Systematic approximation of multi-scale Feynman integrals

arXiv:1804.06824

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Motivation

1. Feynman integrals are often the bottleneck for multi-scale multi-loop calculations, especially with massive propagators!
2. Analytic evaluation of very complicated Feynman integrals not generally understood, although progress is being made!
3. Numerical evaluation often the only option.
4. Producing accurate numerical results over full phase space in an automated way is difficult - divergent nature of loop integrals.
5. An algorithm to analytically approximate Feynman integrals - TayInt!

Philosophy

Three aims: to produce an algebraic integral library with full phase space validity for any kind of integral

1. The idea - the integrand has to be Taylor expanded in the Feynman parameters - otherwise you can't integrate it - the kinematics are not to be touched.
2. The algorithm brings an integral into a form optimised for an accurate Taylor expansion with validity in all kinematic regions.
3. Divide and rule - take all the nastiness in the Feynman integral - distribute it so that it does not hinder a TAYLOR EXPANSION IN THE FEYNMAN PARAMETERS.

Setup I

1. A generic Feynman loop integral G in an arbitrary number of dimensions D at loop level L with N propagators, wherein the propagators P_j with mass m_j can be raised to arbitrary powers ν_j,

$$G^{\mu_1 \dots \mu_R}_{\alpha_1 \dots \alpha_R}(\{p\},\{m\}) = \left(\prod_{\alpha=1}^{L} \int \mathrm{d}^D \kappa_\alpha \right) \frac{k^{\mu_1}_{\alpha_1} \cdots k^{\mu_R}_{\alpha_R}}{\prod_{j=1}^{N} P_j^{\nu_j}(\{k\},\{p\},m_j^2)}$$

$$\mathrm{d}^D \kappa_\alpha = \frac{\mu^{4-D}}{i\pi^{\frac{D}{2}}} \mathrm{d}^D k_\alpha \ , \ P_j(\{k\},\{p\},m_j^2) = q_j^2 - m_j^2 + i\delta \ ,$$

2. The q_j are linear combinations of external momenta p_i and loop momenta k_α.

3. Henceforth scalar integrals considered.

Setup II

1. Rewrite scalar integrals in terms of Feynman parameters t_j, $j = 1 \ldots N$. Integrate the loop momenta to give,

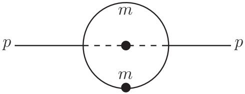
$$G = \frac{(-1)^{N_\nu}}{\prod_{j=1}^{N} \Gamma(\nu_j)} \prod_{j=1}^{N} \int_0^\infty dt_j \; t_j^{\nu_j - 1} \, \delta(1 - \sum_{l=1}^{N} t_l) \, \frac{\mathcal{U}^{N_\nu - (L+1)D/2}}{\mathcal{F}^{N_\nu - LD/2}} \; ,$$

2. $\mathcal{U}$ and $\mathcal{F}$ are the first and second Symanzik polynomials.

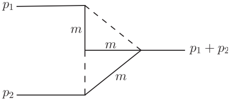
Summary of the method

U1: reduce the Feynman Integral to a quasi-finite basis		
U2: perform an iterated sector decomposition		
below threshold		above threshold
BT1: $t_j \to y_j$	OT1:	$t_j \to \theta_j$, generate $\mathcal{K}$
BT2: Taylor expand and integrate	OT2:	find $\Theta_{o(0),\ldots o(J-1)}$
	OT3:	perform one fold integrations
	OT4:	find θ_j^* and the optimum $\Theta_{o(0),\ldots o(J-2)}$
	OT5:	determine $\mathcal{P}_j$
	OT6:	Taylor expand and integrate

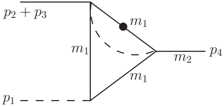
Diagrams



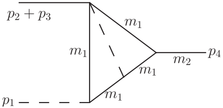
(a) S14^{01220}



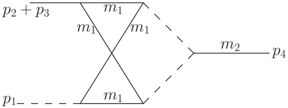
(b) T41



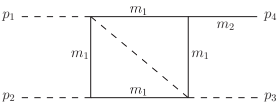
(c) I10



(d) I21



(e) I246



(f) I39

1. The finite sunrise S14_{01220} and the triangle T41 are used to illustrate TayInt. Integrals I10, I21, I246 and I39 enter the Higgs+jet calculation - are computed with TayInt.

Method - U1

Universal step 1 (U1) - Feynman integral G expressed as a superposition of finite Feynman integrals G^{F} multiplying poles in ϵ

1. Quasi-finite basis: (von Manteuffel, Panzer, Schabinger, arXiv:1411.7392),(Panzer, arXiv:1401.4361) these integrals contain no divergences from integration of Feynman parameters - all divergent parts restricted to prefactors.

2. The G^{F} have a shifted number of dimensions or dotted propagators or both.

3. Automated shell script used to direct all required Reduze (von Manteuffel, Panzer, Schabinger, arXiv:1411.7392),(von Manteuffel, Studerus, arXiv:1201.4330) jobs towards generating the quasi-finite basis.

Illustration of Method - U1

1. The divergent sunrise $S14^{01110}$ in terms of the finite integrals $S14^{01220}$, $S14^{01320}$ and the tadpole $S6^{30300}$,

$$S14^{01110} = \frac{8m^2\,(p^2-4\,m^2)(p^2+2\,m^2)}{(-3+D)(-8+3D)(-10+3D)}\cdot S14^{01320}$$
$$+\frac{((4-D)p^4+(-5+D)8\,m^4+(18-5D)4\,p^2m^2)}{(-3+D)(-8+3D)(-10+3D)}\cdot S14^{01220}$$
$$-\frac{16m^4\,((-4+D)\,p^2+2\,(-24+7D)\,m^2)}{(-3+D)(-4+D)^2(-8+3D)(-10+3D)}\cdot S5^{30300}\ ,$$

2. Poles in ϵ: $(-4+D)^{-1}$ terms.

Method - U2

Universal step 2 (U2) - decompose integrals G^{F} to iterated sectors using version 3 of SecDec (Borowka, Heinrich et al., arXiv:1703.09692)

1. Iterated sectors:

$$G_l^{\mathrm{F}} = \prod_{j=2}^{N} \int_0^1 \mathrm{d}t_j \; t_j^{A_{l_j} - B_{l_j}\epsilon} \; \frac{\mathcal{U}_l^{N_\nu - (L+1)D/2}(\vec{t}_j)}{\mathcal{F}_l^{N_\nu - LD/2}},$$

   where $l = 1, \ldots, r$, and r is the number of iterated sector integrals. A_k and B_k are numbers independent of the regulator ϵ.

2. Remap so that j runs from 0 to $J - 1$.

3. Iterated sector integrals G_l^{F} - building blocks of TayInt.

Illustration of the method - U2

1. $\mathcal{O}(\epsilon^0)$ coefficient of an I10 iterated sector,

$$\mathrm{I10}_1 = \prod_{j=0}^{2} \int_0^1 dt_j \frac{1}{(1 + t_0 + t_1 + t_2 + t_1 t_2)}$$
$$\cdot \left[t_0(-u - m_2 t_1) + m_1^2(1 + t_0^2 + t_2 + t_1^2(1 + t_2) + t_1(2 + 2t_2) + t_0(2 + t_2 + t_1(2 + t_2)))\right]^{-1},$$

2. Three Feynman parameters after iterated decomposition, three kinematic scales, m_1, m_2 and u.

Method - BT1

Below Threshold step 1 (BT1) - maximise distance to nearest point of non-analyticity

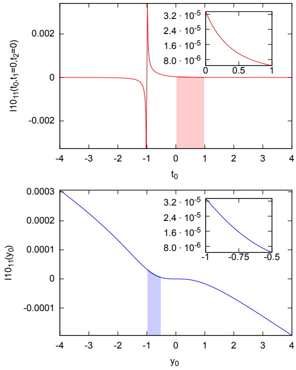
1. The iterated sectors G_l^F - still have non-analytic points outside the integration region.

2. To move these as far away as possible, import G_l^F into Mathematica (Wolfram), apply conformal mappings,

$$t_j = \frac{ay_j + b}{cy_j + d}.$$

3. Thus far: optimum mapping found.

Illustration of the method - BT1



1. Plot of a one dimensional integrand,

$$I10_1(t_0, t_1 = 0, t_2 = 0) = 1/((1 + t_0)(-m_2 t_0 + m_1^2(1 + 2t_0 + t_0^2))),$$

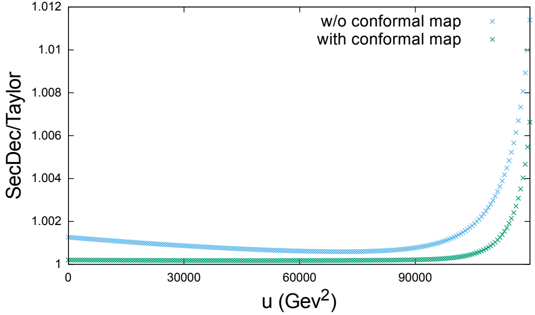
before and after a conformal mapping $y_0 = \frac{-1-t_0}{t_0}$.

Method - BT2 and BT3

Below Threshold steps 2 and 3 (BT2-3) - Taylor expand and integrate in the Feynman parameters

1. BT2 - Taylor expand the integrand in the re-mapped Feynman parameters y_j.
2. BT3 - integrate over the y_j.
3. Done in FORM (Kuipers, Ueda, Vermaseren, arXiv:1310.7007).

Illustration of the method - BT2 and BT3



1. Ratio of SecDec and TayInt calculation of the ϵ^0 coefficient of I10.

Going over threshold (OT1-OT6)

1. Above lowest threshold of an integral - discontinuities on the real axis. A Taylor expansion won't converge!
2. TayInt returns to the result of U2, the iterated sector integrands $G_l^{\mathrm{F}}(t_j)$. The Feynman $+i\delta$ prescription is implemented in Mathematica.
3. TayInt determines the contour configuration in the complex plane to avoid the discontinuities.
4. Over threshold part of TayInt is fully automated in Mathematica.

Method - OT1

Over Threshold step 1 (OT1) - generate all possible contour configurations for each iterated sector integrand

1. The first Over Threshold step (OT1) - transform the Feynman parameters of the $J-1$ iterated sectors, $t_j \to \frac{1}{2} + \frac{1}{2}\exp(i\theta_j)$.

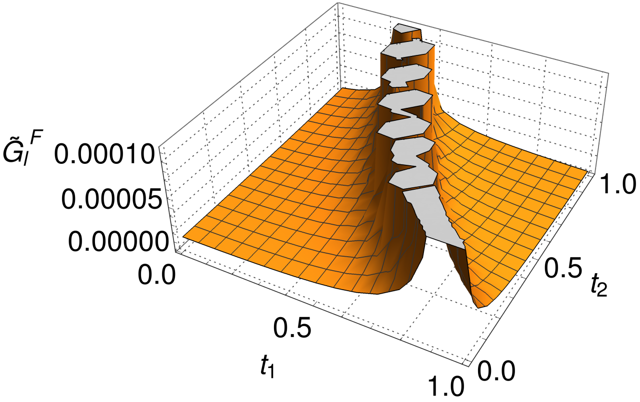
2. Generate representative sample of the kinematic region.

3. A nested list of values $\mathcal{K} = \{\{s_1, \ldots, s_\beta\}_1, \ldots, \{s_1, \ldots, s_\beta\}_\gamma\} = \{\mathcal{K}_1, \ldots, \mathcal{K}_\gamma\}$ for a β scale integral, sample size of γ points.

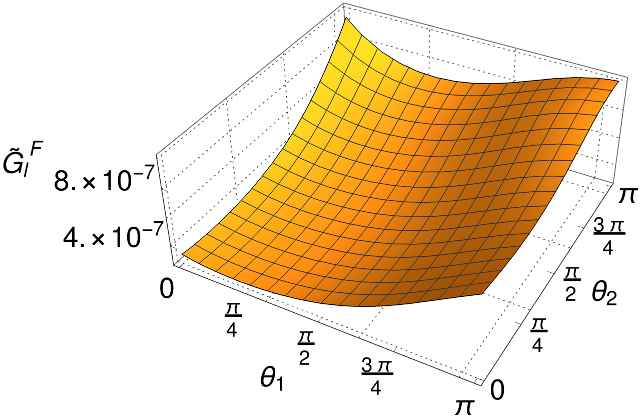
Method - OT2

Over Threshold step 2 (OT2) - select contour configuration optimised for a Taylor expansion

1. OT2 - calculate the mean absolute value of the θ_j derivatives (MAD) of the $G_l^{\mathrm{F}}(\theta_j)$.
2. Kinematic scales first set to the mean of the sample, MAD calculated at the edges.
3. Kinematic scales then set to each sample value, MAD calculated over bulk.
4. MAD calculated for all possible contour configurations, $\Theta_{o(1),\ldots o(J-1)}$ - $o(j) = \pm$ is the orientation of the jth contour in the θ_j.
5. Contour configuration which minimises the MAD selected.



1. Slice of $I10_2$ without a complex mapping...



1. with a complex mapping, contour orientation $\{o(1), ...o(J-1)\}$ determined via TAYINT.

Method - OT3 and OT4

Over Threshold steps 3 and 4 (OT3-4) - generate all possible post-integration contour configurations for each iterated sector integrand - select optimum one.

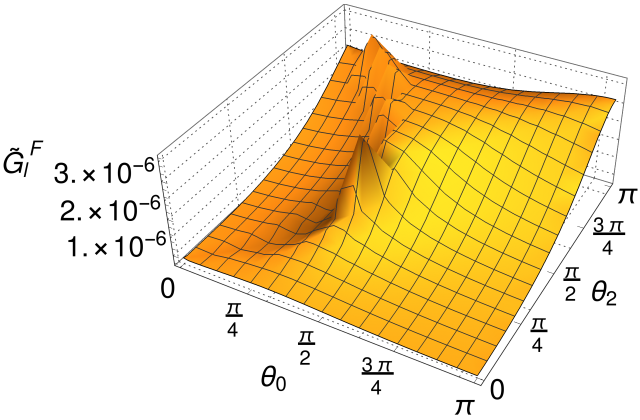
1. OT3 - perform all possible one-fold integrations in the θ_j exactly.

2. In OT4 the resultant $J-2$ variable contour configuration with the lowest MAD is selected - same process as in OT2.

3. If MAD is lower than without integration, this contour configuration is used.

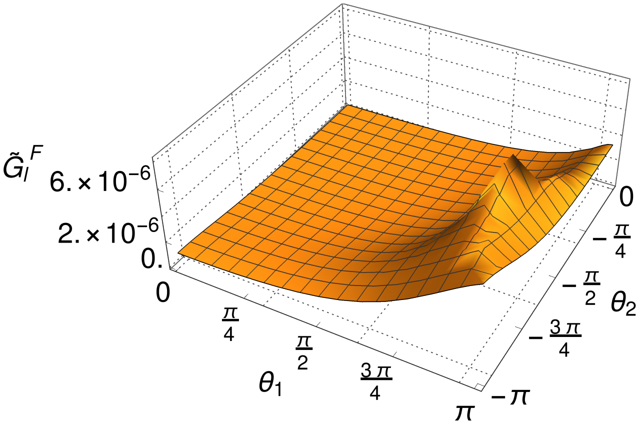
4. The θ_j integrated to yield the selected contour configuration is θ_j^*.

Illustration of the method - OT3 and OT4 I



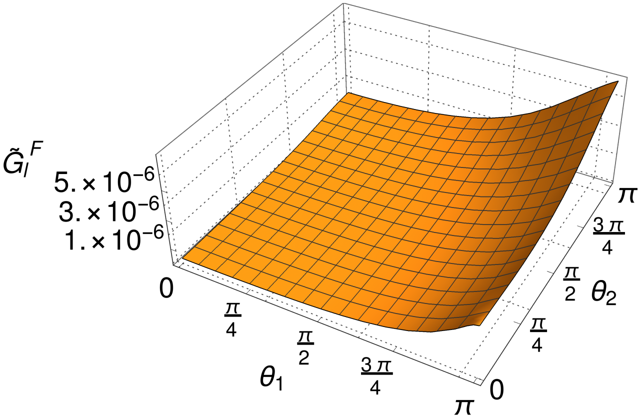
1. $I10_2$: pre-integration contour chosen by TayInt, arbitrary post-integration contour.

Illustration of the method - OT3 and OT4 II



1. $I10_2$: pre-integration contour arbitrary, post-integration contour chosen by TayInt.

Illustration of the method - OT3 and OT4 III



1. I10$_2$: full TayInt algorithm.

Method - OT5 and OT6

Over Threshold steps 5 and 6 (OT5-6) - maximise the convergence of the Taylor expansion

1. OT5 - determine the optimal partitioning, $\mathcal{P}_j = \{(l,h)_1, ..., (l,h)_N\}_j$, of the integrals in θ_j,

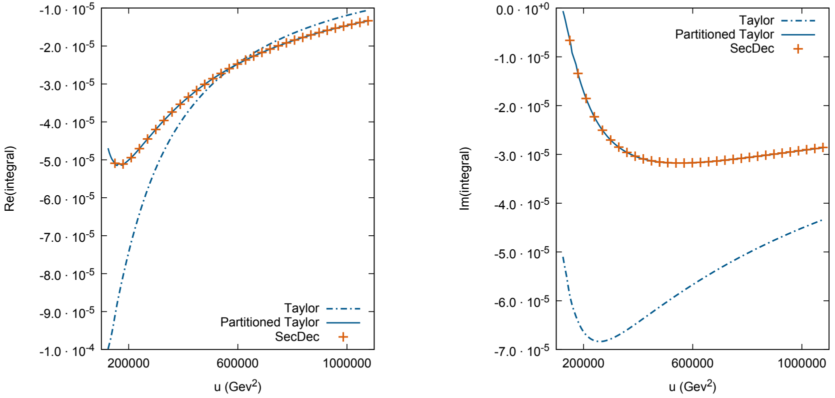
$$\int_0^{\pm\pi} d\theta_j = \sum_{k=1}^{N} \int_{l_{k,j}}^{h_{k,j}} d\theta_j \ ,$$

2. $h_{N,\,j} = \pm\pi$ and $l_{1,\,j} = 0$.

3. New integrands expanded and integrated within each partition in OT6 - allows target precision to be met.

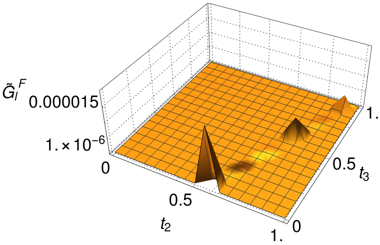
4. Results are functions of the kinematic scales valid everywhere above threshold!

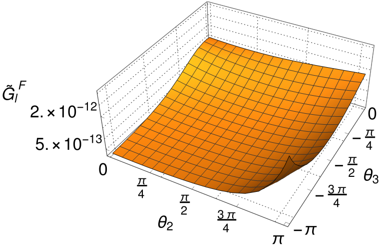
Illustration of the method - OT5 and OT6



1. I10 calculated to ϵ^0 over the $4m_1^2$ threshold with and without partitioning.

Illustration of the method - multiple thresholds I





1. Slice of I246$_1$ at ϵ^0 in the first over threshold region before and after applying TayInt.

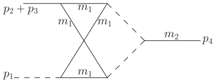
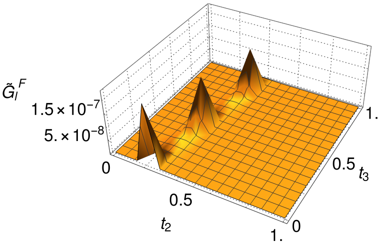
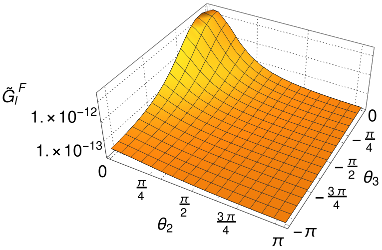
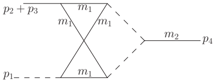


Illustration of the method - multiple thresholds II



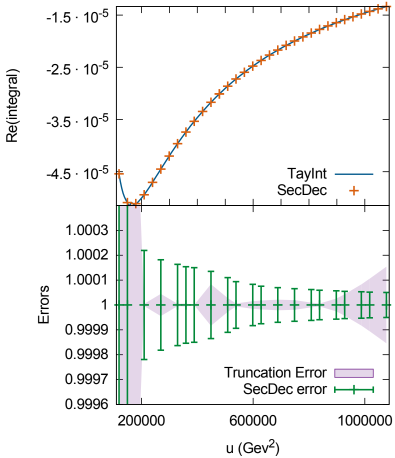


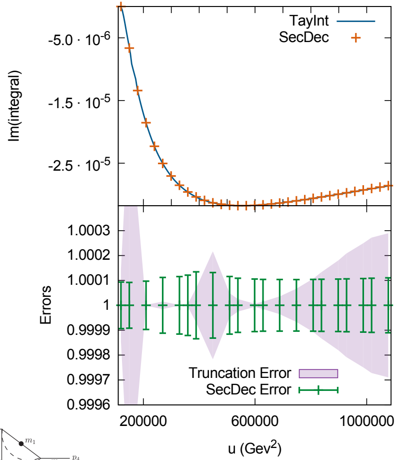
1. Slice of $I246_1$ at ϵ^0 in the second over threshold region before and after applying TAYINT.

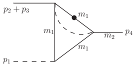


Example of results: I10, $\mathcal{O}\left(\epsilon^0\right)$

I10: $\mathcal{O}\left(\epsilon^0\right)$, $u > 4m_1^2$, $m_2^2 = 0.5m_1^2$, $m_1 = 173\text{GeV}$







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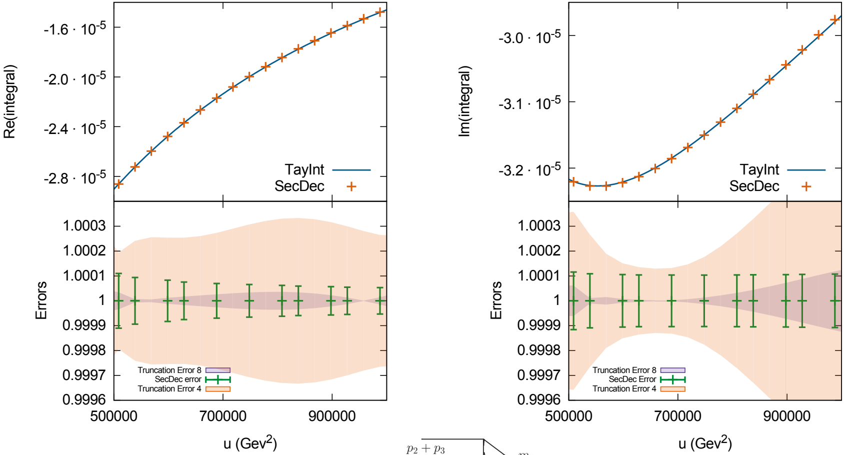

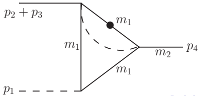

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	OT4: find θ_j^* and hence the optimum $\Theta_{o(1),\dots o(J-2)}$
	OT5: determine $\mathcal{P}_j$
	OT6: Taylor expand and integrate

Conclusions

Three aims: to produce an algebraic integral library with full phase space validity for any kind of integral

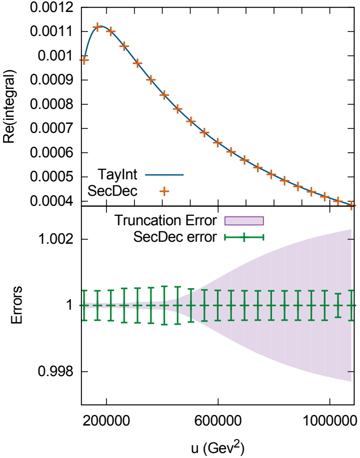
1. TayInt is flexible. Results generated for different:
2. numbers of propagators
3. numbers of external scales
4. ϵ orders
5. kinematic regions, above and below thresholdS.
6. diagrams relevant for Higgs+jet production at two loop.

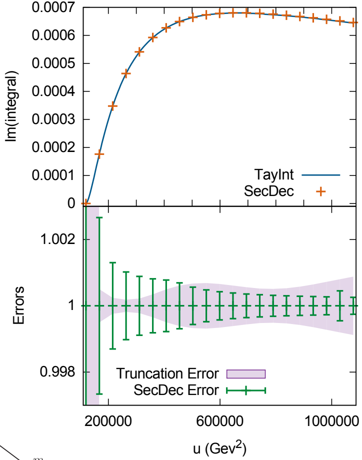
Outlook

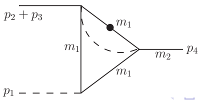
1. Future goals:
2. Automate fully,
3. Apply to phenomenological processes,
4. Apply to five-point integrals.

Thank you very much for listening!

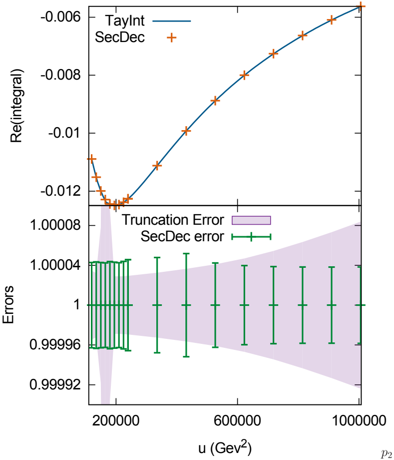
BACKUP-I10 ϵ^1 $u > 4m_1^2,\ m_2^2 = 0.5m_1^2, m_1 = 173\text{GeV}$

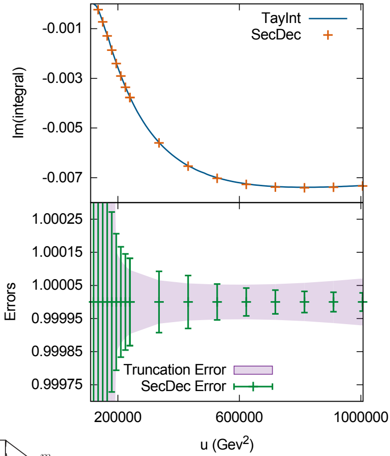


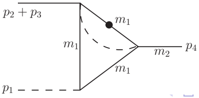




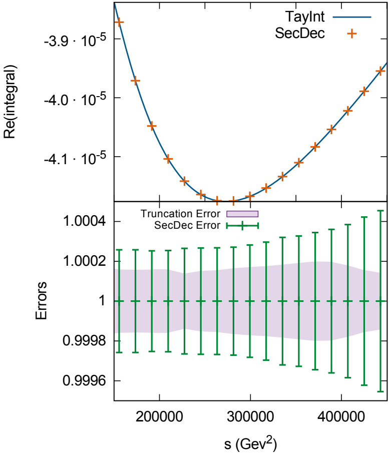
BACKUP-I10 ϵ^2 $u > 4m_1^2$, $m_2^2 = 0.5m_1^2, m_1 = 173\text{GeV}$

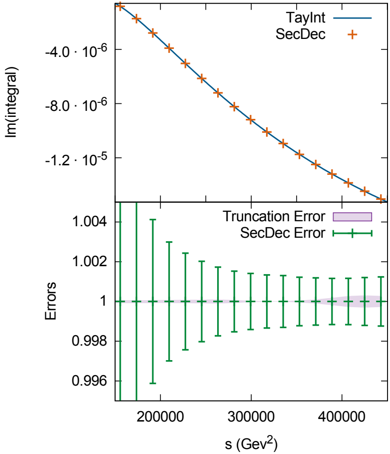


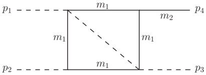




BACKUP-I39 ϵ^0, $s > 4m_1^2$, $m_2^2 = 0.5m_1^2, m_1 = 173\text{GeV}$







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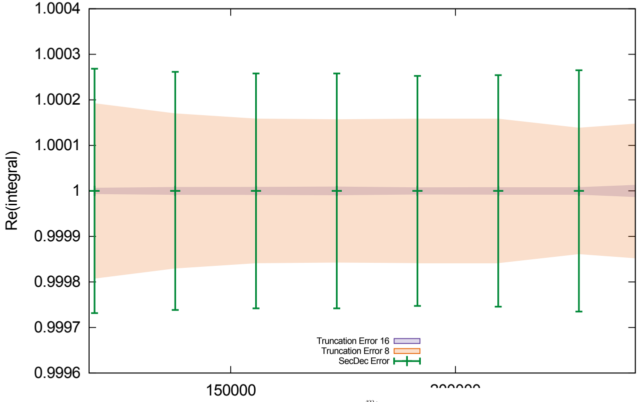

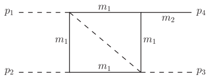

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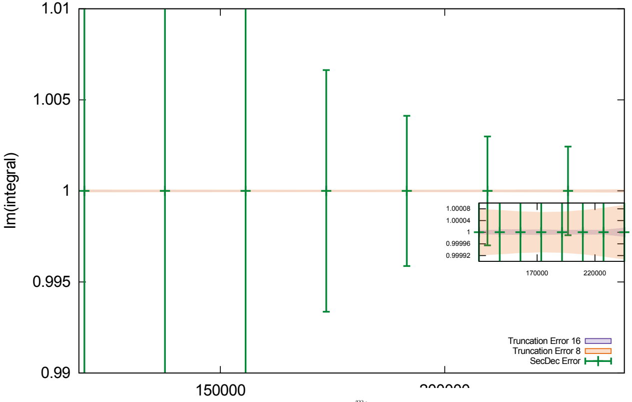
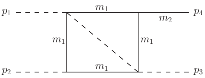
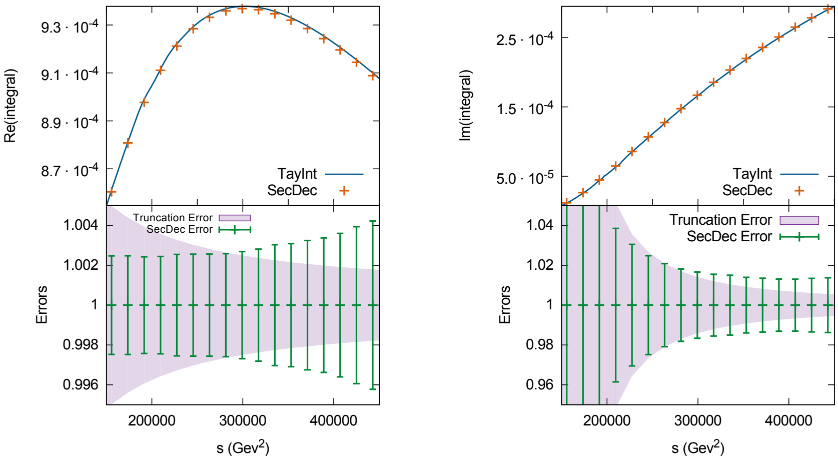
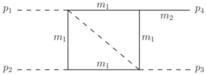


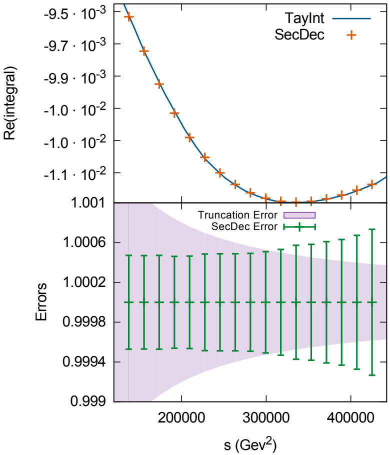


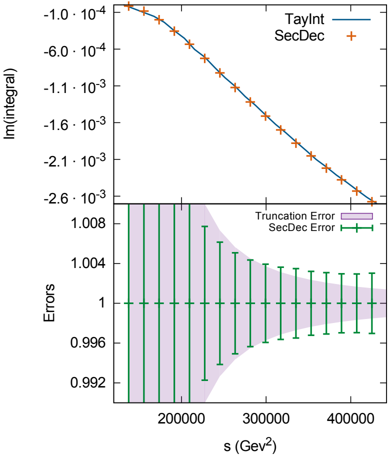
BACKUP-I39 ϵ^1, $s > 4m_1^2$, $m_2^2 = 0.5m_1^2, m_1 = 173$GeV

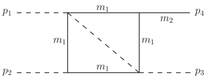




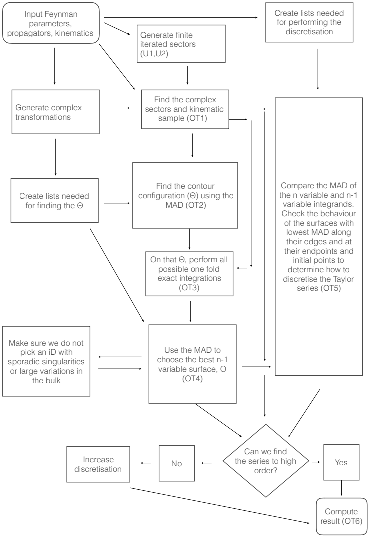
BACKUP-I39 ϵ^2, $s > 4m_1^2$, $m_2^2 = 0.5m_1^2, m_1 = 173\text{GeV}$







BACKUP-Summary of the Method II



BACKUP - Convergence Study

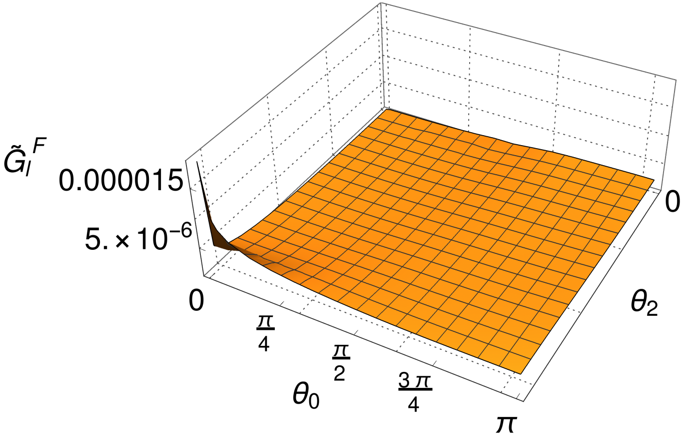
		Mean TayInt Error			
		Number of Partitions			
		4		8	
		Re	**Im**	**Re**	**Im**
Order	0	0.530165	0.623989	0.0812167	0.242449
	2	0.0221554	0.0242271	0.000642405	0.00237282
	4	0.00278254	0.00242541	0.000163342	0.000079292
	6	0.000284179	0.000281809	0.0000239721	0.000038864

BACKUP - Summary of Results

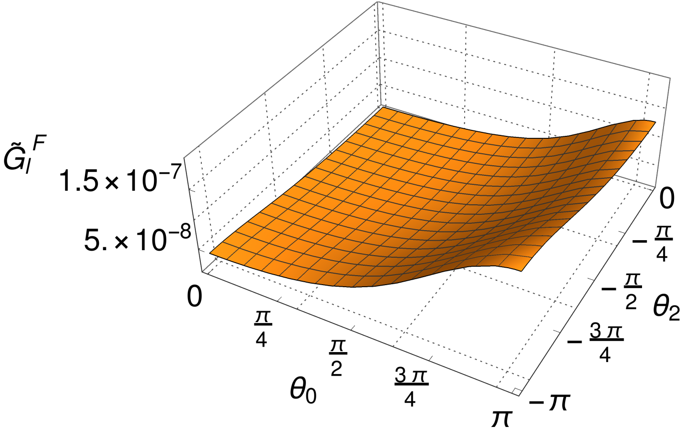
Graph	Re (Δ)	Im (Δ)
I10	0.000658179	0.000270775
I21	0.00126601	0.000277579
I39	0.0000763027	0.0000668706

1. The mean difference Δ between TAYINT, using a sixth order expansion, and SecDec, normalised to the SecDec result.

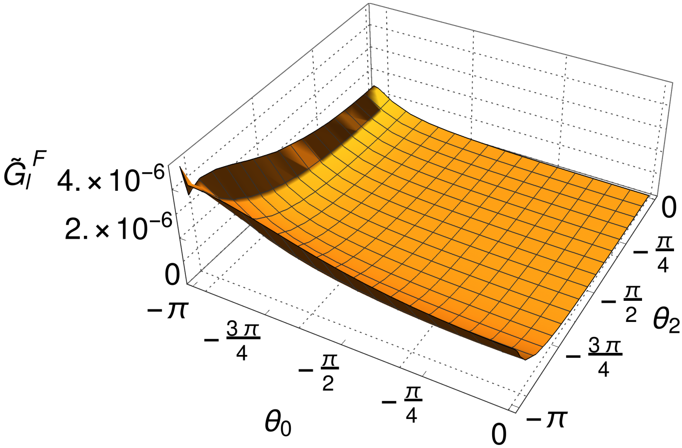
BACKUP - Convergence Sensitivity to Kinematics I



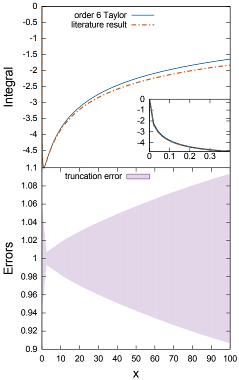
BACKUP - Convergence Sensitivity to Kinematics II



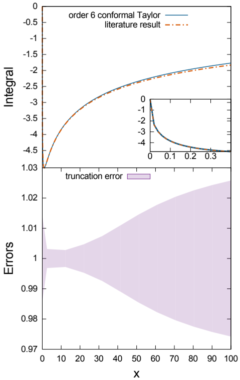
BACKUP - Convergence Sensitivity to Kinematics III



BACKUP - T41



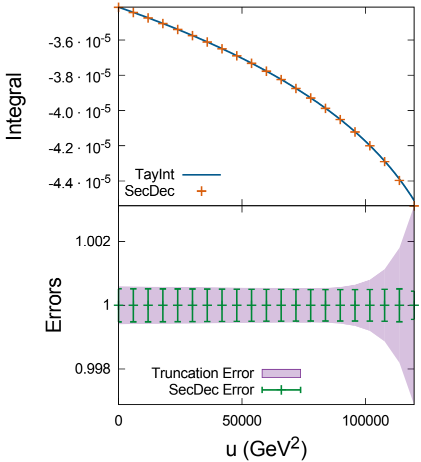
(k)



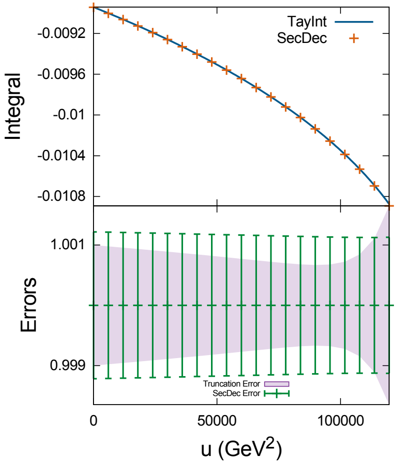
(l)

$$x = \frac{\sqrt{s + 4\,m^2} - \sqrt{s}}{\sqrt{s + 4\,m^2} + \sqrt{s}}$$

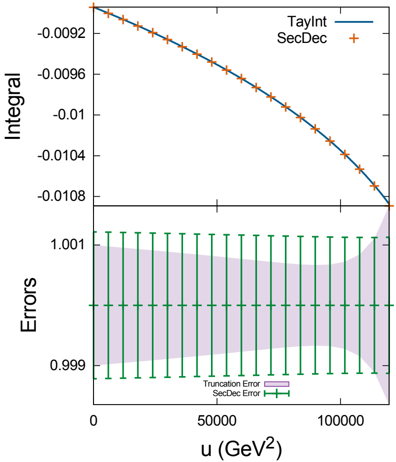
BACKUP - I10 ϵ^0, $u < 4m_1^2$, $m_2^2 = 0.5m_1^2, m_1 = 173$GeV



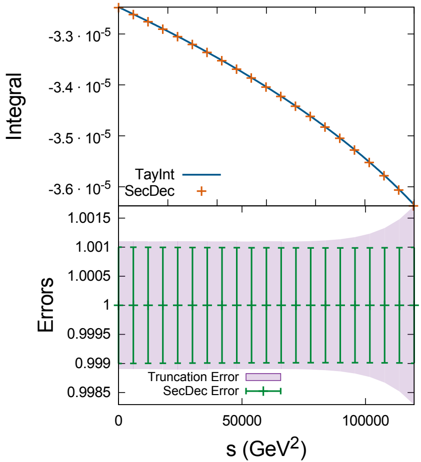
BACKUP - I10 ϵ^1, $u < 4m_1^2$, $m_2^2 = 0.5m_1^2, m_1 = 173\text{GeV}$



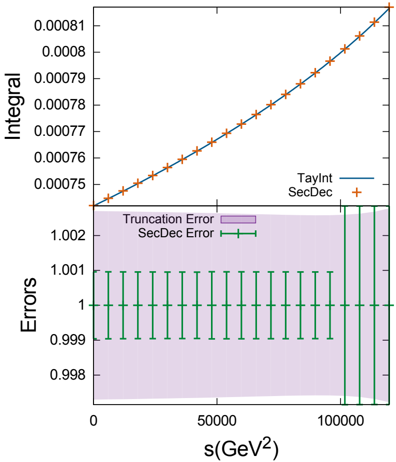
BACKUP - I10 ϵ^2, $u < 4m_1^2$, $m_2^2 = 0.5m_1^2, m_1 = 173\text{GeV}$



BACKUP - I39 ϵ^0, $s < 4m_1^2$, $m_2^2 = 0.5m_1^2, m_1 = 173\text{GeV}$



BACKUP - I39 ϵ^1, $s < 4m_1^2$, $m_2^2 = 0.5m_1^2, m_1 = 173\text{GeV}$



BACKUP - I39 ϵ^2, $s < 4m_1^2$, $m_2^2 = 0.5m_1^2, m_1 = 173\text{GeV}$