Relativistic Corrections within the Integral Transform Approach

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How to extend the reliability of n.r. *ab initio* results for e.w. cross sections to high energy/momentum

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Outline:

- General considerations on the lepton-nucleus hadron tensor and the inclusive response functions $\mathbf{R}^{\mu\nu}(\mathbf{q},\omega)$

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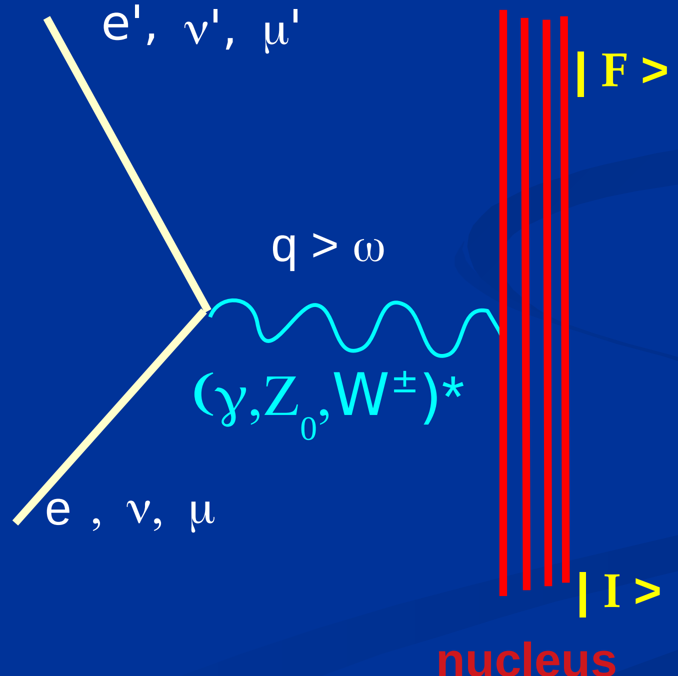
- Test on the (e,e') scattering

Physics of e.w. Interactions **(with nuclei)**



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- δ^4 **expresses the energy-momentum conservation**

If $|\, I >$ is the g.s. $|\, 0 >$ and $|\, F >$ is "inclusive"

r

$$W^{\mu\nu} = < I \,|\, J^{\mu} \,|\, F > < F \,|\, J^{\nu} \,|\, I > \times \delta^4$$

$$R^{\mu\nu}(\vec{q}, \omega) = \sum_n <0|\, J^{\mu}(\vec{q})\,|n><n|\, J^{\nu}(\vec{q})\,|0>|^2 \times$$
$$\times \; \delta(\omega - E_n + E_0)$$

Notice!

- The 3-momentum transfer $\vec{q}$ originates from the 3-momentum delta-function δ^3 which now involves the **c.m. of the nucleus**

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⬇

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- Therefore the non relativistic problem

$$H\,|n\rangle = E_n|n\rangle$$

has to be solved, referred to the "internal" **(i.e. translation/galileian invariant)** dynamics

$$R^{\mu\nu}(\vec{q},\omega) = \sum_n |\langle 0|\,J^\mu(\vec{q})|n\rangle\langle n|\,J^\nu(\vec{q})|0\rangle|^2 \times \delta(\omega - E_n + E_0)$$

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- Take as input an Hamiltonian with protons and neutrons as d.o.f. interacting with *realistic* V_{NN} *(i.e. reproducing NN cross sections with* χ*/datum* ~ 1)

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taking into account the **full many-body dynamics**, respecting **translation/Galileian invariance.** No approximation, but **controlling the numerical accuracy**

The big problems:

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*[**Notice:** due to "good" asymptotic boundary conditions the ground stete **|0>** can be calculated with controlled accuracy, at least up to medium heavy systems]*

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- Up to which energy/momentum can one push the *ab initio* non relativistic treatment of the dynamics in $R^{\mu\nu}(q, \omega)$??

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The solution:

- The integral transform approach

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The solution:

- Analyze the frame dependence, choose the *"right frame"* and the *"proper rel. input kinematics"*

… and test on electromagnetic (e,e’) cross section

The integral transform approach

Integral transform approaches

There are many examples in physics where one uses "integral transform approaches"

$$\Phi = K \; R$$

accessible object

object of interest

There are many classes of problems that are difficult to solve in their original representations. An integral transform "maps" an equation from its **original "domain"** into **another domain**. Manipulating and solving the equation in the **target domain** is sometimes much easier than manipulation and solution in the **original domain**. The solution is then **mapped back** to the original domain with the inverse of the integral transform.

KERNEL

$$\Phi(\tau) = \int d\omega \; K(\omega,\tau)\, R(\omega)$$

One is able to calculate $\Phi(\tau)$ but wants $R(\omega)$, which is the quantity of direct physical meaning.

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Warning:

The "inversion" of $\Phi(\tau)$ may be problematic ("**ill posed problem**") a “good” Kernel has to satisfy two requirements

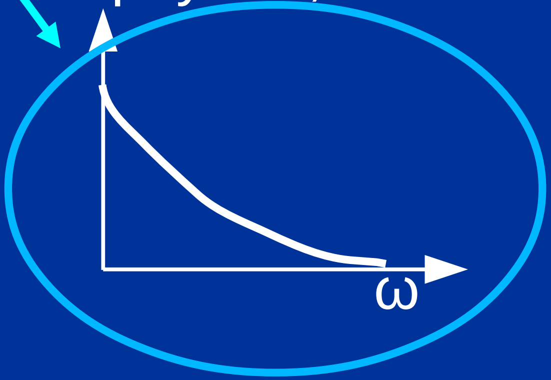
1) one must be able to calculate the integral transform

2) one must be able to invert the transform controlling the instabilities *(“ill-posedeness”)*

Two examples in the literature:

Exponential Kernel: $K(\omega,\tau) = e^{-\omega\tau}$ τ real

- used in condensed matter physics, nuclear physics, lattice QCD,...
- Degree of *ill-posedness* : **high**
- $\Phi(\tau)$ calculated by **GFMC**



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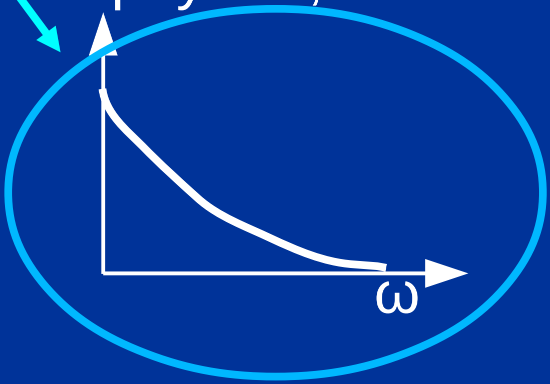

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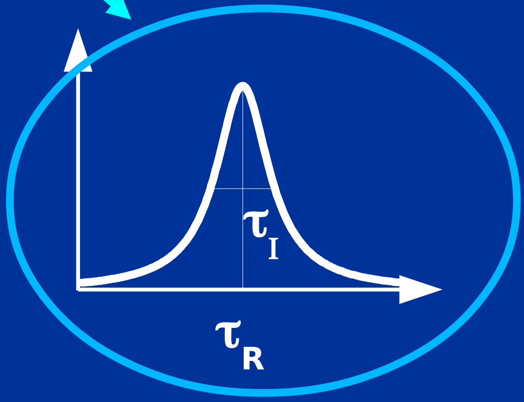
Lorentzian Kernel: $K(\omega,\tau) = [(\omega - \tau)(\omega - \tau)^*]^{-1}$

complex $= \tau_R + \tau_I$



- used in nuclear physics
- Degree of *ill-posedness* : **low**
- $\Phi(\tau)$ calculated via **matrix diagonalization on bound** basis functions

One can calculate

$$\Phi^{\mu\nu}(q, \tau) = \int d\omega \; K(\omega,\tau) \, R^{\mu\nu}(q, \omega)$$

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the many-body scattering problem of calculating **|F>** is avoided!!!

How important are relativistic effects as q increases?

The analysis of frame dependence

One criteria to judge the importance of relativistic effects is the frame dependence of the results

The electron scattering (e,e') response functions in various frames

$$\rho(q)_{fr}$$

$$R_L^{fr}(q^{fr}, \omega^{fr}) = \Sigma_n <0| \; J^0(q)_{fr} |n><n| \; J^0(q)_{fr} |0>|^2 \times \delta(\omega_{fr} - E_n^{fr} + E_0^{fr})$$

$$R_T^{fr}(q^{fr}, \omega^{fr}) = \Sigma_n <0| \; J_\perp(q)_{fr} |n><n| \; J_\perp(q)_{fr} |0>|^2 \times \delta(\omega_{fr} - E_n^{fr} + E_0^{fr})$$

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(in q.e. the final momentum of the "active nucleon" $\mathbf{p}_f \cong \mathbf{q}\,(A-1)/A$ **)**

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ANB: **initially nucleons have momenta** $p_i \cong -q/2$

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They are connected to the response functions in the LAB frame (where they are measured !)

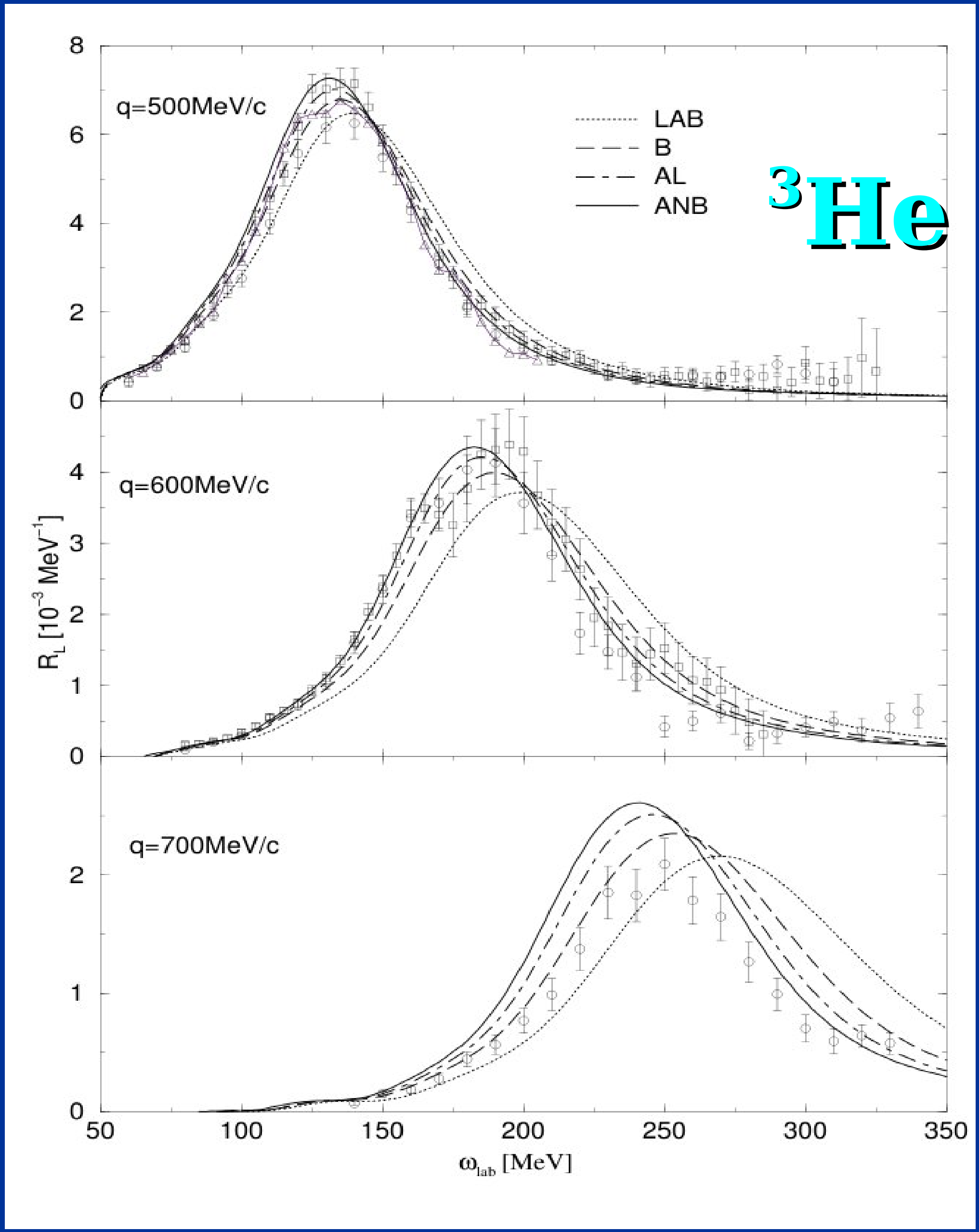
$$R_L^{\mathbf{LAB}}(q,\omega) = \frac{q^2}{q_{fr}^2}\frac{E_i^{fr}}{M_T}R_L^{fr}(q_{fr},\omega_{fr})$$

$$R_T^{\mathbf{LAB}}(q,\omega) = \frac{E_i^{fr}}{M_T}R_T^{fr}(q_{fr},\omega_{fr})$$

Longitudinal response of ^3He

$R_L(q, \omega)$

Large frame dependence!!!



V.Efros, W.Leidemann, G.O., E.L.Tomusiak PRC 72 (2005) 011002

Is there an easy way to cure it?

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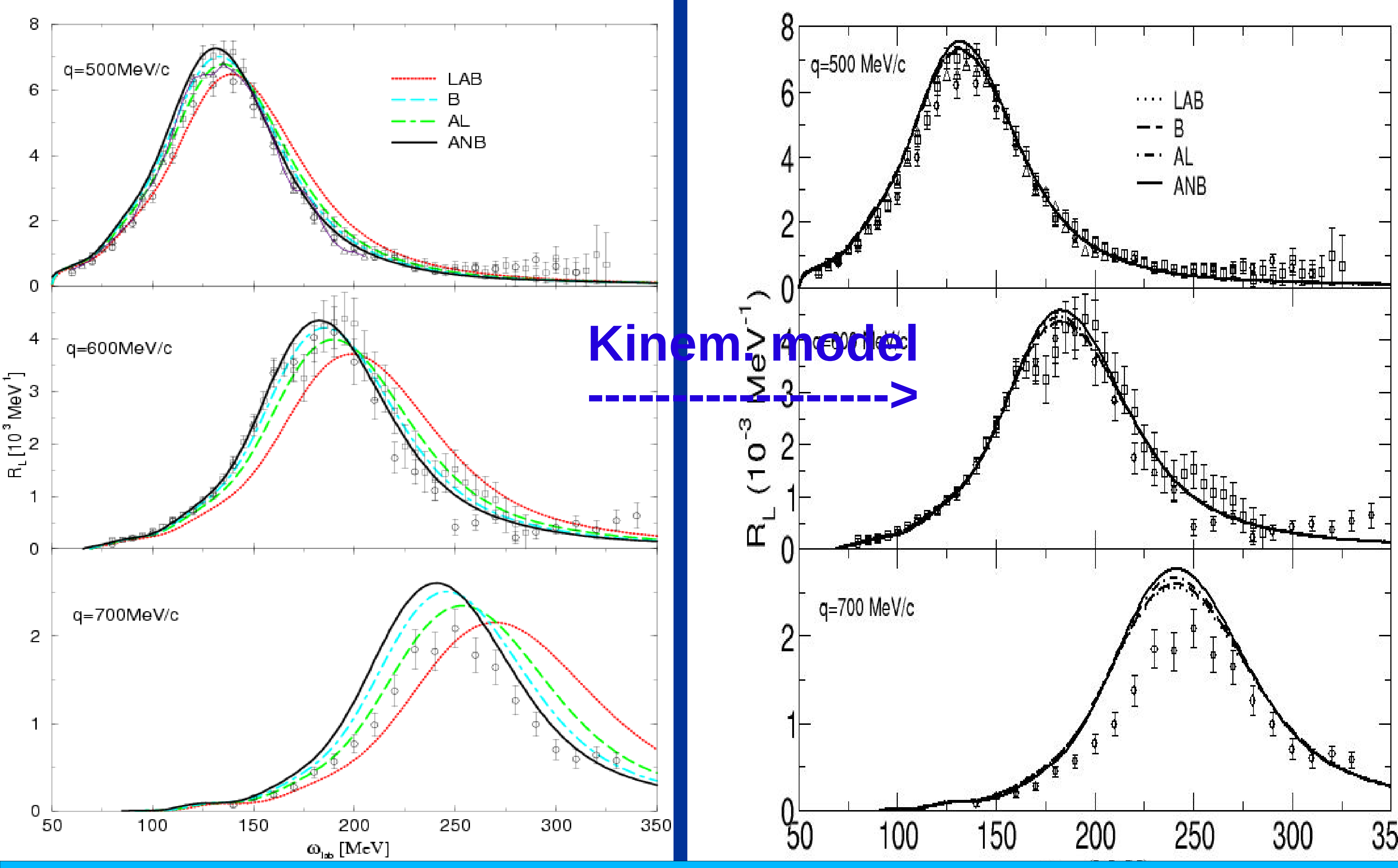
use in each frame the kinematical inputs corresponding to the quasi elastic 2-body assumption i.e.

p + (A-1)-system

The **relative momentum** p_{rel} of the 2 bodies ($p + (A-1)$) can be calculated in each frame in a **relativistically correct** way.

The **energy** of the final state (the input of a **non relativistic** dynamical calculation) is then taken in its **non relativistic form** $p^2_{rel} / 2\mu$

Longitudinal response of ^{3}He $R_L(q, \omega)$



remark:

Of the 4 frames the **ANB** result is the less affected by the relativistically correct kinematical model.

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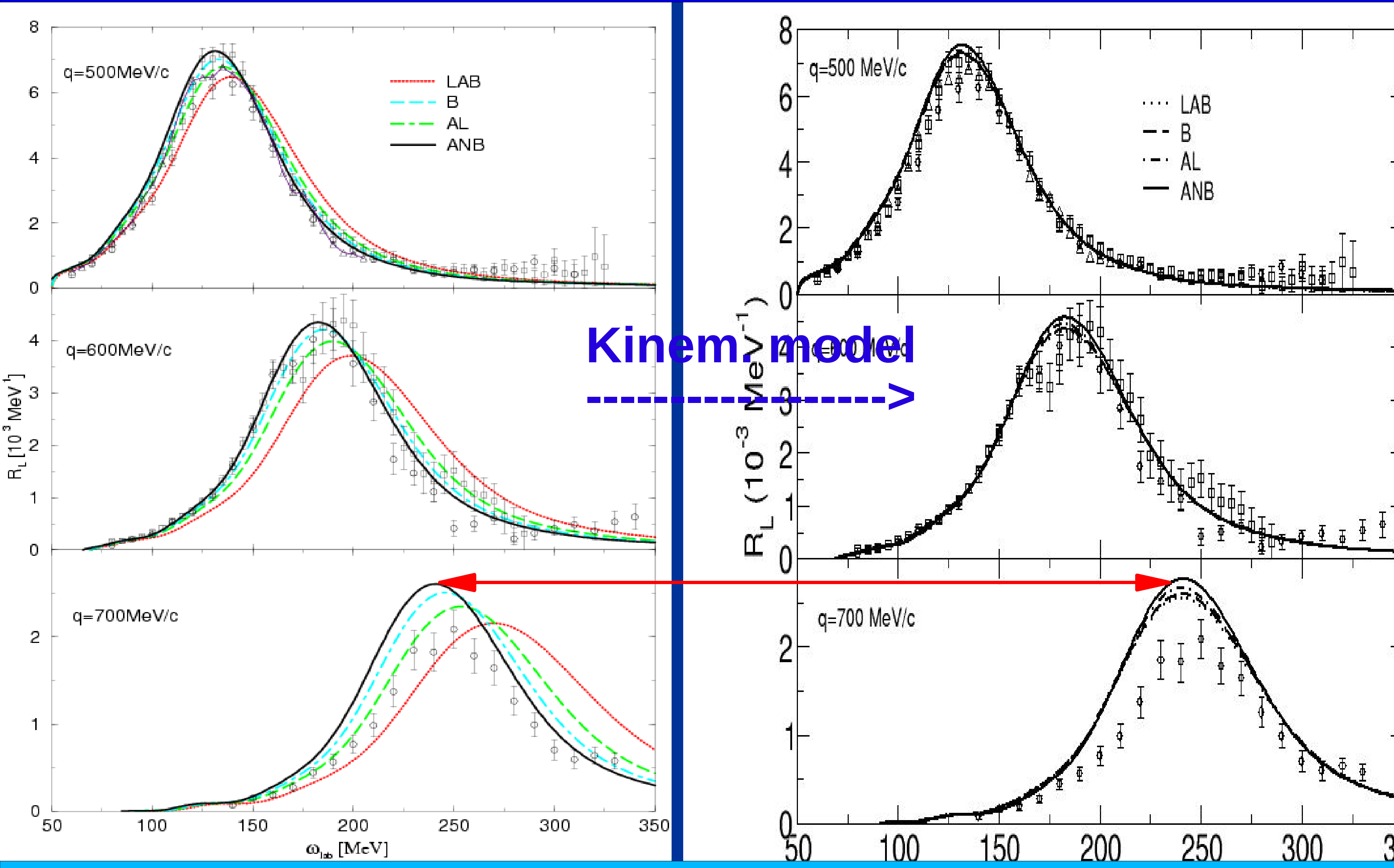

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$$\textbf{LAB}: \quad \frac{\Delta T}{T} \cong \frac{q^2}{4\,m^2} \qquad \textbf{ANB}: \quad \frac{\Delta T}{T} \cong \frac{q^2}{16\,m^2} \quad \textbf{!!!}$$

Moreover: the **peak position** in the **ANB** frame is always relativistically correct, in fact in general:

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ANB : $\omega_{peak} \cong T(q/2) - T(q/2) = 0$

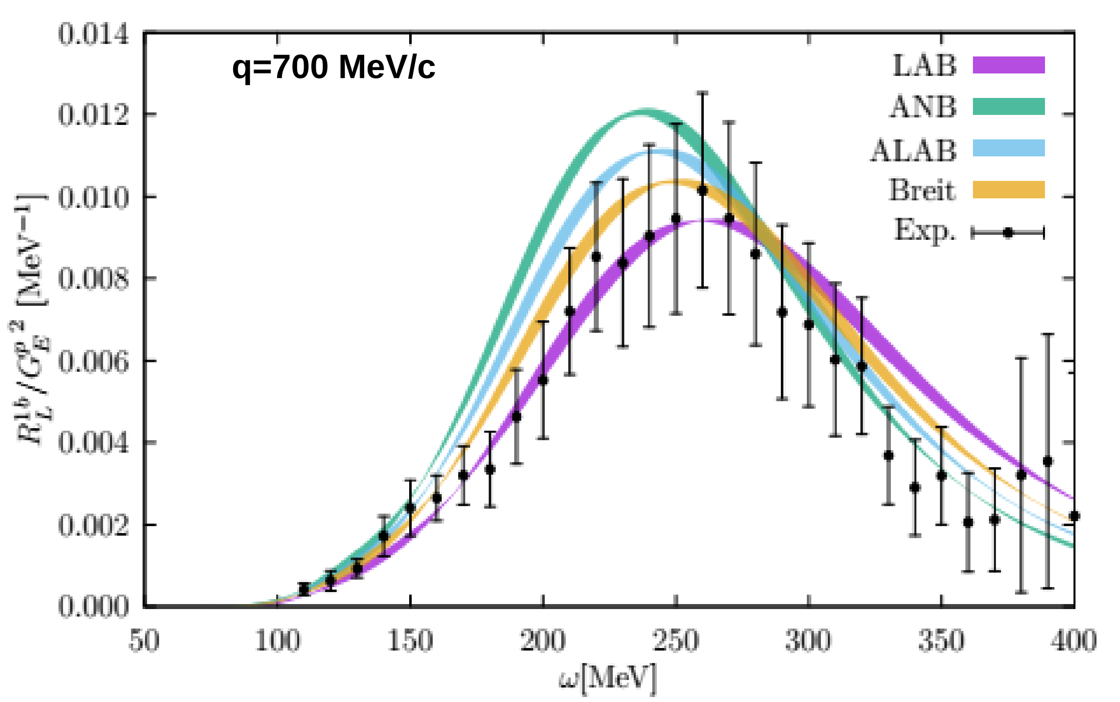
rel. equal to **n.r.**
always correct **!!!**

The test on the ^{4}He(e,e’) cross section

$R_L(q, \omega)$ of ^{4}He

Integral transform calculation with

$K(\omega,\tau) = e^{-\omega\tau}$ τ real, GFMC, **full FSI**



Large frame dependence also in ^{4}He!

N.Rocco, W.Leidemann, A. Lovato, G.O.
Phys. Rev. C 97, 055501 (2018)

Assuming q.e. kinematics ***[2-body break-up 1-(A-1)]***
one can treat the relativistic kinematical inputs
correctly!!

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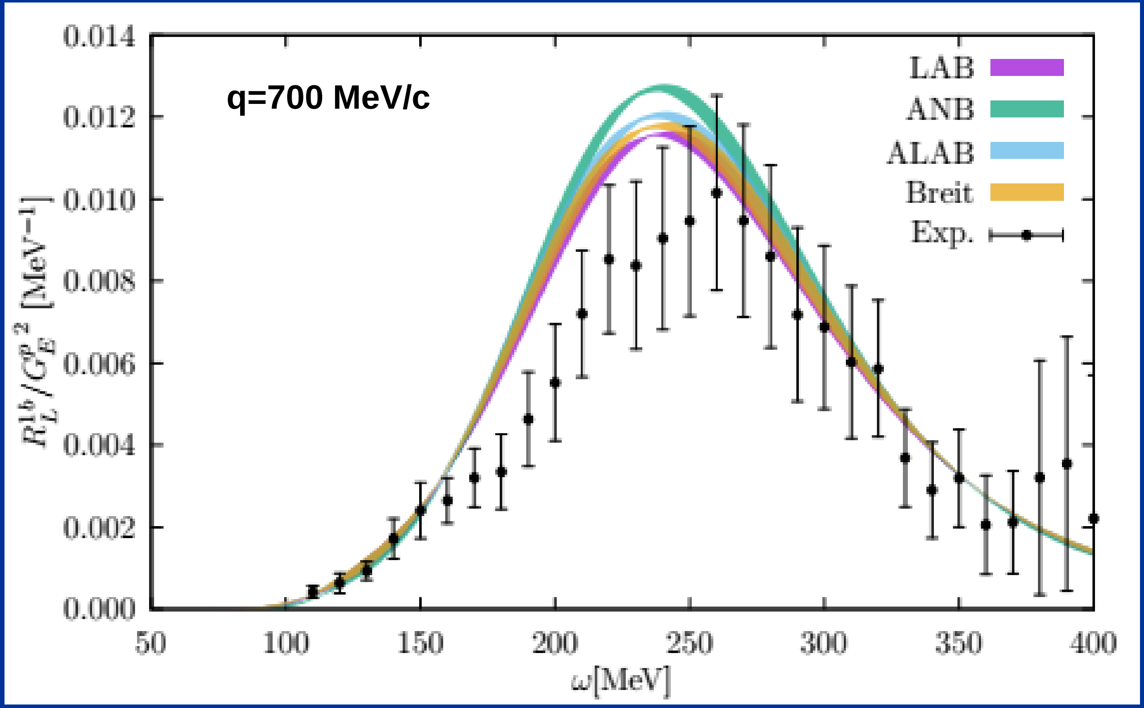

frame dependence much reduced !!!

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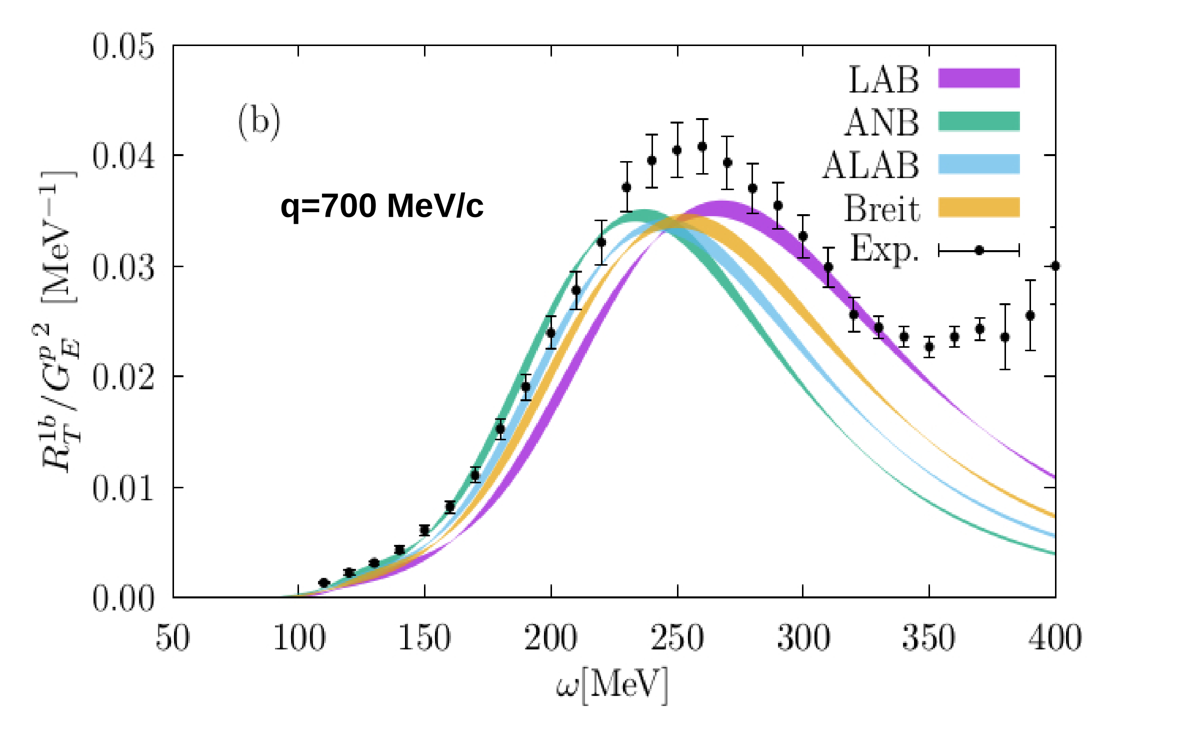

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$R_T(q, \omega)$ of ^{4}He

1-body + 2-body currents
No pion production

Integral transform calculation with
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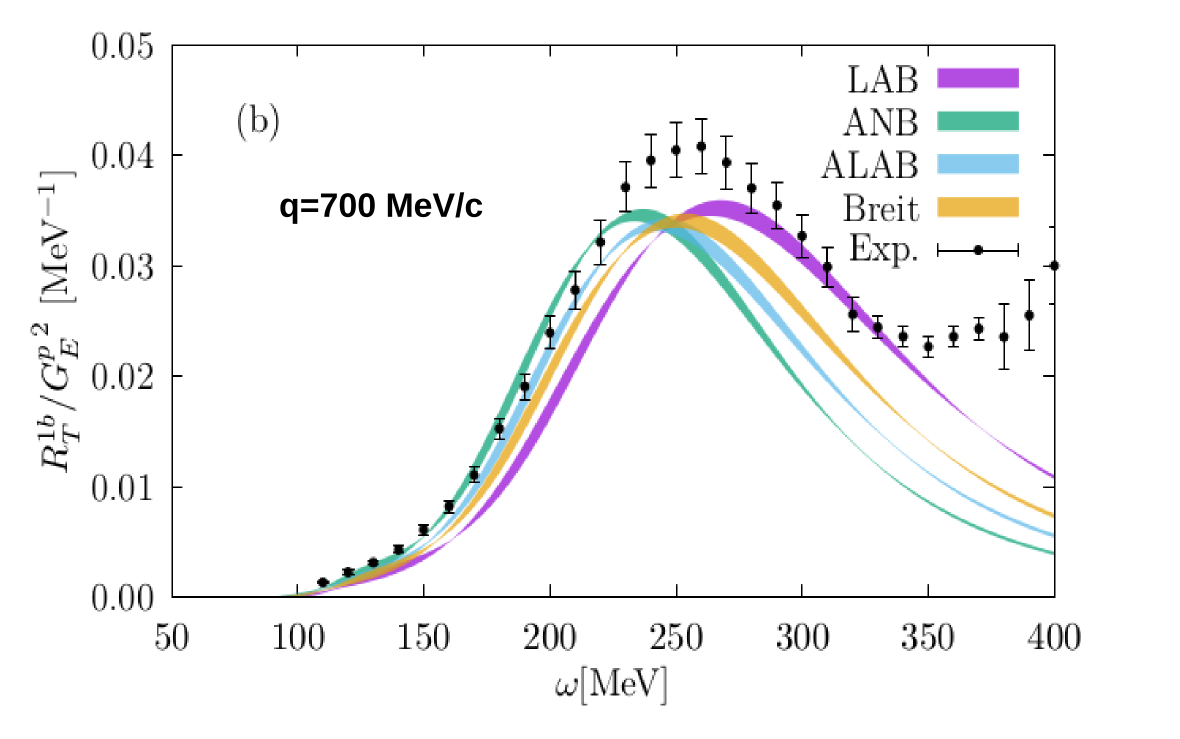

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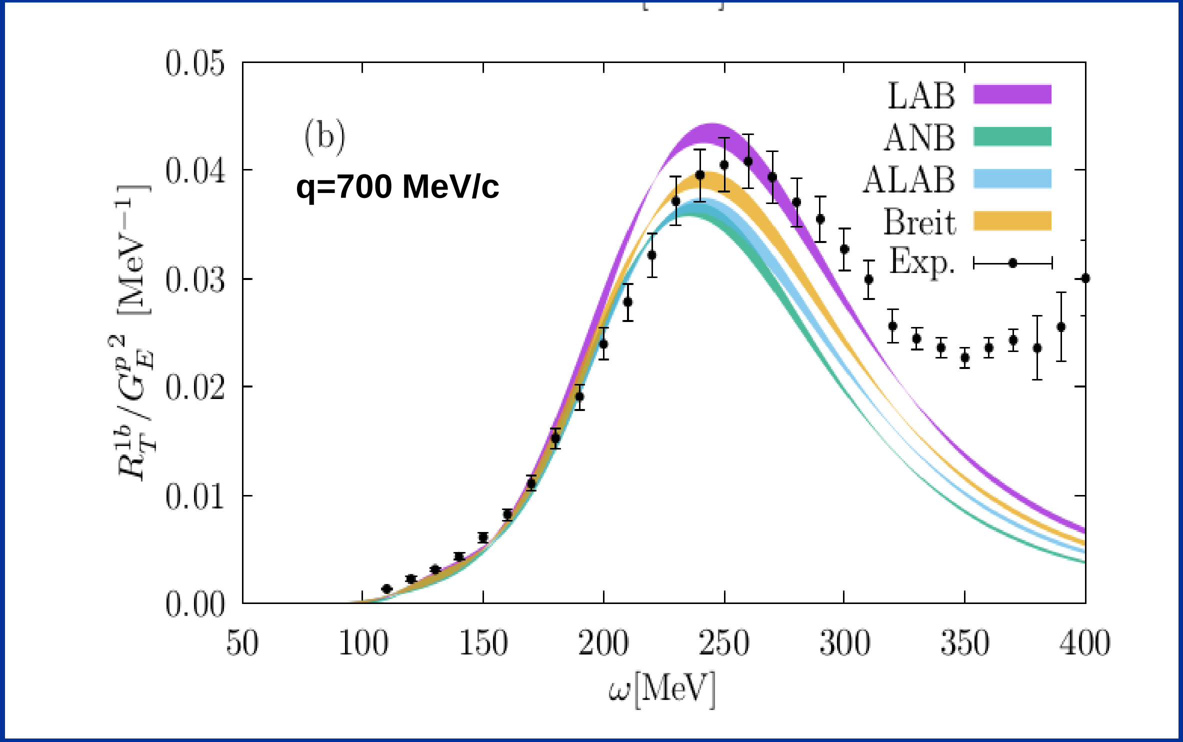

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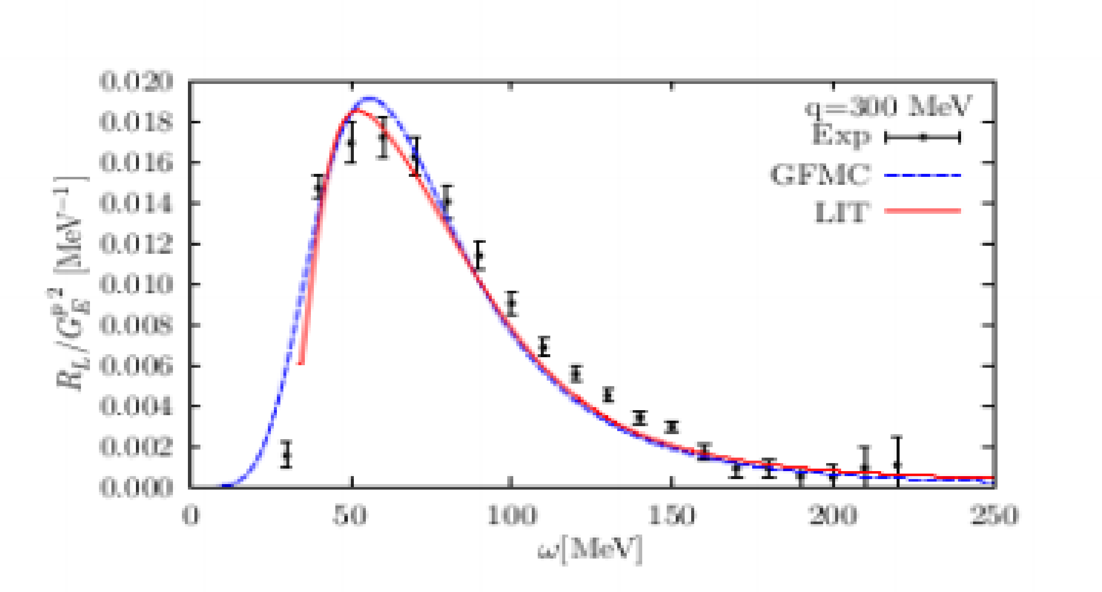
estimation of accuracy



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Blue line: Exponential Kernel

Red line: Lorentzian Kernel:

results on total cross section

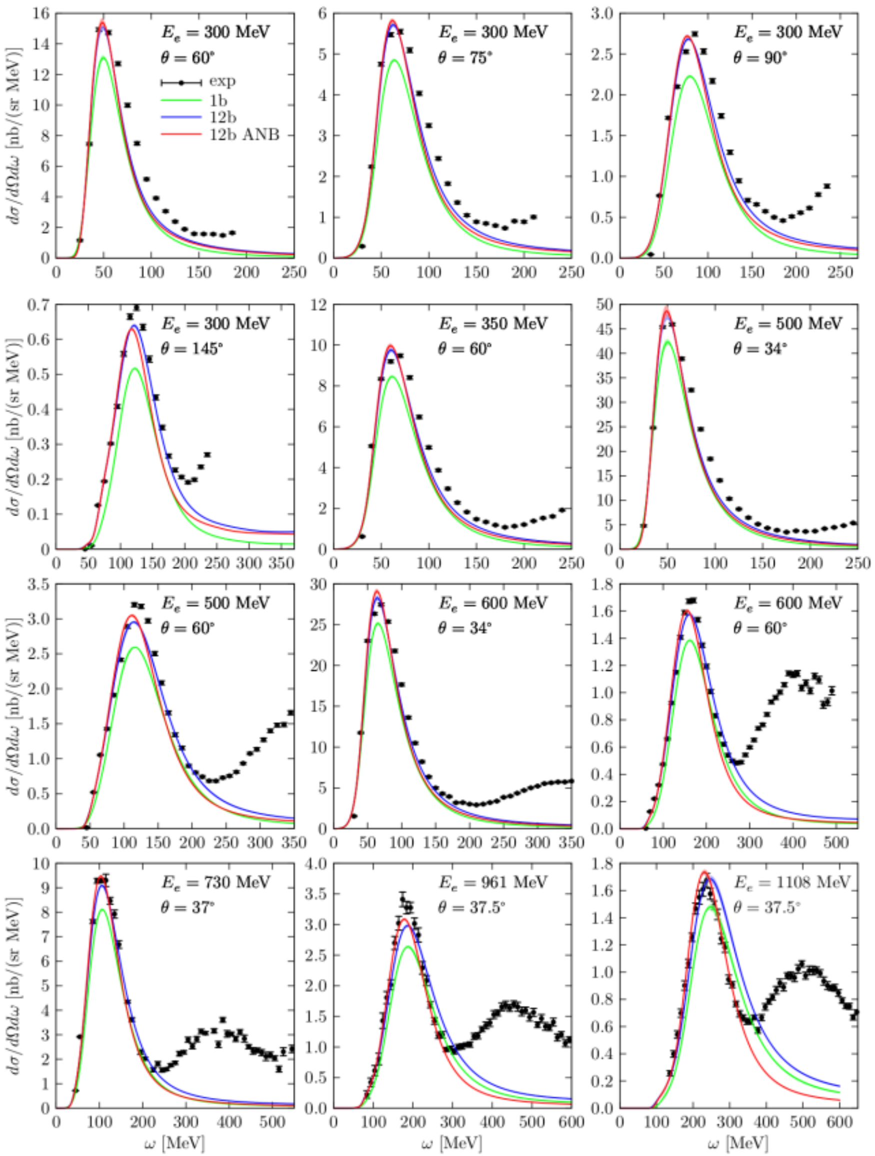

FIG. 7. Double-differential electron-^{4}He cross sections for different values of incident electron energy and scattering angle. The green and blue lines correspond to GFMC calculation were only one- body and one- plus two-body contributions in the electromagnetic currents are accounted for. The red line indicates one plus two-body current results obtained in the ANB frame, employing the two-body fragment model to account for relativistic kinematics. The experimental data are taken from Ref. [?].

- Many contributions at different q !
- Very high computational effort demanded
- Smart interpolation via scaling variable performed
- see

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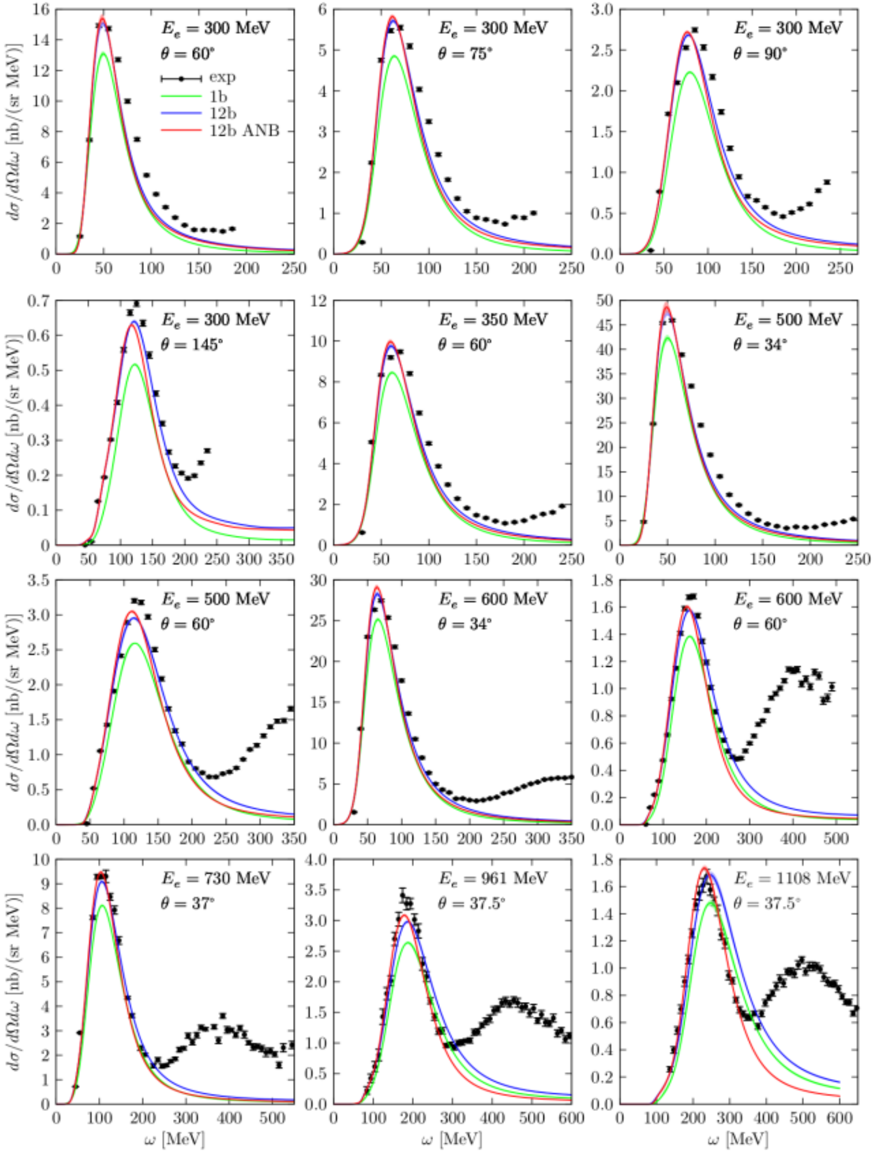

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Test on electron scattering data:

Very good!

N.Rocco, W.Leidemann, A. Lovato, G.O. Phys. Rev. C 97, 055501 (2018)

One can extend to high q the applicability of an *ab initio* n.r. calculation by choosing the right frame

Summarizing:

- Ab initio **n**on **r**elativistic calculations of the (e,e’) nuclear cross section can be performed considering the **full realistic potential dynamics** both in the initial and in the **final** state

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- Ab initio **n**on **r**elativistic calculations of the (e,e') nuclear cross section can be performed considering the **full realistic potential dynamics** both in the initial and in the **final** state
- **One and two-body currents** (with relativistic corrections up to $(q/m)^2$ are included
- **Frame dependence** is much reduced, using quasi elastic kinematics
- In q.e. regime relativistic effects are minimized in the **ANB frame**

Conclusion and outlook

- The **test** of the described approach on **(e,e')** measured cross section turns out to be **very good**
- Then one can use the same approach for **neutrino** scattering
- Heavier targets than ^{4}He can also be treated
- Work on ^{12}C is in progress

Results obtained with

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- Victor Efros (Kurchatov Centre Moscow)
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