

The generating functional reads

$$F[\pi^{ijk}, \hat{t}_{ijk}] \quad = \quad \int d^4x[-\hat{t}^{ijk}\pi_{ijk} - 3k\pi^{ijk}\Delta^{-1}\pi_{ijk}]$$

When expressed in terms of the dual variables, the generating functional depends on the 'old' conjugate momentum π^{ijk} and the 'new' field $\hat{t}_{ijk}$, so the relevant relations are now

$$t_{ijk} = -\frac{\delta F}{\delta \pi^{ijk}} = \hat{t}_{ijk} + 6k\Delta^{-1}\pi_{ijk}, \quad \hat{\pi}^{ijk} = -\frac{\delta F}{\delta \hat{t}_{ijk}} = \pi^{ijk}$$

The action is now expressed in terms of the pair (t_{ijk}, π^{ijk}):

$$S[t_{ijk}, \pi^{ijk}, m_i, m_{ij}] = \int dt d^4x \ \left[\pi^{ijk} \dot{t}_{ijk} - \mathcal{H} - m_i \Gamma^i - m_{ij} \Gamma^{ij}\right]$$

with the Hamiltonian density

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_\Lambda$$

where

$$\begin{aligned} \mathcal{H}_0 &= \frac{1}{2} \partial_i t_{jkl} \partial^i t^{jkl} + \partial_i t_{jkl} \partial^j t^{kjl} - \frac{1}{2} \partial^k t_{ijk} \partial_l t^{ijl} \\ &\quad + \frac{1}{2} \pi_{ijk} \pi^{ijk} - \frac{1}{2} \pi_i{}^{ji} \pi^k{}_{jk} \end{aligned}$$

and

$$\mathcal{H}_\Lambda = 4k\pi^{ijk} t_{ijk} - 6k^2 \pi^{ijk} \Delta^{-1} \pi_{ijk}$$

the term carrying the deformation. The deformed constraints are

$$\begin{aligned} \Gamma^j &= \partial_i \partial_k (t^{ijk} - 6k\Delta^{-1} \pi_{ijk} - \delta_{ijk}) \\ \Gamma^{ij} &= -2\partial_k (\pi^{ijk} + \pi^{kji}) \end{aligned}$$

The introduction of a positive cosmological constant in the Pauli-Fierz theory corresponds to the introduction of spatially non-local terms in the dual theory. This is regarded as a deformation of the Curtright action.

Our analysis relies on the choice of planar coordinates, which rendes the analysis similar to the flat case.

Conceptual questions: what is the interpretation of a cosmological constant int he dual theory? Properties of space-time in the dual picture?

3. Work in progress

Potential formulation of gravity linearized around non-trivial backgrounds, with the intention of studying EM duality (4d) and deformations of dual action principles (d¿4).

AdS duality conjecture: linearized higher-spin theories on AdS_4 spaces possess a generalization of electric-magnetic duality whose holographic image is the natural $SL(2, Z)$ action on boundary two-point functions (Leigh, Petkou)

Kasner: validity of duality near a non maximally symmetric, singular space-time? ($d = 4$) Interpretation of the singularity in the dual picture? ($d > 4$)

Potentials near AdS background:
Make use of the conformally flat character of maximally symmetric space-times: $\bar{g}_{\mu\nu} = e^{\omega}\delta_{\mu\nu}$
Constraints:

$$
\begin{aligned}
C &= \partial_i\partial_j h^{ij} - \Delta h + \partial_i\omega\partial^i h - \partial_0\omega e^{\omega} p, \\
C^i &= \partial_j p^{ij} + \partial_j\omega p^{ij} - \frac{1}{2}\partial^i\omega p - \partial_0\omega e^{-\omega}\partial_k h^{ik} - \frac{1}{2}e^{-\omega}\partial_0\omega\partial_k\omega h^{ik} \\
&+ \frac{1}{2}e^{-\omega}\partial_0\omega\partial^i h
\end{aligned}
\tag{0.-113}
$$

Simplification: work in the gauge $p = h = 0$.
a) Momentum constraint

Using Einstein equations for the background and the gauge condition one gets

$$\partial_j p^{ij} + \partial_j\omega p^{ij} - e^{-\omega}\partial_k(\partial_0\omega h^{ik}) = 0$$

Multiplying the previous expression by e^{ω} renders the terms in the momentum as a total derivative, and the constraint reads

$$\partial_j(e^{\omega} p^{ij} - \partial_0\omega h^{ij}) = 0$$

It is solved as follows:

$$e^{\omega}p^{ij} - \partial_0\omega h^{ij} = \epsilon_{imn}\epsilon_{jkl}\partial^m\partial^k P^{nl}$$

The gauge conditions $p = h = 0$ imply the necessity of projecting the right hand side of (35):

$$\begin{aligned} e^{\omega}p^{ij} - \partial_0\omega h^{ij} &= \partial_i\partial_m P_{jm} + \partial_j\partial_m P_{im} - \frac{1}{2}\partial_i\partial_j P - \frac{1}{2}\partial_i\partial_j\partial_a\partial_b\Delta^{-1}P_{ab} \\ - \quad & \Delta P_{ij} + \frac{1}{2}\delta_{ij}\Delta P - \frac{1}{2}\delta_{ij}\partial_a\partial_b P^{ab} \end{aligned}$$

which amounts to performing the gauge transformation $P_{ij} \to P_{ij} - \frac{1}{2}P\delta_{ij} + \frac{1}{2}\delta_{ij}\partial_a\partial_b\Delta^{-1}P^{ab}$.

b) Hamiltonian constraint:

$$\partial^i \partial^j h_{ij} - \Delta h + \partial_i \omega \partial^i h - e^{\omega} \partial_0 \omega p - \frac{2}{3} e^{\omega} \Lambda h = 0 \tag{0.-118}$$

In the gauge $h = p = 0$ it reads simply

$$\partial^i \partial^j h_{ij} = 0 \tag{0.-118}$$

which is solved as follows:

$$h_{ij} = \epsilon_{iab} \partial^a \phi^b{}_j + \epsilon_{jab} \partial^a \phi^b{}_i \tag{0.-118}$$

which automatically satisfies $h = 0$.

Potentials near a Kasner background:

$g_{00} = -1$, $g_{ij} = t^{2a_i}\delta_{ij}$

a) Momenutm constraint:

$$\partial_j (p^{ij} + h^{im}\bar{\pi}^{\ j}_{m} - \frac{1}{2}\bar{g}^{ij}\bar{\pi}^{mn}h_{mn})$$

This is the divergence of an expression that is not symmetric.

Use the gauge

$$\partial_j h^{im}\bar{\pi}^{\ j}_{m} = \partial_j h^{jm}\bar{\pi}^{\ i}_{m} + \partial_j \bar{g}^{ij}\bar{\pi}^{mn}h_{mn}$$

The solution reads

$$p^{ij} + \frac{1}{2}(h^{im}\bar{\pi}^{\ j}_{m} + h^{jm}\bar{\pi}^{\ i}_{m}) = \epsilon^{imn}\epsilon^{jkl}\partial_m\partial_k P_{nl}$$

b) Hamiltonian constraint:

$$\partial_i\partial_j(h^{ij} - \bar{\pi}\bar{g}^{ij}P + \bar{\pi}P^{ij} + 2\bar{\pi}^{ij}P - 2\bar{\pi}^{im}P^j{}_m - 2\bar{\pi}^{jm}P^i{}_m + \bar{g}^{ij}\bar{\pi}^{ab}P_{ab} - \bar{g}^{ij}h + \bar{g}^{ij}\bar{\pi}^{ab}P_{ab}) = 0$$

Defining

$$j^{ij} \equiv h^{ij} - \bar{\pi}\bar{g}^{ij}P + \bar{\pi}P^{ij} + 2\bar{\pi}^{ij}P - 2\bar{\pi}^{im}P^j{}_m - 2\bar{\pi}^{jm}P^i{}_m + \bar{g}^{ij}\bar{\pi}^{ab}P_{ab}$$

and imposing the usual gauge condition $j = 0$ one finds

$$h^{ij} = \epsilon^{iab}\partial_a\phi_b^{\,j} + \epsilon^{jab}\partial_a\phi_b^{\,i} + \bar{\pi}\bar{g}^{ij} - \bar{\pi}P^{ij} - 2\bar{\pi}^{ij}P + 2\bar{\pi}^{im}P^j{}_m + 2\bar{\pi}^{jm}P^i{}_m - \bar{g}^{ij}\bar{\pi}^{ab}P_{ab}$$

The use of the gauge condition

$$\partial_j h^{im}\bar{\pi}_m^{\,j} = \partial_j h^{jm}\bar{\pi}_m^{\,i} + \partial_j\bar{g}^{ij}\bar{\pi}^{mn}h_{mn}$$

implies the necessity of projecting the potential:

$$\phi \to \phi + \Delta^{-1}\phi + \bar{\pi}\phi + \bar{\pi}\Delta^{-1}\phi + ... \qquad (0.\text{-}125)$$

Note that for maximally symmetric space-times $\bar{\pi}^{ij} \propto \delta^{ij}$.

Thank you