TWO-LOOP AMPLITUDES WITH NUMERICAL UNITARITY

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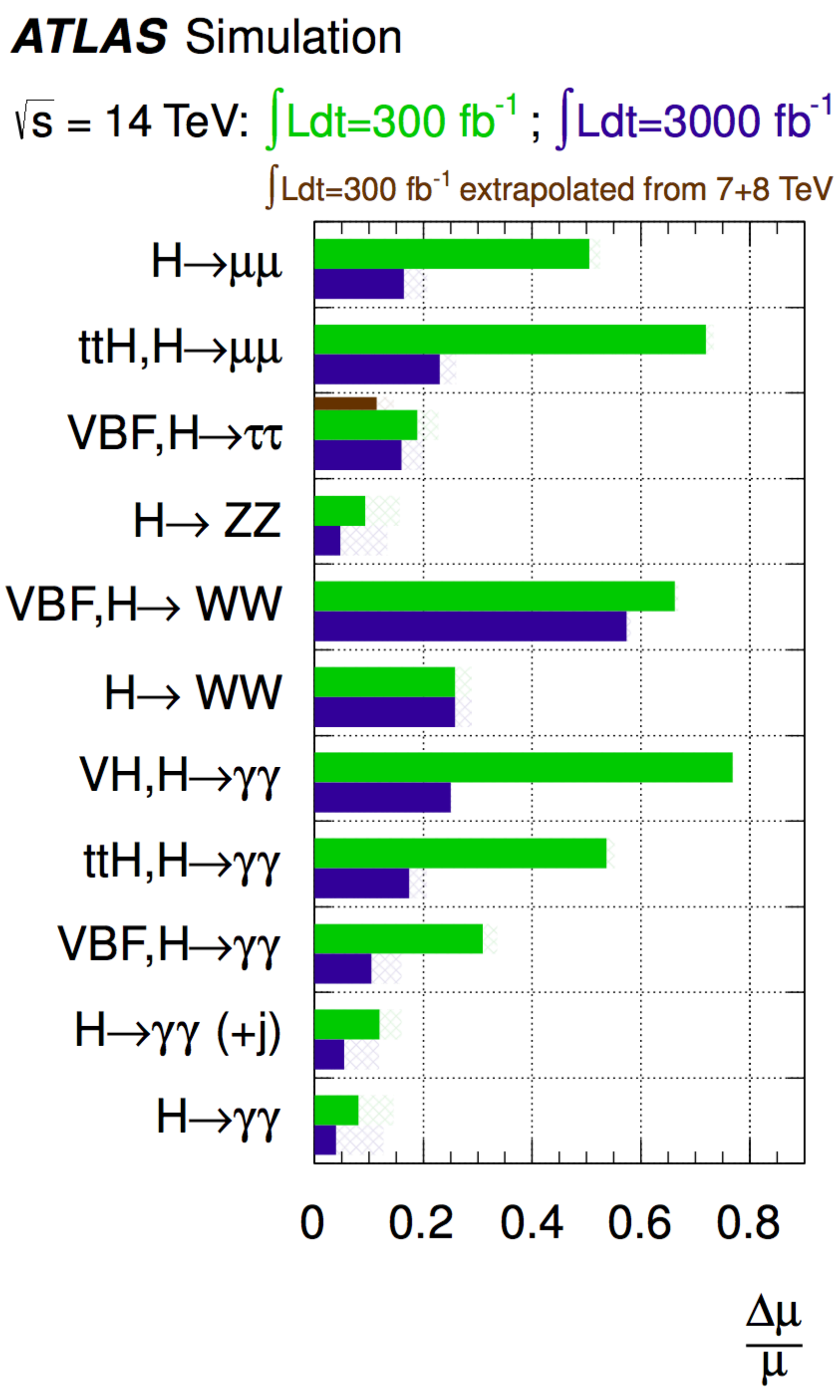
LHC MOTIVATION



- Percent-level precision for many observables: requires NNLO QCD.
- Many 2-to-2 processes known at NNLO.
- Can we probe kinematic dependence of final states? Add recoiling jet?
- Can we add mass effects?

Two key building blocks:

- Handling of IR divergences at NNLO.
- Calculation of two-loop (multi-leg/scale) amplitudes.

TWO-LOOP AMPLITUDE — STANDARD APPROACH

Feynman diagrams

- Tensor reduction [Passarino, Veltman 79]
- IBPs [Tkachov, Chetyrkin 81, Laporta 01]

Master integral decomposition

$$\mathcal{A} = \sum_{\Gamma} \sum_{i \in M_\Gamma} c_{\Gamma,i}(D) I_{\Gamma,i}(D)$$

- Diff. eq. [Kotikov 91, Gehrmann, Remiddi 01; Henn 13]
- Direct integration [..., HyperInt]
- Numeric integration [..., SecDec, Fiesta]

Integrated expression

(very simple compared to size of intermediate expressions)

Completely general approach, but:

- large intermediate expressions
- large IBP system

Bad scaling with number of legs and masses

Numerical unitarity method

1. Numerical approach: suitable for multi-scale processes
2. Reduction & coefficient evaluation done simultaneously

5-POINT INTEGRALS/AMPLITUDE COMPUTATIONS

Analytic 5-point integrals

- Planar massless: full basis [Gehrmann, Henn, lo Presti 15]
- Planar one-mass: partial results [Papadopoulos, Tommasini, Wever 15]
- Non-planar massless: partial results [Chicherin, Henn, Mitev, Sokatchev 17, 18]

Recent progress in 5-point QCD amplitudes

- Benchmark evaluation, pure YM, all helicities [Badger, Brønnum-Hansen, Bayu Hartanto, Peraro 17]
- Numerical reduction to masters, pure YM, all helicities [Abreu, Ita, Febres Cordero, Page, Zeng 17]
- IBP tables [Boels, Jin, Luo 18] [Chawdhry, Lim, Mitov 18]

TWO-LOOP NUMERICAL UNITARITY

TWO-LOOP UNITARITY AT 2-LOOPS

[S. Abreu, F. Febres Cordero, H. Ita, M. Jaquier, B. Page, M. Zeng, 17]

$$\mathcal{A}(\ell_l) = \sum_{\Gamma} \sum_{i=1}^{\dim(\Gamma)} c_{\Gamma,i} \frac{\tilde{m}_{\Gamma,i}(\ell_l)}{\prod_{j\in P_\Gamma} \rho_j}$$

Choosing good variables:
Polynomial and **unitarity compatible** parametrisation of the integrand

Change of integrand basis:
Construct **surface terms** and **master integrands**

$$\mathcal{A}(\ell_l) = \sum_{\Gamma} \sum_{i\in M_\Gamma \cup S_\Gamma} c_{\Gamma,i}(D) \frac{m_{\Gamma,i}(\ell_l, D)}{\prod_{j\in P_\Gamma} \rho_j}$$

Obtain integrated amplitude:
Determine **coefficients with (generalised) unitarity** and insert expression for **master integrals**

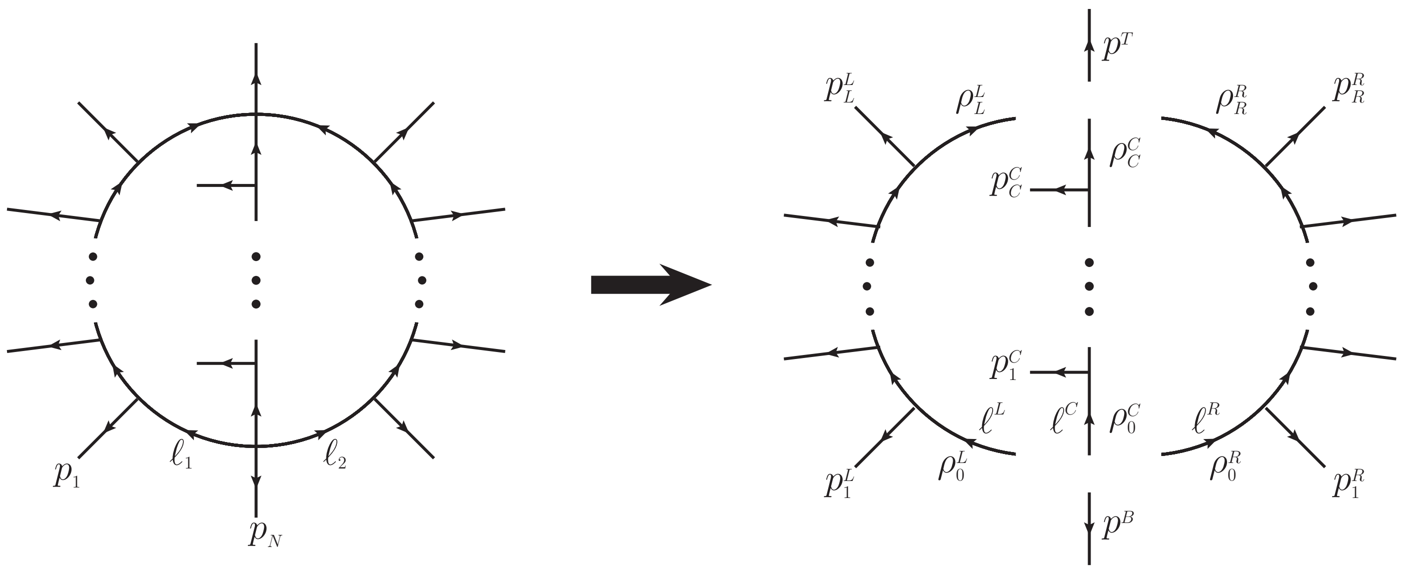
$$\mathcal{A} = \sum_{\Gamma} \sum_{i\in M_\Gamma} c_{\Gamma,i}(D) I_{\Gamma,i}(D)$$

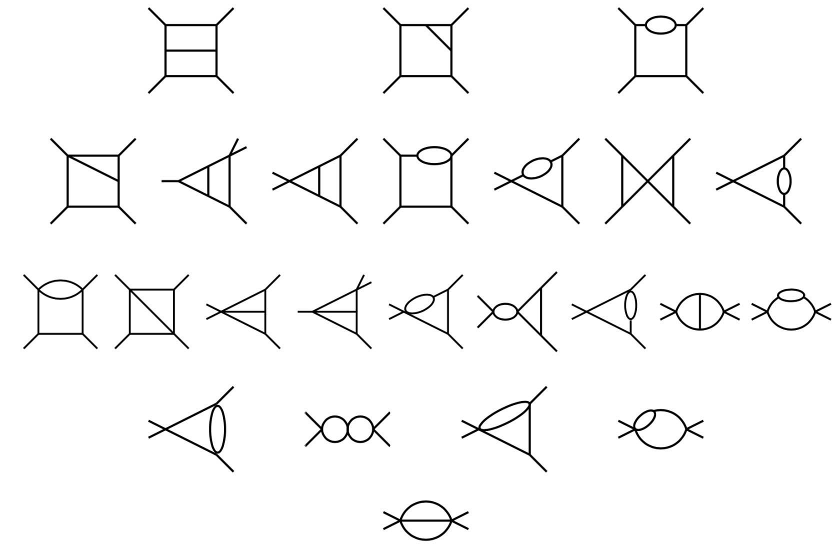
Generalisation of one-loop: Ossola, Papadopoulos, Pittau 07; Ellis, Giele Kunszt 07; Giele Kunszt, Melnikov 08; Berger, Bern, Dixon, Febres Cordero, Ita, Kosower, Maitre 08

Related two-loop approaches: Badger, Frellesvig, Zhang 12, 13; Mastrolia, Mirabella, Ossola, Peraro 12; Mastrolia, Peraro, Primo 16; Badger, Brønnum-Hansen, Hartanto, Peraro 17; Boels, Jin, Luo 18; Chawdhry, Lim, Mitov 18

UNITARITY COMPATIBLE VARIABLES

[H. Ita 15; S. Abreu, F. Febres Cordero, H. Ita, M. Jaquier, B. Page, M. Zeng 17]





- Rung by rung approach to determine suitable variables:

$$\ell_L = \sum_{j\in D^L_\Gamma} v^j_L r^j_L + \sum_{j\in \bar{D}^L_\Gamma} v^j_L \lambda^j_L + \sum_{j\in \bar{D}^4_\Gamma} n^j_\Gamma \alpha^j_L + \sum_{j\in D^\epsilon} n^j_\Gamma \mu^j_L$$

$$p_i \cdot v^j_L = \delta_{ij}$$
$$p_i \cdot n^j_\Gamma = 0$$
$$n^i_\Gamma \cdot n^j_\Gamma = \delta_{ij}$$

Propagators | Irreducible scalar products (ISPs) | Transverse variables | Transverse variables beyond 4d

- Remove redundancy:
  - in each rung: $\rho_{L,0} + m^2_{L,0} = \ell^2_L = c_L(r_L, \lambda_L) + \vec{\mu}_L \cdot \vec{\mu}_L$
  - overall: use momentum conservation at bottom vertex $\alpha^j_C = -\alpha^j_1 - \alpha^j_2$

INTEGRAND PARAMETRISATION & SURFACE TERMS

[H. Ita 15; S. Abreu, F. Febres Cordero, H. Ita, M. Jaquier, B. Page, M. Zeng 17]

Make decomposition in terms of master integrals explicit at the integrand level:

$$\mathcal{A} = \sum_{\Gamma} \sum_{i \in M_\Gamma} c_{\Gamma,i}(D) I_{\Gamma,i}(D) \quad \longrightarrow \quad \mathcal{A}(\ell_l) = \sum_{\Gamma} \sum_{i \in M_\Gamma \cup S_\Gamma} c_{\Gamma,i}(D) \frac{m_{\Gamma,i}(\ell_l, D)}{\prod_{j \in P_\Gamma} \rho_j}$$

Integrand numerator is polynomial in components of loop momentum:

$$m_{\Gamma,i}(\ell_l) \longrightarrow m_{\Gamma,i}(\rho, \lambda, \alpha) \longrightarrow m_{\Gamma,i}(\lambda, \alpha)$$

Monomials that depend on transverse variables: [Passarino-Veltman reduction]

- Monomials with odd powers are surface terms
- Construct surface terms for even powers

$$\alpha_k^i \alpha_k^j \to \left(\alpha_k^i \alpha_k^j - \frac{\mu_{ij}}{D-4} \right)$$

Polynomial in propagators and ISPs

SURFACE TERMS & IBPS

[H. Ita 15; S. Abreu, F. Febres Cordero, H. Ita, M. Jaquier, B. Page, M. Zeng 17]

Surface terms from specific IBP relations: control propagator power

$$\int d^D\ell_l \sum_k \frac{\partial}{\partial \ell_k^\nu}\left[\frac{u_k^\nu}{\prod_{j\in P_\Gamma}\rho_j}\right] = 0 \qquad u_k^\nu \frac{\partial}{\partial \ell_k^\nu}\rho_j = f_j\rho_j$$

[Gluza, Kajda, Kosower 10; Schabinger 11]

Construct IBP-generating vectors as polynomial solution to,

$$\left(u_{ka}^{\text{loop}}\ell_a^\nu + u_{kb}^{\text{ext}}p_b^\nu\right)\frac{\partial}{\partial \ell_k^\nu}\begin{pmatrix}\rho_1\\ \rho_2\\ \vdots\\ \rho_{|\Gamma|}\end{pmatrix} - \begin{pmatrix}f_1\rho_1\\ f_2\rho_2\\ \vdots\\ f_{|\Gamma|}\rho_{|\Gamma|}\end{pmatrix} = \begin{pmatrix}0\\ 0\\ \vdots\\ 0\end{pmatrix}$$

[generating set from Singular]

[related work J. Boehm, A. Georgoudis, K. J. Larsen, M. Schulze, Y. Zhang 16 - 18]

Fill remaining integrand function space with master integrands

$$\mathcal{A}(\ell_l) = \sum_\Gamma \sum_{i\in M_\Gamma\cup S_\Gamma} c_{\Gamma,i}(D)\frac{m_{\Gamma,i}(\ell_l, D)}{\prod_{j\in P_\Gamma}\rho_j}$$

DETERMINING THE COEFFICIENTS WITH UNITARITY

$$\mathcal{A}(\ell_l) = \sum_{\Gamma} \sum_{i \in M_\Gamma \cup S_\Gamma} c_{\Gamma,i}(D) \frac{m_{\Gamma,i}(\ell_l, D)}{\prod_{j \in P_\Gamma} \rho_j}$$

Determine the coefficients from on-shell information [Bern, Dixon, Kosower, Dunbar 94,95]

$$\lim_{\ell_l \to \ell_l^\Gamma} \mathcal{A}(\ell_l) = \frac{\prod \mathcal{A}^{\text{tree}}(\ell_l^\Gamma)}{\prod_{j \in P_\Gamma} \rho_j(\ell_l^\Gamma)} + \mathcal{O}(\rho_j)$$

ℓ^Γ: loop momenta evaluated at $\rho_j = 0\,, \qquad \forall j \in P_\Gamma$

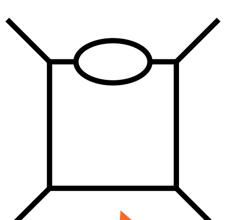
Requires fast evaluation of tree amplitudes: Berends-Giele recursion

Construct and solve linear system for the coefficients

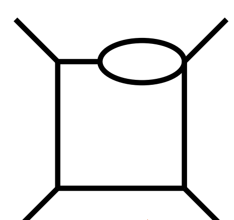
Final results: decomposition in terms of master integrals:

$$\mathcal{A} = \sum_{\Gamma} \sum_{i \in M_\Gamma} c_{\Gamma,i}(D) I_{\Gamma,i}(D)$$

Starting at two-loops, need to worry about subleading poles



Leading pole: standard procedure



Subleading pole: need to be more careful

Construct both numerators together

Same on-shell phase space

[S. Abreu, F. Febres Cordero, H. Ita, M. Jaquier, B. Page 17]

TWO-LOOP NUMERICAL UNITARITY AT 2-LOOPS

$$\mathcal{A}(\ell_l) = \sum_{\Gamma} \sum_{i=1}^{\dim(\Gamma)} c_{\Gamma,i} \frac{m_{\Gamma,i}(\ell_l)}{\prod_{j\in P_\Gamma} \rho_j}$$

Parametrisation of integrand

$$\mathcal{A}(\ell_l) = \sum_{\Gamma} \sum_{i\in M_\Gamma \cup S_\Gamma} c_{\Gamma,i}(D) \frac{m_{\Gamma,i}(\ell_l, D)}{\prod_{j\in P_\Gamma} \rho_j}$$

Master/surface term decomposition

$$\mathcal{A} = \sum_{\Gamma} \sum_{i\in M_\Gamma} c_{\Gamma,i}(D) I_{\Gamma,i}(D)$$

Evaluation of coefficients

Analytic expressions

Numerics

Floating point arithmetics:

- good behaviour with increase in number of scales
- can compute at arbitrary (even non-rational) PS point
- need to worry about loss of precision

Used in 4-gluon 2-loop calculation [S. Abreu, F. Febres Cordero, H. Ita, M. Jaquier, B. Page, M. Zeng, 17]

Finite-field arithmetics:

[Schabinger, von Manteuffel 14], [Peraro 16]

- exact calculation, no loss of precision
- opens the door to possible analytic reconstruction
- only rational PS points
- \`complicated' PS points take longer (more FF needed)
- update algorithm to avoid non-rational numbers at intermediate steps

Used in 5-gluon 2-loop calculation [S. Abreu, F. Febres Cordero, H. Ita, B. Page, M. Zeng, 17]

FINITE FIELDS AND NUMERICAL UNITARITY

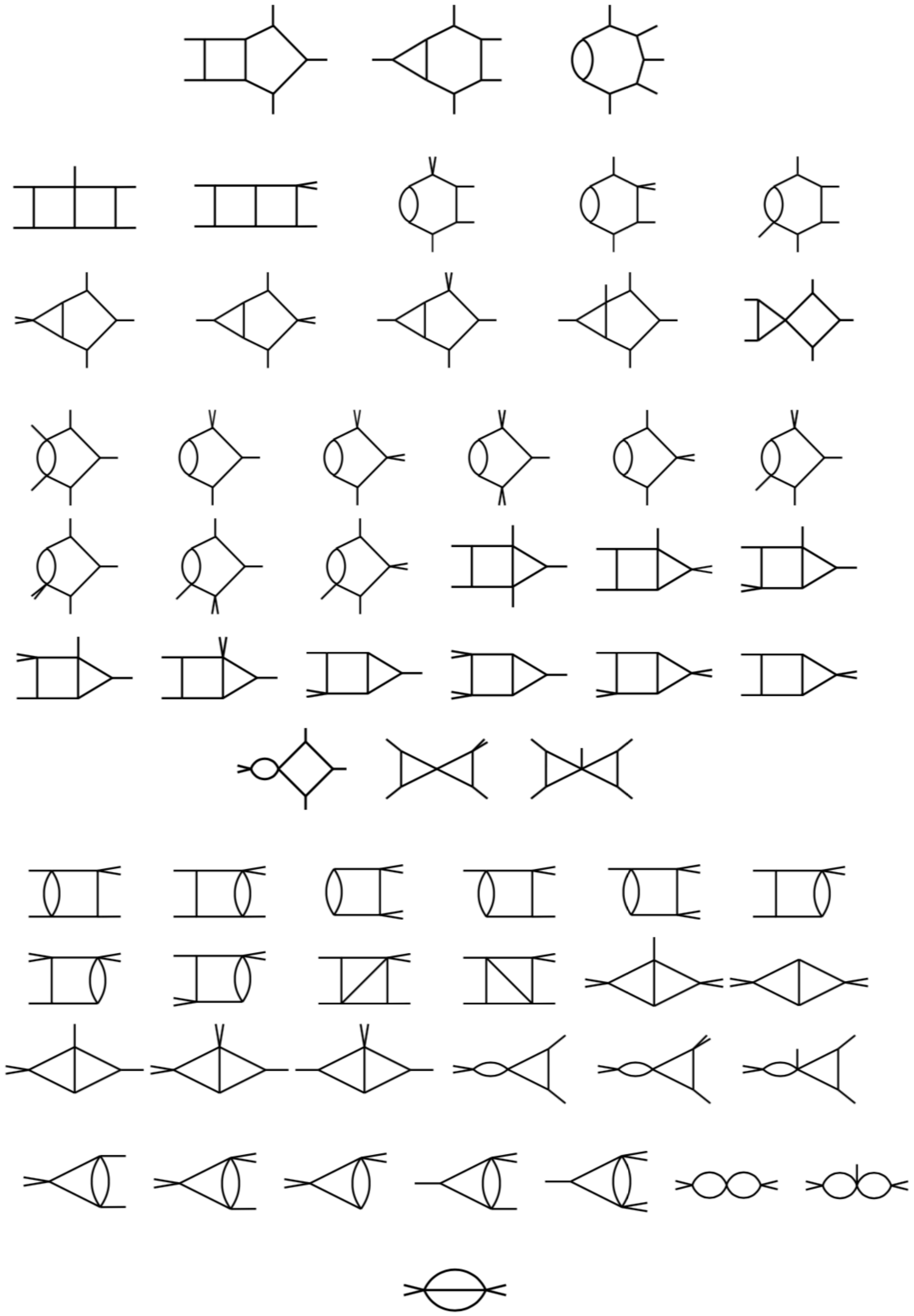
[Peraro 16] [S. Abreu, F. Febres Cordero, H. Ita, B. Page, M. Zeng 17]

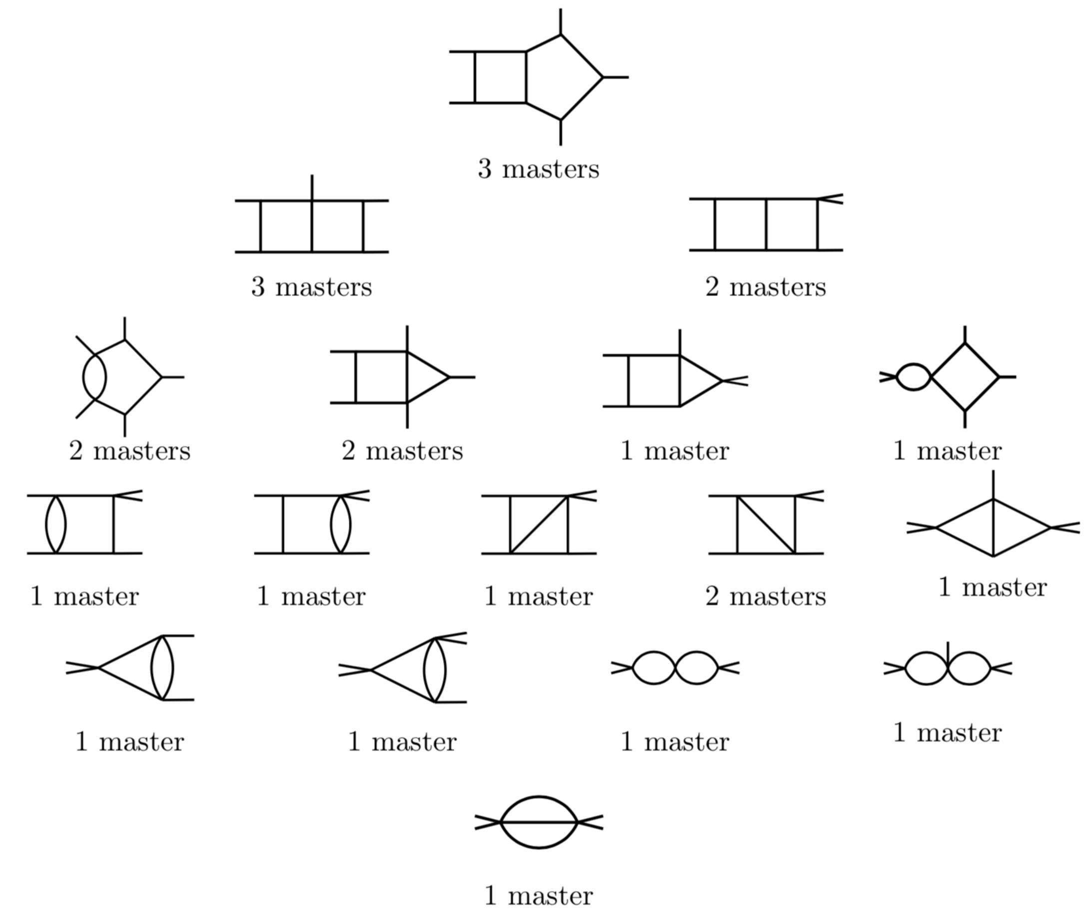
- Work in a finite field: $\mathbb{F}_p = \{0, \ldots, p-1\}$
  - Integers modulo *p*
  - Operations modulo *p*, define inverse through Euclidean algorithm
  - Algorithms exist for mapping from $\mathbb{F}_p$ to $\mathbb{Q}$. Might require several finite fields.
- Requires:
  - Rational external kinematics (e.g. use mom. twistors for 5 points)
  - Avoid complex numbers (use +-+- metric)
  - Rational on-shell momenta (use orthogonal vectors)
  - Avoid constructing polarisation states in BG recursion

5-GLUON PLANAR AMPLITUDE

PROPAGATOR STRUCTURES AND MASTER INTEGRANDS

[S. Abreu, F. Febres Cordero, H. Ita, B. Page, M. Zeng 17]





- Much more complicated calculation than four-gluon amplitude
- Avoid loss of precision: finite field calculation gives exact numeric results for master integral coefficients

INGREDIENTS FOR AMPLITUDE

[S. Abreu, F. Febres Cordero, H. Ita, B. Page, M. Zeng 17]

- Surface terms/master integrand decomposition:
  - obtained using `slimgb' algorithm of Singular, expressions cleaned with FORM.
  - example: penta-box topology takes < 1 sec.
  - validated with FIRE on numerical point.
  - implemented in a C++ code, about 3.5MB of data.
- Rational coefficients – 155 coefficients to determine:
  - Reconstructed from finite-field calculation (use Givaro [Gauthier, Roch, Villard])
  - 2.5 core minutes per finite field
  - Full D_s dependence (quadratic, evaluate at three D_s values) [Giele, Kunszt, Melnikov 08]
- Master integrals:
  - public results from [Papadopoulos, Tommasini, Wever 15]
  - validated with FIESTA on numerical point

RESULTS — FOUR INDEPENDENT HELICITIES

[S. Abreu, F. Febres Cordero, H. Ita, B. Page, M. Zeng 17]

- All plus and single minus

$\mathcal{A}^{(2)}/\left(\mathcal{A}^{(1)}(\epsilon=0)\right)$	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
$(1^+,2^+,3^+,4^+,5^+)$	-5.000000000	-3.8931790255	5.9810885816
$(1^-,2^+,3^+,4^+,5^+)$	-5.000000000	-16.322002103	-10.383813287

$$s_{12}=-1,\quad s_{23}=-8,\quad s_{34}=-10$$
$$s_{45}=-7,\quad s_{51}=-3$$

- MHV amplitudes

$\mathcal{A}^{(2)}/\mathcal{A}^{(0)}$	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
$(1^-,2^-,3^+,4^+,5^+)$	12.5000000	25.46246919	-1152.843107	-4072.938337	-3637.249567
$(1^-,2^+,3^-,4^+,5^+)$	12.5000000	25.46246919	-6.121629624	-90.22184215	-115.7836685

Exact numeric results: precision depends on number of digits of GiNaC output when evaluating master integrals

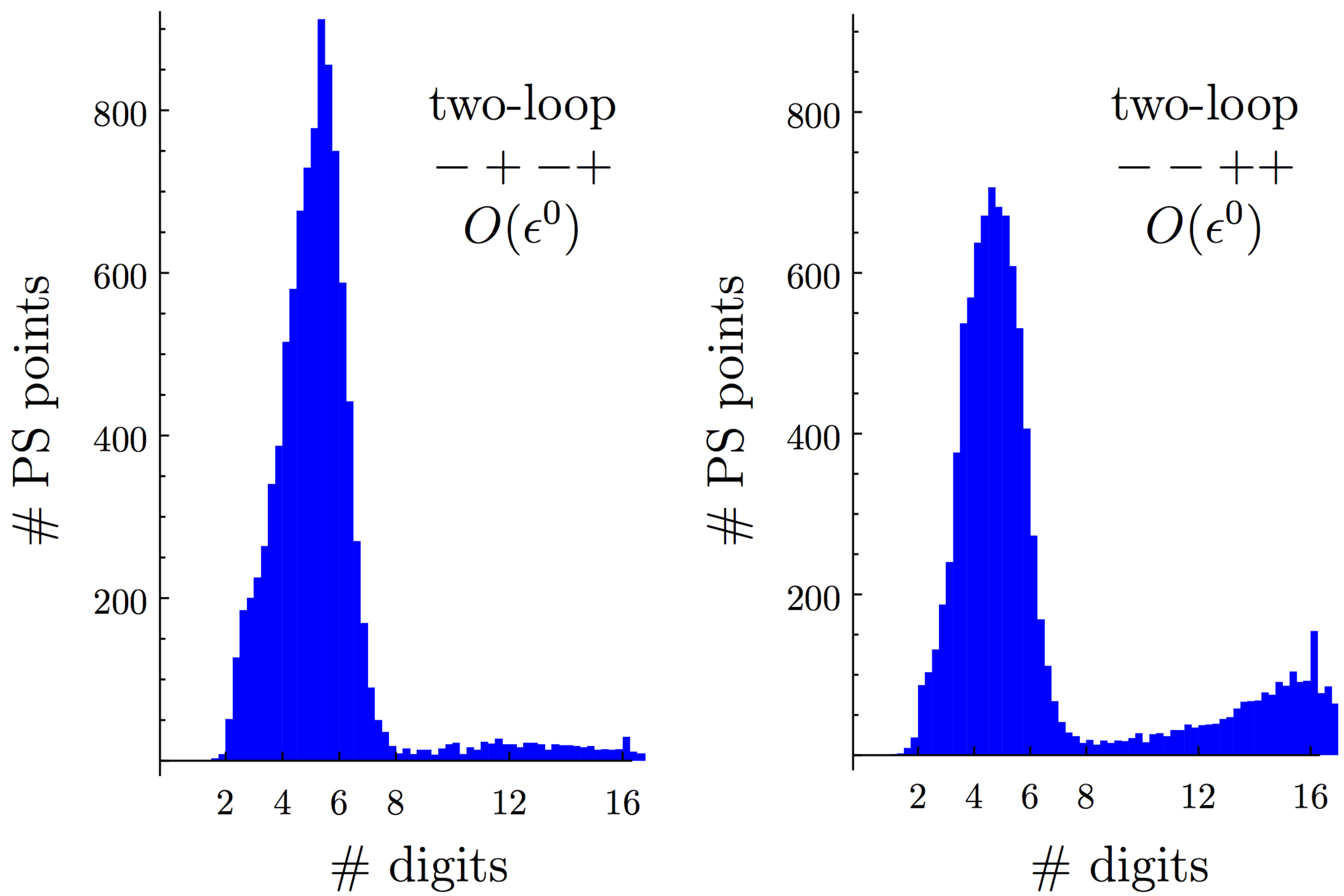
Checks: reproduce universal pole structure [Catani 98], 5-gluon all-plus [Badger, Frellesvig, Zhang 13; Gehrmann, Henn, Lo Presti 15; Dunbar, Perkins, Jehu 16], and recent 5-gluon QCD [Badger, Brønnum-Hansen, Hartanto, Peraro 17]

SUMMARY AND CONCLUSIONS

- Numerical unitarity approach is ready for two-loop amplitudes
- Provides a clean path towards final result, with compact intermediate steps
- We can fully reduce the 5-gluon QCD amplitudes
- Numerical approach
  - computer cluster compatible setup
- Complemented with finite fields, no issues with precision loss
  - opens the door to analytic reconstruction of result
- We look forward to computing new multi-scale processes

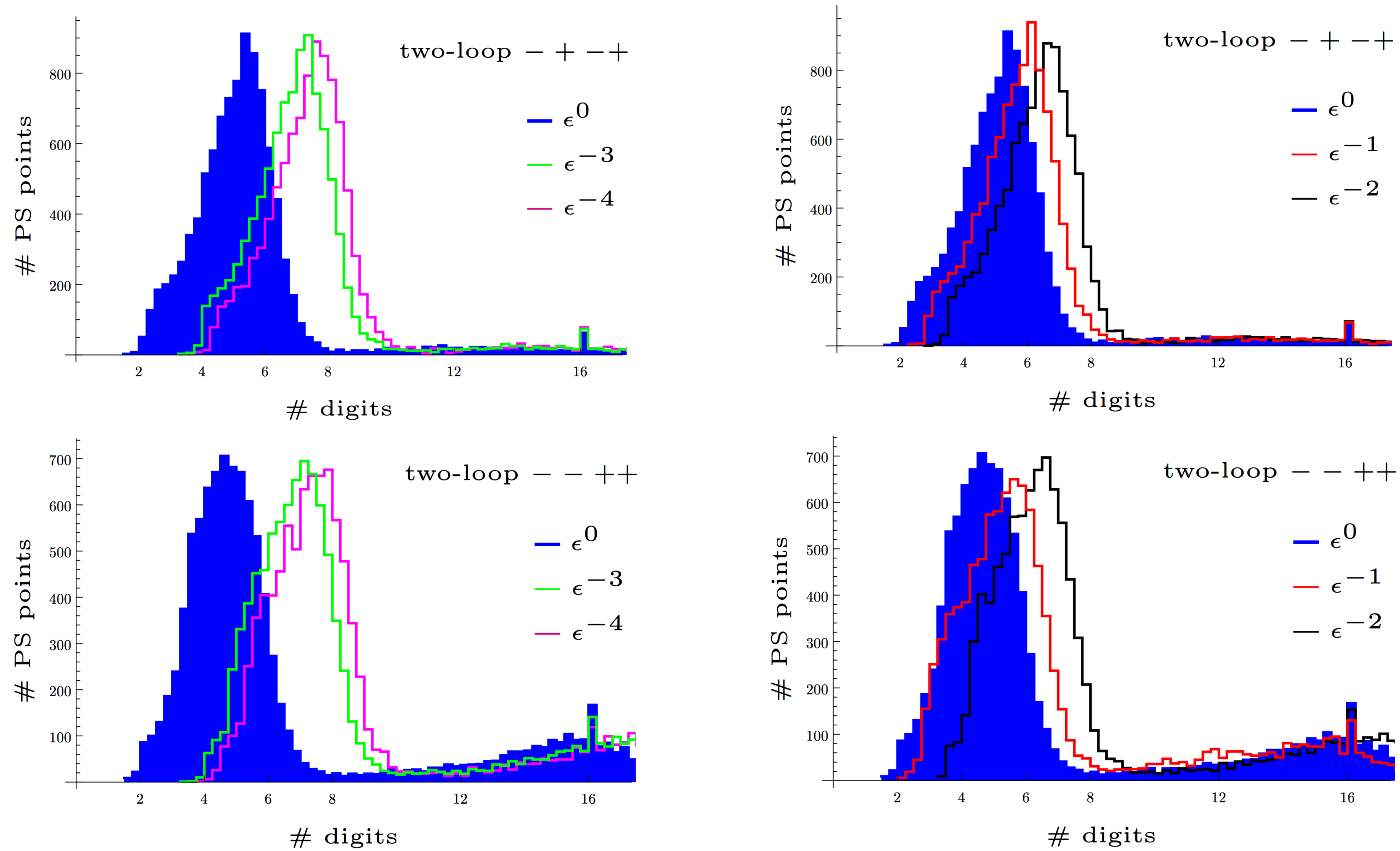
THANK YOU!

FLOATING POINT — NUMERICAL STABILITY



- Even distribution of 10000 phase-space points
- Number of digits compared to analytic result
- Coefficients have ~8 digit precision, cancelation when combining with master integrals
- Maximal cuts evaluated with double-double precision

FLOATING POINT — NUMERICAL STABILITY



- Precision of poles correlated with that of finite pieces
- Use single pole with threshold of 2 digits for real-time recovery system
- Rescued points (22% --++; 7% -+-+) re-evaluated with 32-digit precision