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Spatial Resolution of Pad/Strip Structures

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Goal of the Talk

- Understand better how spatial resolution of pad/strip structures depends on
  - The geometrical properties (width, shape) of the signal
  - The type of readout (binary / analog)
  - The noise

- What I will show is the result of *many years* of ‘playing around’ with the mathematics.
  - I never had the time / patience to write a paper so far…

- Warning:
  - We need to do some math, but it’s all ‚classical‘ stuff

Overview

1. Warm-up:
   - Resolution with 'binary' readout
   - Optimal signal width

2. Error of Center-of-Gravity reconstruction:
   - When do we need a fit?

3. Influence of noise on spatial resolution
   - Resolution as a function of noise (quantitatively)
   - Higher Moments of noise distribution
   - Correlated Noise
   - 2D structures

4. Error when doing 'Eta-reconstruction'
   - Search for 'best' response function

BINARY READOUT OF BOX SIGNALS

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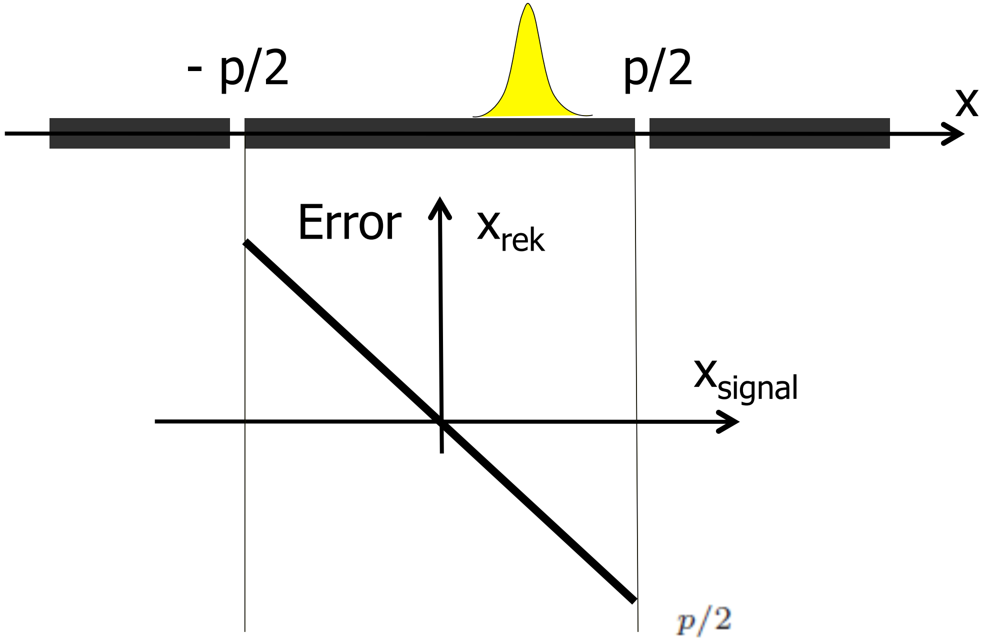
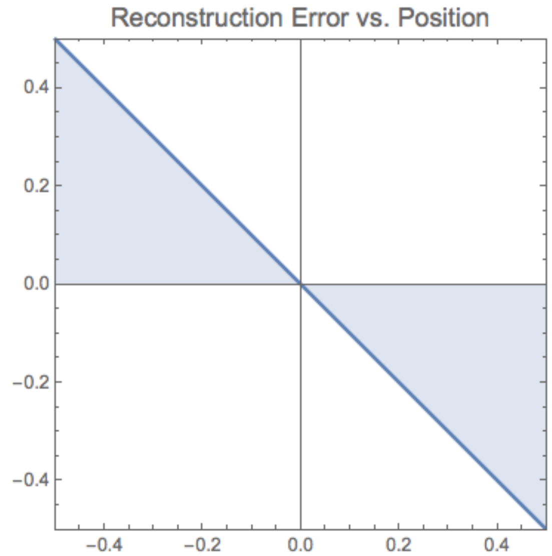
Spatial Resolution of Narrow Signals

- Consider **very** narrow signal on a strip structure
- $\rightarrow$ Only **one** strip is hit $\rightarrow$ Binary 'yes/no' - readout
- Reconstructed position = strip center. Error = offset in strip.





- Sigma of Error $\sigma^2_{\text{position}} = \frac{\int_{-p/2}^{p/2} x_r^2 \cdot 1 \cdot dx_r}{\int_{-p/2}^{p/2} 1 \cdot dx_r} = \frac{p^2}{12}$

$$\boxed{\sigma_{\text{position}} = \frac{p}{\sqrt{12}}}$$

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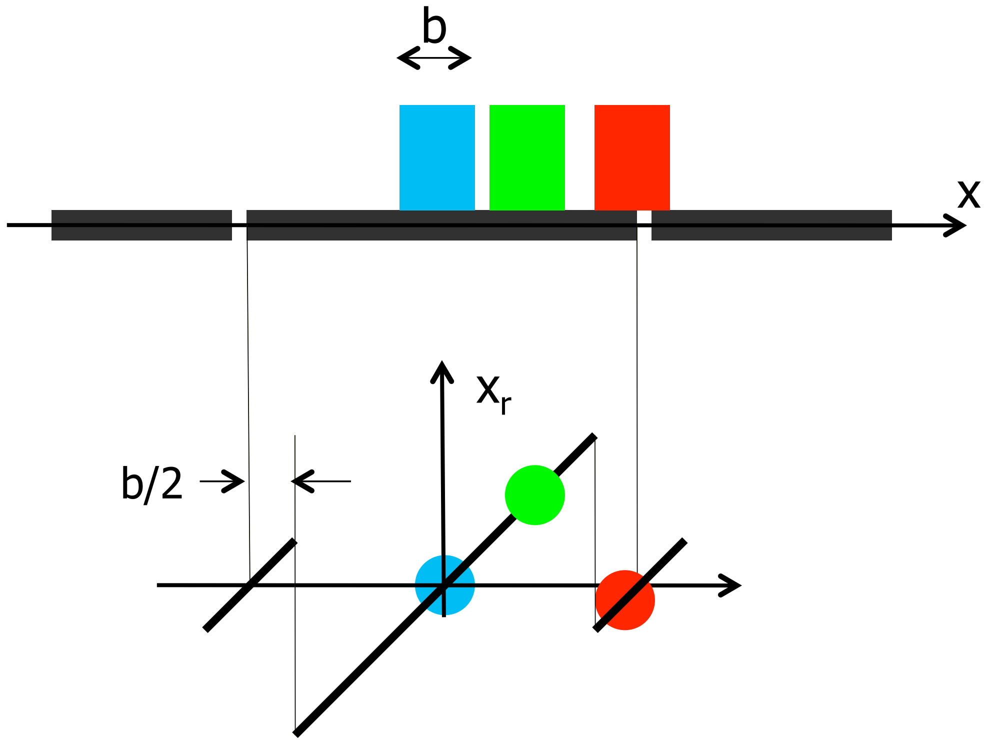
Resolution with Wider Signals (‘Binary’ Readout!)

- Consider ‘Box’ Signals of width b for simplicity.



- When 2 strips are hit → reconstruct at edge → small error
- Signal sharing is GOOD

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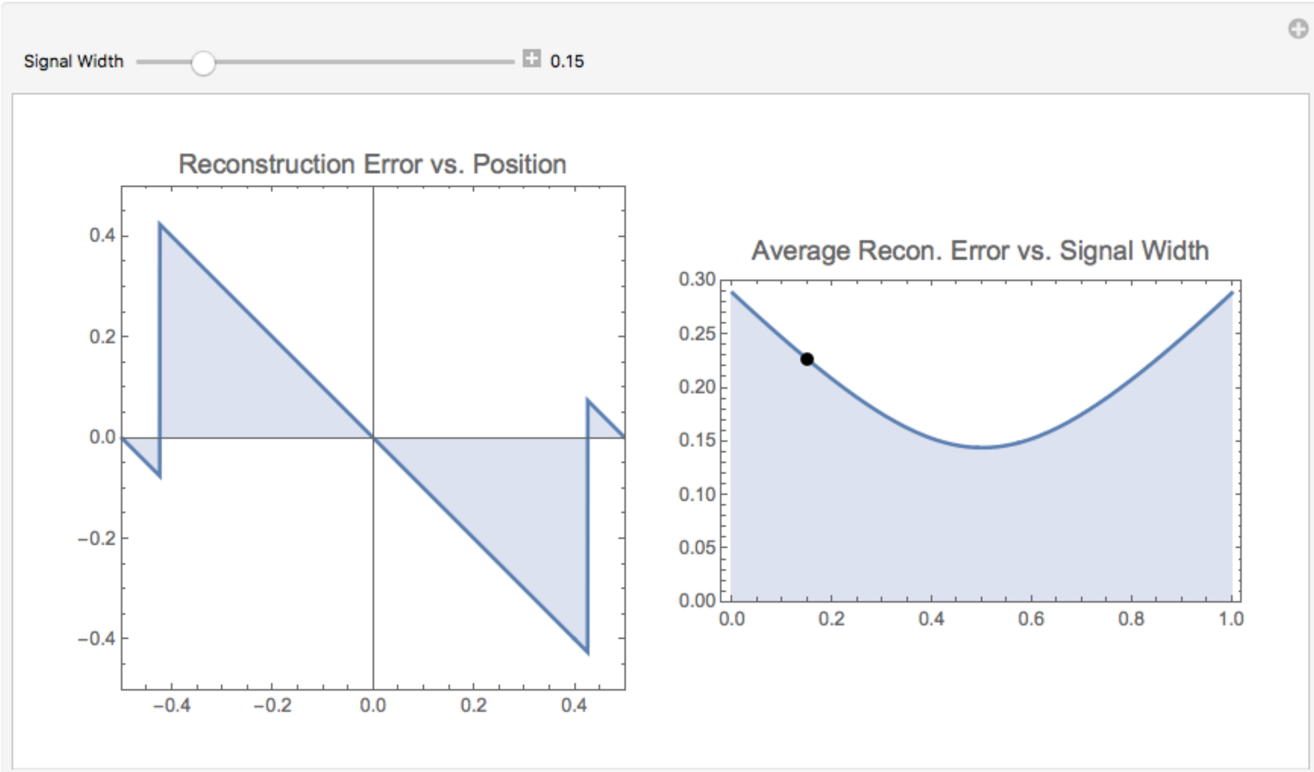
Resolution vs. (Box) Signal Width



- Minimum Error for b = p/2. Error becomes half: s = ½ p/√12
- We then have 50% single and 50% double hits,
  i.e. an average ‘cluster size’ of 1.5
- Signals of Width = Pitch are stupid: We *always* fire 2 strips!

Adding Amplitude Information

- So far, our detector only provides a binary yes/no signal per strip
- Obviously we can do better if we have the *analogue* signal *amplitude*
  - But this is MUCH more effort / data
  - Is it worth it ?

- How do we exploit the analogue information ?

FIT

Centre of Gravity (COG)

Neural Net

…

SIMPLE

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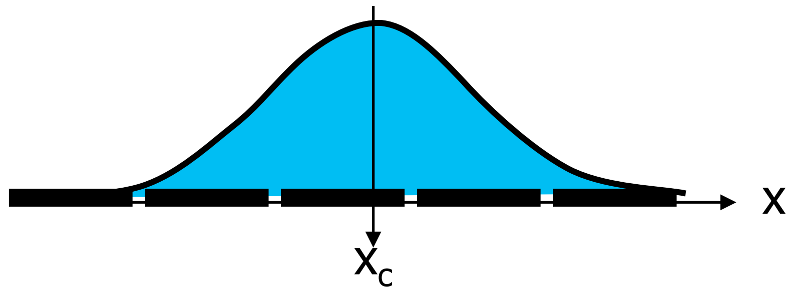
CENTER-OF-GRAVITY RECONSTRUCTION: WHEN IS IT SUFFICIENT - OR - WHEN DO WE NEED A FIT?

The question:

- A 1D signal with (spatial) shape f(x) falls onto a strip structure with pitch *a*
  - We assume $\int$**f(x)dx = 1** and **f(x) symmetric**.
- This generates (analogue) signals on several strips.
- We assume for now that **noise = 0**.
- Question:
  What is the reconstruction error for CoG reconstruction?
  - More precisely: Error for a single event? Average error? Sigma?



- We expect the answer to depend on
  - signal shape
  - Strip pitch a
  - signal position (for single events)

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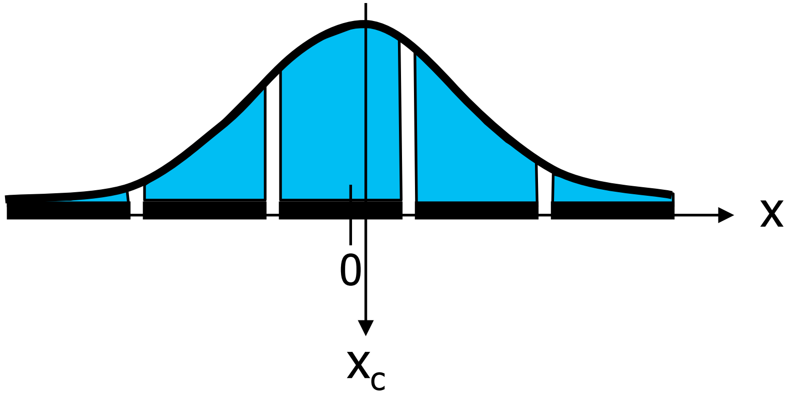
Remark

- The following calculation involves partial integrals over arbitrary function.
- Normally we must give up soon analytically (consider Gaussians..)
- But it turns out that we can go quite a way…

- Maybe showing the derivation would not really be necessary, but I like the fact that so many 'simple' aspects of basic Analysis show up…

1. Signal on Strips



We **assume** the signal on a strip is the integral of f(x)

Strip centers are at **m·a**

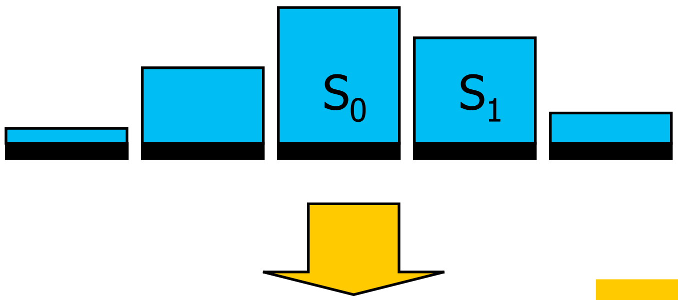
$$S_m(x_c) = \int_{(m-1/2)a}^{(m+1/2)a} f(x - x_c)\, dx$$

This is the signal in strip m when the charge cloud is centered around x_c

$$= \int_{-\infty}^{\infty} f(x) \cdot \mathrm{Rect}_a(x - (ma - x_c))\, dx$$

Box of width **a** centered at **m·a-x$_c$**

2. Reconstructed Position



x_{rek} ?

Position of Strip m

Signal on Strip m

$$x_{rek}(x_c) = \sum_m ma \cdot S_m(x_c)$$

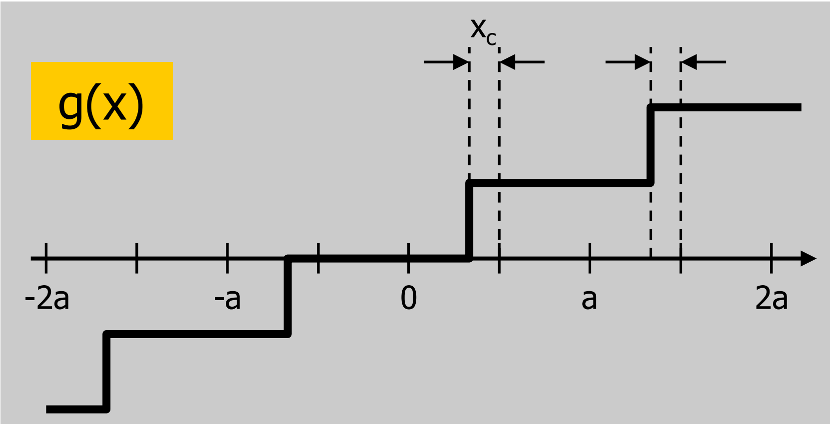
$$= a \int f(x) \underbrace{\sum_m m \cdot \mathrm{Box_a}(x + x_c - ma)} \, dx$$

$$= a \int f(x)\, g(x)\, dx$$

position reconstructed by CoG (normalized signal!)

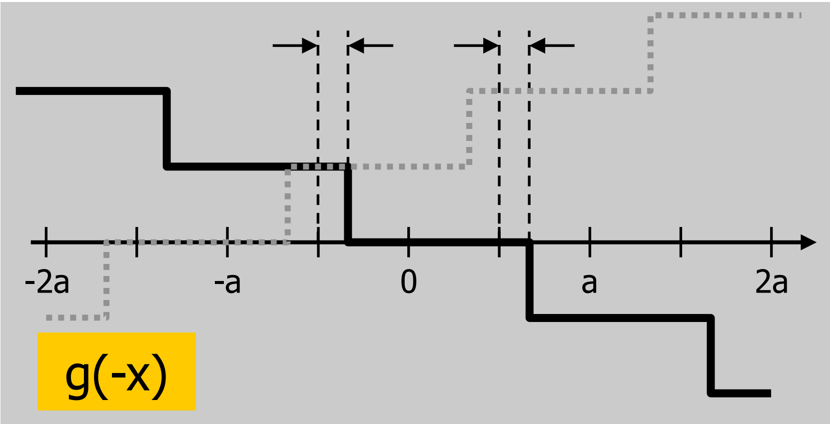
Staircase = g(x)

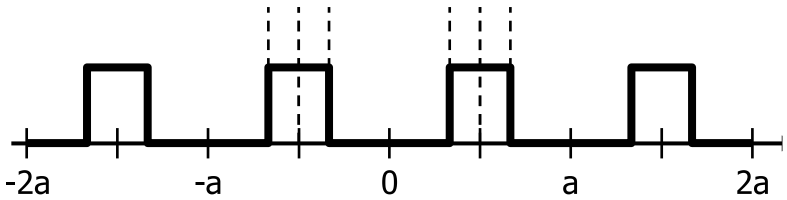
3. Divide Staircase in sym. / antisym. parts



$$g(x) = g_{\text{sym}}(x) + g_{\text{antisym}}(x)$$

$$g_{\text{sym}}(x) = \frac{g(x) + g(-x)}{2}$$





NB: this is only valid for $a-2x_c > 0$, i.e. $x_c < a/2$. This will be sufficient.

$$g_{\text{sym}}(x) = \frac{1}{2} - \frac{1}{2}\sum_m \text{Box}_{a-2x_c}(x - ma)$$

4. Simplify the integrals. Move to Fourier Space

- Integral of $g_{antisym}(x)$ is zero because f is assumed symmetric
- We are left with

$$x_{rek}(x_c) = a \int f(x)\, g_{\mathrm{sym}}(x)\, dx$$

$$= \frac{a}{2} - \frac{a}{2} \int f(x) \sum_m \mathrm{Box}_{\mathrm{a}-2\mathrm{x_c}}(x - ma)\, dx$$

$$= \frac{a}{2} - \frac{a}{2} \int f(x) \left[\mathrm{Box}_{\mathrm{a}-2\mathrm{x_c}}(x) \star \mathrm{Comb}_{\mathrm{a}}(x)\right] dx$$

Write Sum of Boxes as convolution of a single Box with Dirac Comb

- To solve this, move to Fourier Space with

$$\tilde{f}(k) := \int f(x)\, e^{-2\pi i k x} dx$$

- We can use $\int a(x)\, b(x)\, \mathrm{d}x = \int \tilde{a}(k)\, \tilde{b}(k)\, \mathrm{d}k$ and $\widetilde{a \star b} = \tilde{a} \cdot \tilde{b}$ (for symmetrical a, b)

5. Get rid of the Integral

$$x_{rek}(x_c) = \frac{a}{2} - \frac{a}{2}\int f(x)\left[\mathrm{Box}_{\mathrm{a-2x_c}}(x) \star \mathrm{Comb}_{\mathrm{a}}(x)\right] dx$$

$$= \frac{a}{2} - \frac{a}{2}\int \widetilde{f}(k)\cdot \widetilde{\mathrm{Box}}_{a-2x_c}(k)\cdot \widetilde{\mathrm{Comb}}_a(k)\, dk$$

This is again a Dirac Comb, i.e. a **sum** of peaks at distances 1/a

sinc

integral over k can be carried out. The sum from the comb is left over

$$= \frac{a}{2} - \frac{a}{2}\sum_{m=-\infty}^{\infty} \widetilde{f}\left(\frac{m}{a}\right)\frac{\sin\left(m\pi - \frac{2\pi m x_c}{a}\right)}{m\pi}$$

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6. Use Symmetry, Simplify the Sin() function

$$x_{rek} = \frac{a}{2} - \frac{a}{2} \sum_{m=-\infty}^{\infty} \tilde{f}\left(\frac{m}{a}\right) \frac{\sin\left(m\pi - \frac{2\pi m x_c}{a}\right)}{m\pi}$$

Use symmetry
m ⇔ -m.

Treat m=0:
~f(0) = 1,
Sin(m k)/m → k

$$= x_c - \frac{a}{\pi} \sum_{m=1}^{\infty} \tilde{f}\left(\frac{m}{a}\right) \frac{\sin\left(m\pi - \frac{2\pi m x_c}{a}\right)}{m}$$

Center position
x_c shows up !

$$\begin{aligned} \sin(m\pi - x) &= \sin(m\pi)\cos(x) - \cos(m\pi)\sin(x) \\ &= -(-1)^m \sin(x) \end{aligned}$$

$$\boxed{x_{err}(x_c) = \frac{a}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \tilde{f}\left(\frac{m}{a}\right) \sin\left(\frac{2\pi m x_c}{a}\right)}$$

PUH.....

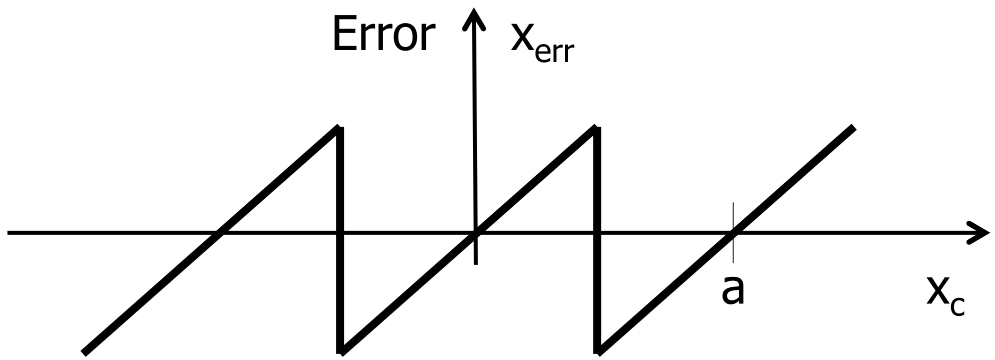
A First Check

$$x_{err}(x_c) = \frac{a}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^m}{m}\tilde{f}\left(\frac{m}{a}\right)\sin\left(\frac{2\pi m x_c}{a}\right)$$

- For very narrow f(x), f(x) → Dirac(x) and therefore $\tilde{f}(k) \to 1$ so that

$$x_{err}(x_c) = \frac{a}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^m}{m}\sin\left(m\pi\frac{2x_c}{a}\right)$$

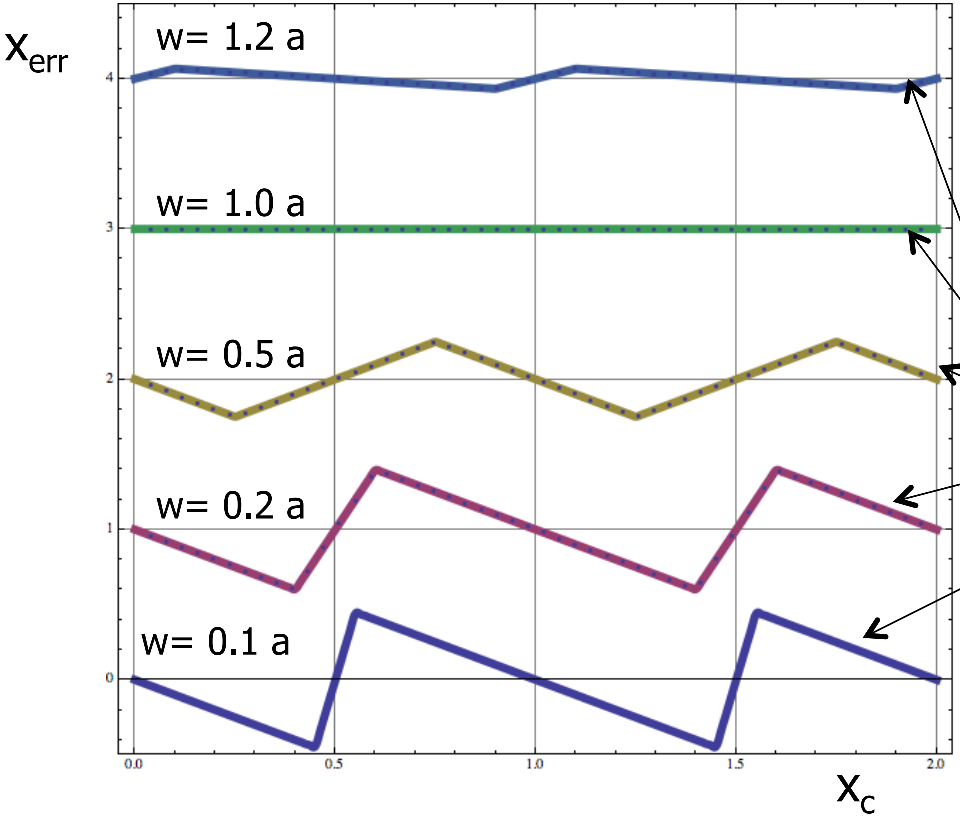
- This is the Fourier Series of a Saw-Tooth, as expected!

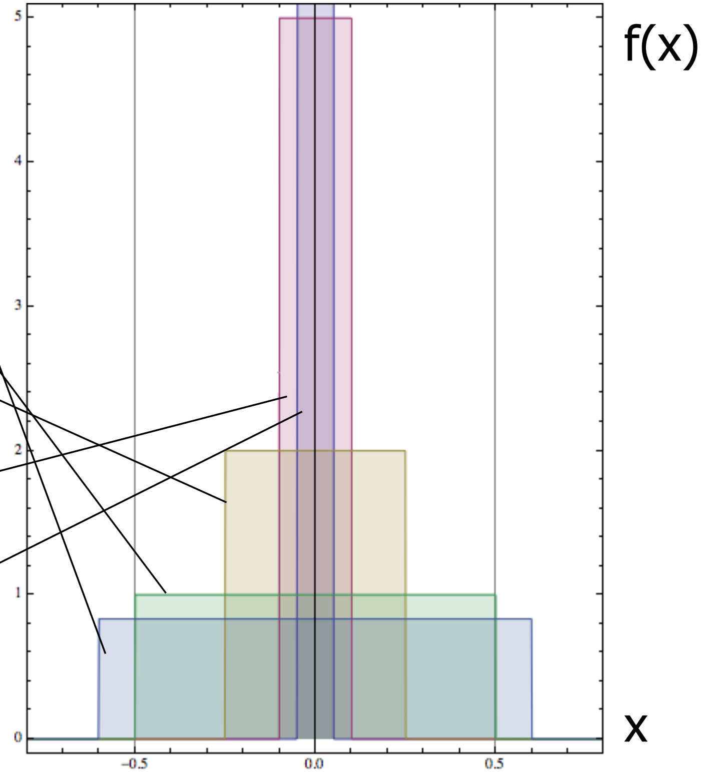


Check with f(x) = Box

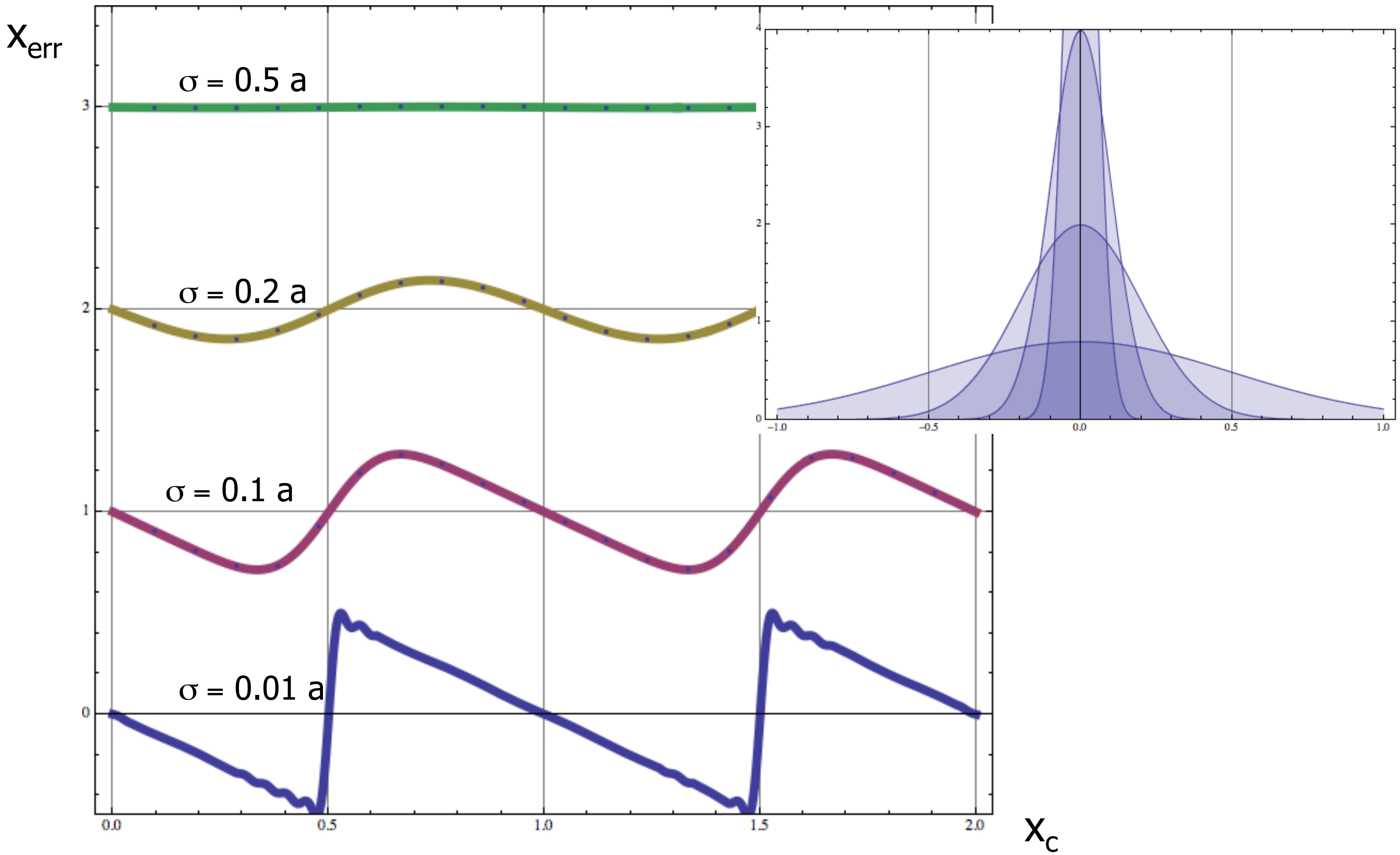
$$x_{err}(x_c) = \frac{a}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \tilde{f}\left(\frac{m}{a}\right) \sin\left(\frac{2\pi m x_c}{a}\right)$$

- For a box of width a, $\tilde{f}\left(\frac{m}{a}\right) = \frac{\sin m\pi}{m\pi}$ is zero for m∈N.
  → reconstruction is perfect. Same for width = multiple of a.





Check with Gaussians



- Error already very small for σ = 0.5a

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Going Further: **Sigma** of x_{err}?

$$x_{err}(x_c) = \frac{a}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \tilde{f}\left(\frac{m}{a}\right) \sin\left(\frac{2\pi m x_c}{a}\right)$$

$$\begin{aligned}
\sigma^2_{\mathrm{rec}} &= \frac{1}{a} \int_{-a/2}^{a/2} x^2_{\mathrm{err}}(x_c)\, dx_c \\
&= \frac{a}{\pi^2} \sum_{n,m=1}^{\infty} \frac{(-1)^{n+m}}{nm} \tilde{f}\left(\frac{n}{a}\right) \tilde{f}\left(\frac{m}{a}\right) \int_{-a/2}^{a/2} \sin\frac{2\pi n x_c}{a} \sin\frac{2\pi m x_c}{a}\, dx_c \\
&= \frac{a}{\pi^2} \sum_{n,m=1}^{\infty} \frac{\cancel{(-1)^{n+m}}}{nm} \tilde{f}\left(\frac{n}{a}\right) \tilde{f}\left(\frac{m}{a}\right) \frac{a}{2} \boxed{\delta_{n,m}}
\end{aligned}$$

sin() are orthorgonal!

$$\boxed{\sigma^2_{\mathrm{rec}} = \frac{a^2}{2\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2} \tilde{f}^2\left(\frac{m}{a}\right).}$$

Another Check

$$\boxed{\sigma_{\mathrm{rec}}^2 \quad = \quad \frac{a^2}{2\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\,\tilde{f}^2\left(\frac{m}{a}\right).}$$

- For very narrow signals, we have again $\tilde{f}(k) \rightarrow 1$ so that

$$\left(\frac{\sigma_{\mathrm{err}}}{a}\right)^2 = \frac{1}{2\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2} = \frac{1}{12}$$

as expected….

$\pi^2/6$

- This is probably the most complicated way to get the 1/12…

f(x) = Gaussian(x) or Box(x)

- For a Gaussian signal with width σ

$$G(x) = \frac{1}{\sqrt{2\pi}\sigma_s} \exp\left(-\frac{x^2}{2\sigma_s^2}\right) \qquad \text{with} \qquad \tilde{G}(k) = \exp\left(-2\pi^2 k^2 \sigma_s^2\right)$$

we get

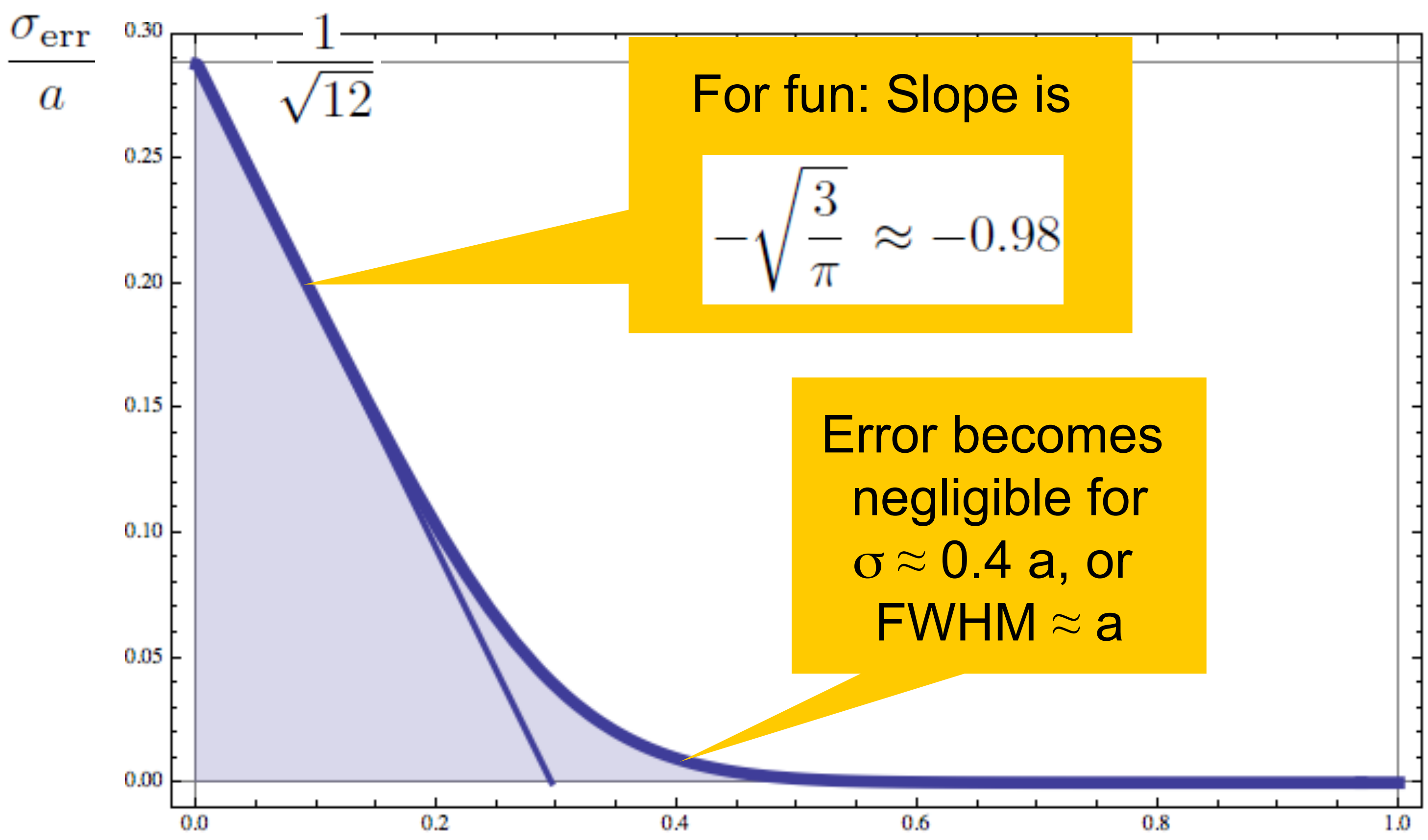
$$\boxed{\left(\frac{\sigma_{\text{err}}}{a}\right)^2 = \frac{1}{2\pi^2} \sum_{m=1}^{\infty} \frac{\exp\left(-\frac{4m^2\pi^2\sigma_s^2}{a^2}\right)}{m^2}}$$

- For a Box of width s·a

$$\left(\frac{\sigma_{\text{err}}}{a}\right)^2 = \frac{1}{360 s^2} - \frac{1}{4\pi^4 s^2} \sum_{m=1}^{\infty} \frac{\cos(2\pi m s)}{m^4}$$

which is zero for integer s thanks to $\Sigma(1/n^4) = \pi^4/90$..

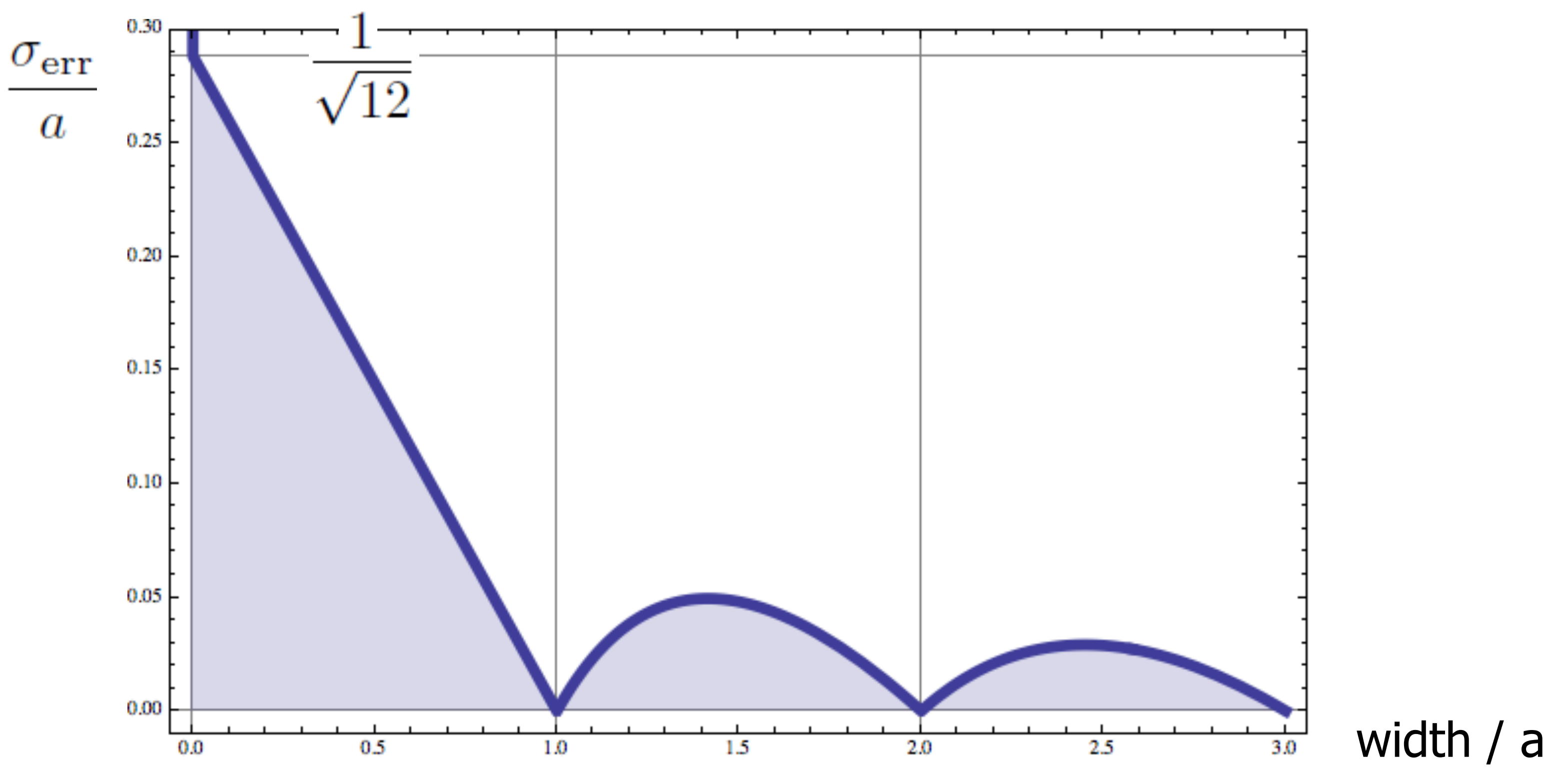
Plot this for f(x) = Gauss(x)



The result 'Error $\approx$ 0 for FWHM $\approx$ a' can be found for many pulse shapes. We knew this… but now we know *for sure*…

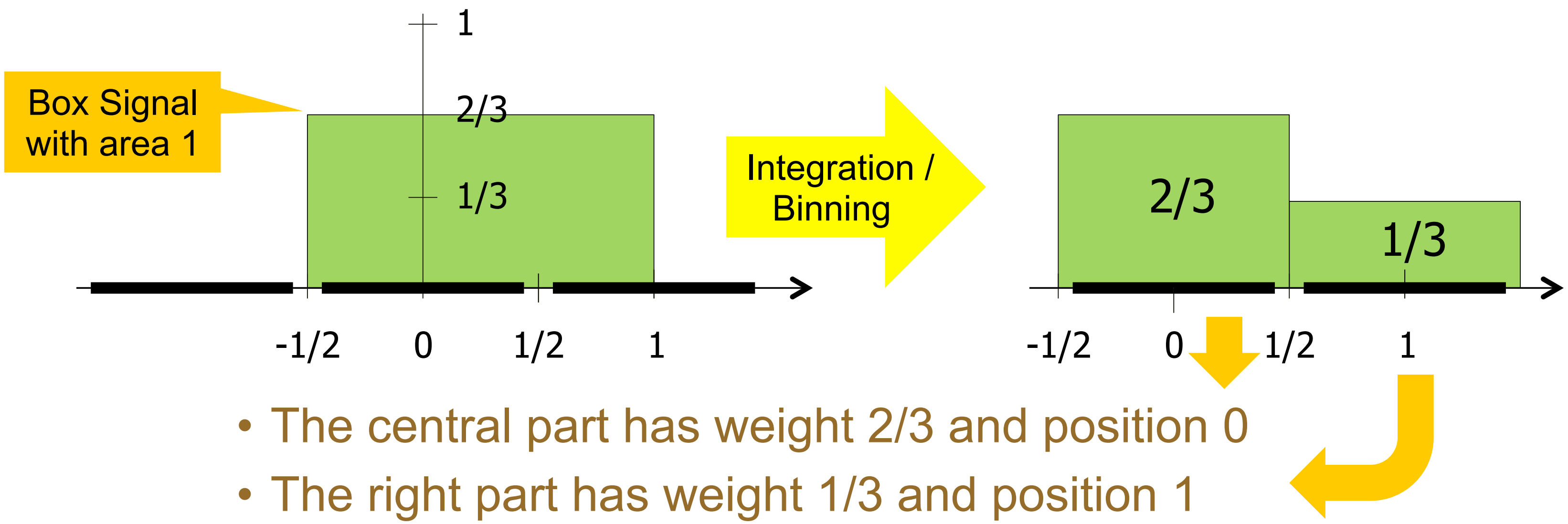
Plot this for f(x) = Box(x)

- Error is zero for integer box width.
- Behavior in-between is not trivial (see next slide)…



(Understanding the BOX-Behavior)

- Why does the error → 0 for wider Gauss while it is ≠ 0 also for wide boxes?
- We consider an example case:
  $a = 1$, $b = 1.5$, $x_c = 0.25$



- The central part has weight 2/3 and position 0
- The right part has weight 1/3 and position 1
  → reconstructed position is 1/3 and NOT 1/4

- This shows: the ‘central parts’ of the box carry no information, the edges are badly assigned to bins

The answer to our question

- When do we need a fit ?
- We ONLY need a fit when the signal is more narrow than the strips.
  - For a more quantitative assessment, use the formulae derived above…
- My impression is that people fit too much…

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LIMIT OF SPATIAL RESOLUTION FROM NOISE

The Question

- How is spatial resolution degraded by noise?
- We all 'know' $\sigma_{\mathrm{err}} = \kappa \cdot \sigma_{\mathrm{n}} = \dfrac{\kappa}{SNR}$.

  This states, that the resolution degrades with noise 'linearly to first order'.
- The proportionality κ is empirical. We want to *calculate* it
- We also want to check what happens with *correlated noise*
- We want to see what happens to *higher order*
  - What is this here? It is the distribution of the noise…
- We *assume* we can reconstruct with CoG (more later…)
- We restrict on a 1D treatment, but 2D is straight forward

1. Write down x_{rek} with noise

- A Signal at $\vec{x}$ is distributed over N strips at positions $\vec{x}_\mathrm{i}$
- Signal on i-th strip is $S_\mathrm{i}(\vec{x})$
- The sum of all signals shall be normalized to 1 ('trivial'):

$$\sum S_\mathrm{i} = 1$$

1

- ***Assume*** we can perfectly reconstruct the position as center of gravity:

$$\vec{x} = \frac{\sum S_\mathrm{i}\vec{x}_\mathrm{i}}{\sum S_\mathrm{i}} = \sum S_\mathrm{i}\vec{x}_\mathrm{i}$$

2

- ***Now*** assume noise n_i on all strips → signals are S_i+n_i
- The reconstructed position is:

$$\vec{x}_\mathrm{rek}(\vec{x}) = \frac{\sum (S_\mathrm{i} + n_\mathrm{i})\vec{x}_\mathrm{i}}{\sum (S_\mathrm{i} + n_\mathrm{i})} = \frac{\vec{x} + \sum n_\mathrm{i}\vec{x}_\mathrm{i}}{1 + \sum n_\mathrm{i}}$$

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2. Assume noise is small. Get the standard dev.

- This becomes (Taylor Expansion of Denominator):

$$\vec{x}_{\mathrm{rek}}(\vec{x}) = \frac{\vec{x} + \sum n_{\mathrm{i}} \vec{x}_{\mathrm{i}}}{1 + \sum n_{\mathrm{i}}} = \left(\vec{x} + \sum n_{\mathrm{i}} \vec{x}_{\mathrm{i}}\right)\left(1 - \sum n_{\mathrm{i}} + \mathcal{O}(n^2)\right)$$

- The reconstruction *error* is:

$$\vec{x}_{\mathrm{err}}(\vec{x}) = \sum_i n_{\mathrm{i}}(\vec{x}_{\mathrm{i}} - \vec{x}) + \mathcal{O}(n^2).$$

- We need the standard deviation:

$$\sigma^2_{\mathrm{err}} = \langle \vec{x}^{\,2}_{\mathrm{err}} \rangle - \langle \vec{x}_{\mathrm{err}} \rangle^2 = \langle \vec{x}^{\,2}_{\mathrm{err}} \rangle$$

Average error is zero!

We need to average over
- ALL possible positions
- ALL noises

3. Do the averaging

$$\vec{x}_{\mathrm{err}}(\vec{x}) = \sum_i n_{\mathrm{i}}(\vec{x}_{\mathrm{i}} - \vec{x}) + \mathcal{O}(n^2).$$

$$\begin{aligned}\sigma^2_{\mathrm{err}} &= \langle \vec{x}^{\,2}_{\mathrm{err}} \rangle \\ &= \sum_{i,j} \langle n_{\mathrm{i}} n_{\mathrm{j}} \rangle \langle (\vec{x}_{\mathrm{i}} - \vec{x})(\vec{x}_{\mathrm{j}} - \vec{x}) \rangle + \langle \mathcal{O}(n^3) \rangle \\ &= \sigma^2_{\mathrm{n}} \cdot \sum_i \langle (\vec{x}_{\mathrm{i}} - \vec{x})^2 \rangle + \mathcal{O}(\sigma^3_{\mathrm{n}})\end{aligned}$$

For **un**correlated noise

$$\langle n_{\mathrm{i}} n_{\mathrm{j}} \rangle = \delta_{ij} \cdot \sigma^2_{\mathrm{n}}$$

- If we chose the origin such that

$$\sum_i \vec{x}_{\mathrm{i}} = \vec{0}$$

this simplifies to:

This is the proportionality factor κ^2 we are looking for!

$$\boxed{\sigma^2_{\mathrm{err}} = \sigma^2_{\mathrm{n}} \left(\sum_{i=1}^{N} \vec{x}^{\,2}_{\mathrm{i}} + N \langle \vec{x}^{\,2} \rangle \right) + \mathcal{O}(\sigma^3_{\mathrm{n}}).}$$

3

Example: Strips

- Consider two strips at x_1 = -a/2 and x_2 = +a/2 (N = 2)
- Signals for a hit at x are

$$S_1(x) = (x_2 - x)/a \text{ and } S_2(x) = (x + x_2)/a$$

- 1 , 2 and 3 are fulfilled:

$S_1 + S_2 = 1$; $x_1 S_1 + x_2 S_2 = x$; $x_1 + x_2 = 0$

- We get $$\left(\frac{\sigma_{\mathrm{err}}}{\sigma_{\mathrm{n}}}\right)^2 \approx x_1^2 + x_2^2 + \frac{2}{a}\int_{x_1}^{x_2} x^2 dx = \frac{2}{3}a^2$$

- Or $$\boxed{\sigma_{\mathrm{err}} = 0.816 \cdot a \cdot \sigma_{\mathrm{n}}}$$

- For σ_n **= 0.1** (Signal/Noise = 10), **resolution = 8% · a**
- Resolution is better than optimal binary readout for S/N>5.6

Correlated Noise ?

- For FULLY correlated noise, $n_i == n_j$ and $\langle n_i n_j \rangle = \sigma_n^2$
- We get $\sigma_{err}^2 \approx \sigma_n^2 N^2 \langle \vec{x}^2 \rangle$
- For the strip example

$$\sigma_{err} = a\,\sigma_n / \sqrt{3} = 0.57\, a\,\sigma_n \quad \text{(instead of 0.816..)}$$

- Correlated noise is *less* harmful than ‘normal’ noise
- Note: For mixed noise, superimpose both components
- Note: If the Amplitude of the signal is *KNOWN* (X-ray), noise becomes correlated and resolution improves!

Higher Orders (in noise)

- Noise can have different distributions.
- They have different higher moments:

$$\begin{aligned}\langle n_i^2 \rangle &= \sigma_{\mathrm{n}}^2, \\ \langle n_i^4 \rangle &= \beta \cdot \sigma_{\mathrm{n}}^4.\end{aligned}$$

- They are $\beta := \dfrac{\int n^4\, p(n)\, dn}{\left(\int n^2\, p(n)\, dn\right)^2} = \begin{cases} 3 : \text{Gauss} \\ 9/5 : \text{Box} \\ 1 : \text{Peaks} \end{cases}$

- We need then higher order correlations (not trivial..):

$$\begin{aligned}\langle n_{\mathrm{i}} \rangle &= 0 \\ \langle n_{\mathrm{i}} n_{\mathrm{j}} \rangle &= \delta_{ij} \sigma_{\mathrm{n}}^2 \\ \langle n_{\mathrm{i}} n_{\mathrm{j}} n_{\mathrm{k}} \rangle &= 0 \\ \langle n_{\mathrm{i}} n_{\mathrm{j}} n_{\mathrm{k}} n_{\mathrm{l}} \rangle &= \delta_{ij}\delta_{jk}\delta_{kl}\,(\beta - 3)\,\sigma_{\mathrm{n}}^4 \quad + (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\,\sigma_{\mathrm{n}}^4\end{aligned}$$

Higher Orders

- Repeating the derivation yields

Previous result

$$\sigma_{\mathrm{err}}^2 = \sigma_{\mathrm{n}}^2 \cdot (A + N\,B) \cdot \left(1 + 3\left[\beta - 3 + N\frac{A + 3\,N\,B}{A + N\,B}\right]\sigma_{\mathrm{n}}^2\right)$$

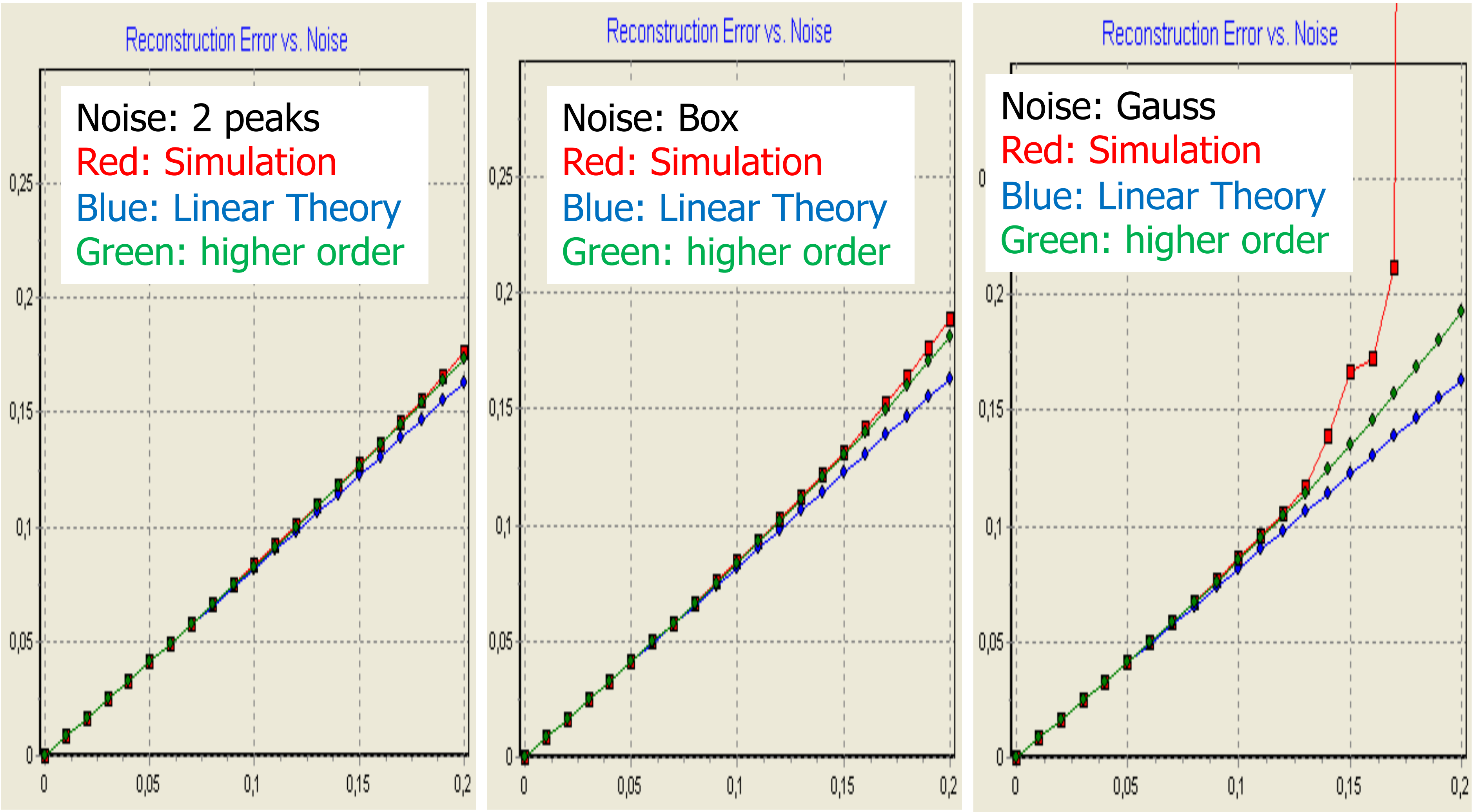
Correction

with

$$A := \sum_{i=1}^{N} \vec{x}_{\mathrm{i}}^{\,2} \quad \text{and} \quad B := \langle \vec{x}^{\,2} \rangle$$

- Only the *correction* depends on the ‘type’ (shape) of noise.
- NOTE:
  - For small noise, there is *no need* to simulate Gaussian noise
  - Randomly adding or subtracting $\pm\,\sigma_{\mathrm{n}}$ has the same effect!

Is this true? → Small Monte Carlo: Error vs. Noise



- Reconstruction for Gauss noise fails completely in few cases due to very high noise values

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2D Structures

- Can be treated similarly
- Observations:
  - Small number of electrodes is good
  - Well confined acceptance is good ('circle')

$$\sigma_{\rm err}^2 = \sigma_{\rm n}^2 \left(\sum_{i=1}^{N} \vec{x}_{\rm i}^{\,2} + N \langle \vec{x}^{\,2} \rangle \right) + \mathcal{O}(\sigma_{\rm n}^3).$$

Geometry	$\sigma_{1\mathrm{D}}$ theory linear	correction	numerical (A=p=1)
strips	$\sqrt{\frac{2}{3}} \cdot p$	$\sqrt{1 + 3\beta\sigma_{\rm n}^2}$	0.8165
square	$\frac{2}{\sqrt{3}} \cdot \sqrt{A}$	$\sqrt{1 + 3(3+\beta)\sigma_{\rm n}^2}$	1.1547
hexagon	$\frac{\sqrt{5} \cdot 3^{1/4}}{6} \cdot \sqrt{A}$	$\sqrt{1 + 3\left(\frac{6}{5} + \beta\right)\sigma_{\rm n}^2}$	0.4905

- Hexagons are best (least sensitive to noise!)

BACK TO GOG
NOW WITH NOISE

Problems with Centroid

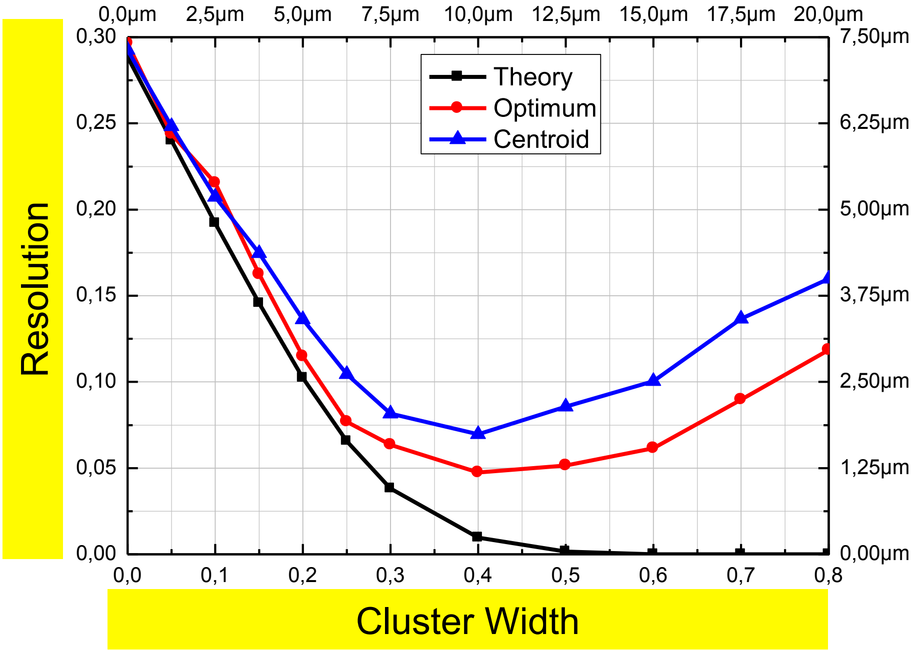
- Resolution for small σ is bad $\rightarrow$ better make f(x) wide
- BUT: Summing up many strips (large N) increases noise
- Must chose N small but such that reconstruction is 'just' ok.
  - Obviously N ~ σ
- The choice is fairly arbitrary
- And:
  - In real system, there is often a **threshold** (hits below this are not read out)
  - The reconstructed amplitude is wrong (signals below threshold are lost)
  - Broken pixels need special treatment

Monte Carlo Simulation

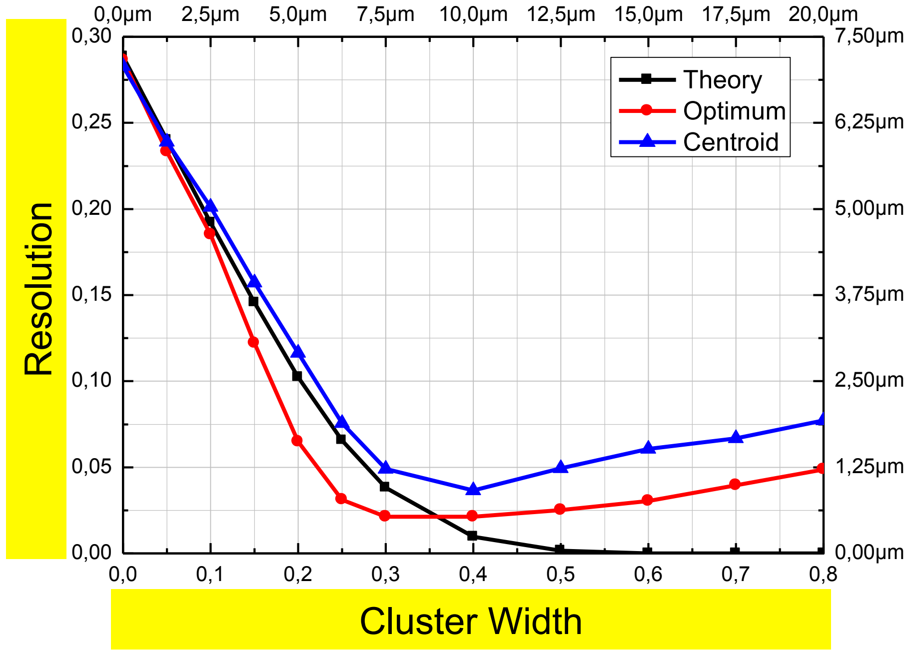
- Many possibilities… I do not go in details

Ignore red curves…

S/N = 40



S/N = 80



- The **optimum signal width is still close to FWHM = a!**

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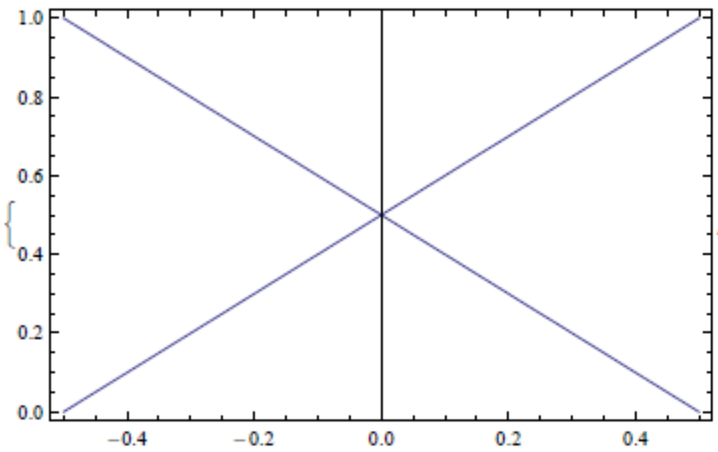
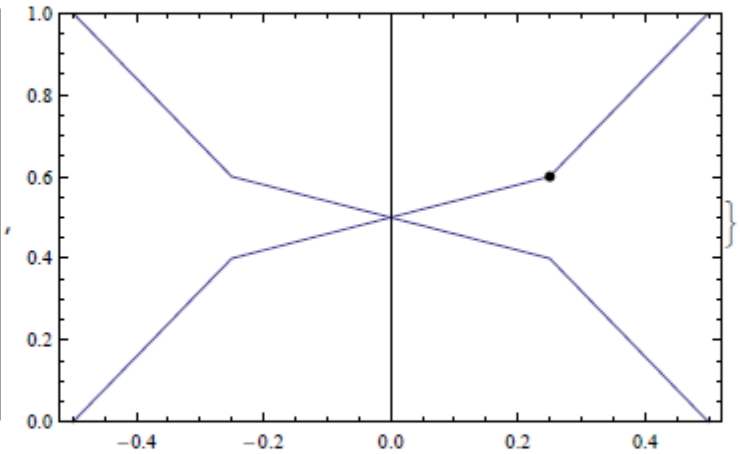
ETA FUNCTION

Motivation

- Often the Signals Distribution function (e.g. on 2 strips) is not linear.
- This is related to the 'famous' eta-function.



Linear Charge Sharing



Arbitrary function

- The position then cannot be calculated by GoG, but by using the inverse function (or the 'eta'-lookup table)
- Question: How does resolution depend on f(x)?

1D Case: Reconstruction with Inverse Function

- The signals on the two strips shall be

$$\begin{aligned} S_1(x) &= Q\, f(x) \\ S_2(x) &= Q - S_1(x) = Q\ (1 - f(x)) \end{aligned}$$

  (we assume no signal is lost, i.e. we require $S_1+S_2 = Q$

- We require
  - f(x) is strictly monotonic (obvious)
  - f(x) shall be symmetric in x (may not always be the case)

- Obviously

$$x_{rek} = f^{-1}\left[\frac{S_1}{S_1 + S_2}\right]$$

Adding Noise

- With Noise on S_1 and S_2 we get

$$x_{rek} = f^{-1}\left[\frac{f(x)+n_1}{1+n_1+n_2}\right]$$

$$\approx f^{-1}\left[(f(x)+n_1)(1-n_1-n_2)\right]$$

$$\approx f^{-1}\left[f(x)+n_1(1-f(x))-n_2 f(x)\right]$$

$$\approx x + \boxed{\left.\frac{\mathrm{d}f^{-1}(s)}{\mathrm{d}s}\right|_{f(x)}} \cdot \left[n_1(1-f(x))-n_2 f(x)\right]$$

$$x_{err} = \frac{n_1(1-f(x))-n_2 f(x)}{\boxed{f'(x)}}$$

Add noise

Taylor (as before)

1^{st} order in noise

Taylor Series for f^{-1} around f(x)

A 'forgotten' math theorem: The derivative of the inverse function is the inverse of the derivative

Sigma – Averaging over Noise

- To get

$$x_{err} = \frac{n_1(1 - f(x)) - n_2 f(x)}{f'(x)}$$

$$\sigma_{err}^2 = \langle x_{err}^2 \rangle - \cancel{\langle x_{err} \rangle}^2$$

- we average first over noise. We get

$$\sigma_{err}^2 = \sigma_n^2 \left\langle \frac{1 - 2f + 2f^2}{f'^2} \right\rangle + 2 \langle n_1 n_2 \rangle \left\langle \frac{f^2 - f}{f'^2} \right\rangle$$

  - Coefficients depend on the shape of the response function
  - They are small where the response function is steep (obvious..)
  - Vice versa: Flat parts in eta are bad.

- For *uncorrelated* noise, only the first term matters

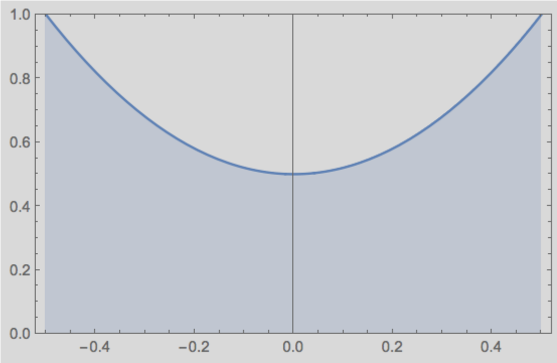
$$\frac{\sigma_{err}^2}{\sigma_n^2} = \left\langle \frac{1 - 2f + 2f^2}{f'^2} \right\rangle$$

Average over position

Back to linear Interpolation

- What does this mean for linear interpolation, f(x) = x+0.5 ?
- Let us first look at the *position dependent* error

$$\frac{\sigma^2_{err}}{\sigma^2_n} = \frac{1 - 2f + 2f^2}{f'^2} = \frac{1}{2} + 2x^2$$



- This is NOT constant. It *double*s at the edges !!!

- The average error is

$$\frac{\sigma^2_{err}}{\sigma^2_n} = \int_{-1/2}^{1/2} \left(\frac{1}{2} + 2x^2 \right) = \frac{2}{3}$$

as before. (s. page 30)

Finding New Distribution functions

- Very exciting: Can we find a f(x) such that the integral is ***better*** than with linear interpolation
  - Probably not (?) But let's see…

- Easier: Can we find a distribution function so that the error is *independent of position*?
- One line of Mathematica is enough:

$$f_{flat}(x) = \frac{1}{2}\left(1 - \mathrm{Sinh}[2x\mathrm{ArcSinh}(1)]\right)$$

The average σ^2 is 0.643, which is (a little bit) ***better*** than 2/3=0.66 !!

We found a distribution which is better than linear interpolation!
(it is less noise sensitive)

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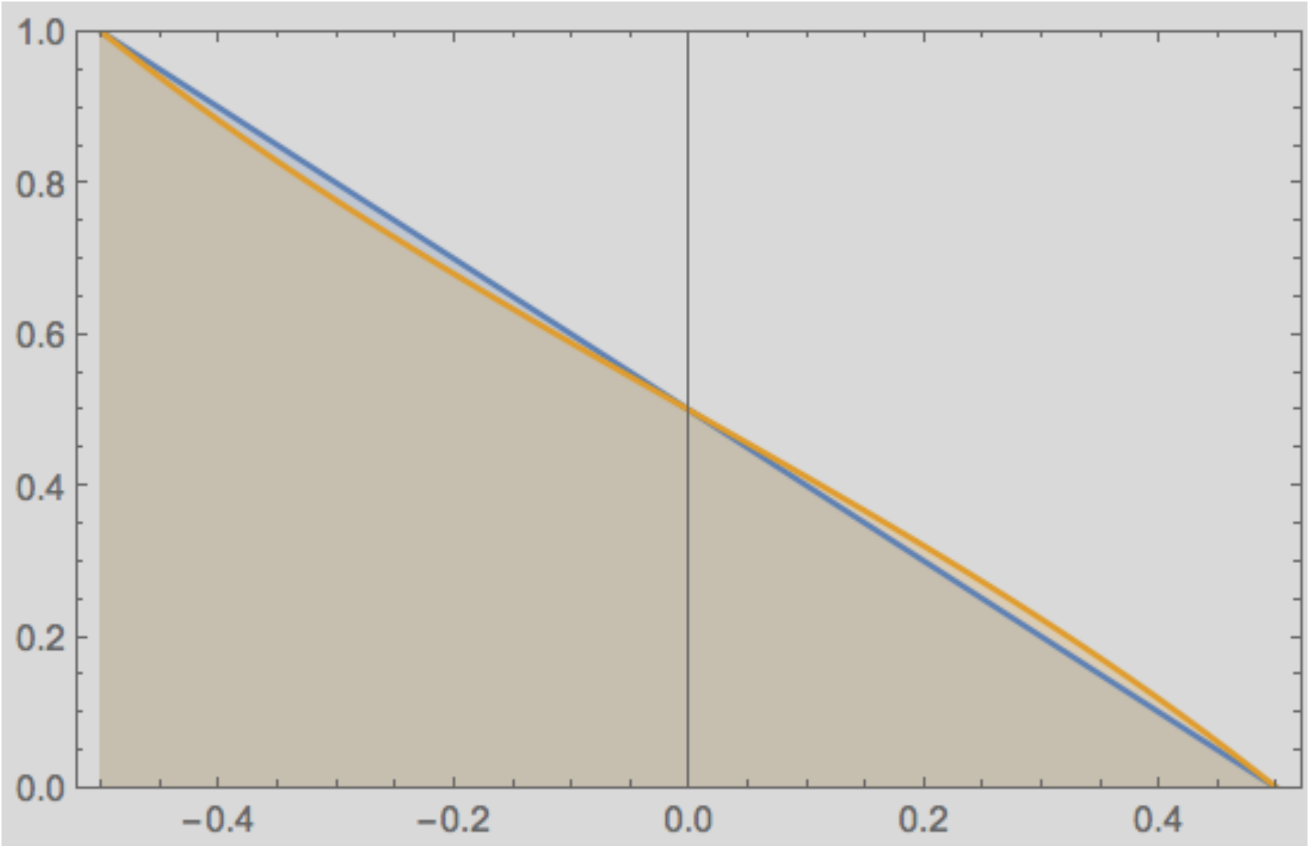
Better! (but just a little…)



- Are there better functions???

Summary: What did we learn ?

- Basic Algebra is fun…..

- CoG is 'perfect' as soon as signal width >≈ strip width
- Wider (too wide) signals are more sensitive to noise
- Ideal κ for strips is 0.816
- Analogue readout for S/N<6 is useless.
- Noise shape (distribution) does not matter for S/N > 10
- Correlated noise is less harmful
- Hexagons have better res. and are less sensitive to noise
- Linear interpolation has more error at the edges
- There ***is*** a better reconstruction function than linear
  - but the difference is negligible….
  - I did not find better so far…

Thank you for your attention!