LoopFest XVII, Michigan State University

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Higgs + Jet Production at NLO with full top quark mass dependence

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Motivation

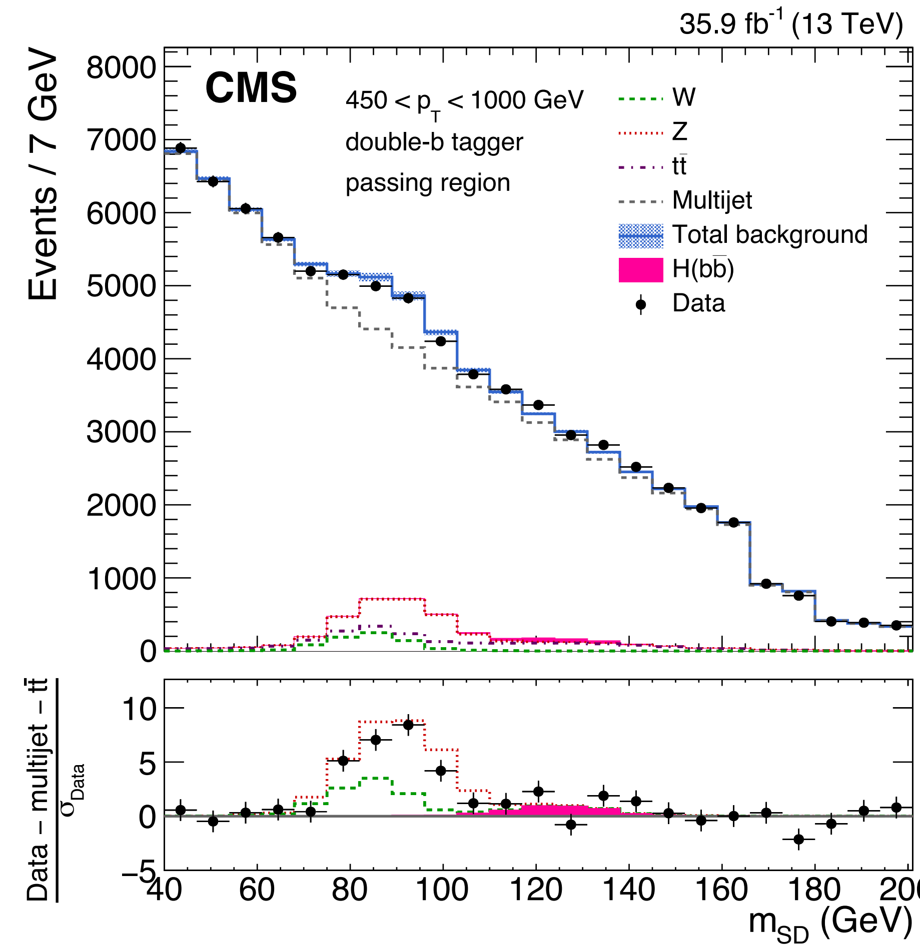
Higgs physics has transformed from discovery to precision study

Data are becoming more precise also for more differential observables

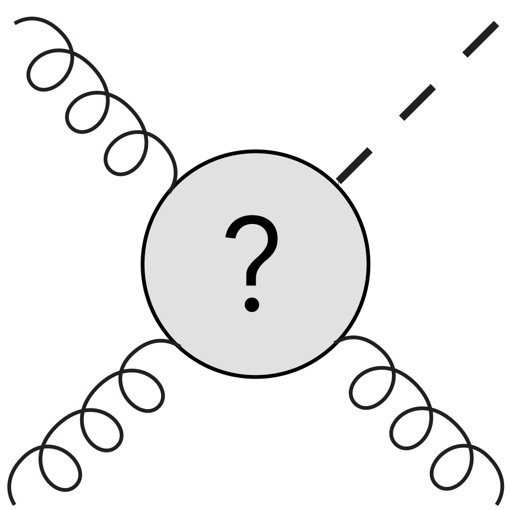
CMS 17: Search for boosted $H \to b\bar{b}$

$$\boxed{p_T > 450 \text{ GeV}}$$

$$\sigma(H \to b\bar{b}) = 74 \pm 48(\text{stat})^{-10}_{+17}(\text{syst})\ \text{fb}$$



CMS 17

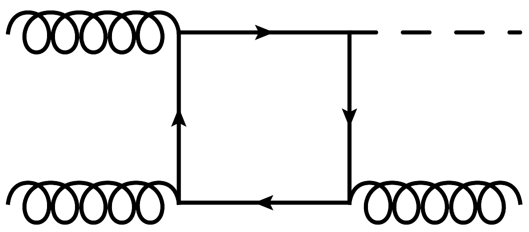


At moderate/large pT:

- Particles in the loop can be resolved
- May disentangle modified top quark Yukawa coupling from (BSM) point-like ggH coupling
- Top quark mass can be important

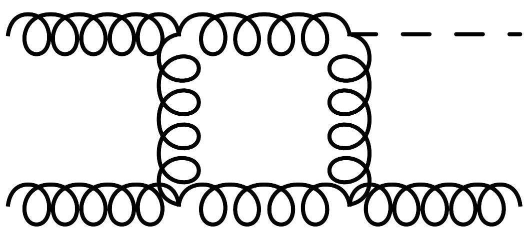
HJ Production History

1. LO (full m_T dependence)
   Ellis, Hinchliffe, Soldate, van der Bij 87
   Baur, Glover 89



2. NLO **K≈1.8**

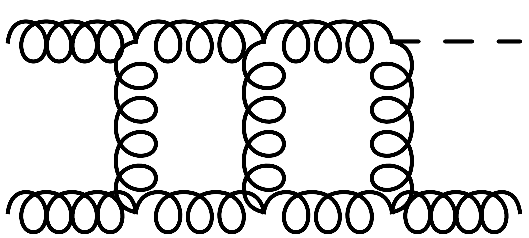
   Heavy Top Quark Limit
   de Florian, Grazzini, Kunszt 99; Glosser, Schmidt 02;
   Ravindran, Smith, van Neerven 02



   Approximate m_T dependence
   Harlander, Neumann, Ozeren, Wiesemann 12; Neumann, Wiesemann 14;
   Frederix, Frixione, Vryonidou, Wiesemann 16; Neumann, Williams 16;
   Caola, Forte, Marzani, Muselliand, Vita 16; Braaten, Zhang, Zhang 17;
   Lindert, Kudashkin, Melnikov, Wever 18; Neumann 18;

   Top quark/Bottom quark interference
   (Lindert,) Melnikov, Tancredi, Wever 16, 17;
   Caola, Lindert, Melnikov, Monni, Tancredi, Wever 18

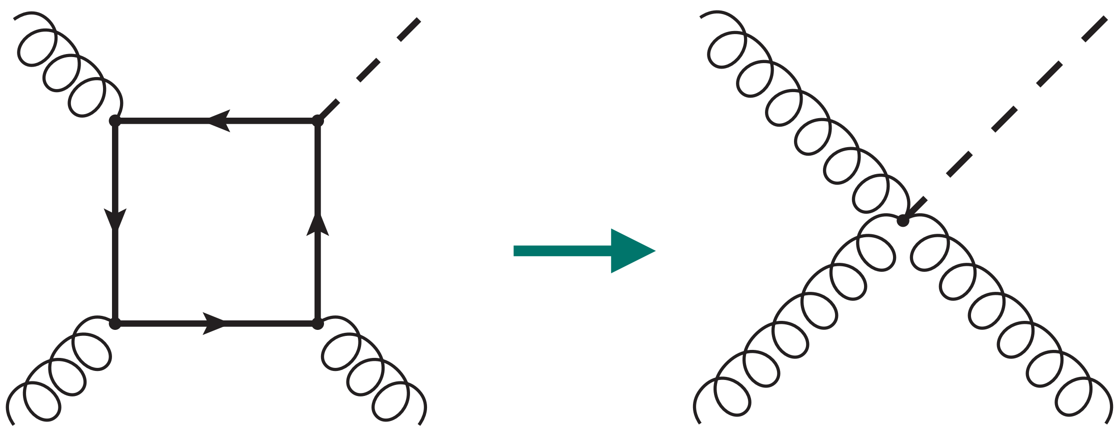
3. NNLO Heavy Top Quark Limit **K≈1.2**
   Boughezal, Caola, Melnikov, Petriello, Schulze 13, 14
   Chen, (Martinez,) Gehrmann, Glover, Jaquier 14, 16
   Boughezal, Focke, Giele, Liu, Petriello 15



The Challenge

Heavy top quark limit (HTL): $m_T \to \infty$

Introduces effective tree-level coupling between Higgs and gluons



$$\mathcal{L}_{\rm eff} \supset -\frac{\lambda}{4} H G^a_{\mu\nu} G^{a,\mu\nu}$$
$$\lambda = -\frac{\alpha_s}{3\pi v} + \mathcal{O}(\alpha_s^2)$$

HTL valid for: $s \ll 4m_T^2$

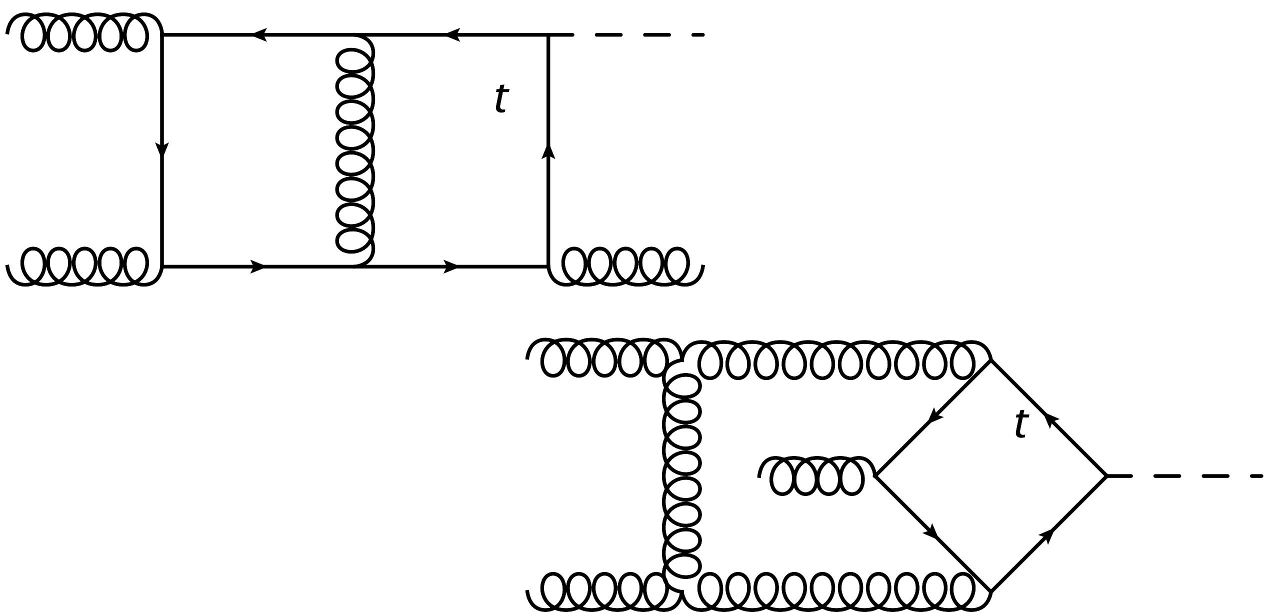
May not describe well high p_T or invariant mass region

Full NLO Calculation

2-loops 4 scales (s, t, m_T, m_H)

Involved integral reduction

Complicated integrals



Analytic results (so far)

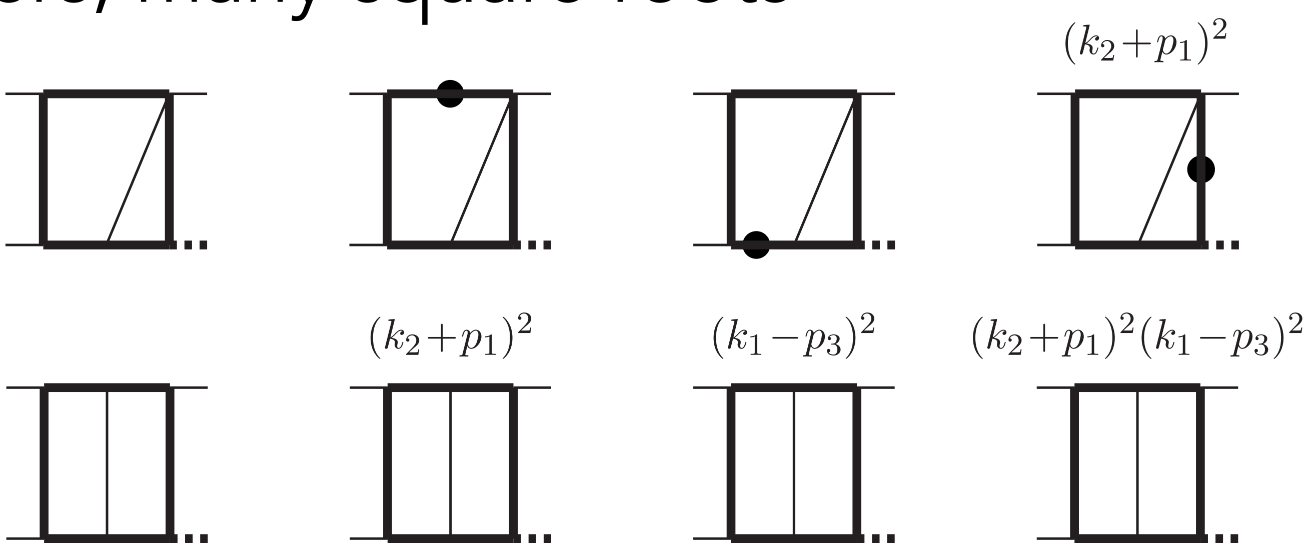
Planar integrals contributing to HJ at 2-loop are known analytically
Bonciani, Del Duca, Frellesvig, Henn, Moriello, Smirnov 16

Most integrals:

- Canonical basis found
- $\log, \mathrm{Li}_2$ up to weight 2
- 1-fold integrals at weights 3,4
- Alphabet with 3 variables, 49 letters, many square roots

Two sectors expressed in terms of elliptic functions:

- 2- and 3-fold iterated integrals
- Elliptic kernel



Non-planar integrals not currently known analytically, but subject to ongoing work by other group(s)
Bonciani, Del Duca, Frellesvig, Henn, Moriello, Smirnov, …

Expansion results

Alternatively can consider Higgs boson & top quark masses as small

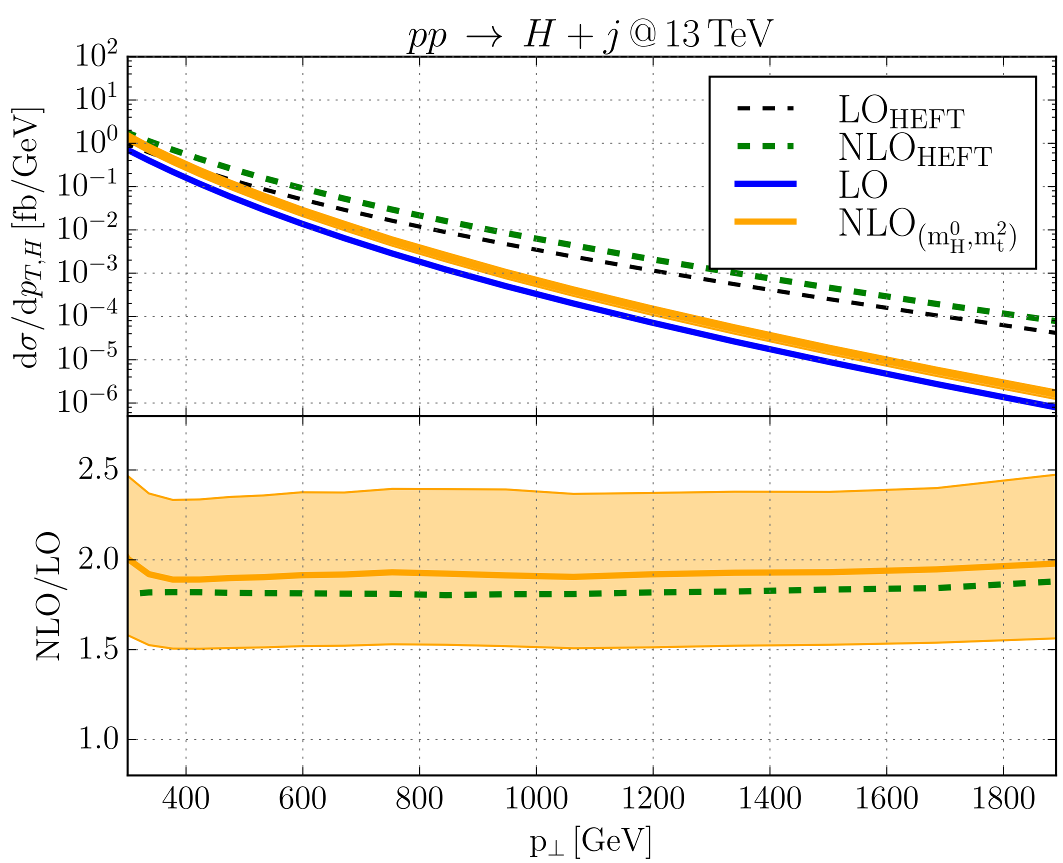
Introduce variables:

$$\eta = -\frac{m_H^2}{4m_T^2}, \quad \kappa = -\frac{m_T^2}{s}, \quad z = \frac{u}{s}$$

Expand integrals to $\mathcal{O}(\eta^0\kappa^1)$ justified for $m_H^2, m_T^2 \ll |s| \sim |t| \sim |u|$,

For example at large $p_T^2 = ut/s$

Kudashkin, Melnikov, Wever 17



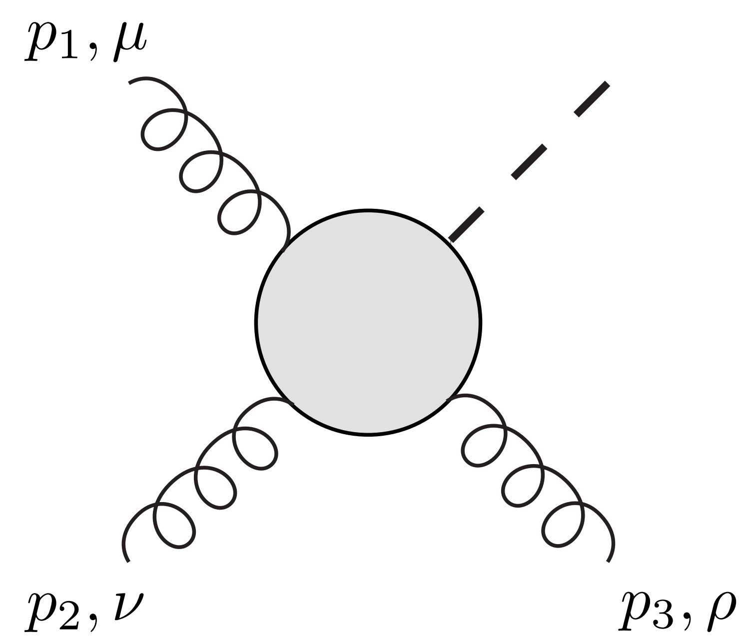
Expanded 2-loop virtuals can be combined with full reals to predict Higgs boson p_T distribution above top threshold

$$\frac{K^{\mathrm{SM}}}{K^{\mathrm{HTL}}} = 1.04 \ldots 1.06$$

Lindert, Kudashkin, Melnikov, Wever 18

Form Factor Decomposition (Gluon)

Expose tensor structure: $\mathcal{M} = \epsilon_\mu(p_1)\epsilon_\nu(p_2)\epsilon_\tau(p_3)\mathcal{M}^{\mu\nu\tau}$



$$s_{ij} = (p_i + p_j)^2$$

Form Factors (Contain integrals)

$$\mathcal{M}^{\mu\nu\tau}_{\text{physical}} = F_{212}T^{\mu\nu\tau}_{212} + F_{332}T^{\mu\nu\tau}_{332} + F_{311}T^{\mu\nu\tau}_{311} + F_{312}T^{\mu\nu\tau}_{312}$$

Choose tensor basis (constructed from external momenta & metric):

$$T^{\mu\nu\tau}_{212} = (s_{12}g^{\mu\nu} - 2p_2^\mu p_1^\nu)(s_{23}p_1^\tau - s_{13}p_2^\tau)/(2s_{13})$$
$$T^{\mu\nu\tau}_{332} = (s_{23}g^{\nu\tau} - 2p_3^\nu p_2^\tau)(s_{13}p_2^\mu - s_{12}p_3^\mu)/(2s_{12})$$
$$T^{\mu\nu\tau}_{311} = (s_{13}g^{\tau\mu} - 2p_1^\tau p_3^\mu)(s_{12}p_3^\nu - s_{23}p_1^\nu)/(2s_{23})$$
$$T^{\mu\nu\tau}_{312} = \Big(g^{\mu\nu}(s_{23}p_1^\tau - s_{13}p_2^\tau) + g^{\nu\tau}(s_{23}p_2^\mu - s_{12}p_3^\mu) + g^{\tau\mu}(s_{12}p_3^\nu - s_{23}p_1^\nu) + 2p_3^\mu p_1^\nu p_2^\tau - 2p_2^\mu p_3^\nu p_1^\tau\Big)/2$$

Build projectors P such that: $P^{212}_{\mu\nu\tau}\mathcal{M}^{\mu\nu\tau} = F_{212}$, etc…

Thanks: Glover, Frellesvig

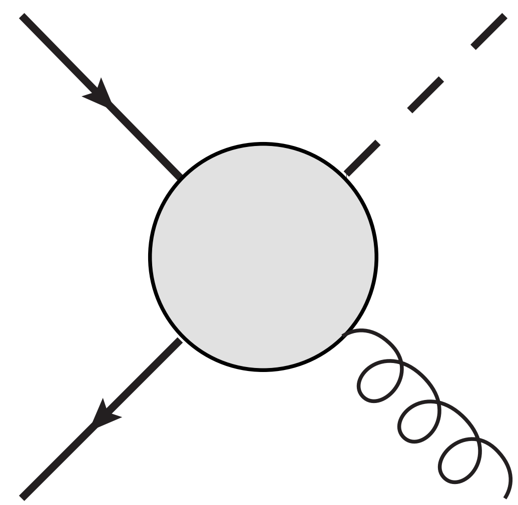
With this choice form factors are gauge invariant and related by cyclic permutations:

$$\boxed{\begin{aligned} F_{311}(s_{23}, s_{12}, s_{13}) &= F_{332}(s_{13}, s_{23}, s_{12}) = F_{212}(s_{12}, s_{13}, s_{23}) \\ F_{312}(s_{23}, s_{12}, s_{13}) &= F_{312}(s_{13}, s_{23}, s_{12}) = F_{312}(s_{12}, s_{13}, s_{23}) \end{aligned}}$$

Note: Need to generate code only for 2 form factors and use it to compute all
In reality we generate code for all form factors and use this symmetry as a cross-check

Form Factor Decomposition (Quark)

We compute $q\bar{q}gH$ and obtain other channels by crossing



Form Factors (Contain integrals)

$$\mathcal{M}^{\mu}_{\beta\alpha} = F_1 T^{\mu}_{1,\beta\alpha} + F_2 T^{\mu}_{2,\beta\alpha}$$

Choose basis:

$$T^{\mu}_{1,\beta\alpha} = \left(\bar{v}_\beta(p_2)\not{p}_3 u_\alpha(p_1) p_1^\mu - \bar{v}_\beta(p_2)\gamma^\mu u_\alpha(p_1) p_1 \cdot p_3\right)$$

$$T^{\mu}_{2,\beta\alpha} = \left(\bar{v}_\beta(p_2)\not{p}_3 u_\alpha(p_1) p_2^\mu - \bar{v}_\beta(p_2)\gamma^\mu u_\alpha(p_1) p_2 \cdot p_3\right)$$

Build projectors P such that: $(P^1_{\alpha\beta})_\mu \mathcal{M}^{\mu}_{\beta\alpha} = F_1, \ (P^2_{\alpha\beta})_\mu \mathcal{M}^{\mu}_{\beta\alpha} = F_2$

Gehrmann, Glover, Jaquier, Koukoutsakis 11

Integral Families

All 2-loop HJ integrals can be written in terms of 3 integral families:

	F1	F2	F3
D_1	$k_1^2 - m_T^2$	k_2^2	$k_1^2 - m_T^2$
D_2	$(k_1 + p_1)^2 - m_T^2$	$(k_2 + p_1)^2$	$(k_1 + p_1)^2 - m_T^2$
D_3	$(k_1 - p_2)^2 - m_T^2$	$(k_2 - p_2)^2$	$(k_1 - p_2 - p_3)^2 - m_T^2$
D_4	$(k_1 - p_2 - p_3)^2 - m_T^2$	$(k_2 - p_2 - p_3)^2$	$k_2^2 - m_T^2$
D_5	$k_2^2 - m_T^2$	$k_1^2 - m_T^2$	$(k_2 + p_1)^2 - m_T^2$
D_6	$(k_2 + p_1)^2 - m_T^2$	$(k_1 + p_1)^2 - m_T^2$	$(k_2 - p_3)^2 - m_T^2$
D_7	$(k_2 - p_2)^2 - m_T^2$	$(k_1 - p_2)^2 - m_T^2$	$(k_1 - k_2)^2$
D_8	$(k_2 - p_2 - p_3)^2 - m_T^2$	$(k_1 - p_2 - p_3)^2 - m_T^2$	$(k_1 - k_2 - p_2)^2$
D_9	$(k_1 - k_2)^2$	$(k_1 - k_2)^2 - m_T^2$	$(k_1 - k_2 - p_2 - p_3)^2$

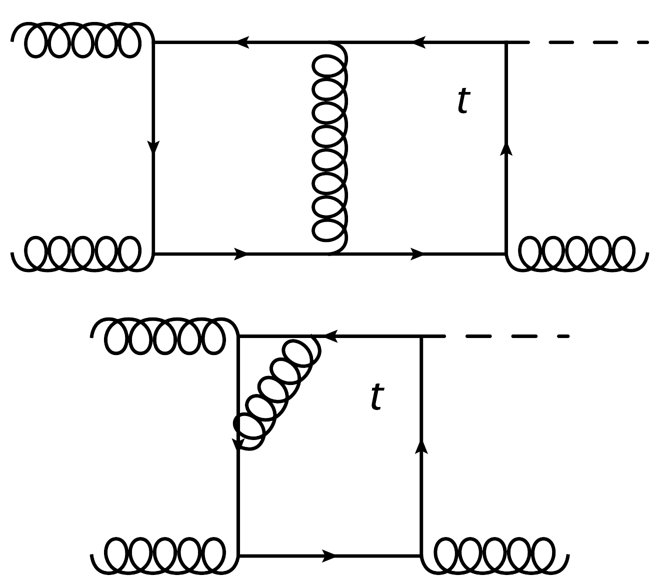
Melnikov, Tancredi, Wever 16

Integrals written as:

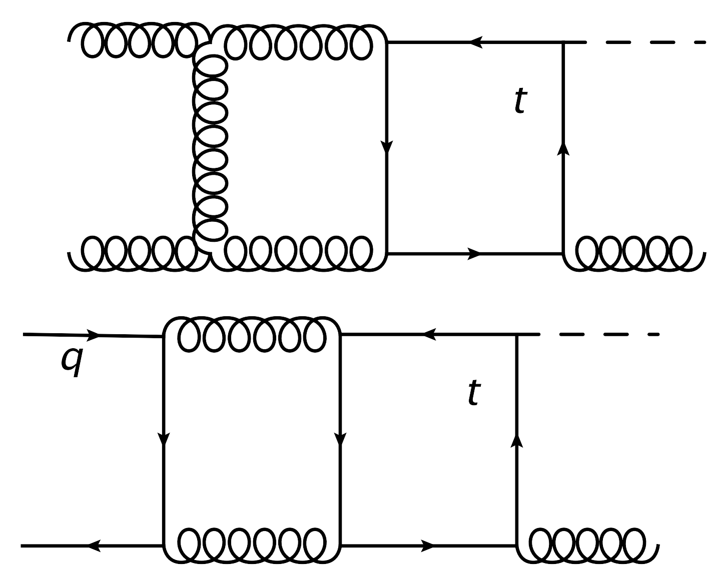
$$I^{F_j}_{\alpha_1,\ldots,\alpha_9} = \int \mathrm{d}^d k_1 \int \mathrm{d}^d k_2 \frac{1}{D_1^{\alpha_1} \cdots D_9^{\alpha_9}}$$

Integral Families (II)

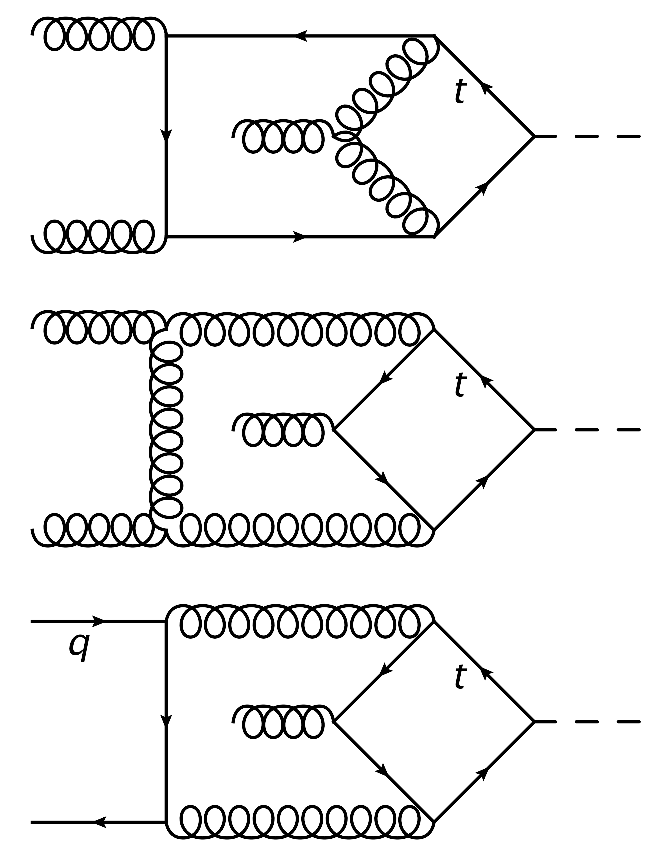
F1



F2



F3



Channel	#Diagrams @ 2-loop
$gggH$	354
$q\bar{q}gH$	57

Quark and gluon channel integrals belong to same integral families

Integral Reduction

Full IBP reduction achieved with (modified) `Reduze2`

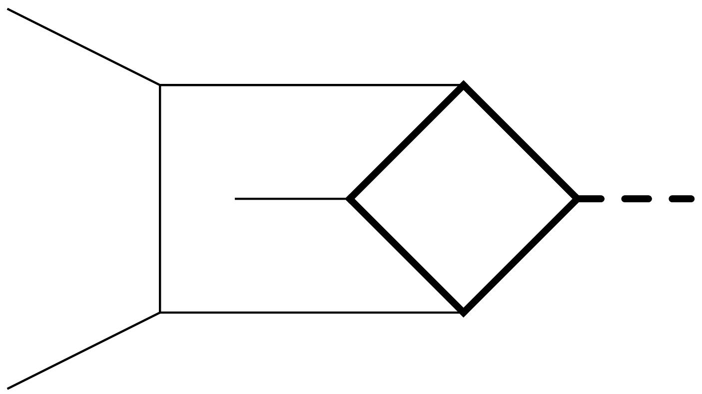
Chetyrkin, Tkachov 81; Laporta 01; von Manteuffel, Studerus 12

Unreduced Amplitude

- 3767 Integrals
- Up to 3 inverse propagators for 7-propagator integrals
- Up to 4 inverse propagators for factoring 6-propagator integrals

Reduced Amplitude

- 458 Integrals
- Up to 6 master integrals per sector, e.g:



Sector also known to be elliptic

Frellesvig, Loops & Legs 2018

IBP reduction allows us to choose a basis of master integrals

We choose quasi-finite master integrals

Panzer 14; von Manteuffel, Panzer, Schabinger 15

Integral Reduction (II)

`Reduze2` Modifications:

- Change order of solving system of equations, sort by number of unreduced integrals (prefer fewer)
- Specify list of required integrals, consider only equations containing these integrals
- Prefer solving smaller equations first
- Improve mechanism for pausing/resuming reductions

Reduction achieved using 2 different setups:

Current calculation uses only this result

Symbolic mass dependence

Reduction directory size: **1.1 TB**

Can include m_B, Γ_T

Fix mass ratio

$$\frac{m_H^2}{m_T^2} = \frac{12}{23} \to \begin{array}{l} m_H = 125 \text{ GeV} \\ m_T = 173.055 \text{ GeV} \end{array}$$

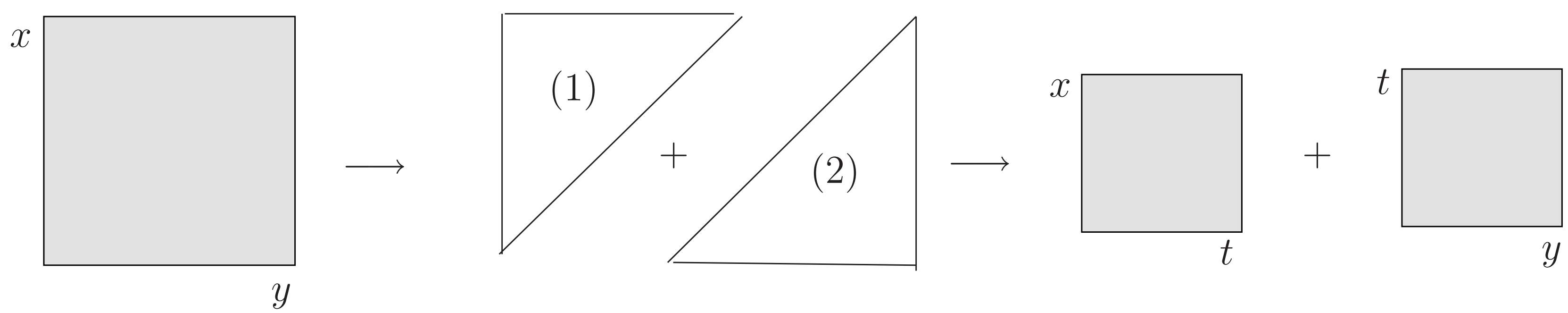
Reduction directory size: **0.25 TB**

Can not include m_B, Γ_T

Sector Decomposition

1) Feynman parametrise integral and compute momentum integrals
2) Apply sector decomposition to factorize overlapping singularities

Hepp 66; Denner, Roth 96; Binoth, Heinrich 00



3) Subtract poles and expand in ϵ
4) Use contour deformation (analytic continuation to physical region)

Soper 99; Binoth, Guillet, Heinrich, Pilon, Schubert 05; Nagy, Soper 06; Anastasiou, Beerli, Daleo 07; Beerli 08; Borowka, Carter, Heinrich 12; Borowka 14;

5) Numerically integrate resulting parameter integrals

All master integrals processed with SecDec Borowka, Heinrich, Jahn, SJ, Kerner, Schlenk, Zirke

interface to amplitude based on calculation of HH production

Borowka, Greiner, Heinrich, SJ, Kerner, Schlenk, Schubert, Zirke 16

Finite Basis

Always possible to pick finite basis of integrals using:

- Dimension Shifts Tarasov 96; Lee 10
- Dots

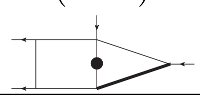
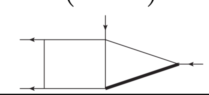
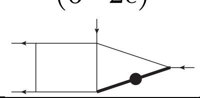
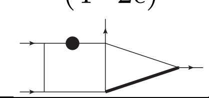
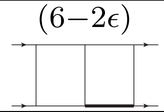
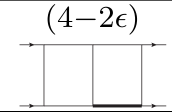
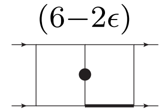
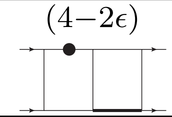
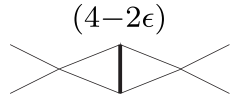
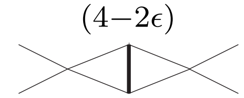
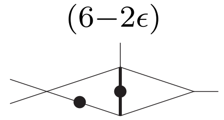
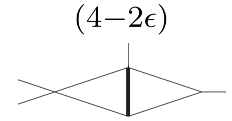
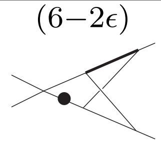
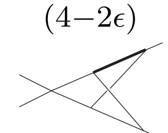
Panzer 14; von Manteuffel, Panzer, Schabinger 15

Finding finite integrals and basis change implemented in `Reduze2.1`

Finite basis greatly improves numerical performance
(but requires reduction of integrals with up to 2 inverse props and 2 dots)

Two-loop EW-QCD Drell-Yan

von Manteuffel, Schabinger 17

Finite Basis...			Conventional...		
$(6-2\epsilon)$  (s,t)	201 s	2.34×10^{-4}	$(4-2\epsilon)$  (s,t)	384 s	8.12×10^{-4}
$(6-2\epsilon)$  (s,t)	150 s	4.83×10^{-4}	$(4-2\epsilon)$  (s,t)	56538 s	1.67×10^{-2}
$(6-2\epsilon)$  (s,t)	280 s	1.00×10^{-3}	$(4-2\epsilon)$  (s,t)	214135 s	8.29×10^{-3}
$(6-2\epsilon)$  (s,t)	294 s	1.21×10^{-3}	$(4-2\epsilon)$  (s,t)	3484378 s	30.9
$(4-2\epsilon)$  (s)	91 s	3.76×10^{-4}	$(4-2\epsilon)$  (s)	87 s	3.76×10^{-4}
$(6-2\epsilon)$  (s)	17 s	5.15×10^{-4}	$(4-2\epsilon)$  (s)	20 s	1.95×10^{-4}
$(6-2\epsilon)$  (s)	119 s	2.32×10^{-3}	$(4-2\epsilon)$  (s)	118 s	2.12×10^{-3}
Total/Max:	3995 s	5.84×10^{-3}	Total/Max:	5136862 s	30.9

Rel. Err.

Amplitude Evaluation

Use Quasi-Monte Carlo (QMC) integration $\mathcal{O}(n^{-1})$ error scaling

Li, Wang, Yan, Zhao 15; Review: Dick, Kuo, Sloan 13;

Implemented in OpenCL, evaluated on GPUs

Entire 2-loop amplitude evaluated with a single code

$$F = \sum_i \underbrace{\left(\sum_j C_{i,j} \epsilon^j \right)}_{\text{coeff.}} \underbrace{\left(\sum_k I_{i,k} \epsilon^k \right)}_{\text{integral}} = \epsilon^{-2} \left[C^{(L)}_{1,-2} I^{(L)}_{1,0} + \ldots \right] + \epsilon^{-1} \left[C^{(L)}_{1,-1} I^{(L)}_{1,0} + \ldots \right] + \ldots$$

compute once

Dynamically set target precision for each sector, minimising time:

$$T = \sum_i t_i + \bar{\lambda} \left(\sigma^2 - \sum_i \sigma_i^2 \right), \quad \sigma_i \sim t_i^{-e}$$

$\bar{\lambda}$ – Lagrange multiplier

σ – precision goal

σ_i – error estimate

Comparison of HJ and HH

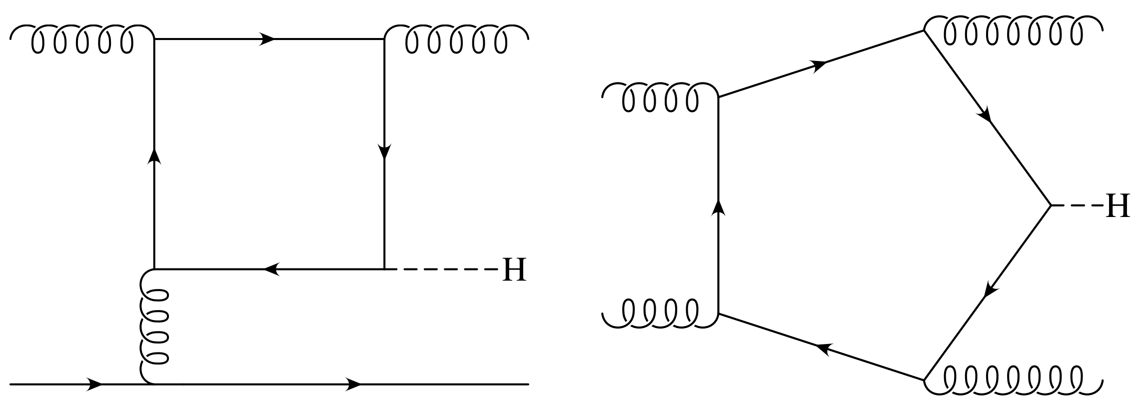
	HJ production	HH production
#Form factors	4+2	2
Full reduction	✓	only planar
(quasi-) finite basis	✓	only planar
#Master integrals including crossings	458	327*
#Master integrals neglecting crossings	120	215*
#Integrals after sector decomposition and expansion in ϵ	22675	11244
Code size coefficients	~340 MB	~80 MB
Code size integrals	~330 MB	~580 MB
Compile time coefficients	~ 2 weeks	few days
Compile time integrals	~4 hours	~1-2 days
Time for linking the program	~3-4 days	few hours

Slide: Matthias Kerner, Radcor 2017

* HH Non-planar not fully reduced

Phase-Space & Real Radiation

Real Radiation



Known analytically

Del Duca, Kilgore, Oleari, Schmidt, Zeppenfeld 01

We use an upgraded `GoSam` setup:
Cullen et al. 14

- Generate quadruple precision copy of the code
- Rescues unstable points on-the-fly with `Ninja` (quad) Mastrolia, Mirabella, Peraro 12; Peraro 14
- `OneLoop` for scalar integrals van Hameren 11

Implemented in `POWHEG-BOX-V2`, uses FKS subtraction
Nason 04; Frixione, Nason Oleari 07; Alioli, Nason, Oleari, Re 10; Frixione, Kunszt, Signer 96

Virtual Phase Space

1. Apply VEGAS algorithm to LO matrix element
2. Using LO events generate unweighted events via accept/reject

For p_T distribution we include a reweighing factor to sample sufficiently also at large transverse momenta

Results: Total Cross Section

$$m_H = 125 \text{ GeV},$$
$$m_T = \sqrt{23/12}\, m_H \approx 173.05 \text{ GeV}$$
$$p_{T,j} > 30 \text{ GeV}, \quad \text{anti}-k_T \; R = 0.4,$$
$$\mu = \frac{H_T}{2} = \frac{1}{2}\left(\sqrt{m_H^2 + p_{t,H}^2} + \sum_i |p_{t,i}|\right)$$

`PDF4LHC15_nlo_30_pdfas`

Butterworth et al. 16; Dulat et al. 16;
Harland-Lang et al. 15; Ball et al. 15

$$\mathrm{d}\sigma_{\mathrm{NLO}}^{\mathrm{FT_{approx}}} = \int \mathrm{d}\phi_2 \left(\mathrm{d}\sigma_B + \frac{\mathrm{d}\sigma_{\mathrm{B}}}{\mathrm{d}\sigma_{\mathrm{B}}^{\mathrm{HEFT}}}\mathrm{d}\sigma_{\mathrm{V}}^{\mathrm{HEFT}}\right) + \int \mathrm{d}\phi_3 \; \mathrm{d}\sigma_{\mathrm{R}}$$

$\mathrm{FT_{approx}}$:

Full Born & Reals

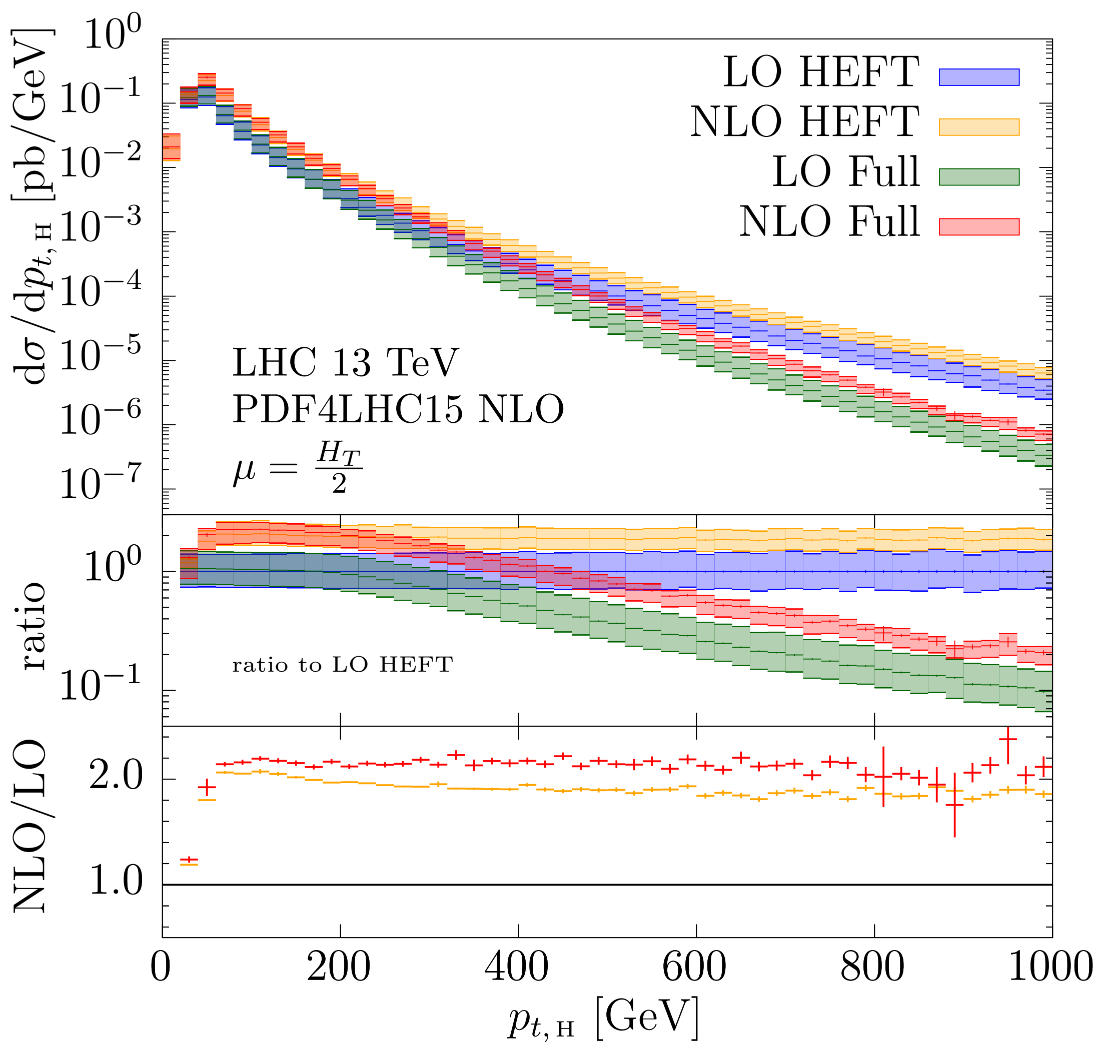
Reweight virtuals event-by-event

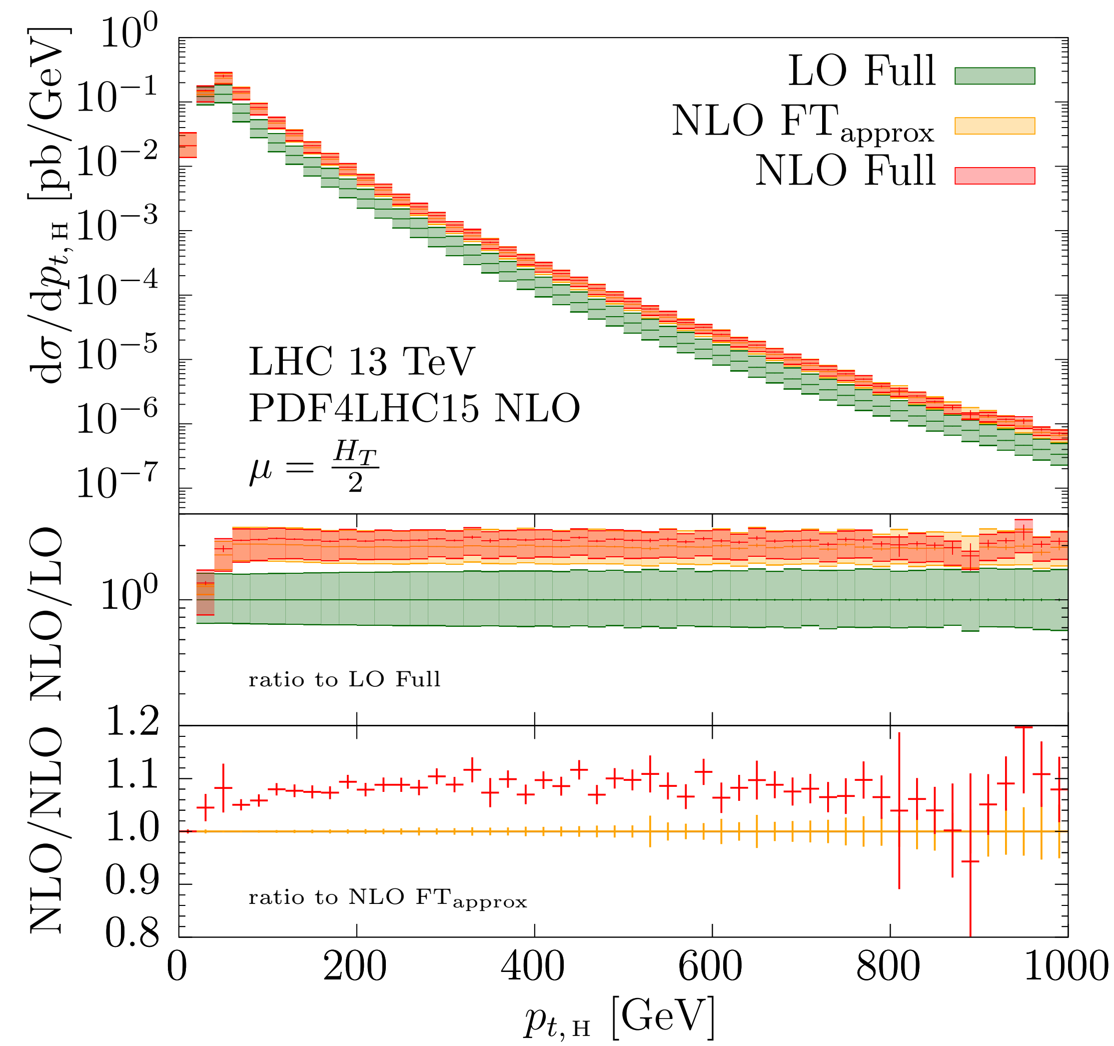
Theory	LO [pb]	NLO [pb]
HEFT:	$\sigma_{\mathrm{LO}} = 8.22^{+3.17}_{-2.15}$	$\sigma_{\mathrm{NLO}} = 14.63^{+3.30}_{-2.54}$
$\mathrm{FT_{approx}}$:	$\sigma_{\mathrm{LO}} = 8.57^{+3.31}_{-2.24}$	$\sigma_{\mathrm{NLO}} = 15.07^{+2.89}_{-2.54}$
Full:	$\sigma_{\mathrm{LO}} = 8.57^{+3.31}_{-2.24}$	$\sigma_{\mathrm{NLO}} = 16.01^{+1.59}_{-3.73}$

Note: Non-negligible contribution from top-bottom interference known at NLO but not included here

(Lindert,) Melnikov, Tancredi, Wever 16, 17

Results: Higgs Boson pT (I)





Confirm expected scaling of $\mathrm{d}\sigma/\mathrm{d}p_T^2$ in HEFT and full theory at NLO

$\sim p_T^{-2}$ in HEFT

$\sim p_T^{-4}$ in full theory

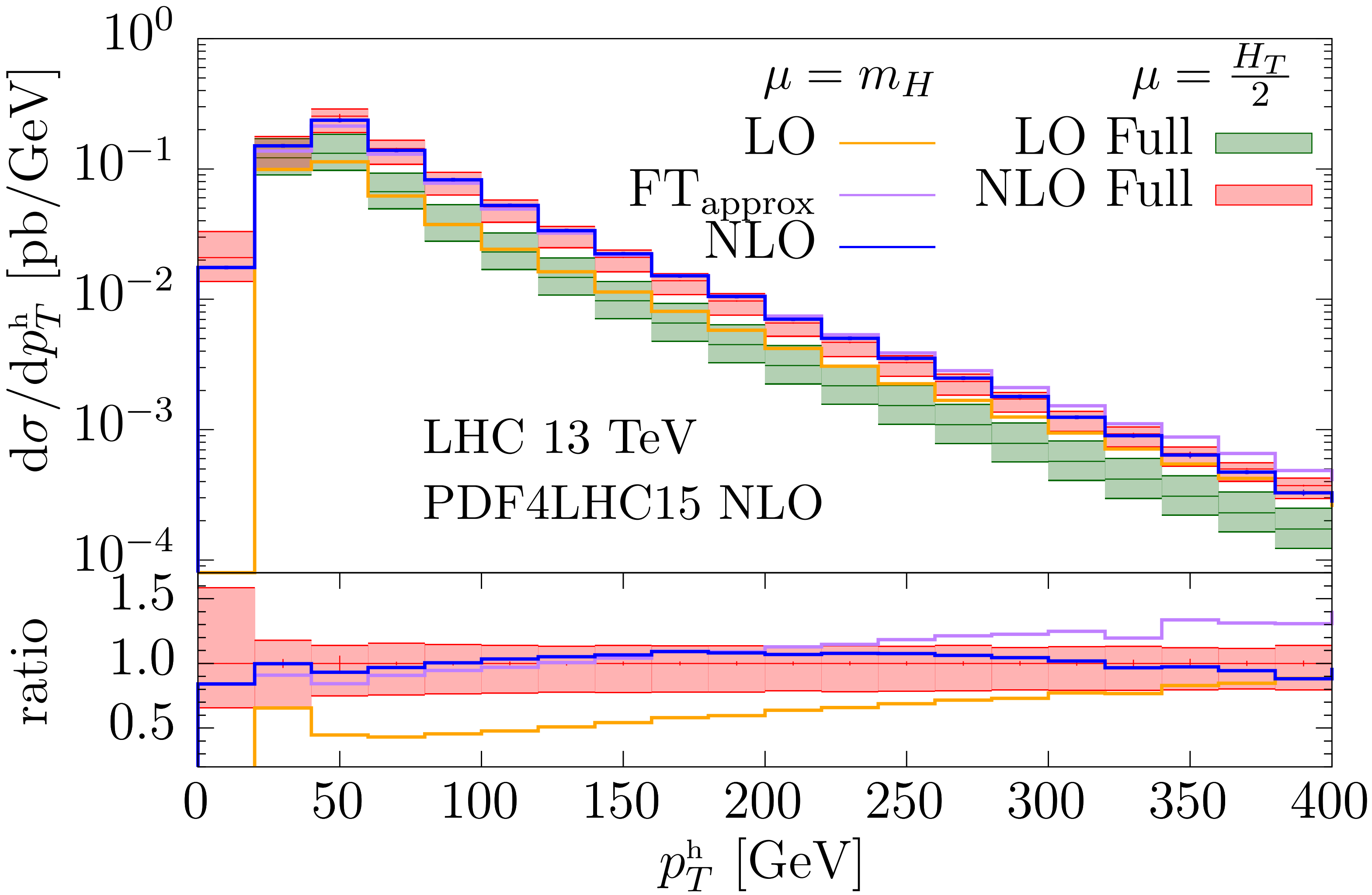
Forte, Muselli 15; Caola, Forte, Marzani, Muselli, Vita 16;

$\mathrm{FT}_{\mathrm{approx}}$ predicts similar p_T distribution shape to full theory

Full theory predicts nearly flat K factor at large p_T

~8% increase in tail by including top quark mass dependence in virtuals

Results: Different Scale Choices



Full NLO result with $\mu = m_H$ and $\mu = H_T/2$ in good agreement

With fixed scale $\mathrm{FT_{approx}}$ has different shape to full theory (overestimates tail)

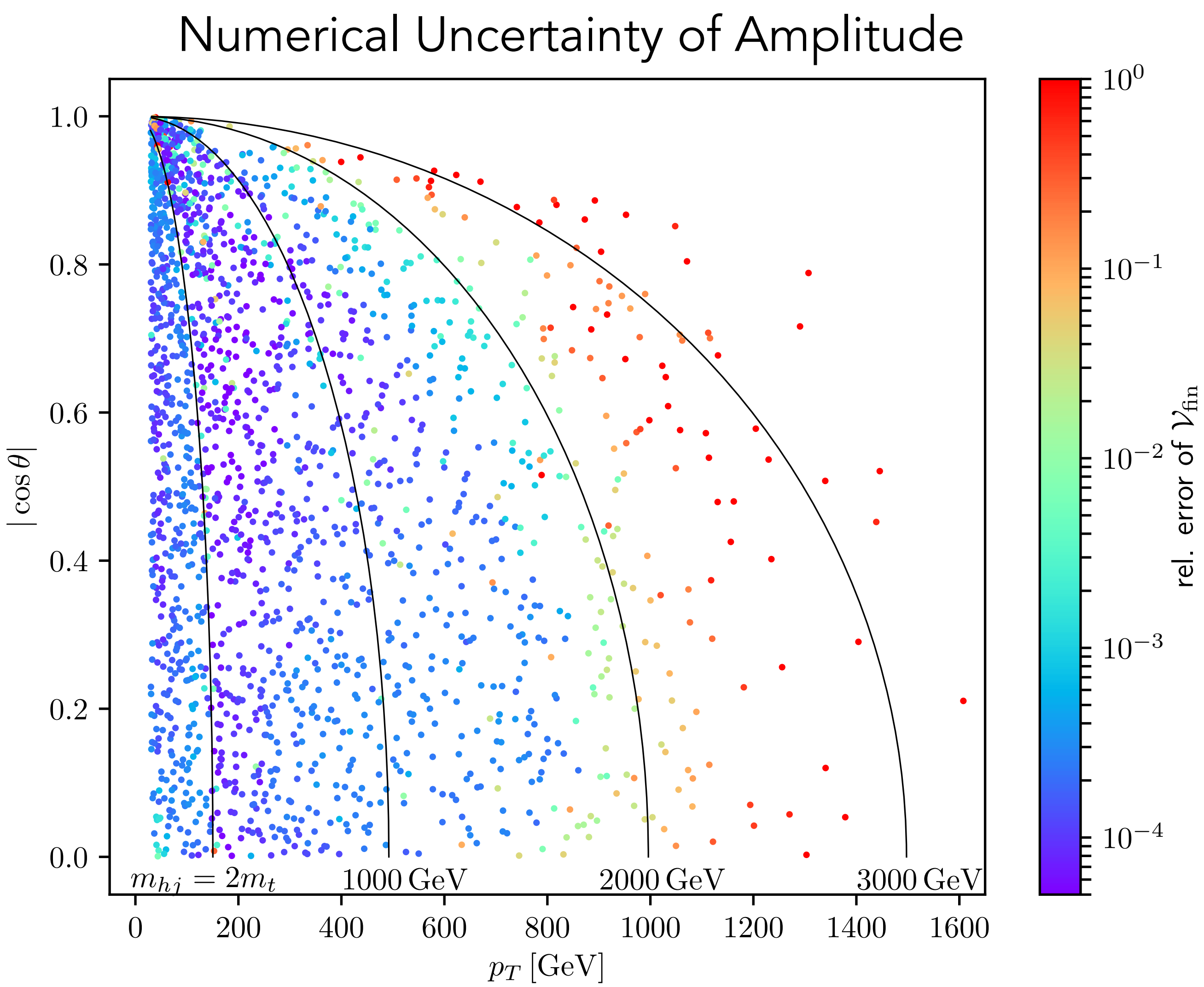
Note: K-factor only flat for dynamic scale choice, not for fixed scale

Numerical Stability

Numerical evaluation of virtual amplitude:

- accuracy goal: 0.5% for each form factor
- wall-clock limit: 2d GPU-time (Tesla K20X GPU)

Thanks: MPCDF



Accuracy reached for $|\mathcal{M}|^2$

- Better than per-mill for most points below $m_{hj} = 1.5$ TeV
- Region $m_{hj} \geq 2$ TeV numerically challenging
- Forward region challenging

Run time per point:

- Minimum: 1.3h
- Median: 15h

Improving Numerical Stability

MI Basis Change:

- Consider quasi-finite integrals (prefer finite integrals)
- Brute force combinations of masters, factor denominators and check that dimension factorises (achieved)
- Prefer simple denominator factors
- Prefer computing fewer orders in epsilon for each master
- Prefer simpler numerators (check number of terms/file size)

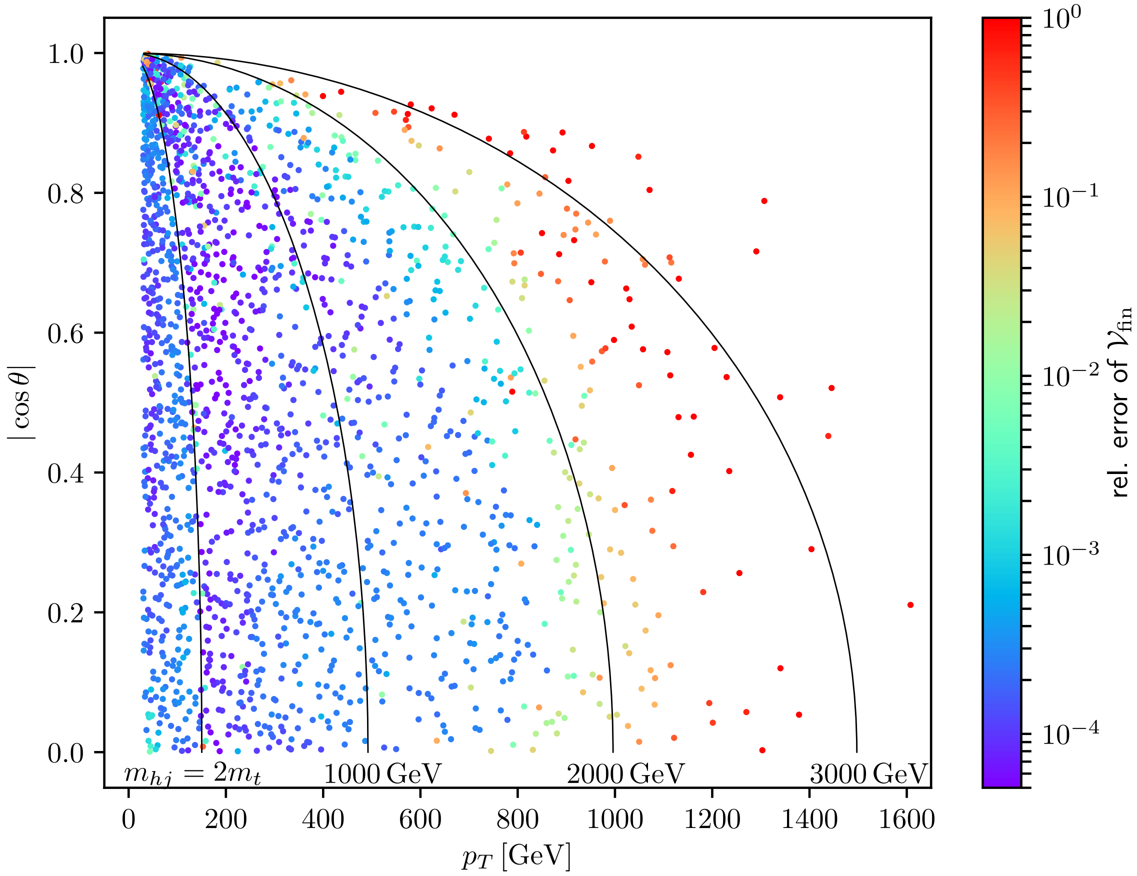
See: Matthias Kerner, Loops and Legs Proceedings 2018

Numerical Improvements:

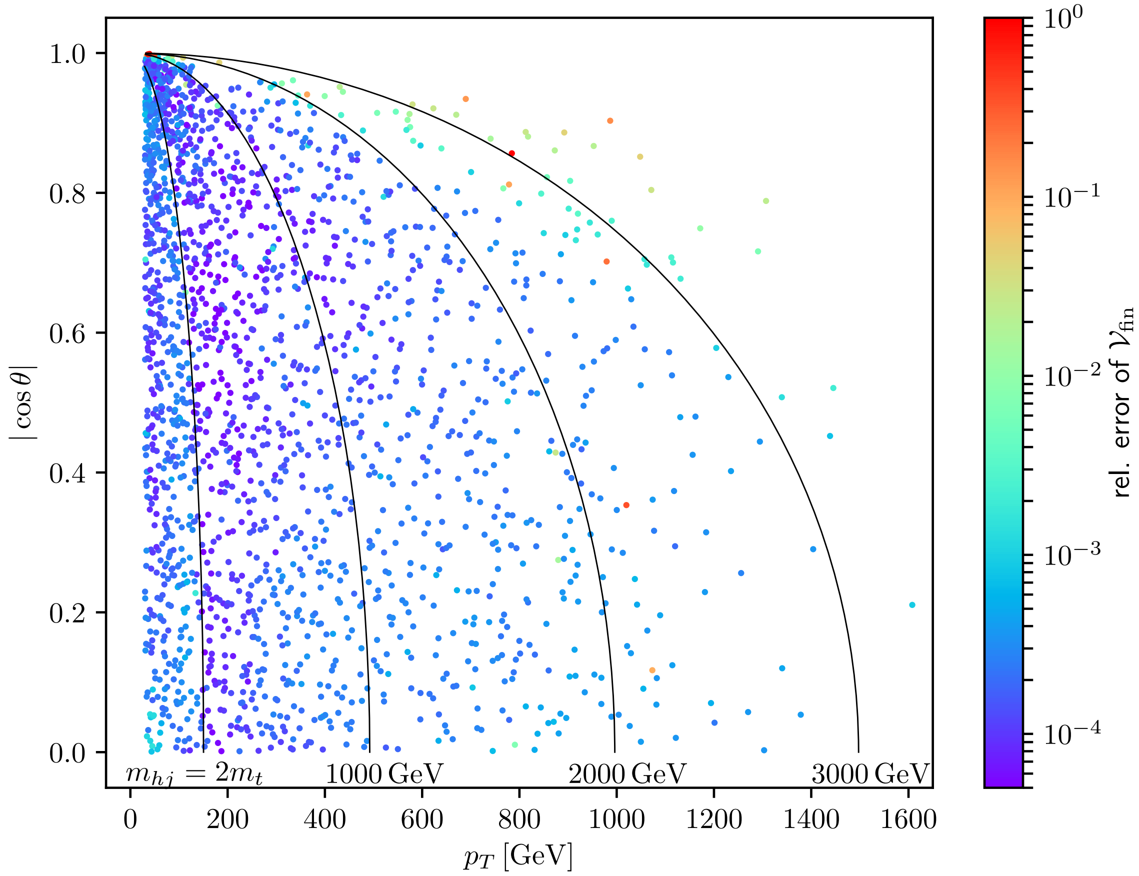
- Improve modular arithmetic implementation (needed for larger lattices)
- Do not try to further evaluate integrals if $\mathrm{rel.\ err.} < 10^{-14}$
- Adjust time spent integrating when iteration may exceed wall clock limit

Improved Numerical Stability

Before basis change:



After basis change:



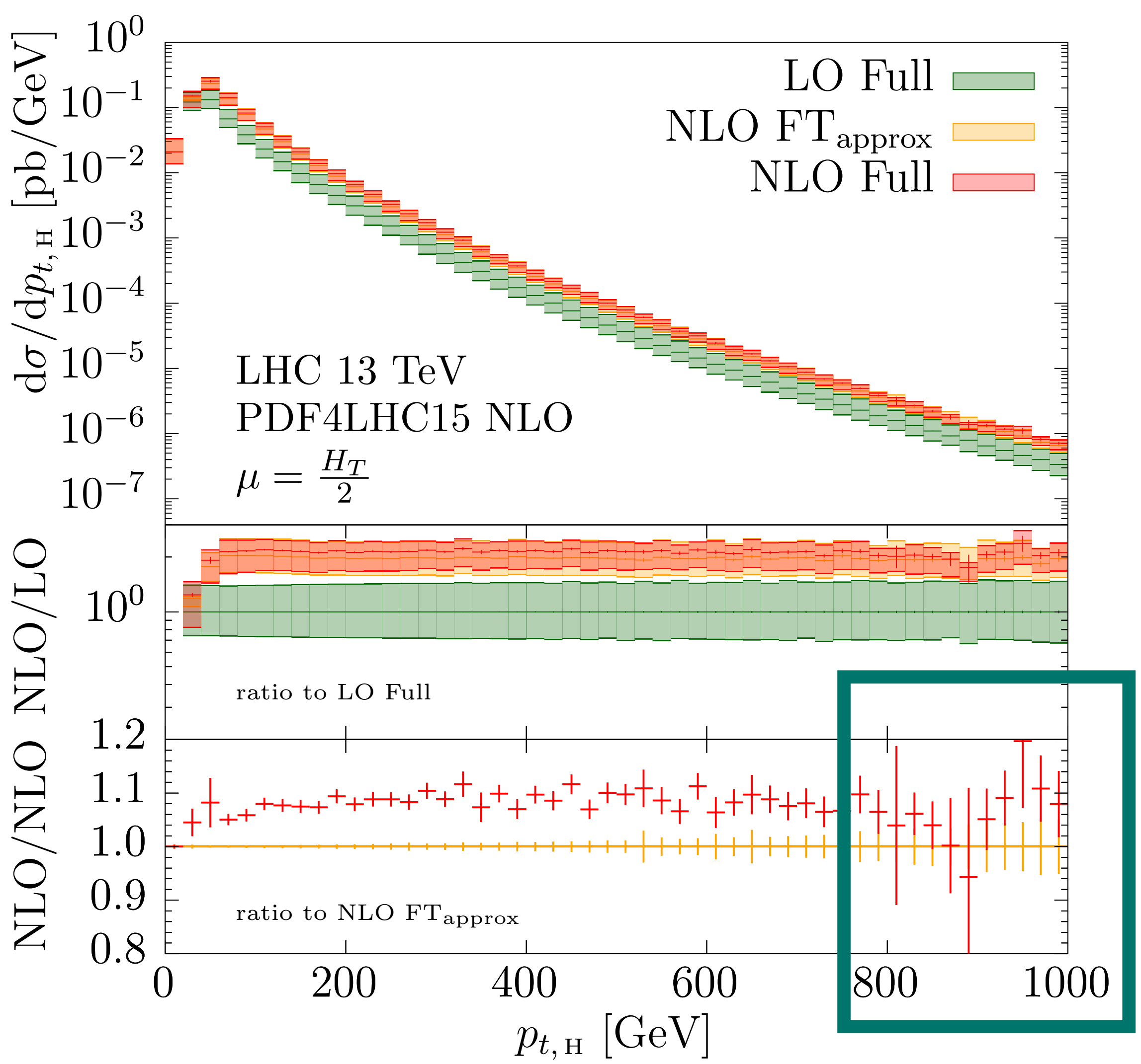
Phase-space points significantly more stable:

- Good accuracy around top quark threshold
- Huge improvement in accuracy at larger invariant mass (2-3 TeV)
- Improvement in the forward region

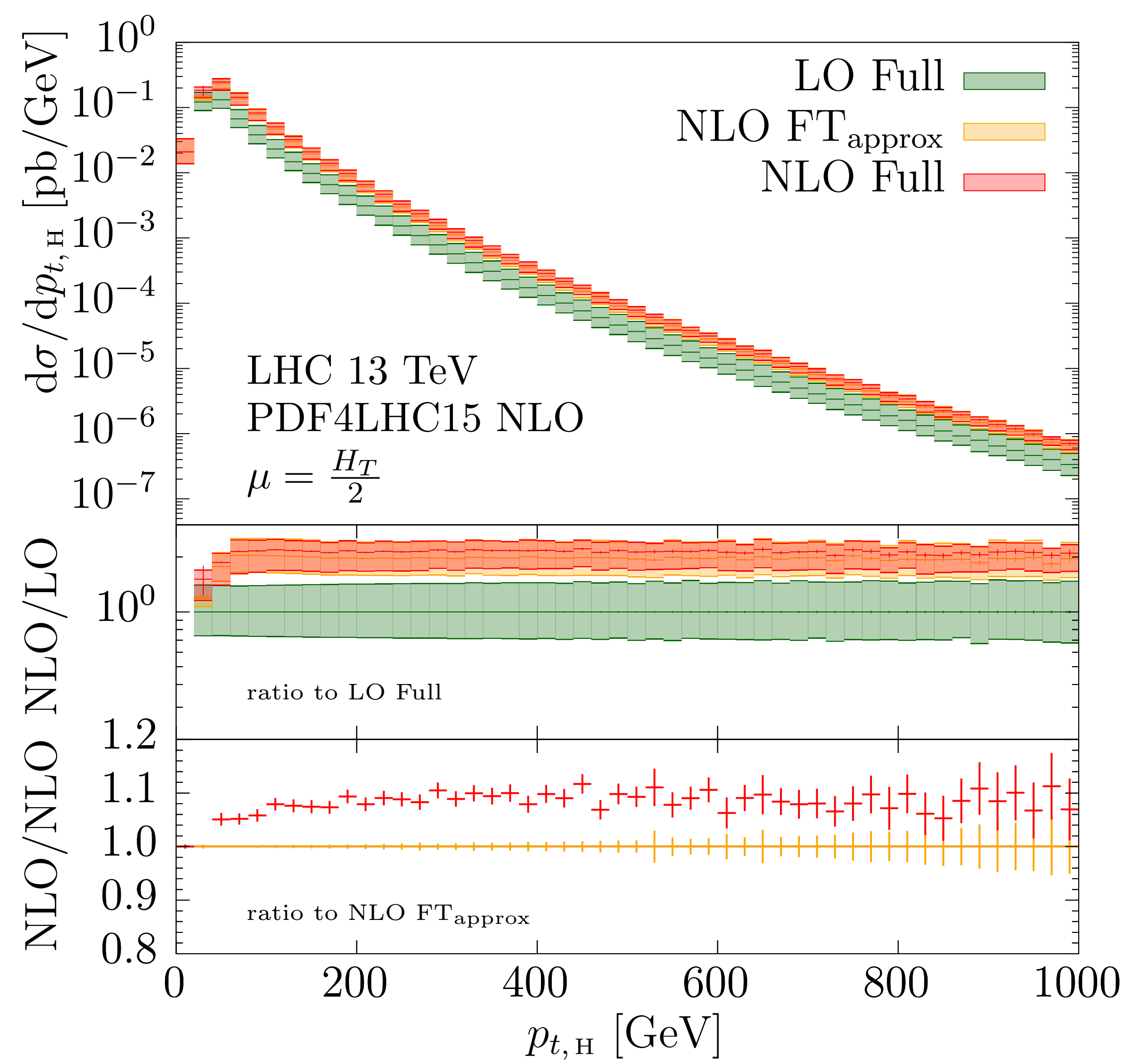
Coefficient code size: 340 MB → 100 MB

Results: Higgs Boson pT (II)

Before basis change:



After basis change:



Recomputing unstable points improves fluctuations in tail

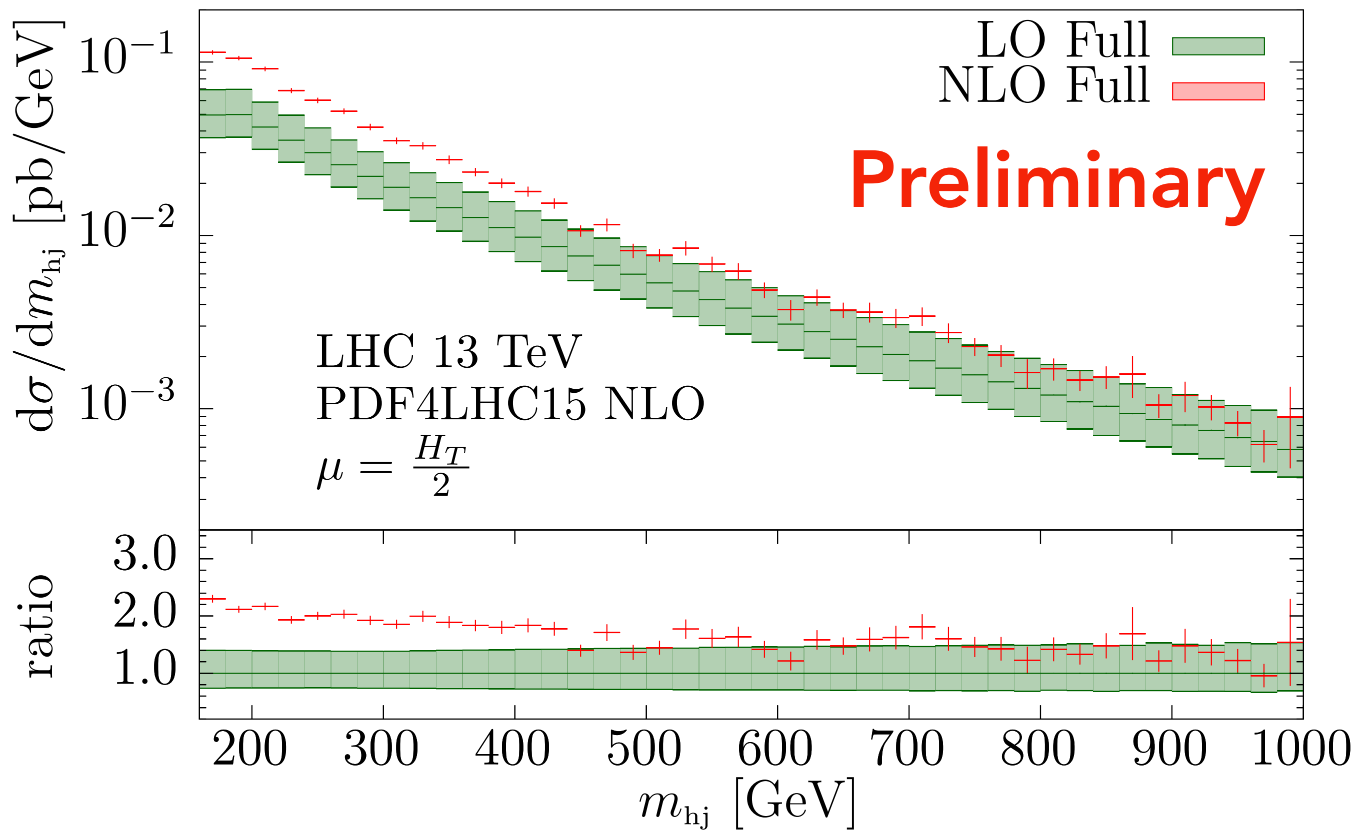
Low fraction of points excluded 3/2004

Median run time per point 15h → 2h

Ongoing Work

Starting to look at other distributions…

Basis change improves numerical stability of invariant mass distribution



Note:

- Produced with fairly low statistics (817 points)
- No scale variations yet

Conclusion

Higgs + Jet

- Computed at NLO with full top quark mass dependence
- 2-loop virtual amplitude calculated numerically
- Discussed basis choice which improves numerical performance

Future

- Need detailed comparison with expanded result
- Complete more distributions (e.g. invariant mass)
- Produce 2D grid of results (similar to `hhgrid`)
- Combine with a parton shower (`Sherpa`, `POWHEG-BOX`)
- Combination with NNLO HTL (similar to recent HH combination)

Grazzini, Heinrich, SJ, Kallweit, Kerner, Lindert, Mazzitelli 18

Thank you for listening!

Backup

Amplitude Structure

Write integrals with r propagators and s inverse propagators as

Arbitrary scale

$$I_{r,s}(\hat{s},\hat{t},m_h^2,m_t^2) = (M^2)^{-L\epsilon}(M^2)^{2L-r+s} I_{r,s}\left(\frac{\hat{s}}{M^2},\frac{\hat{t}}{M^2},\frac{m_h^2}{M^2},\frac{m_t^2}{M^2}\right)$$

We renormalize strong coupling, a, in $\overline{\mathrm{MS}}$ scheme and top quark mass in OS scheme, each renormalized form factor can be written as:

$$\mathrm{F} = a^{\frac{3}{2}}\left[\mathrm{F}^{(1)} + a(\frac{n_g}{2}\delta Z_A + \frac{3}{2}\delta Z_a)\mathrm{F}^{(1)} + a\delta m_t^2 \mathrm{F}^{ct,(1)} + a\mathrm{F}^{(2)} + O(a^2)\right]$$

$$\mathrm{F}^{(1)} = \left(\frac{\mu_R^2}{M^2}\right)^{\epsilon}\left[b_0^{(1)} + b_1^{(1)}\epsilon + b_2^{(1)}\epsilon^2 + \mathcal{O}(\epsilon^3)\right]$$ ← 1-loop

$$\mathrm{F}^{ct,(1)} = \left(\frac{\mu_R^2}{M^2}\right)^{\epsilon}\left[c_0^{(1)} + c_1^{(1)}\epsilon + \mathcal{O}(\epsilon^2)\right]$$ ← Mass Counter-Terms

$$\mathrm{F}^{(2)} = \left(\frac{\mu_R^2}{M^2}\right)^{2\epsilon}\left[\frac{b_{-2}^{(2)}}{\epsilon^2} + \frac{b_{-1}^{(2)}}{\epsilon} + b_0^{(2)} + \mathcal{O}(\epsilon)\right]$$ ← 2-loop

Scale variations do not require any re-computation of **red** terms

pySecDec

`pySecDec`: a program to numerically evaluate dimensionally regulated parameter integrals (Rewrite of `SecDec` in `python` & `c++`)

https://github.com/mppmu/secdec/releases

Borowka, Heinrich, Jahn, SJ, Kerner, Schlenk, Zirke

Supports:

Contour deformation, Arbitrary loops/legs (within reason)

General parameter integrals (not just loop integrals)

Arbitrary number of regulators (not just ϵ)

Flexible numerators (contracted Lorentz vectors, inverse propagators)

Generates `c++` Library (can be linked to your own program)

Coming soon:

Quasi-Monte Carlo integration & CUDA GPU Support

Li, Wang, Yan, Zhao 15; Review: Dick, Kuo, Sloan 13;

QMC (Rank 1 Shifted Lattices)

Efficient algorithm for numerical integration:

$$I_s[f] \equiv \int_{[0,1]^s} \mathrm{d}^s x f(\vec{x})$$

$$f : \mathbb{R}^s \to \mathbb{C}$$

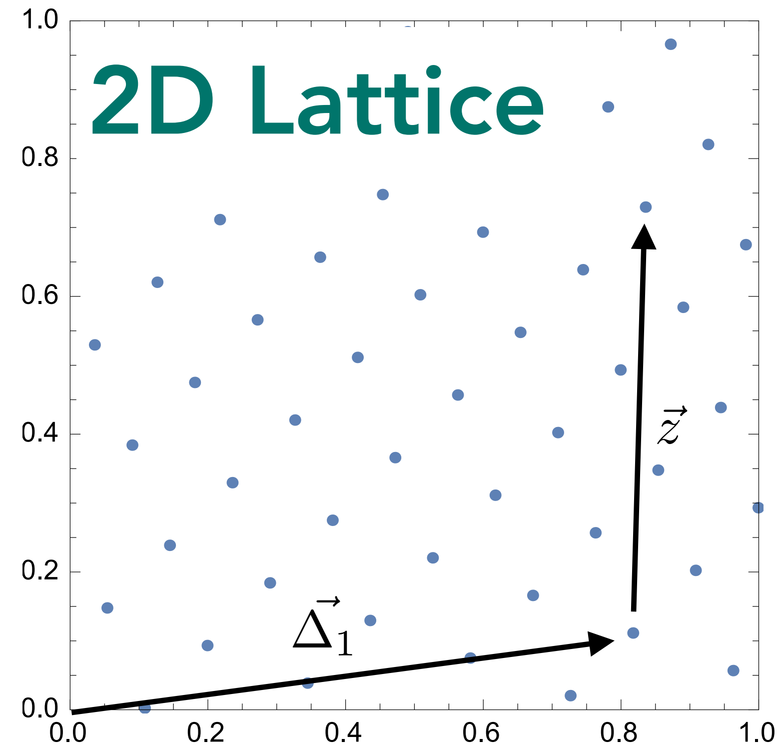
$$\boxed{I_s[f] \approx \bar{Q}_{s,n,m}[f] \equiv \frac{1}{m} \sum_{k=1}^{m} \frac{1}{n} \sum_{i=0}^{n-1} f\left(\left\{\frac{i\vec{z}}{n} + \vec{\Delta}_k\right\}\right)}$$

$\vec{\Delta}_k$ - Random shift vec.

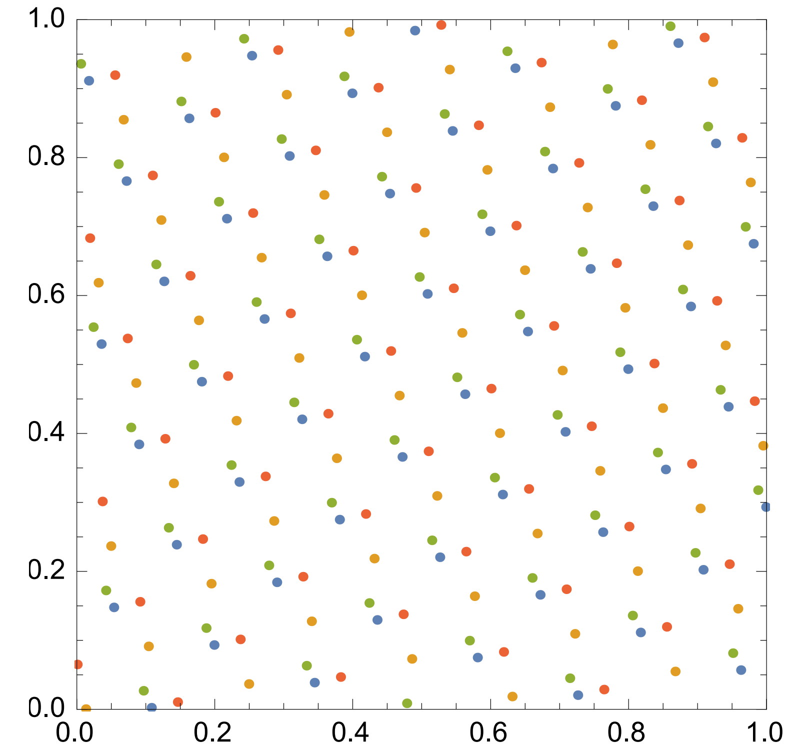
$\{\}$ - Fractional part

n - # Lattice points

m - # Random shifts



1 Shift



4 Shifts

Precomputed generating vector $\vec{z}$ chosen to minimise worst-case error, depends on number of lattice points n

Review: Dick, Kuo, Sloan 2013

Error Estimate

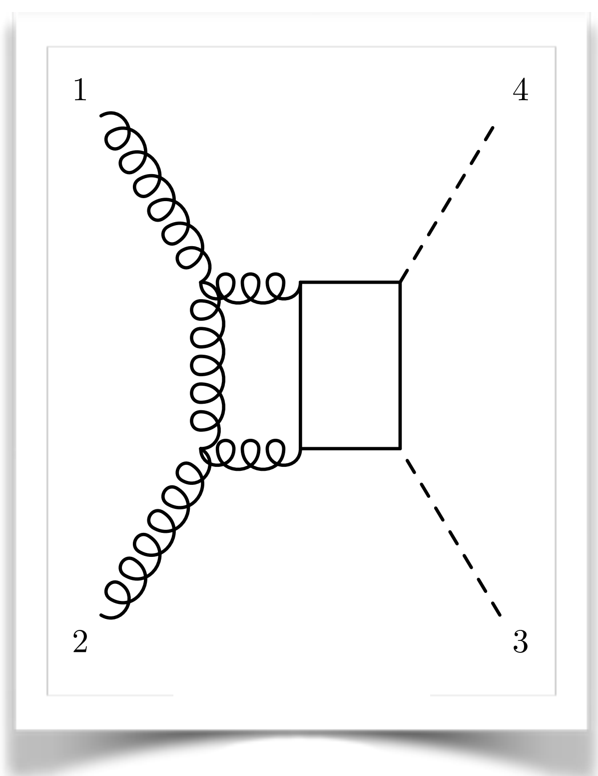
Unbiased error estimate computed from random shifts:

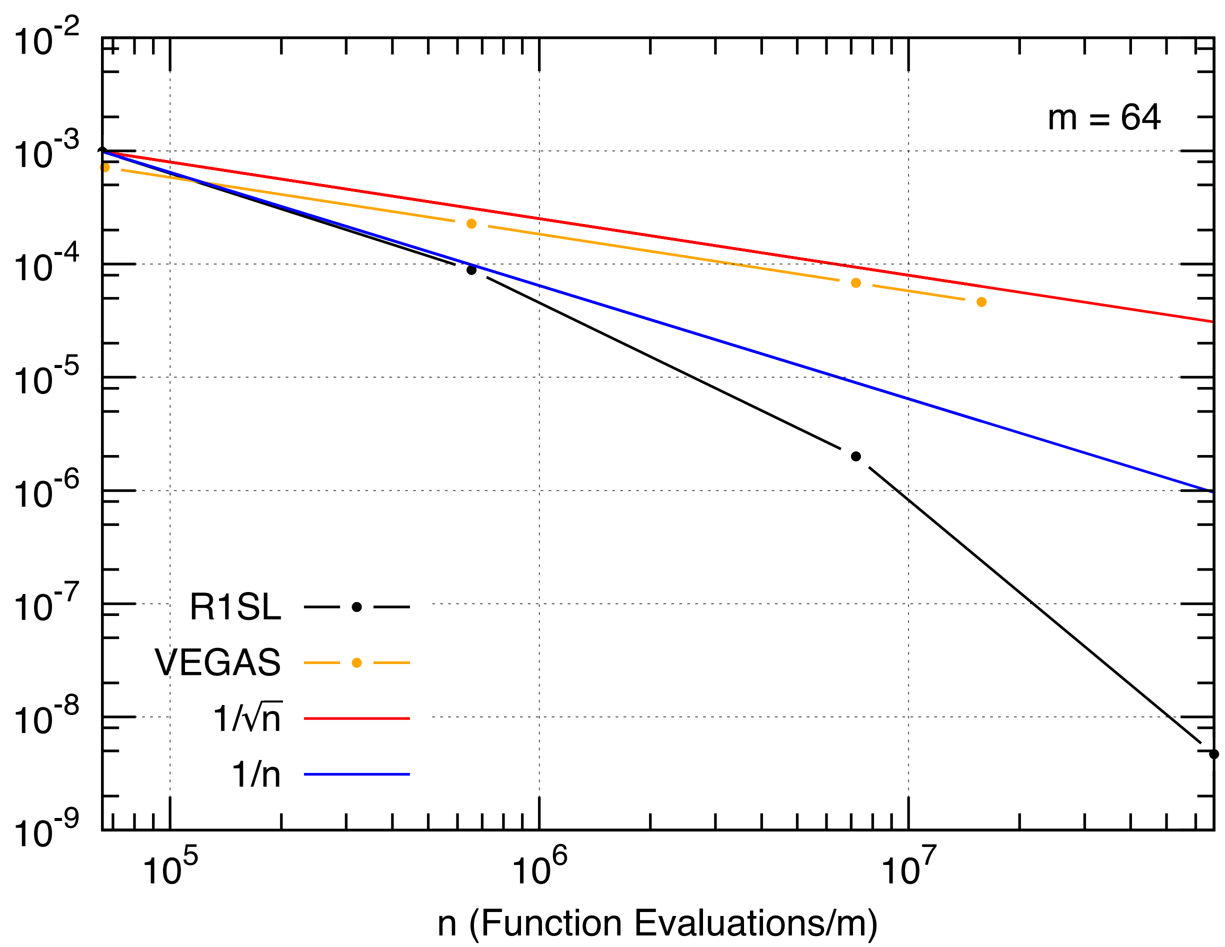
$$\mathrm{Var}[\bar{Q}_{s,n,m}[f]] \approx \frac{1}{m(m-1)} \sum_{k=1}^{m} (Q_{s,n,k} - \bar{Q}_{s,n,m})^2$$

Shift k

Application of QMC/R1SL to sector decomposed functions suggested by: Li, Wang, Yan, Zhao 15

Example: Sector Decomposed Loop Integral



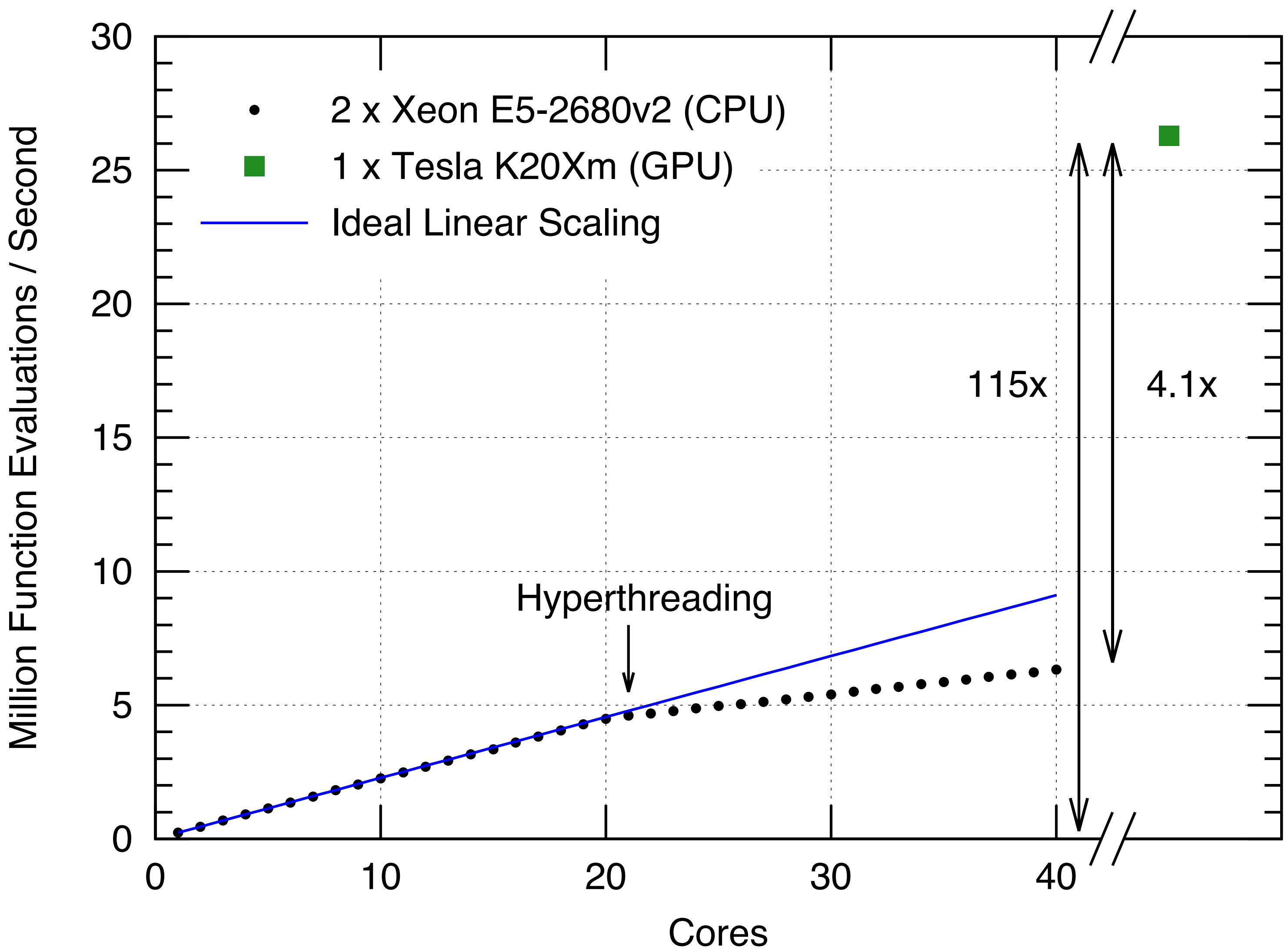


Monte Carlo Scaling

\`\`Guaranteed" Lattice Scaling

Accuracy limited by number of function evaluations

Implemented in OpenCL 1.2 for CPU & GPU



2 CPUs (20 Cores + HT)

1 GPU

n	CPU (s)	GPU(s)	C/G
655357	6.63	1.60	**4.1**
7208951	72.3	16.4	**4.4**
67264993	674.2	152.2	**4.4**

Sector Decomposition (I)

One technique **Iterated Sector Decomposition** repeat:

Binoth, Heinrich 00

$$\int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \frac{1}{(x_1+x_2)^{2+\epsilon}} \longleftarrow \textbf{Overlapping singularity for } \; x_1, x_2 \to 0$$

$$= \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \frac{1}{(x_1+x_2)^{2+\epsilon}} \left(\theta(x_1 - x_2) + \theta(x_2 - x_1)\right)$$

$$= \int_0^1 \mathrm{d}x_1 \int_0^{x_1} \mathrm{d}x_2 \frac{1}{(x_1+x_2)^{2+\epsilon}} + \int_0^1 \mathrm{d}x_2 \int_0^{x_2} \mathrm{d}x_1 \frac{1}{(x_1+x_2)^{2+\epsilon}}$$

$$= \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}t_2 \frac{x_1}{(x_1+x_1 t_2)^{2+\epsilon}} + \int_0^1 \mathrm{d}x_2 \int_0^1 \mathrm{d}t_1 \frac{x_2}{(x_2 t_1+x_2)^{2+\epsilon}}$$

$$= \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}t_2 \frac{x_1^{-1-\epsilon}}{(1+t_2)^{2+\epsilon}} + \int_0^1 \mathrm{d}x_2 \int_0^1 \mathrm{d}t_1 \frac{x_2^{-1-\epsilon}}{(t_1+1)^{2+\epsilon}} \longleftarrow \textbf{Singularities factorised}$$

If this procedure terminates depends on order of decomposition steps

An alternative strategy **Geometric Sector Decomposition** always terminates; both strategies are implemented in `pySecDec`.

Kaneko, Ueda 10; See also: Bogner, Weinzierl 08; Smirnov, Tentyukov 09

Sector Decomposition (II)

Expand in ϵ (simple case $a = -1$ "Logarithmic Divergence"):

$$\int_0^1 \mathrm{d}x^{-1-b\epsilon} g(x) = \frac{g(0)}{-b\epsilon} + \int_0^1 \mathrm{d}x x^{-b\epsilon} \left[\frac{g(x) - g(0)}{x}\right]$$

Finite

Poles

`subtraction' of $g(0)$

By Definition: $g(0) \neq 0, g(0)$ finite

Can now numerically integrate

Key Point: Sector Decomposed integrals can be expanded in ϵ and numerically integrated